

# The study of the dynamics of quantum bipartite entangled systems using Feynman path integrals

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*This version is not to be published. It is my own notes during the research.*

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# Introduction

## 1.1 Historical backgrounds<sup>1</sup>

Light, the first gateway to the quantum world of the humankind. I've always been intrigued by it. And, we've been able to confirm that light can be entangled, causing correlations in its polarization axes. Most modern development of physics has been into that field. But often, we tend to forget what we left behind four hundred years ago: the principle of least action. What happens when light passes through different medium? Refraction, everyone can answer that. But what happens if light is entangled?

To resolve this problem, I originally turned to the Schrödinger's equation for two particles. However, it's quite complex, and the mathematics behind it doesn't really elude me that much. So, I turned to the path integrals method. Its ability to intuitively produce the classical world, governed by the principle of least action, is captivating. The action principle appears so naturally in path integrals that I thought I was hallucinating. Question arises: what does the principle of least action looks like for a many body systems? Is it any different if they're entangled? I was literally mashing random words together at that point, but it has led me into the realm of physics that I don't think anyone really cares before: the dynamics of an entangled system.

Entanglement is usually defined as the inseparability of states, commonly used to describe the spin of a particle, which has been studied in great extent. It has found its use in quantum computer, giving its blazing speed. However, from my knowledge, no literatures has mentioned the effect of entanglement on the movement of particles. Will entangled particle move together in the same direction? If I separate a pair of entangled particles, put one in a potential, and let the other be free, does the particle in the potential have any effect on the free particle? And thus, this research (*project*) was born.

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<sup>1</sup>This introduction is rewritten after the school project has passed. I've declared this project to be independent of the school ever since. So, it's going to be much more authentic and straightforward.

This research started out by two people; me, and the other one that shall not be named. However, I have a lot of conflicts with that person. Long story short, he quitted; and thus, I'm on my own journey.

## List of Symbols (Still incomplete)

$1$	Identity unit
$\tilde{A}$	Extension of arbitrary operator $A$ defined in $\mathcal{E}_n$ onto $\mathcal{E}$
$\mathcal{D}[q]$	Path integral measure on a variable
$e$	The Euler's constant
$\mathcal{E}$	Hilbert space of the whole system
$\mathcal{E}_n$	The $n^{\text{th}}$ subspace of Hilbert space
$h$	Planck's constant
$\hbar$	Reduced Planck's constant. $\hbar = h/2\pi$
$\mathcal{H}$	Hamiltonian of a system
$i$	The imaginary unit
$\mathbb{I}$	The identity matrix/identity operator
$K_{\mathcal{A}}(q_F, t_F; q_0, t_0)$	Propagator from point $q_F, t_F$ to $q_0, t_0$ using the Lagrangian of particle $\mathcal{A}$
$K_0(q', t'; q_0, t_0)$	Propagator of a separable system. Extra arguments represents the usage on bipartite systems.
$p$	Generalized momentum
$q$	Generalized position
$\{ u_i\rangle\}$	Set of arbitrary basis $ u_1\rangle,  u_2\rangle, \dots$ of a Hilbert space with index $i$
$\hat{U}(t_B, t_A)$	Time evolution operator. Evolves the time of the system from $t_A$ to $t_B$
$\alpha_\psi$	Eigenvalue of arbitrary ket $ \psi\rangle$
$ \eta\rangle$	Ket of the whole system. Used when describing a bipartite system
$ \xi\rangle$	Ket of the subsystems. Used when describing a monopartite system
$\pi$	The Archimedes' constant
$\otimes$	Tensor product

## The generalized Gaussian integrals

A Gaussian integral is the integral of the Gaussian function  $\exp[-x^2]$ :

$$\int_{-\infty}^{\infty} \exp[-x^2] dx = \sqrt{\pi}. \quad (2.1)$$

However, the path integral method usually demands a more generalized form of the Gaussian integral denoted in this study with  $\Omega_n$ :

$$\Omega_n(a, b, c; x) = \int_{-\infty}^{\infty} x^n \exp[-(ax^2 + bx + c)] dx, \quad (2.2)$$

where  $a$ ,  $b$ , and  $c$  are constants and  $n$  is a positive integer. Because we're mostly going to be dealing with indefinite integrals, all integrals without a bound is assumed to be evaluated from  $-\infty$  to  $\infty$ .

To evaluate  $\Omega_n$  (eq. (2.2)), we shall complete the square first, then substitute the exponents to fit the form of the standard Gaussian integral.

$$\begin{aligned} \int x^n e^{-(ax^2+bx+c)} dx &= \int x^n \exp\left[-a\left(x + \frac{b^2}{2a}\right)^2 + \frac{b^2}{4a} - c\right] dx \\ &= e^{\frac{b^2}{4a}-c} \int x^n \exp\left[-a\left(x + \frac{b^2}{2a}\right)^2\right] dx. \end{aligned}$$

Let

$$-u^2 = -a\left(x + \frac{b^2}{2a}\right)^2 \quad (2.3)$$

$$x = \frac{u}{\sqrt{a}} - \frac{b^2}{2a} \quad (2.4)$$

$$\frac{dx}{du} = \frac{d}{du} \left( \frac{u}{\sqrt{a}} - \frac{b^2}{2a} \right) \quad (2.5)$$

$$dx = \frac{1}{\sqrt{a}} du. \quad (2.6)$$

Then,

$$\Omega_n = e^{\frac{b^2}{4a}-c} \frac{1}{\sqrt{a}} \int \left( \frac{u}{\sqrt{a}} - \frac{b^2}{2a} \right)^n e^{-u^2} du. \quad (2.7)$$

$$= e^{\frac{b^2}{4a}-c} \frac{1}{\sqrt{a}} \int \sum_{k=0}^n \binom{n}{k} \left(\frac{b^2}{2a}\right)^{n-k} \left(\frac{u}{\sqrt{a}}\right)^k e^{-u^2} du. \quad (2.8)$$

$$= e^{\frac{b^2}{4a}-c} \frac{1}{\sqrt{a}} \sum_{k=0}^n \left[ \binom{n}{k} \left(\frac{b^2}{2a}\right)^{n-k} \left(\frac{1}{\sqrt{a}}\right)^k \int u^k e^{u^2} du \right], \quad (2.9)$$

i.e.,  $\Omega_n$  can be written as a sum of

$$\omega = \int u^k e^{-u^2} du. \quad (2.10)$$

Notice that when  $n$  is odd, the integrand of  $\omega$  is an odd function; thus,  $\omega$  is zero when  $n$  is odd.

When  $n$  is even however, the integrand is even and thus can be simplified into

$$\omega = 2 \int_0^\infty u^{2m} e^{-u^2} du \quad (2.11)$$

where  $k = 2m$ . We then do another substitution by letting  $-t = -u^2$ . Thus,  $u = \sqrt{t}$  and  $du = dt/2\sqrt{t}$ . The bound of the integral isn't affected by a radical. Our integral then becomes

$$\omega = 2 \times \int_0^\infty t^m e^{-t} \frac{1}{2\sqrt{t}} dt. \quad (2.12)$$

$$= \int_0^\infty t^{m-\frac{1}{2}} e^{-t} dt \quad (2.13)$$

$$= \int_0^\infty t^{m+\frac{1}{2}-1} e^{-t} dt \quad (2.14)$$

$$= \Gamma\left(m + \frac{1}{2}\right) \quad (2.15)$$

$$= \Gamma\left(\frac{k+1}{2}\right). \quad (2.16)$$

Since the integral evaluates to the gamma function for only even numbers, we add a term that makes  $\omega = 0$  when  $n$  is odd:

$$\omega = \frac{1}{2}((-1)^k + 1) \Gamma\left(\frac{k+1}{2}\right). \quad (2.17)$$

Thus, from eq. (2.9),

$$\Omega_n = e^{\frac{b^2}{4a}-c} \frac{1}{\sqrt{a}} \sum_{k=0}^n \left[ \binom{n}{k} \left(\frac{b^2}{2a}\right)^{n-k} \left(\frac{1}{\sqrt{a}}\right)^k \frac{1}{2}((-1)^k + 1) \Gamma\left(\frac{k+1}{2}\right) \right] \quad (2.18)$$

