Dynamical analysis of interacting bipartite entangled systems via the Feynman path integral method

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This version is not to be published. It is my own notes during the research.

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CHAPTER 1

Introduction

1.1 Historical backgrounds[1]

Light, the first gateway to the quantum world of the humankind. I've always been intrigued by it. And, we've been able to confirm that light can be entangled, causing correlations in its polarization axes. Most modern development of physics has been into that field. But often, we tend to forget what we left behind four hundred years ago: the principle of least action. What happens when light passes through different medium? Refraction, everyone can answer that. But what happens if light is entangled?

To resolve this problem, I originally turned to the Schrödinger's equation for two particles. However, it's quite complex, and the mathematics behind it doesn't really elude me that much. So, I turned to the path integrals method. Its ability to intuitively produce the classical world, governed by the principle of least action, is captivating. The action principle appears so naturally in path integrals that I thought I was hallucinating. Question arises: what does the principle of least action looks like for a many body systems? Is it any different if they're entangled? I was literally mashing random words together at that point, but it has led me into the realm of physics that I don't think anyone really cares before: the dynamics of an entangled system.

Entanglement is usually defined as the inseparability of states, commonly used to describe the spin of a particle, which has been studied in great extent. It has found its use in quantum computer, giving its blazing speed. However, from my knowledge, no literatures has mentioned the effect of entanglement on the movement of particles. Will entangled particle move together in the same direction? If I separate a pair of entangled particles, put one in a potential, and let the other be free, does the particle in the potential have any effect on the free particle? And thus, this research (*project*) was born.

[1]This introduction is rewritten after the school project has passed. I've declared this project to be independent of the school ever since. So, it's going to be much more authentic and straightforward.

This research started out by two people; me, and the other one that shall not be named. However, I have a lot of conflicts with that person. Long story short, he quitted; and thus, I'm on my own journey.

List of Symbols (Still incomplete)

1	Identity unit
$\tilde{A}$	Extension of arbitrary operator A defined in $\mathcal{E}_n$ onto $\mathcal{E}$
$\mathcal{D}[q]$	Path integral measure on a variable
e	The Euler's constant
$\mathcal{E}$	Hilbert space of the whole system
$\mathcal{E}_n$	The n^{th} subspace of Hilbert space
h	Planck's constant
$\hbar$	Reduced Planck's constant. $\hbar = h/2\pi$
$\mathcal{H}$	Hamiltonian of a system
i	The imaginary unit
$\mathbb{I}$	The identity matrix/identity operator
$K_{\mathcal{A}}(q_F, t_F; q_0, t_0)$	Propagator from point q_F, t_F to q_0, t_0 using the Lagrangian of particle $\mathcal{A}$
$K_0(q\prime, t\prime; q_0, t_0)$	Propagator of a separable system. Extra arguments represents the usage on bipartite systems.
p	Generalized momentum
q	Generalized position
$\{\lvert u_i \rangle\}$	Set of arbitrary basis $\lvert u_1 \rangle, \lvert u_2 \rangle, \ldots$ of a Hilbert space with index i
$\hat{U}(t_B, t_A)$	Time evolution operator. Evolves the time of the system from t_A to t_B
α_ψ	Eigenvalue of arbitrary ket $\lvert \psi \rangle$
$\lvert \eta \rangle$	Ket of the whole system. Used when describing a bipartite system
$\lvert \xi \rangle$	Ket of the subsystems. Used when describing a monopartite system
π	The Archimedes' constant
$\otimes$	Tensor product

CHAPTER 2

The generalized Gaussian integrals

A Gaussian integral is the integral of the Gaussian function $\exp\left[-x^2\right]$:

$$\int_{\infty}^{-\infty} \exp\left[-x^2\right] \mathrm{d}x = \sqrt{\pi}. \tag{2.1}$$

However, the path integral method usually demands a more generalized form of the Gaussian integral denoted in this study with Ω_n:

$$\Omega_n(a, b, c; x) \equiv \int_{\infty}^{-\infty} x^n \exp\left[-(ax^2 + bx + c)\right] \mathrm{d}x\,, \tag{2.2}$$

where a, b, and c are constants and n is a positive integer. Because we're mostly going to be dealing with indefinite integrals, all integrals without a bound is assumed to be evaluated from $-\infty$ to ∞.

