$X \sim \text{Uniform}(a,b)$

$E[X]$?

$$E[X] = \int_{-\infty}^{\infty} x \cdot P(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{x^2}{2}\left(\frac{1}{b-a}\right)\Bigg|_a^b$$

$$= \frac{b^2}{2(b-a)} - \frac{a^2}{2(b-a)} = \frac{b^2-a^2}{2(b-a)} = \frac{(b-a)(b+a)}{2(b-a)} = \boxed{\frac{b+a}{2}}$$

$Var(X)$?

$$Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot P(x)\, dx = \int_a^b (x-\mu)^2 \frac{1}{(b-a)}\, dx$$

$$= \frac{1}{b-a}\int_a^b \left(x - \frac{(b+a)}{2}\right)^2 dx = \frac{1}{b-a}\int_a^b x^2 - (b+a) + \left(\frac{b+a}{2}\right)^2 dx$$

$$= \frac{1}{b-a}\int_a^b \left[x^2 - (b+a) + \frac{(b+a)^2}{4}\right] dx = \frac{1}{b-a}\left(\frac{x^3}{3} - (b+a)x + \frac{(b+a)^2}{12}x\right)\Bigg|_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - (b+a)b + \frac{(b+a)^2 b}{12} - \frac{a^3}{3} + (b+a)a - \frac{(b+a)^2 a}{12}\right]$$

$$= \frac{1}{b-a}\left[\frac{b^3}{3} - b^2 - ab + \frac{b(b+a)^2}{12} - \frac{a^3}{3} + ba + a^2 - \frac{a(b+a)^2}{12}\right]$$

$\vdots$

$$= \boxed{\frac{(b-a)^2}{12}}$$

$$f_{xy}(x,y) = \begin{cases} x + \frac{3}{2}y^2 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$$

$$P(0 \le x \le \tfrac{1}{2},\ 0 \le y \le \tfrac{1}{2}) = \int_0^{1/2}\int_0^{1/2} f_{xy}(x,y)\,dx\,dy$$

$$= \int_0^{1/2}\int_0^{1/2} \left[x + \frac{3}{2}y^2\right] dx\,dy$$

$$= \int_0^{1/2} \frac{x^2}{2} + \frac{3}{2}xy^2 \Big|_0^{1/2} dy$$

$$= \int_0^{1/2} \frac{1}{8} + \frac{3}{4}y^2\,dy$$

$$= \frac{1}{8}y + \frac{y^3}{4} \Big|_0^{1/2}$$

$$= \frac{1}{16} + \frac{1}{32} = \boxed{\frac{3}{32}}$$

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$$

$$= \int_0^1 x + \frac{3}{2}y^2\,dy$$

$$= xy + \frac{y^3}{2} \Big|_0^1$$

$$= \boxed{x + \frac{1}{2}}$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx$$

$$= \int_0^1 x + \frac{3}{2}y^2\,dx$$

$$= \frac{x^2}{2} + \frac{3y^2}{2}x \Big|_0^1$$

$$= \boxed{\frac{1}{2} + \frac{3y^2}{2}}$$

Given iid coin tosses $x_1 \dots x_n \sim \text{Bernoulli}(q)$

What is the MLE?

$$\text{MLE}(\theta) = \underset{\theta}{\text{argmax}}\ P(x_1 \dots x_n \mid \theta)$$

$$= \underset{\theta}{\text{argmax}} \prod_{i=1}^{n} P(x_i \mid \theta) \qquad x_i \text{ independent } \forall x_i \in x_1 \dots x_n$$

$$= \underset{q}{\text{argmax}} \prod_{i=1}^{n} q^{x_i} (1-q)^{1-x_i}$$

$$= \underset{q}{\text{argmax}} \sum_{i=1}^{n} \left[\log(q)^{x_i} + \log(1-q)^{1-x_i} \right]$$

$$= \underset{q}{\text{argmax}} \sum_{i=1}^{n} \left[x_i \log(q) + (1-x_i) \log(1-q) \right]$$

$$= \underset{q}{\text{argmax}}\ \log(q) \sum_{i=1}^{n} x_i + \log(1-q) \sum_{i=1}^{n} (1-x_i)$$

~ take derivative and ~
set to 0

$$\frac{1}{q} \sum_{i=1}^{n} x_i + \frac{-1}{1-q} \sum_{i=1}^{n} 1-x_i = 0$$

$$\boxed{\hat{q}_{ML} = \frac{\sum_{i=1}^{n} x_i}{n}}$$

given iid samples $x_1 \dots x_n \sim \text{Exponential}(\beta)$ what is the MLE?

$$\underset{\theta}{\text{argmax}}\ P(x_1 \dots x_n \mid \theta)$$

$$= \underset{\beta}{\text{argmax}} \prod_{i=1}^{n} P(x_i \mid \beta)$$ independent samples

$$= \underset{\beta}{\text{argmax}} \prod_{i=1}^{n} \frac{1}{\beta} e^{-\frac{x_i}{\beta}}$$ $0 \leq x < \infty$

$$= \underset{\beta}{\text{argmax}} \sum_{i=1}^{n} \log\left(\frac{1}{\beta} e^{-\frac{x_i}{\beta}}\right)$$ $\log\left(\prod_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} \log(x_i)$

$$= \underset{\beta}{\text{argmax}} \sum_{i=1}^{n}\left(\log\left(\frac{1}{\beta}\right) + \log\left(e^{-\frac{x_i}{\beta}}\right)\right)$$ "

$$= \underset{\beta}{\text{argmax}} \sum_{i=1}^{n}\left(\underbrace{\log(1)}^{0} - \log(\beta) - \frac{x_i}{\beta}\right)$$ $\log\left(\frac{x_1}{x_2}\right) = \log(x_1) - \log(x_2)$

$\log(e^{x}) = x$

$$= \underset{\beta}{\text{argmax}} -n\log\beta - \sum_{i=1}^{n} \frac{x_i}{\beta}$$

$$= \underset{\beta}{\text{argmax}} -n\log\beta - \frac{1}{\beta}\sum_{i=1}^{n} x_i$$ *

~ take derivative ~

$$-n\left(\frac{1}{\beta}\right) - \left(\frac{-1}{\beta^2}\right)\left[\sum_{i=1}^{n} x_i\right] = 0$$

$$-\frac{n}{\beta} + \left(\frac{1}{\beta^2}\right)\sum_{i=1}^{n} x_i = 0$$

$$\left(\frac{1}{\beta^2}\right)\sum_{i=1}^{n} x_i = \frac{n}{\beta}$$

$$\frac{1}{\beta}\sum_{i=1}^{n} x_i = n$$

$$\hat{\beta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i$$