To continue this, we then expand the combinatorics and use the formulas of arguments with half-integer real part to get

$$e^{\frac{b^2}{4a}-c} \frac{1}{\sqrt{a}} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \left(\frac{b^2}{2a}\right)^n \left(\frac{2a}{b^2}\right)^k \left(\frac{1}{\sqrt{a}}\right)^k \left( \sqrt{\pi} \frac{k!}{4^{\frac{k}{2}} (k/2)!} \right) \times \frac{1}{2}((-1)^k + 1) \quad (2.19)$$

$$= e^{\frac{b^2}{4a}-c} n! \sqrt{\frac{\pi}{a}} \left(\frac{b^2}{2a}\right)^n \sum_{k=0}^n \frac{1}{(n-k)!(k/2)!} \left(\frac{\sqrt{a}}{b^2}\right)^k \frac{1}{2}((-1)^k + 1) \quad (2.20)$$

To simplify the summation, we replace  $k$  with  $2k$ , and the upper limit with  $m$  where  $m = \lfloor n/2 \rfloor$ .

Thus, we get

$$\left[ \sum_{k=0}^m \frac{1}{(n-2k)!k!} \left(\frac{\sqrt{a}}{b^2}\right)^{2k} \right] e^{\frac{b^2}{4a}-c} n! \left(\frac{b^2}{2a}\right)^n \sqrt{\frac{\pi}{a}} \quad (2.21)$$

As a convention in this study, I shall define

$$\int_{-\infty}^{\infty} x^n \exp[-ax^2 - bx - c] dx = \Omega_n(a, b, c; x) \equiv \zeta_n(a, b, c) \kappa(a, b, c) \quad (2.22)$$

where

$$\zeta_n(a, b, c) \equiv \left[ \sum_{k=0}^n \frac{1}{(n-2k)!k!} \left( \frac{\sqrt{a}}{b^2} \right)^{2k} \right] n! \left( \frac{b^2}{2a} \right)^n \sqrt{\frac{\pi}{a}}, \quad (2.23)$$

$$\kappa(a, b, c) \equiv \exp \left[ \frac{b^2}{4a} - c \right]. \quad (2.24)$$

I shall refer to  $\zeta$  as the coefficient of the Gaussian integral, and  $\kappa$ , the resulting exponent. The following section is a table for values of  $\zeta_n$  from  $n = 1$  to 10.

**Comment on the generalized Gaussian integral** The generalized Gaussian integral can be cast in the form of Meijer's  $G$  function. For  $n \geq 0$ ,

$$\begin{aligned} & \int x^n \exp[-ax^2 - bx - c] dx \\ &= \frac{e^{-c}}{2\sqrt{\pi}ab} \left[ -\frac{2^{n+1}a}{b^n} G_{2,1}^{1,2} \left( \frac{1-n}{2}, -\frac{n}{2} \middle| \frac{4ae^{-2i\pi}}{b^2} \right) + a^{\frac{1-n}{2}} b G_{1,2}^{2,1} \left( \frac{1-n}{2} \middle| \frac{b^2}{4a} \right) \right] \end{aligned} \quad (2.25)$$

However, this form isn't that helpful because the Meijer's  $G$  function is defined with integral of products.

Originally, I was going to evaluate everything using `SymPy`. However, `SymPy` doesn't bother integrating any Gaussian integrals that have complex arguments. My own implementation is written in [chapter 5](#)

## 2.1 Tables of Gaussian integrals with varying orders

$$\zeta_0: +\frac{1}{1}\sqrt{\frac{\pi}{a}}$$

$$\zeta_1: -\frac{b}{2}\sqrt{\frac{\pi}{a^3}}$$

$$\zeta_2: +\frac{1}{4}\sqrt{\frac{\pi}{a^5}}(2a + b^2)$$

$$\zeta_3: -\frac{b}{8}\sqrt{\frac{\pi}{a^7}}(6a + b^2)$$

$$\zeta_4: +\frac{1}{16}\sqrt{\frac{\pi}{a^9}}(12a^2 + 12ab^2 + b^4)$$

$$\zeta_5: -\frac{b}{32}\sqrt{\frac{\pi}{a^{11}}}(60a^2 + 20ab^2 + b^2)$$

$$\zeta_6: +\frac{1}{64}\sqrt{\frac{\pi}{a^{13}}}(120a^3 + 180a^2b^2 + 30ab^4 + b^6)$$



$$\zeta_7: -\frac{b}{128} \sqrt{\frac{\pi}{a^{15}}} (840a^3 + 420a^2b^2 + 42ab^4 + b^6)$$

$$\zeta_8: +\frac{1}{256} \sqrt{\frac{\pi}{a^{17}}} (1680a^4 + 3360a^3b^2 + 840a^2b^4 + 56ab^6 + b^8)$$

$$\zeta_9: -\frac{b}{512} \sqrt{\frac{\pi}{a^{19}}} (15120a^4 + 10080a^3b^2 + 1512a^2b^4 + 72ab^6 + b^8)$$

# Propagator and joint probability distribution in the position basis

## 3.1 General theory for the propagator of two particles

### 3.1.1 Non-interacting systems

The propagator for a single particle system is used to find the probability distribution  $\tilde{P}(x)$  of a state at anytime, i.e., for a state  $|\psi(t_0)\rangle$  of a single particle,

$$\tilde{P}(q_F, t_F) = \langle q_F | \psi(t_F) \rangle = \int K(q_F, q_0; \Delta t) \psi(q_0, t_0) dq_0. \quad (3.1)$$

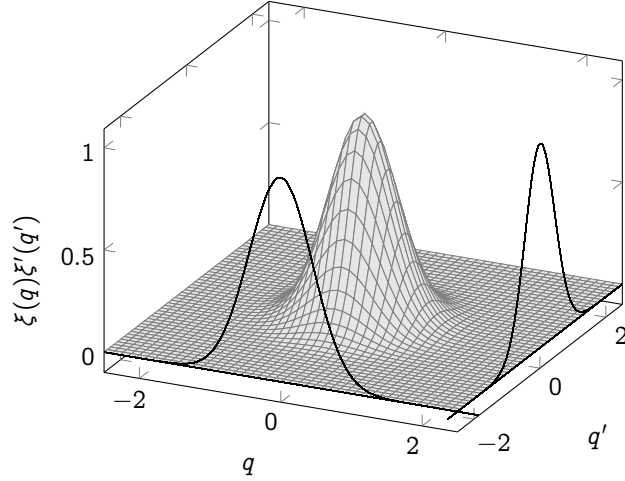
The probability of finding the particle at any point  $(q_F, t_F)$  in spacetime is defined as

$$P(q_F, t_F) \equiv |\tilde{P}(q_F, t_F)|^2 \quad (3.2)$$

This idea can be easily extended to describe any multiparticle system. But we shall only concern two particles systems in this study. Because the conventional notation is quite cumbersome, I shall introduce my own concise notation. From now on, every unprimed variable belong to particle one, and every unprimed variable, to particle one. E.g., the first particle Hilbert space shall be written with  $\mathcal{E}$ , and for the second particle,  $\mathcal{E}_2$ . This extends to all physical variables, e.g., position and momenta. The combined state shall be referred as  $\eta$ , and the separate states as  $\xi$ . Note that primed time do not exist; the interaction must happen simultaneously at a point in time; therefore, all the time variables  $t$  will not be primed.

