icoFoamH - incompressible solver on a C-grid with a Hodge operator

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The continuous equations solved are the incompressible Euler equations

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\cdot(\mathbf{u}\mathbf{u}) = -2\mathbf{\Omega}\times\mathbf{u} - \nabla p \tag{1}$$

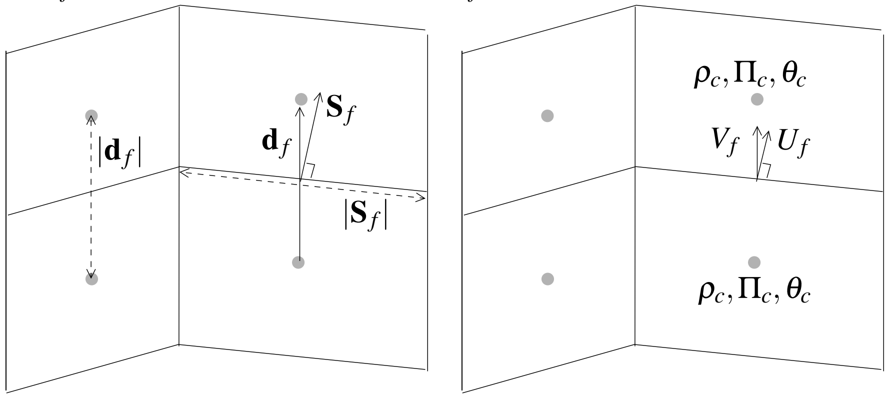
subject to the constraint

$$\nabla\cdot\mathbf{u} = 0 \tag{2}$$

where $\mathbf{u}$ is the velocity and p is the pressure. The numerical solution uses the prognostic variable

$$V_f = \mathbf{u}_f\cdot\mathbf{d}_f \tag{3}$$

where $\mathbf{u}_f$ is the velocity on face f and $\mathbf{d}_f$ is the vector between cell centres either side of face f.



(a) 2D projection of geometry (b) Placement of variables

We also define a diagnostic variable for the volumetric flux across face f:

$$U_f = \mathbf{u}\cdot\mathbf{S}_f \tag{4}$$

where $\mathbf{S}_f$ is the normal vector to face f with magnitude equal to the area of face f. The pressure, p, is an auxiliary variable.

We use the convection that U is a vector of all values of U_f on all faces and V is a vector of all values of V_f.

The numerical solution uses projection method: the momentum equation (1) is solved without the pressure gradient term to find an intermediate value of the velocity, then this velocity is projected into divergence free space by solving a pressure equation and then adding the pressure gradient term to the intermediate velocity. To discretise the momentum equation (1), we take the dot product with $\mathbf{d}$:

$$\frac{\partial V}{\partial t} + (\nabla\cdot(\mathbf{u}\mathbf{u}))\cdot\mathbf{d} = -(2\mathbf{\Omega}\times\mathbf{u})\cdot\mathbf{d} - \nabla_d p \tag{5}$$

where $\nabla_d p = \mathbf{d} \cdot \nabla p$ so that the simplest discretisation of $\nabla_d p$ is simply the difference between the two values of the pressure, p, either side of the face. We can now discretise in time using trapezoidal-implicit time-stepping with deferred correction of explicit terms:

$$V^{n+1} = V^n + (1-\alpha)\Delta t \left(\frac{\partial V}{\partial t}\right)^n - \alpha\Delta t \left\{ \left(\nabla\cdot(\mathbf{uu})^\ell\right)\cdot\mathbf{d} + \left(2\mathbf{\Omega}\times\mathbf{u}^\ell\right)\cdot\mathbf{d} + \nabla_d p^{n+1} \right\} \tag{6}$$

where Δt is the time-step, α is the off-centering parameter and ℓ represents values of variables at the most recent iteration within each time-step but not at the latest time since they are not solved for implicitly. Next we define an intermediate value, V^i, (which shares the same storage as V), using explicitly defined values:

$$V^i = V^n + (1-\alpha)\Delta t \left(\frac{\partial V}{\partial t}\right)^n - \alpha\Delta t \left(\nabla\cdot(\mathbf{uu})^\ell\right)\cdot\mathbf{d}. \tag{7}$$

Variables V, V^i and $\partial V/\partial t$ share the same boundary conditions at zero flux boundaries (ie they are all set to zero at zero flux boundaries). Next we apply the Hodge operator, H, to find the intermediate value of the flux, before pressure gradients are applied:

$$U^i = HV^i - \alpha\Delta t\, 2\left(\mathbf{\Omega}\times\mathbf{u}^\ell\right)\cdot\mathbf{S}. \tag{8}$$

$$= HV^i - \alpha\Delta t\, 2\left(\mathbf{S}\times\mathbf{\Omega}\right)\cdot\mathbf{u}^\ell. \tag{9}$$

Alternatively, if we had applied the Hodge operator to V^{n+1} from eqn (6) we would get an equation for the flux:

$$U^{n+1} = U^i - \alpha\Delta t\, H\nabla_d p^{n+1}. \tag{10}$$

We require the velocity field to be divergence free at every time-step, ie $\nabla\cdot U^{n+1} = 0$ where $\nabla\cdot U^{n+1}$ represents the discrete calculation of $\nabla\cdot\mathbf{u}$ and is calculated as:

$$\nabla\cdot U = \frac{1}{v_c}\sum_{f\in c} U_f \tag{11}$$

where v_c is the volume of cell c and $f\in c$ means all the faces, f, of cell c. Therefore, substituting eqn (10) into $\nabla\cdot U = 0$ gives:

$$\nabla\cdot U^i - \alpha\Delta t\nabla\cdot H\nabla_d p^{n+1} = 0. \tag{12}$$

This is the pressure equation, a Helmholtz equation that can be solved implicitly for p. In order to simplify the construction of the matrix for the implicit solution of (12), we can separate $H\nabla_d p$ into diagonal and off-diagonal parts:

$$H\nabla_d p = H_d\nabla_d p + H_{off}\nabla_d p \tag{13}$$

$$= \frac{|\mathbf{S}|}{|\mathbf{d}|}\nabla_d p + H_{off}\nabla_d p \tag{14}$$

and only solve the diagonal part implicitly. Then U^i can be re-defined to include the off diagonal part of the Hodge operator:

$$U^i = HV^i - \alpha\Delta t\, 2\left(\mathbf{S}\times\mathbf{\Omega}\right)\cdot\mathbf{u}^\ell - \alpha\Delta t\, H_{off}\nabla_d p^\ell \tag{15}$$

leaving the pressure equation simpler:

$$\nabla\cdot U^i - \alpha\Delta t\nabla\cdot\frac{|\mathbf{S}|}{|\mathbf{d}|}\nabla_d p^{n+1} = 0. \tag{16}$$

Equation (16) is solved implicitly for p and then we can calculate divergence free velocity components, U and V (the back-substitution step):

$$V^{n+1} = V^i - \alpha\Delta t \left\{ \left(2\mathbf{\Omega} \times \mathbf{u}^{\ell}\right) \cdot \mathbf{d} + \nabla_d p^{n+1} \right\} \tag{17}$$

$$U^{n+1} = U^i - \alpha\Delta t \, H \nabla_d p^{n+1}. \tag{18}$$

In order to ensure no flow at boundaries, geostrophic balance can be imposed, ie:

$$\nabla_d p = \left(2\mathbf{\Omega} \times \mathbf{u}^{\ell}\right) \cdot \mathbf{d} \tag{19}$$

at boundaries.

References