EC414 Homework 1: Probability and Decision Rules

by Natalia Frumkin with the help of B. Kulis, A. Tsiligkaridis, R. Manzelli, and K. Chauhan

Due: Wednesday, Feb. 6th at 2:30pm

Words of wisdom: Begin early and show your work. You will get credit for the problems where you show work. Even if you get to the right answer, if the steps do not make logical sense, you will be deducted points. Good luck!

1 Expectation and Covariance

We are given X $\sim$ Poisson(λ) and Y $\sim$ Uniform(a,b) sampled IID from known distributions. We define:

$$A = 2X+Y$$
$$B = X-2Y$$

(a) What is E[A] and E[B]?

(b) What is var(A) and var(B)? Helpful identity: $E[X^2] = \lambda^2 + \lambda$

(c) What is covariance and what is correlation? Explain in your own words.

(d) What is the covariance of A and B?

(e) Define independence of two random variables. Are A and B independent? Explain.

2 Joint Gaussians and Conditional Distributions

We are given jointly gaussian random variables X and Y with means μ_x and μ_y and variances σ_x and σ_y. The correlation coefficient between the two random variables is ρ.

(a) Write the covariance matrix in terms of σ_x, σ_y, and ρ.

(b) Given $\mu_x = 0$ and $\sigma_x = 1$, write out the marginal distribution $f_x(x)$. Hint: What do we need in order to specify a marginal distribution? This is a conceptual question.

(c) Given that $\mu_x = 0$, $\mu_y = 0$, $\sigma_x = 1$, $\sigma_y = 1$, and ρ=0.25, what is the conditional distribution $f_{Y|X}(y|x)$?

Hint: There are two helpful properties of joint gaussian conditionals mean and variances as described here: `https://en.wikipedia.org/wiki/Multivariate_normal_distribution#Conditional_distributions`

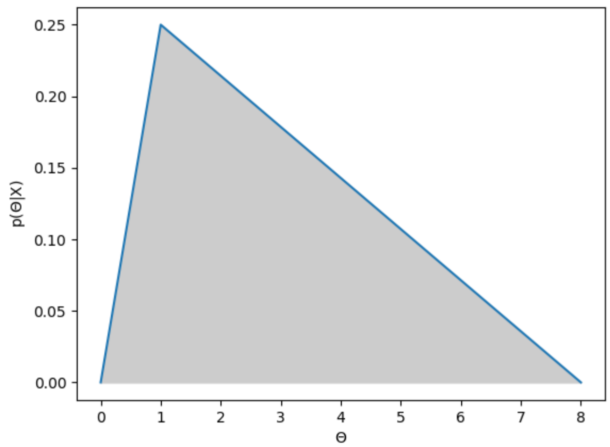
3 ML and MAP Estimates

(a) State the ML and MAP estimates of a random variable X and parameters θ. In your own words, describe the fundamental difference between the ML and MAP estimates.

(b) Given the pmf below, what is the MAP estimate? Use interval notation.

(c) What is the MMSE?

(d) What is the ML estimate if the prior, P(θ), is uniform(0,8)?



4 Closed-Form Maximum Likelihood

Assume that we are given n iid samples $(x_1, ..., x_n)$ from each $P(X \mid \theta)$ given below. Compute the maximum likelihood estimates (MLEs) for the parameter θ of the given distributions.

(a) $P(X \mid \theta) = \frac{2x}{\theta^2} e^{\frac{-x^2}{\theta^2}}$ for $x \geq 0$

(b) $P(X \mid \theta) = N(\theta, 1)$

(c) $P(X \mid \theta) = \text{Poisson}(\theta)$

5 Decision Rules: Empirical Risk Minimization

Let's say we are given a dataset of iid features x_j and labels y_j . Beginning with the basic formula for the MAP decision rule, derive the empirical risk minimization formula:

$$\arg\min_\theta \left[\frac{1}{n}\sum_{j=1}^{n} -ln(p(x_j, y_j|\theta)) - \frac{1}{n} ln(p(\theta))\right]$$

Be complete in your derivation. You should have a minimum of 5 logical steps and write a brief phrase next to each step explaining what basic principle that step is using.

logical step example:	$\arg\min_\theta \left[P(X)\right] = \arg\min_\theta \left[\ln P(X)\right]$	log transformation