Let there be a combined state ket of two subsystems  $|\eta\rangle$ . If  $\eta$  is separable at any point in time  $t_0$ , it follows that

$$|\eta\rangle = |\xi\rangle \otimes |\xi'\rangle. \quad (3.3)$$



**FIG. 3.1** | EXAMPLE OF THE JOINT PROBABILITY AMPLITUDE OF A GAUSSIAN WAVE PACKET.

If there are no interactions between the subsystems following  $t_0$ , the Hamiltonian operator must be separable which also implies that the time evolution operator  $\hat{U}$  is also separable. From the property of tensor product, the quantity  $\langle q, q' | \eta(t_0) \rangle$  can be written as

$$\langle q, q' | \eta(t_0) \rangle = [\langle q | \otimes \langle q' |] [|\xi\rangle \otimes |\xi'\rangle] = \langle q | \xi(t_0) \rangle \langle q' | \xi'(t_0) \rangle = \xi(q, t_0) \xi'(q', t_0), \quad (3.4)$$

which represents the product of the probability that  $\xi$  is at  $(q, t_0)$ , and  $\xi'$  is at  $q', t_0$ . We call  $\langle q, q' | \eta(t_0) \rangle$ , the **joint probability distribution**. The modulus squared of the joint probability distribution gives the **joint probability amplitude**,  $|\langle q, q' | \eta(t_0) \rangle|^2$  which represents the probability to find  $\xi$  at  $(q, t_0)$  and  $\xi'$  at  $(q', t_0)$ .

In the position basis, the joint probability amplitude is a function that takes in the position of both particles and outputs the probability that both  $\xi$  and  $\xi'$  will be there. Therefore, the joint probability amplitude encodes a surface, which can be plotted as shown in fig. 3.1. The probability amplitude of each particle can be found by projecting the surface onto the axis of each corresponding position basis.

### 3.1.2 Interacting systems

Let there be a state ket  $|\eta\rangle$  which describes the quantum state of two particles. When the two particles are interacting, the time-evolution operator cannot be factored into tensor products of two operators. Therefore, there can only be one combined time-evolution operator for both of the systems:

$$\hat{U}(t_F, t_0) = \exp \left[ -i\Delta t \left( \frac{\hat{p}^2}{2m} + \frac{\hat{p}'^2}{2m'} + V(\hat{q}, \hat{q}', t) \right) \right] \quad (3.5)$$

The consequence of the combined time-evolution operator is that, we cannot calculate the joint probability distribution between different time of the subsystem. For simplicity' sake, let the index

$j = i + 1$ , where  $t_j = t_i + \Delta t$  in which  $\Delta t \rightarrow 0$ .

$$\langle q_j, q'_j | \eta(t_j) \rangle \quad (3.6)$$

$$= \langle q_j, q'_j | \hat{U}(t_j, t_i) | \eta(t_i) \rangle. \quad (3.7)$$

$$= \iint dq_i dq'_i \langle q_j, q'_j | \hat{U}(t_j, t_i) | q_i, q'_i \rangle \langle q_i, q'_i | \eta(t_i) \rangle. \quad (3.8)$$

As seen, the form of transition element is

$$\langle q_j, q'_j | \exp \left[ -i\Delta t \left( \frac{\hat{p}^2}{2m} + \frac{\hat{p}'^2}{2m'} + V(\hat{q}, \hat{q}', t) \right) \right] | q_i, q'_i \rangle. \quad (3.9)$$

The terms in the exponents can be separated due to the vanishing commutator when  $\Delta t \rightarrow 0$ . By separating the terms in the exponents and inserting two complete sets of momentum basis, the equation above turns into

$$\begin{aligned} & \frac{1}{(2\pi)^2} \iint dp dp' \langle q_j, q'_j | \exp \left[ -i\Delta t \frac{\hat{p}^2}{2m} \right] | p, p' \rangle \langle p, p' | \exp \left[ -i\Delta t \frac{\hat{p}'^2}{2m'} \right] \exp \left[ -i\Delta t V(\hat{q}, \hat{q}', t) \right] | q_i, q'_i \rangle \\ &= \frac{1}{(2\pi)^2} \iint dp dp' \exp \left[ -iu\Delta t \left( \frac{p^2}{2m} + \frac{p'^2}{2m'} + V(q_i, q_i, \Delta t) \right) \right] \langle q_j | p \rangle \langle q'_j | p' \rangle \langle q_i | p \rangle^* \langle q'_i | p' \rangle^* \\ &= \frac{1}{(2\pi)^4} e^{-i\Delta t V(q_i, q'_i, t)} \int dp \exp \left[ -i\Delta t \frac{p^2}{2m} + ip(q_j - q_i) \right] \int dp' \exp \left[ -i\Delta t \frac{p'^2}{2m'} + ip'(q'_j - q'_i) \right] \\ &= \frac{(mm')^{\frac{1}{2}}}{8\pi^3 i \Delta t} \exp \left[ -i\Delta t V(q_i, q'_i, t) + \frac{im'}{2\Delta t} (q'_j - q'_i)^2 + \frac{im}{2\Delta t} (q_j - q_i)^2 \right]. \end{aligned} \quad (3.10)$$

To find the propagator for an interacting system, we need to perform successive integrals on  $q$  and  $q'$ , i.e.,

$$K_\eta = \int \cdots \int dq_N dq'_N \cdots dq_1 dq'_1 \langle q_F, q'_F | \hat{U}(t_F, t_N) | q_N, q'_N \rangle \cdots \langle q_1, q'_1 | \hat{U}(t_1, t_0) | q_0, q'_0 \rangle \quad (3.11)$$

Notice that when there is no interaction between the two systems ( $V = 0$ ), the integrals become separable and reduces down to the form of the non-interacting system's propagator but off by a normalization factor.

There are two common forms of interaction, which is the spring interaction and the coulomb interaction. Both of which includes the term  $(q_i - q'_i)^2$  in  $V(q, q')$ , which causes major problems in integration. When expanded, there is a  $q_i q'_i$  term that makes the integral inseparable which causes the integral pattern to not repeat; therefore, we resort to perturbation.

### 3.1.3 Form of the two particle perturbation series

The perturbation series for one particle are already given by Feynman in his path integrals textbook:

$$K_n(F, 0) = (-i)^n \int \cdots \int K_0(F, n) \prod_{i=1}^n V(i) K_0(i, i-1) d\tau_i \quad (3.12)$$

where  $K_n$  is the  $n$ 'th order perturbation,  $K(j, k) = K(q_j, t_j; q_k, t_k)$ ,  $K = \sum_n K_n$  and  $d\tau_i = dq_i dq'_i dt_i$ . To apply it with two particles, we extend it:

$$K_n(F, 0; F', 0') = (-i)^n \int \dots \int K_0(F, n) K_0(F', n') \left[ \prod_{i=1}^n V(i) K_0(i, i-1) K_0(i', i'-1) dq_i dq'_i dt_i \right]. \quad (3.13)$$

We can also expand the joint probability distribution in the form of the perturbation series.

$$\eta(q_F, q'_F; t_F) = \int K(q_F, q'_F; q_0, q'_0; t_F - t_0) \eta(q_0, q'_0; t_0) dq_0 dq'_0 \quad (3.14)$$

$$= \int \sum_n K_n(q_F, q'_F; q_0, q'_0; t_F - t_0) \eta(q_0, q'_0; t_0) dq_0 dq'_0 \quad (3.15)$$

$$= \sum_n \int K_n(q_F, q'_F; q_0, q'_0; t_F - t_0) \eta(q_0, q'_0; t_0) dq_0 dq'_0 \quad (3.16)$$

### 3.2 Initial states of choice

From now on, I shall use the letter  $\xi$  to refer to a state of a single particle, and the letter  $\eta$  to refer to the combined state of two particles.

