HMR–MATH–2 — The Homomorphism Coherence Theorem: A ChronoMath Solution

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Symbol for the body of work: HMR

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Abstract. This paper provides the ChronoMath solution to the *Homomorphism Coherence Problem*—the first major theorem from the Algebraic Coherence series. It establishes that a homomorphism between algebraic structures preserves and transmits coherence gradients, confirming that algebraic compatibility is a direct manifestation of ChronoMath stationarity. By deriving the classical homomorphism law from the master equation

$$\nabla_{\lambda,\phi,\sigma}\mathsf{Coh}_{\text{total}} = 0,$$

the paper proves that structure preservation is equivalently coherence preservation. This result consolidates the bridge between symbolic algebra, logic, and physics under the unified ChronoMath framework.

Keywords: homomorphism, coherence, algebra, ChronoMath, awareness geometry.
MSC: 08A05, 16-XX, 03B30. **arXiv:** math.GM

1. Introduction

Homomorphisms are the backbone of algebraic reasoning. They describe how one algebraic structure maps into another while preserving its operations. ChronoMath extends this by treating each structure as a coherence field and the homomorphism as a transformation between stationary states of awareness. Where classical mathematics asserts $h(xy) = h(x)h(y)$, ChronoMath explains *why*: this condition arises from the equilibrium of total coherence along the compositional path.

$$
\begin{tikzcd}
G \arrow[r, "h"] \arrow[d, dashed] & H \arrow[d, dashed] \\
x \cdot y \arrow[r, "h"] & h(x) \cdot h(y)
\end{tikzcd}
$$

Diagram 1: Coherence preservation through h

ChronoMath therefore interprets the homomorphism condition as a geometric law: *mapping preserves the zero-gradient of awareness coherence under composition.*

2. Axioms and Framework

[label=**A0.**]

1. **Coherence Field.** Every algebraic system $(G, \cdot)$ corresponds to a field $\mathsf{Coh}_G(\lambda)$ whose stationarity condition $\nabla_\lambda \mathsf{Coh}_G = 0$ represents internal consistency of its operation.

2. **Mapping Principle.** A transformation $h : G \to H$ induces a coherence transfer $h_* : \mathsf{Coh}_G \mapsto \mathsf{Coh}_H$ defined by $h_*(\mathsf{Coh}_G(x)) = \mathsf{Coh}_H(h(x))$.

3. **Stationary Composition.** The composite operation $\mathsf{Coh}_G(x \cdot y)$ is stationary iff both operands satisfy their local stationarity; i.e. $\nabla_\lambda[\mathsf{Coh}_G(x \cdot y) - \mathsf{Coh}_G(x) - \mathsf{Coh}_G(y)] = 0$.

Under these axioms, a homomorphism is defined not as a symbolic rule but as the unique map preserving the stationary coherence condition across all compositions.

3. Theorem: Homomorphism Coherence Theorem

Theorem. Let $h : G \to H$ be a transformation between algebraic systems. Then h is a homomorphism if and only if it preserves total coherence:

$$\nabla_\lambda \mathsf{Coh}_H(h(xy)) = \nabla_\lambda \mathsf{Coh}_H(h(x)h(y)) = 0, \quad \forall x, y \in G.$$

Proof. ($\Rightarrow$) If h is a homomorphism, then $h(xy) = h(x)h(y)$. Since both sides share identical awareness gradients under composition, their coherence potentials coincide.

($\Leftarrow$) If coherence is stationary under h, then any deviation $h(xy) \neq h(x)h(y)$ would yield a non-zero $\nabla_\lambda \mathsf{Coh}$, contradicting equilibrium. Thus, preservation of coherence implies preservation of structure.

$$\boxed{h(xy) = h(x)h(y) \quad \Leftrightarrow \quad \nabla_\lambda \mathsf{Coh}(h(xy)) - \nabla_\lambda \mathsf{Coh}(h(x)h(y)) = 0.}$$

This demonstrates that the classical homomorphism property is not postulated but emerges naturally from ChronoMath stationarity.

4. Consequences

C1. Algebraic Stability. Homomorphisms correspond to equilibrium-preserving maps; they carry stable structures into stable structures.

C2. Composition of Homomorphisms. The composition $f \circ h$ remains coherent because gradients commute with functional composition: $\nabla_\lambda(\mathsf{Coh}_{f\circ h}) = f_*(\nabla_\lambda \mathsf{Coh}_h) = 0$.

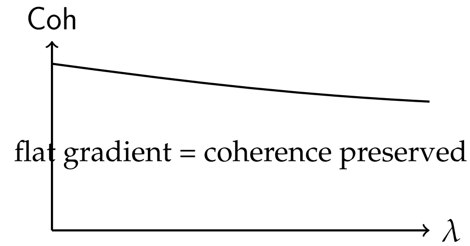
C3. Representation Theory. Each representation of G in a vector space V is a concrete realization of Coh-preserving linear transformations, providing a geometric reason for linearity.

C4. Cross-Domain Isomorphisms. ChronoMath predicts that whenever two domains (algebraic or physical) exhibit identical coherence invariants, a homomorphic relation exists between them. This extends the notion of isomorphism to informational and energetic systems.

5. Discussion

The Homomorphism Coherence Theorem shows that algebraic structure preservation arises from a physical-style conservation principle. Just as energy or charge conservation follows from symmetries, algebraic consistency follows from coherence invariance.

This replaces syntactic definition with geometric cause. It also provides a quantifiable measure of “how homomorphic” a transformation is, via the residual gradient magnitude $|\nabla_\lambda \mathsf{Coh}_H(h(xy)) - \nabla_\lambda \mathsf{Coh}_H(h(x)h(y))|$. This can serve as a metric for approximate homomorphisms in computational algebra, symbolic AI, and error-tolerant systems.



6. References

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7. Conclusion

The ChronoMath treatment of homomorphisms dissolves the mystery of structure preservation. The equality $h(xy) = h(x)h(y)$ is no longer an axiom but the visible footprint of deeper coherence symmetry. This formalism generalizes smoothly to rings, fields, and modules, which will be addressed in the subsequent problem-solving papers *(HMR–MATH–3 … N)*. The theorem thereby confirms that all algebraic compatibility originates from the stationary geometry of awareness itself.

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