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Solving Kinematic Equations for θ

We are trying to aim a projectile at a target. Assume we are in 2 dimensional space (higher dimensional cases can be simplified to the 2 dimensional one.) Then let $\langle 0, 0\rangle$ be the initial position of the projectile and let $\mathbf{x}_t = \langle x_t, y_t\rangle \in [0, \infty)^2$ be the position of the target. The projectile has a launch speed of $v_i \in \mathbb{R}^{>0}$, and the acceleration due to gravity is $a \in \mathbb{R}^{>0}$. The projectile is launched at some angle $\theta \in \left[0, \frac{\pi}{2}\right]$. Let $\mathbf{x}_\theta(t) : [0, \inf) \to \mathbb{R}^2$ be the path of the projectile. We want the largest theta such that there exists a $t : \mathbf{x}_\theta(t) = \mathbf{x}_t$.

The equation for $\mathbf{x}_\theta(t)$ is below:

$$\mathbf{x}_\theta(t) = \left\langle v_i t\cos\theta, -\frac{a}{2}t^2 + v_i t\sin\theta\right\rangle$$

Letting $\mathbf{x}_\theta(t) = \mathbf{x}_t$:

$$\langle x_t, y_t\rangle = \left\langle v_i t\cos\theta, -\frac{a}{2}t^2 + v_i t\sin\theta\right\rangle$$

This gives us two equations. Our two unknowns are t and θ, but we only care about θ. So, let's write t in terms of θ using the first equation:

$$t = \frac{x_t}{v_i\cos\theta}$$

Therefore,

$$y_t = -\frac{a}{2}\left(\frac{x_t}{v_i\cos\theta}\right)^2 + x_t\tan\theta$$

$$y_t\cos^2\theta = -\frac{a}{2}\left(\frac{x_t^2}{v_i^2}\right) + x_t\sin\theta\cos\theta$$

$$y_t\cos^2\theta - x_t\sin\theta\cos\theta + \frac{ax_t^2}{2v_i^2} = 0$$

Let $C = \frac{ax_t^2}{2v_i^2}$. Then the desired value of θ is a root of f, where

$$f(\theta) = y_t\cos^2\theta - x_t\sin\theta\cos\theta + C$$

Finding a root

We can approximate the above equation using Taylor polynomials:

$$f(\theta) \approx y_t\left(1 - \frac{\theta^2}{2}\right)^2 - x_t\theta\left(1 - \frac{\theta^2}{2}\right) + C$$

$$= \frac{y_t}{4}\theta^4 + \frac{x_t}{2}\theta^3 - y_t\theta^2 - x_t\theta + (y_t + C)$$

This is a quartic equation, which has calculatable roots. We can then use the largest root in $\left[0, \frac{\pi}{2}\right]$ as a guess for the root of f and use Newton's method to find an approximation within the requested tolerance. If a root is found, we have our angle. If no root is found, the target must be out of range. Either way, we are done.