We are interested in the time evolution of a Gaussian wave packet due to its mathematical simplicity. The normalized state of a Gaussian wave packet is given by

$$\sqrt[4]{\frac{1}{2\pi\sigma}} \exp \left[ ipq - \frac{(q-s)^2}{4\sigma^2} \right]. \quad (3.17)$$

For two separable particles, its joint probability is

$$\langle q, q' | \eta(t_0) \rangle = \sqrt[4]{\frac{1}{2\pi\sigma^2}} \sqrt[4]{\frac{1}{2\pi\sigma'^2}} \exp \left[ ipq - \frac{(q-s)^2}{4\sigma^2} \right] \exp \left[ ip'q' - \frac{(q'-s')^2}{4\sigma'^2} \right] \quad (3.18)$$

$$= \sqrt{\frac{1}{2\pi\sigma\sigma'}} \exp \left[ ipq - \frac{(q-s)^2}{4\sigma^2} \right] \exp \left[ ip'q' - \frac{(q'-s')^2}{4\sigma'^2} \right] \quad (3.19)$$

As we'll see, the higher order perturbation terms (at least of the spring system) are expressed as the integral of the product between the wave function and some polynomials, which we'll find later.

$$\langle q | \xi(t) \rangle \quad (3.20)$$

$$= \int K_0(q_F, q_0; t_F) \langle q_0 | \xi(t) \rangle dq_0 \quad (3.21)$$

$$= \sqrt{\frac{m}{2\pi i t_F}} \int \exp \left[ \frac{im}{2t_F} (q_F - q_0)^2 \right] \left( \sqrt[4]{\frac{1}{2\pi\sigma}} \exp \left[ ipq_0 - \frac{(q_0 - s)^2}{4\sigma^2} \right] \right) dq_0 \quad (3.22)$$

$$= \sqrt[4]{-\frac{m^2}{8\pi^3 t_F^2 \sigma^2}} \int \exp \left[ \frac{im}{2t_F} (q_F - q_0)^2 + ipq_0 - \frac{(q_0 - s)^2}{4\sigma^2} \right] dq_0 \quad (3.23)$$

$$= \sqrt[4]{-\frac{m^2}{8\pi^3 t_F^2 \sigma^2}} \int \exp \left[ \left( \frac{imq_F^2}{2t_F} - \frac{s^2}{4\sigma^2} \right) + q \left( -\frac{imq_F}{t_F} + ip + \frac{s}{2\sigma^2} \right) + q^2 \left( \frac{im}{2t_F} - \frac{1}{4\sigma^2} \right) \right] dq_0 \quad (3.24)$$

$$\begin{aligned}
 &= \sqrt[4]{-\frac{m^2}{8\pi^3 t_F^2 \sigma^2}} \cdot 2\sigma \sqrt{\frac{\pi t_F}{-2im\sigma^2 + t_F}} \exp \left[ \frac{4impq_F \sigma^2 - mq_F^2 + 2mq_F s - ms^2 - 2ip^2 t_F \sigma^2 - 2pst_F}{4m\sigma^2 + 2it_F} \right] \\
 &= \sqrt[4]{\frac{2m^2 \sigma^2}{\pi(2m\sigma^2 + it_F)^2}} \exp \left[ \frac{4impq_F \sigma^2 - mq_F^2 + 2mq_F s - ms^2 - 2ip^2 t_F \sigma^2 - 2pst_F}{4m\sigma^2 + 2it_F} \right]. \quad (3.25)
 \end{aligned}$$

To plot its probability amplitude, we have to take its modulus squared, which equals to

$$\frac{m\sigma}{\sqrt{4m^2 \sigma^4 + t_F^2}} \exp \left[ \frac{-2m^2 q_F^2 \sigma^2 + 4m^2 q_F s \sigma^2 - 2m^2 s^2 \sigma^2 + 4mpq_F t_F \sigma^2 - 4mpst_F \sigma^2 - 2p^2 t_F^2 \sigma^2}{4m^2 \sigma^4 + t_F^2} \right] \quad (3.26)$$

### 3.3 The spring system

The spring potential is given by  $V(q, q') = \frac{k}{2}(q - q')^2$ . I shall let  $\alpha = \frac{k}{2}$  to simplify the notation a bit.

#### 3.3.1 First order perturbation term

From eq. (3.13), set  $t_1 = t, t_0 = 0$

$$\begin{aligned}
 &K_1(F, 0; F', 0) \\
 &= (-i)^1 \iiint K_0(F, 1)K_0(F', 1')K_0(1, 0)K_0(1, 0')\alpha(q_1 - q'_1)^2 dq_1 dq'_1 dt. \quad (3.27) \\
 &= -i\alpha \int_0^{t_F} \left[ \iint K_0(q_F, q_1; t_F - t)K_0(q'_F, q'_1; t_F - t)K_0(q_1, q_0; t)K_0(q'_1, q'_0; t)(q_1 - q'_1)^2 dq_1 dq'_1 \right] dt.
 \end{aligned}$$

We then let the terms in the square bracket,

$$\iint K_0(q_F, q_1; t_F - t)K_0(q'_F, q'_1; t_F - t)K_0(q_1, q_0; t)K_0(q'_1, q'_0; t)(q_1 - q'_1)^2 dq_1 dq'_1 \quad (3.28)$$

equals  $I$ ; therefore,  $K_1 = -i\alpha \int_0^{t_F} I dt$ .

We then separate  $I$  into three integrals:

$$I_{P1} = \int q_1^2 K_0(q_F, q_1; t_F - t)K_0(q_1, q_0; t) dq_1 \int K_0(q'_F, q'_1; t_F - t)K_0(q'_1, q'_0; t) dq'_1, \quad (3.29)$$

$$I_{P2} = \int K_0(q_F, q_1; t_F - t)K_0(q_1, q_0; t) dq_1 \int q_1'^2 K_0(q'_F, q'_1; t_F - t)K_0(q'_1, q'_0; t) dq'_1, \quad (3.30)$$

$$I_{P3} = \int q_1 K_0(q_F, q_1; t_F - t)K_0(q_1, q_0; t) dq_1 \int q'_1 K_0(q'_F, q'_1; t_F - t)K_0(q'_1, q'_0; t) dq'_1, \quad (3.31)$$

where  $I = I_{P1} + I_{P2} + 2I_{P3}$ . The integrals without the factor  $q_1$  and  $q_1'^2$  can be reduced into the kernel for the free particle:

$$I_{P1} = K_0(q'_F, q'_0; t_F) \int q_1^2 K_0(q_F, q_1; t_F - t)K_0(q_1, q_0; t) dq_1 \quad (3.32)$$

$$I_{P2} = K_0(q_F, q_0; t_F) \int q_1'^2 K_0(q'_F, q'_1; t_F - t)K_0(q'_1, q'_0; t) dq'_1 \quad (3.33)$$

Since  $I_{P_2}$  can be obtained by switching all the primed variables with the corresponding unprimed in  $I_{P_1}$ , we're left with two family of integrals:

$$\int q_1 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) dq_1 \quad \text{and,} \quad \int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) dq_1. \quad (3.34)$$

To evaluate these, we first simplify the product of kernel under the assumption that  $t_F > t$ .

$$\begin{aligned} & K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) \\ &= \sqrt{\frac{m}{2\pi i(t_F - t)}} \sqrt{\frac{m}{2\pi i t}} \exp \left[ \frac{im}{2(t_F - t)} (q_F - q_1)^2 + \frac{im}{2t} (q_1 - q_0)^2 \right] \\ &= \frac{m}{2\pi} \sqrt{-\frac{1}{t(t_F - t)}} \exp \left[ q_1^2 \left( \frac{im}{2t} + \frac{im}{2(t_F - t)} \right) - q_1 \left( \frac{im q_0}{t} + \frac{im q_F}{t_F - t} \right) + \left( \frac{im q_0^2}{2t} + \frac{im q_F^2}{2(t_F - t)} \right) \right] \\ &= \frac{m}{2\pi} \sqrt{-\frac{1}{t(t_F - t)}} \exp \left[ -q_1^2 \left( \frac{mt_F}{2it(t_F - t)} \right) - q_1(im) \left( \frac{q_0}{t} + \frac{q_F}{t_F - t} \right) - \left( \frac{m}{2i} \right) \left( \frac{q_0^2}{t} + \frac{q_F^2}{t_F - t} \right) \right] \end{aligned} \quad (3.35)$$

The normalization factor are pulled out. Both integrals in eq. (3.34) can be evaluated with

$$a = \frac{mt_F}{2it(t_F - t)}, \quad b = im \left( \frac{q_0}{t} + \frac{q_F}{t_F - t} \right) \quad \text{and,} \quad c = \left( \frac{m}{2i} \right) \left( \frac{q_0^2}{t} + \frac{q_F^2}{t_F - t} \right) \quad (3.36)$$

in which,

$$\exp \left[ \frac{b^2}{4a} - c \right] = \exp \left[ \frac{im}{2t_F} (q_F - q_0)^2 \right] = \sqrt{\frac{2\pi i t_F}{m}} K_0(q_F, q_0; t_F). \quad (3.37)$$

To summarize,

$$\begin{aligned} \int q_1 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) dq_1 &= -\frac{b}{2} \sqrt{\frac{\pi}{a^3}} \times \frac{m}{2\pi} \sqrt{-\frac{1}{t(t_F - t)}} \times \sqrt{\frac{2\pi i t_F}{m}} K_0(q_F, q_0; t_F) \\ &= -\frac{b}{2} \sqrt{\frac{\pi}{a^3}} \times \sqrt{\frac{mt_F}{2\pi i t(t_F - t)}} K_0(q_F, q_0; t_F), \end{aligned} \quad (3.38)$$

and

$$\int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) dq_1 = \frac{1}{4} \sqrt{\frac{\pi}{a^5}} (2a + b^2) \times \sqrt{\frac{mt_F}{2\pi i t(t_F - t)}} K_0(q_F, q_0; t_F). \quad (3.39)$$

On the Gaussian integral with degree one,

$$-\frac{b}{2} \sqrt{\frac{\pi}{a^3}} = \sqrt{\frac{2\pi t}{mt_F^3}} \frac{\sqrt{-i(t_F - t)^3}}{i(t_F - t)} \times [q_0(t_F - t) + q_F t]; \quad (3.40)$$

thus from eq. (3.38),

$$\int q_1 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) = \sqrt{\frac{2\pi t}{mt_F^3}} \frac{\sqrt{-i(t_F - t)^3}}{i(t_F - t)} \times [q_0(t_F - t) + q_F t] \quad (3.41)$$

$$\begin{aligned} & \times \sqrt{\frac{mt_F}{2\pi i t(t_F - t)}} K_0(q_F, q_0; t_F) \\ &= -\frac{1}{t_F} [q_0(t_F - t) + q_F t] K_0(q_F, q_0; t_F). \end{aligned} \quad (3.42)$$

On the Gaussian integral with degree two,

$$\frac{1}{4} \sqrt{\frac{\pi}{a^5}} (2a + b^2) = -\sqrt{\frac{2\pi t}{m^3 t_F^5}} \frac{\sqrt{i(t_F - t)^5}}{(t_F - t)^2} \left[ m (q_0(t_F - t) + q_F t)^2 + i t t_F (t_F - t) \right]; \quad (3.43)$$

and thus,

$$\int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) dq_1 \quad (3.44)$$

$$\begin{aligned} &= \sqrt{\frac{m t_F}{2\pi i t(t_F - t)}} K_0(q_F, q_0; t_F) \times -\sqrt{\frac{2\pi t}{m^3 t_F^5}} \frac{\sqrt{i(t_F - t)^5}}{(t_F - t)^2} \left[ m (q_0(t_F - t) + q_F t)^2 + i t t_F (t_F - t) \right] \\ &= -\frac{1}{m t_F^2} \left[ m (q_0(t_F - t) + q_F t)^2 + i t t_F (t_F - t) \right] K_0(q_F, q_0; t_F). \end{aligned} \quad (3.45)$$

We're now in the place to finally construct the first order propagator term for the spring system. Recall that

$$K_1 = -i\alpha \int_0^{t_F} I dt = -i\alpha \left[ \int_0^{t_F} I_{P1} dt + \int_0^{t_F} I_{P2} dt + 2 \int_0^{t_F} I_{P3} dt \right] \quad (3.46)$$

From earlier,

$$\begin{aligned} I_{P1} &= K_0(q'_F, q'_0; t_F) \int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) dq_1 \\ &= -\frac{1}{m t_F^2} \left[ m (q_0(t_F - t) + q_F t)^2 + i t t_F (t_F - t) \right] K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F), \end{aligned} \quad (3.47)$$

$$\begin{aligned} I_{P2} &= K_0(q_F, q_0; t_F) \int q_1'^2 K_0(q'_F, q'_1; t_F - t) K_0(q'_1, q'_0; t) dq'_1 \\ &= -\frac{1}{m t_F^2} \left[ m (q'_0(t_F - t) + q'_F t)^2 + i t t_F (t_F - t) \right] K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F), \end{aligned} \quad (3.48)$$

$$\begin{aligned} I_{P3} &= -\frac{1}{t_F} [q_0(t_F - t) + q_F t] K_0(q_F, q_0; t_F) \times -\frac{1}{t_F} [q'_0(t_F - t) + q'_F t] K_0(q'_F, q'_0; t_F) \\ &= \frac{1}{t_F^2} [q_0(t_F - t) + q_F t] [q'_0(t_F - t) + q'_F t] K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F). \end{aligned} \quad (3.49)$$

Thus,

$$\begin{aligned} K_1 &= -i\alpha K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F) \frac{1}{t_F^2} \left[ -\frac{1}{m} \int_0^{t_F} m (q_0(t_F - t) + q_F t)^2 + i t t_F (t_F - t) dt \right. \\ &\quad \left. - \frac{1}{m} \int_0^{t_F} m (q'_0(t_F - t) + q'_F t)^2 + i t t_F (t_F - t) dt \right. \\ &\quad \left. + \int_0^{t_F} [q_0(t_F - t) + q_F t] [q'_0(t_F - t) + q'_F t] dt \right] \end{aligned} \quad (3.50)$$

The integrals are then taken out to be evaluated term by term. Since the integrand of these integrals are all polynomials, we can just plug it into `SymPy.jl`:

$$\int_0^{t_F} m (q_0(t_F - t) + q_F t)^2 dt = \frac{t_F^3}{6} (2m(q_0 + q_F)^2 + i t_F) \quad (3.51)$$

$$\int_0^{t_F} m (q'_0(t_F - t) + q'_F t)^2 dt = \frac{t_F^3}{6} (2m(q'_0 + q'_F)^2 + i t_F) \quad (3.52)$$



$$\int_0^{t_F} [q_0(t_F - t) + q_F t] [q'_0(t_F - t) + q'_F t] dt = \frac{t_F^3}{6} (2q_0 q'_0 + q_0 q'_F + q'_0 q_F + 2q_F q'_F) \quad (3.53)$$

Therefore, the first order perturbation term of the spring system takes the form

$$\begin{aligned} K_1(q_F, q_0; q'_F, q'_0; t_F) &= -i \frac{\alpha t_F}{6} K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F) \\ &\times [-2(m(q_0 + q_F)^2 + m(q'_0 + q'_F)^2 + it_F) + 2q_0 q'_0 + q_0 q'_F + q'_0 q_F + 2q_F q'_F]. \end{aligned} \quad (3.54)$$

**Interpretation of the propagator** As of now, my knowledge of the propagator is pretty limited.

### 3.3.2 Joint probability contribution from the first order perturbation term

Let us first analyze the time evolution of a separable wave packet with the initial state

$$\eta(q_0, q_0; t_0) \equiv \frac{1}{2\pi\sqrt{\sigma\sigma'}} \exp\left[ipq_0 + ip'q'_0 - \frac{(q_0 - s)^2}{4\sigma^2} - \frac{(q'_0 - s')^2}{4\sigma'^2}\right]. \quad (3.55)$$

For simplicity's sake, let

$$\begin{aligned} \xi &\equiv \exp\left[ipq_0 - \frac{(q_0 - s)^2}{4\sigma^2}\right], \quad \xi' \equiv \exp\left[ip'q'_0 - \frac{(q'_0 - s')^2}{4\sigma'^2}\right], \\ K &\equiv \sqrt{\frac{m}{2\pi it_F}} \exp\left[\frac{i}{2t_F}(q_F - q_0)^2\right], \quad K' \equiv \sqrt{\frac{m}{2\pi it_F}} \exp\left[\frac{i}{2t_F}(q'_F - q'_0)^2\right]. \end{aligned}$$

Let us also set  $t_0$  to be 0. Using the first order perturbation term from eq. (3.54), the state at any following time  $t_F$  is then

$$\eta(q_F, q'_F; t_F) = \iint (K_0(q_F, q_0; q'_F, q'_0; t_F - t_0) + K_1(q_F, q_0; q'_F, q'_0; t_F - t_0)) \eta(q_0, q'_0; t_0) dq_0 dq'_0 \quad (3.56)$$

$$\begin{aligned} &= \frac{1}{2\pi\sqrt{\sigma\sigma'}} \iint K_0(q_F, q_0; q'_F, q'_0; t_F - t_0) \left\{ 1 - i \frac{\alpha t_F}{6} [-2(m(q_0 + q_F)^2 \right. \\ &\quad \left. + m(q'_0 + q'_F)^2 + it_F) + 2q_0 q'_0 + q_0 q'_F + q'_0 q_F + 2q_F q'_F] \right\} \end{aligned} \quad (3.57)$$

$$\begin{aligned} &\exp\left[ipq_0 + ip'q'_0 - \frac{(q_0 - s)^2}{4\sigma^2} - \frac{(q'_0 - s')^2}{4\sigma'^2}\right] dq_0 dq'_0 \\ &= \frac{1}{2\pi\sqrt{\sigma\sigma'}} \iint K K' \xi \xi' \left\{ 1 - i \frac{\alpha t_F}{6} [-2(m(q_0 + q_F)^2 \right. \\ &\quad \left. + m(q'_0 + q'_F)^2 + it_F) + 2q_0 q'_0 + q_0 q'_F + q'_0 q_F + 2q_F q'_F] \right\} dq_0 dq'_0 \end{aligned} \quad (3.58)$$

Note that this method of separation will work for all separable  $\eta(t_0)$ . Therefore,  $\eta(t_0)$  doesn't have to be Gaussian: we just select it to be. The integral above can then be further expanded and broken down into smaller integrals:

$$\begin{aligned} \eta(q_F, q'_F; t_F) &= \frac{1}{2\pi\sqrt{\sigma\sigma'}} \left[ \iint K K' \xi \xi' dq_0 dq'_0 \right. \\ &\quad \left. - i \frac{\alpha t_F}{6} \iint K K' \xi \xi' \left\{ -2m(q_0^2 + 2q_0 q_F + q_F^2 + q_0'^2 + 2q'_0 q'_F + q_F'^2) \right. \right. \\ &\quad \left. \left. + 2q_0 q'_0 + 2q_F q'_F + q_0 q'_F + q'_0 q_F + it_F \right\} dq_0 dq'_0 \right] \end{aligned} \quad (3.59)$$

(3.60)

I shall let  $I$  equals the term in the big square bracket:

$$I \equiv \iint K K' \xi \xi' dq_0 dq'_0 - i \frac{\alpha t_F}{6} \iint K K' \xi \xi' \left\{ -2m(q_0^2 + 2q_0 q_F + q_F^2 + q_0'^2 + 2q_0' q'_F + q_F'^2) + 2q_0 q'_0 + 2q_F q'_F + q_0 q'_F + q_0' q_F + i t_F \right\} dq_0 dq'_0 \quad (3.61)$$

and thus,

$$\eta(q_F, q'_F; t_F) \equiv \frac{1}{2\pi\sqrt{\sigma\sigma'}} I \quad (3.62)$$

We then pull the terms that doesn't include  $q_0$  or  $q'_0$  out:

$$\begin{aligned} I &= \left[ \iint K K' \xi \xi' dq_0 dq'_0 \right] + \left[ \iint K K' \xi \xi' dq_0 dq'_0 \right] \left( -i \frac{\alpha t_F}{6} \right) (-2m(q_F^2 + q_F'^2) + 2q_F q'_F + i t_F) \\ &\quad - \left( \frac{i \alpha t_F}{6} \right) \iint K K' \xi \xi' \left\{ -2m(q_0^2 + q_0'^2 + 2q_0 q_F + 2q_0' q'_F) + 2q_0 q'_0 + q_0 q'_F + q_0' q_F \right\} dq_0 dq'_0 \\ &= \left[ \iint K K' \xi \xi' dq_0 dq'_0 \right] \left[ 1 + \left( \frac{i \alpha t_F}{6} \right) (2m(q_F^2 + q_F'^2) - 2q_F q'_F - i t_F) \right] \\ &\quad - \left( \frac{i \alpha t_F}{6} \right) \left[ \iint K K' \xi \xi' \left\{ -2m(q_0^2 + q_0'^2 + 2q_0 q_F + 2q_0' q'_F) + 2q_0 q'_0 + q_0 q'_F + q_0' q_F \right\} dq_0 dq'_0 \right] \\ &= \left[ \int K \xi dq_0 \int K' \xi' dq'_0 \right] \left[ 1 + \left( \frac{i \alpha t_F}{6} \right) (2m(q_F^2 + q_F'^2) - 2q_F q'_F - i t_F) \right] \\ &\quad - \left( \frac{i \alpha t_F}{6} \right) \left\{ -2m \left[ \int q_0^2 K \xi dq_0 \int K' \xi' dq'_0 + \int K \xi dq_0 \int q_0'^2 K' \xi' dq'_0 \right] \right. \\ &\quad \left. - 4m q_F \left[ \int q_0 K \xi dq_0 \int K' \xi' dq'_0 + \int K \xi dq_0 \int q_0' K' \xi' dq'_0 \right] \right. \\ &\quad \left. + q_F \left[ \int K \xi dq_0 \int q_0' K' \xi' dq'_0 \right] + q'_F \left[ \int q_0 K \xi dq_0 \int K' \xi' dq'_0 \right] \right. \\ &\quad \left. + 2 \left[ \int q_0 K \xi dq_0 \int q_0' K' \xi' dq'_0 \right] \right\} \end{aligned} \quad (3.63)$$

For the purpose of legibility, let

$$I_0 = \int K \xi dq_0, \quad I_1 = \int q_0 K \xi dq_0, \quad I_2 = \int q_0^2 K \xi dq_0, \\ I'_0 = \int K' \xi' dq'_0, \quad I'_1 = \int q_0' K' \xi' dq'_0 \quad \text{and} \quad I'_2 = \int q_0'^2 K' \xi' dq'_0. \quad (3.64)$$

Thus,  $I$ ; and hence  $\eta$ , becomes

$$\begin{aligned} \eta(q_F, q'_F; t_F) &= \frac{1}{2\pi\sqrt{\sigma\sigma'}} \left\{ I_0 I'_0 \left[ 1 + \left( \frac{i \alpha t_F}{6} \right) (2m(q_F^2 + q_F'^2) - 2q_F q'_F - i t_F) \right] \right. \\ &\quad \left. - \left( \frac{i \alpha t_F}{6} \right) [-2m(I_2 I'_0 + I_0 I'_2) - 4m q_F (I_1 I'_0 + I_0 I'_1) + q_F I_0 I'_1 + q'_F I_1 I'_0 + 2I_1 I'_1] \right\} \end{aligned} \quad (3.65)$$

As said earlier, this equation is general for the spring propagator with separable initial state. When  $\alpha = 0$ , this joint probability distribution reduces down to the joint probability distribution for the free particle.

The integrals listed in eq. (3.64) are then evaluated:

$$I_0 = \int K \xi \, dq_0 = \sqrt{\frac{m}{2\pi i t_F}} \sqrt[4]{\frac{1}{2\pi\sigma}} \int \exp \left[ i p q_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{i m}{2 t_F} (q_F - q_0)^2 \right] dq_0 \quad (3.66)$$

$$I_1 = \int q_0 K \xi \, dq_0 = \sqrt{\frac{m}{2\pi i t_F}} \sqrt[4]{\frac{1}{2\pi\sigma}} \int q_0 \exp \left[ i p q_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{i m}{2 t_F} (q_F - q_0)^2 \right] dq_0 \quad (3.67)$$

$$I_2 = \int q_0^2 K \xi \, dq_0 = \sqrt{\frac{m}{2\pi i t_F}} \sqrt[4]{\frac{1}{2\pi\sigma}} \int q_0^2 \exp \left[ i p q_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{i m}{2 t_F} (q_F - q_0)^2 \right] dq_0 \quad (3.68)$$

All of these integrals are Gaussian integrals with the exponential argument

$$i p q_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{i m}{2 t_F} (q_F - q_0)^2 \quad (3.69)$$

$$= - \left( \frac{1}{4\sigma^2} - \frac{i m}{2 t_F} \right) q_0^2 - \left( \frac{i m q_F}{t_F} - i p - \frac{s}{2\sigma^2} \right) q_0 - \left( \frac{s^2}{4\sigma^2} - \frac{i m q_F^2}{2 t_F} \right), \quad (3.70)$$

with

$$a = \frac{1}{4\sigma^2} - \frac{i m}{2 t_F}, \quad b = \frac{i m q_F}{t_F} - i p - \frac{s}{2\sigma^2} \quad \text{and} \quad \frac{s^2}{4\sigma^2} - \frac{i m q_F^2}{2 t_F}. \quad (3.71)$$

From section 3.2,

$$\exp \left[ \frac{b^2}{4a} - c \right] = \exp \left[ \frac{4i m p q_F \sigma^2 - m q_F^2 + 2m q_F s - m s^2 - 2i p^2 t_F \sigma^2 - 2p s t_F}{4m \sigma^2 + 2i t_F} \right]. \quad (3.72)$$

I shall let

$$\Xi \equiv \sqrt{\frac{m}{2\pi i t_F}} \sqrt[4]{\frac{1}{2\pi\sigma}} \exp \left[ \frac{4i m p q_F \sigma^2 - m q_F^2 + 2m q_F s - m s^2 - 2i p^2 t_F \sigma^2 - 2p s t_F}{4m \sigma^2 + 2i t_F} \right], \quad (3.73)$$

$$\Xi' \equiv \sqrt{\frac{m}{2\pi i t_F}} \sqrt[4]{\frac{1}{2\pi\sigma'}} \exp \left[ \frac{4i m p' q_F' \sigma'^2 - m q_F'^2 + 2m q_F' s' - m s'^2 - 2i p'^2 t_F \sigma'^2 - 2p' s' t_F}{4m' \sigma'^2 + 2i t_F} \right] \quad (3.74)$$

which represents the unperturbed state of a particle. And,

$$\sqrt{\frac{\pi}{a}} = 2\sigma \sqrt{\frac{\pi t_F}{t_F - 2i m \sigma^2}} \equiv G_0, \quad (3.75)$$

$$-\frac{b}{2} \sqrt{\frac{\pi}{a^3}} = 2\sigma \sqrt{\frac{\pi t_F}{(t_F - 2i m \sigma^2)^3}} (-2i q_F + 2i p t_F \sigma^2 + s t_F) \equiv G_1, \quad (3.76)$$

$$\frac{1}{4} \sqrt{\frac{\pi}{a^5}} = 2\sigma \sqrt{\frac{\pi t_F}{(t_F - 2i m \sigma^2)^5}} (-4i m t_F \sigma^4 + 2t_F^2 \sigma^2 + (-2i m q_F \sigma^2 + 2i p t_F \sigma^2 + s t_F)^2) \equiv G_2. \quad (3.77)$$

Therefore, with the change of primed and unprimed variables,

$$I_0 = G_0 \Xi, \quad I_1 = G_1 \Xi, \quad I_2 = G_2 \Xi, \quad I_0' = G_0' \Xi', \quad I_1' = G_1' \Xi', \quad \text{and} \quad I_2' = G_2' \Xi' \quad (3.78)$$

$\eta$  then becomes

$$\begin{aligned} \eta = \frac{1}{2\pi \sqrt{\sigma \sigma'}} & \left\{ G_0 G_0' \Xi \Xi' \left[ 1 + \left( \frac{i \alpha t_F}{6} \right) (2m(q_F^2 + q_F'^2) - 2q_F q_F' - i t_F) \right] \right. \\ & - \left( \frac{i \alpha t_F}{6} \right) [-2m(G_2 G_0' \Xi \Xi' + G_0 G_2' \Xi \Xi') - 4m q_F (G_1 G_0' \Xi \Xi' + G_0 G_1' \Xi \Xi') \\ & \left. + q_F G_0 G_1' \Xi \Xi' + q_F' G_1 G_0' \Xi \Xi' + 2G_1 G_1' \Xi \Xi'] \right\} \end{aligned} \quad (3.79)$$

$$\begin{aligned}
&= \frac{\Xi\Xi'}{2\pi\sqrt{\sigma\sigma'}} \left\{ G_0 G'_0 \left[ 1 + \left( \frac{i\alpha t_F}{6} \right) (2m(q_F^2 + q_F'^2) - 2q_F q_F' - it_F) \right] \right. \\
&\quad \left. - \left( \frac{i\alpha t_F}{6} \right) [-2m(G_2 G'_0 + G_0 G'_2) - 4mq_F(G_1 G'_0 + G_0 G'_1) + q_F G_0 G'_1 + q_F' G_1 G'_0 + 2G_1 G'_1] \right\} \\
&= \frac{\Xi\Xi'}{2\pi\sqrt{\sigma\sigma'}} \left\{ G_0 G'_0 \left[ 1 + \left( \frac{i\alpha t_F}{6} \right) (2m(q_F^2 + q_F'^2) - 2q_F q_F' - it_F) \right] \right. \\
&\quad \left. - \left( \frac{i\alpha t_F}{6} \right) [G_0(q_F G'_1(1 - 4m) - 2m G'_2) + G'_0(q_F G_1(1 - 4m) - 2m G_2) + 2G_1 G'_1] \right\} \quad (3.80)
\end{aligned}$$

We then proceed to normalize the state  $\eta$

### 3.3.3 Entanglement entropy

### 3.3.4 Joint probability contribution from the first order perturbation term on an entangled state

### 3.3.5 Second order perturbation term

The second order perturbation term,  $K_2$  is

$$K_2 = \int \dots \int K_0(F, 2)V(2)K_0(2, 1)V(1)K_0(1, 0) dq_1 dq'_1 dt_1 dq_2 dq'_2 dt_2 \quad (3.81)$$

$$= \int \dots \int K_0(q_F, q_2; t_F - t_2)K_0(q'_F, q'_2; t_F - t_2)K_0(q_2, q_1; t_2 - t_1)K_0(q_2, q_1; t_2 - t_1) \quad (3.82)$$

$$\begin{aligned}
&\times K_0(q_1, q_0; t_1 - t_0)K_0(q_1, q_0; t_1 - t_0)(q_2 - q'_2)^2(q_1 - q'_1)^2 dq_1 dq'_1 dt_1 dq_2 dq'_2 dt_2 \\
&= \int_{t_1}^{t_F} \int_{t_0}^{t_F} \left[ \int \dots \int K_0(q_F, q_2; t_F - t_2)K_0(q'_F, q'_2; t_F - t_2)K_0(q_2, q_1; t_2 - t_1) \right. \\
&\quad \times K_0(q_2, q_1; t_2 - t_1)K_0(q_1, q_0; t_1 - t_0)K_0(q_1, q_0; t_1 - t_0) \left( q_1^2 q_2^2 - 2q_1^2 q_2 q'_2 \right. \\
&\quad \left. + q_1^2 (q'_2)^2 - 2q_1 q_2^2 q'_1 + 4q_1 q_2 q'_1 q'_2 - 2q_1 q'_1 (q'_2)^2 + q_2^2 (q'_1)^2 \right. \\
&\quad \left. \left. - 2q_2 (q'_1)^2 q'_2 + (q'_1)^2 (q'_2)^2 \right) dq_1 dq'_1 dq_2 dq'_2 \right] dt_1 dt_2 \quad (3.83)
\end{aligned}$$

The integral once again can be broken into nine integrals that must be integrated w.r.t. time twice later on. All those nine integrals have a product of propagator as a multiplier. We shall evaluate those first, separating the primed and unprimed variables.

$$\begin{aligned}
&K_0(q_F, q_2; t_F - t_2)K_0(q'_F, q'_2; t_F - t_2)K_0(q_2, q_1; t_2 - t_1)K_0(q_2, q_1; t_2 - t_1)K_0(q_1, q_0; t_1 - t_0) \\
&\quad \times K_0(q_1, q_0; t_1 - t_0) \\
&= \frac{im^3}{8\pi^3(t_F - t_2)(t_2 - t_1)(t_1 - t_0)} \exp \left[ \frac{im}{2(t_F - t_2)} ((q_F - q_2)^2 + (q'_F - q'_2)^2) \right. \\
&\quad \left. + \frac{im}{2(t_2 - t_1)} ((q_2 - q_1)^2 + (q'_2 - q'_1)^2) + \frac{im}{2(t_1 - t_0)} ((q_1 - q_0)^2 + (q'_1 - q'_0)^2) \right] \quad (3.84)
\end{aligned}$$

## 3.4 Anharmonic oscillator

An anharmonic oscillator's potential is in the form



# CHAPTER 4

## **Correlation function**



## Implementation of simulations

To represent the joint probability distribution of a two particle system, we need four axes. Two for the position of each Hilbert space that those particles belong in, one for the probability amplitude, and one for the time.

$$P(q_F, q'_F; t_F) = |\langle q_F, q'_F | \eta(t_F) \rangle|^2 \quad (5.1)$$

$$= \left| \iint K(q_F, q'_F, t_F; q_0, q'_0, t_0) \eta(q_0, q'_0, t_0) dq_0 dq'_0 \right|^2 \quad (5.2)$$

$$= \left| \iint \left\{ \sum_{i=0}^{\infty} K_i(q_F, q'_F, t_F; q_0, q'_0, t_0) \right\} \eta(q_0, q'_0, t_0) dq_0 dq'_0 \right|^2 \quad (5.3)$$

For this equation to work with a discrete simulation, we need to exchange the integral sign with the summation sign:

$$= \left| \sum_{q_0} \sum_{q'_0} \left\{ \sum_{i=0}^{\infty} K_i(q_F, q'_F, t_F; q_0, q'_0, t_0) \right\} \eta(q_0, q'_0, t_0) \right| \quad (5.4)$$

The sum of propagators can then be truncated to our liking. I've written the code in Julia to test for the free particle propagator, and it looks something like this:

```

1 using Plots; plotlyjs()
2 using LinearAlgebra
3 using SymPy
4 using BenchmarkTools
5
6 q, q0, q1, q2, qf = symbols("q q_0 q_1 q_2 q_F", real = true)
7 m, t, t0, t1, t2, tf = symbols("m t t_0 t_1 t_2 t_F", real = true, positive = true)
8 q0p, q1p, q2p, qfp = symbols("q^{\\prime}_0 q^{\\prime}_1 q^{\\prime}_2 q^{\\prime}_F",
  ↪ real = true)
```



```

9  a, b, c, d, e, f = symbols("a b c d e f")
10 s1, s2, p1, p2, σ1, σ2 = symbols("s_1 s_2 p_1 p_2 σ_1 σ_2")
11
12 freePropagator(finPos, startPos, finTime, startTime = 0, m = 1) = sqrt(m / (2 * pi * im *
    ↪ (finTime - startTime))) * exp(im * m / (2 * (finTime - startTime)) * (finPos -
    ↪ startPos)^2)
13 freePropagatorC(qf, qfp, q0, q0p, tf, t0) = freePropagator(qf, q0, tf, t0) +
    ↪ freePropagator(qfp, q0p, tf, t0)
14
15 initStateFunction(q0, q0p, σ1, s1, p1, σ2, s2, p2) = (1//2 * pi * σ1)^(1//4) * exp(-(q0 -
    ↪ s1)^2 / (4 * σ1^2) + im * p1 * q0) * (1//2 * pi * σ2)^(1//4) * exp(-(q0p - s2)^2 / (4 *
    ↪ σ2^2) + im * p2 * q0p)
16
17 maxPos = 5
18 minPos = -5
19 stepPos = 0.25
20
21 pos1Vect = collect(minPos:stepPos:maxPos)
22 pos2Vect = collect(minPos:stepPos:maxPos)
23 posVectSize = size(pos1Vect, 1)
24 posMat = [(i, j) for i in pos1Vect, j in pos2Vect]
25
26 posToIndex(pos) = Int32((pos - minPos) / stepPos + 1)
27
28 initState(q) = initStateFunction(q[1], q[2], 1, 2, 2, 1, -2, -2)
29 initMat = initState.(posMat)
30 initMat = round.(initMat, digits = 7)
31
32 surface(pos1Vect, pos2Vect, abs.(initMat))
33
34 finalTime = 1
35 finalMat = Matrix{ComplexF32}(undef, posVectSize, posVectSize)
36
37 for xf in pos1Vect, xfp in pos2Vect
38     sumPos = 0
39     for i in 1:posVectSize, j in 1:posVectSize

```

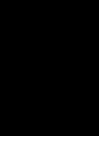
```
40     x0 = pos1Vect[i]
41     x0p = pos2Vect[j]
42     sumPos += freePropagatorC(xf, xfp, x0, x0p, finalTime, 0) * initMat[i, j]
43     end
44     finalMat[posToIndex(xf), posToIndex(xfp)] = sumPos
45 end
46
47 surface(pos1Vect, pos2Vect, abs2.(finalMat))
```



CHAPTER 6

**The Schrödinger's equation for two  
particles**





# **Propagator and joint probability distribution in the momentum basis**



## Integrals evaluation code

All of these codes that I've written are in the Julia language, which I've imported three packages: `SymPy`, `OffsetArrays`, and `Plots`; `plotlyjs()`

### A.1 Polynomial extraction

Since we're going to be doing a lot of polynomials rearranging, I've implemented the polynomial extraction function as follows:

```
1 function extractPolynomial(expr, arg)
2     expr isa Sym ? nothing : expr = sympify(1)
3     expr = expand(expr)
4     polyDegree = Int(degree(expr, arg))
5     extractOneSet = []
6     for i in 0:polyDegree
7         temp = expr.coeff(arg, i)
8         push!(extractOneSet, temp)
9     end
10    polyExt = OffsetVector(extractOneSet, 0:polyDegree)
11    return polyExt
12 end
```

This function accepts two arguments: `expr`, which is the expression you want to extract, and `arg`, the variable that you extract with respect to. For example, inputting  $ax^2 + bx + c$ , `x` would give



out

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \quad (\text{A.1})$$

which is a zero index matrix.

## A.2 Code for the spring problem

### A.2.1 First order perturbation

The first order perturbation uses the following symbols:

```

1 @syms q0 q1 q2
2 @syms m::(real, positive) t::(real, positive)
3 q0', q1', q2', qf', qf = symbols("q_{0}^{\prime}, q_{1}^{\prime}, q_{2}^{\prime},
   ↪ q_{F}^{\prime}, q_F")
4 t0, t1, t2, tf = symbols("t_0, t_1, t_2, t_F", real = true, positive = true)

```

The following code is used to aid the evaluation of integrals.

```

1 # Simplifying the product of propagators in eq. 2.25
2 idenFunc = IM*m/(2 * (tf - t)) * (qf - q1)^2 + IM*m/(2*t) * (q1 - q0)^2
3
4 idenPoly = extractPolynomial(idenFunc, q1)
5 idenPolyA = -idenPoly[2]
6 idenPolyB = -idenPoly[1]
7 idenPolyC = -idenPoly[0]
8
9 display(idenPolyA)
10 display(idenPolyB)
11 display(idenPolyC)
12

```