To evaluate Ω_n (eq. (2.2)), we shall complete the square first, then substitute the exponents to fit the form of the standard Gaussian integral.

$$\begin{aligned}
\int x^n \mathrm{e}^{-(ax^2+bx+c)} \,\mathrm{d}x &= \int x^n \exp\left[-a\left(x + \frac{b^2}{2a}\right)^2 + \frac{b^2}{4a} - c\right] \mathrm{d}x \\
&= \mathrm{e}^{\frac{b^2}{4a} - c} \int x^n \exp\left[-a\left(x + \frac{b^2}{2a}\right)\right] \mathrm{d}x\,.
\end{aligned}$$

Let

$$-u^2 = -a\left(x + \frac{b^2}{2a}\right)^2 \tag{2.3}$$

$$x = \frac{u}{\sqrt{a}} - \frac{b^2}{2a} \tag{2.4}$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{u}{\sqrt{a}} - \frac{b^2}{2a}\right) \tag{2.5}$$

$$\mathrm{d}x = \frac{1}{\sqrt{a}} \,\mathrm{d}u\,. \tag{2.6}$$

Then,

$$\Omega_n = \mathrm{e}^{\frac{b^2}{4a} - c} \frac{1}{\sqrt{a}} \int \left(\frac{u}{\sqrt{a}} - \frac{b^2}{2a}\right)^n \mathrm{e}^{-u^2} \,\mathrm{d}u\,. \tag{2.7}$$

$$= \mathrm{e}^{\frac{b^2}{4a}-c}\frac{1}{\sqrt{a}}\int\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b^2}{2a}\right)^{n-k}\left(\frac{u}{\sqrt{a}}\right)^{k}\mathrm{e}^{-u^2}\,\mathrm{d}u\,. \tag{2.8}$$

$$= \mathrm{e}^{\frac{b^2}{4a}-c}\frac{1}{\sqrt{a}}\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{b^2}{2a}\right)^{n-k}\left(\frac{1}{\sqrt{a}}\right)^{k}\int u^k\mathrm{e}^{u^2}\,\mathrm{d}u\right], \tag{2.9}$$

i.e., Ω_n can be written as a sum of

$$\omega = \int u^k\mathrm{e}^{-u^2}\,\mathrm{d}u\,. \tag{2.10}$$

Notice that when n is odd, the integrand of ω is an odd function; thus, ω is zero when n is odd. When n is even however, the integrand is even and thus can be simplified into

$$\omega = 2\int_0^\infty u^{2m}\mathrm{e}^{-u^2}\,\mathrm{d}u \tag{2.11}$$

where $k = 2m$. We then do another substitution by letting $-t = -u^2$. Thus, $u = \sqrt{t}$ and $\mathrm{d}u = \mathrm{d}t/2\sqrt{t}$. The bound of the integral isn't affected by a radical. Our integral then becomes

$$\omega = 2\times\int_0^\infty t^m\mathrm{e}^{-t}\frac{1}{2\sqrt{t}}\,\mathrm{d}t\,. \tag{2.12}$$

$$= \int_0^\infty t^{m-\frac{1}{2}}\mathrm{e}^{-t}\,\mathrm{d}t \tag{2.13}$$

$$= \int_0^\infty t^{m+\frac{1}{2}-1}\mathrm{e}^{-t}\,\mathrm{d}t \tag{2.14}$$

$$= \Gamma\left(m+\frac{1}{2}\right) \tag{2.15}$$

$$= \Gamma\left(\frac{k+1}{2}\right). \tag{2.16}$$

Since the integral evaluates to the gamma function for only even numbers, we add a term that makes $\omega = 0$ when n is odd:

$$\omega = \frac{1}{2}((-1)^k+1)\,\Gamma\left(\frac{k+1}{2}\right). \tag{2.17}$$

Thus, from eq. (2.9),

$$\Omega_n = \mathrm{e}^{\frac{b^2}{4a}-c}\frac{1}{\sqrt{a}}\sum_{k=0}^{n}\left[\binom{n}{k}\left(\frac{b^2}{2a}\right)^{n-k}\left(\frac{1}{\sqrt{a}}\right)^{k}\frac{1}{2}((-1)^k+1)\,\Gamma\left(\frac{k+1}{2}\right)\right] \tag{2.18}$$

To continue this, we then expand the combinatorics and use the formulas of arguments with half-integer real part to get

$$\mathrm{e}^{\frac{b^2}{4a}-c}\frac{1}{\sqrt{a}}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(\frac{b^2}{2a}\right)^{n}\left(\frac{2a}{b^2}\right)^{k}\left(\frac{1}{\sqrt{a}}\right)^{k}\left(\sqrt{\pi}\frac{k!}{4^{\frac{k}{2}}\,(k/2)!}\right) \times\frac{1}{2}((-1)^k+1) \tag{2.19}$$

$$= \mathrm{e}^{\frac{b^2}{4a}-c}n!\sqrt{\frac{\pi}{a}}\left(\frac{b^2}{2a}\right)^{n}\sum_{k=0}^{n}\frac{1}{(n-k)!(k/2)!}\left(\frac{\sqrt{a}}{b^2}\right)^{k}\frac{1}{2}((-1)^k+1) \tag{2.20}$$

To simplify the summation, we replace k with $2k$, and the upper limit with m where $m = \lfloor n/2 \rfloor$. Thus, we get

$$\left[\sum_{k=0}^{m}\frac{1}{(n-2k)!k!}\left(\frac{\sqrt{a}}{b^2}\right)^{2k}\right]\mathrm{e}^{\frac{b^2}{4a}-c}n!\left(\frac{b^2}{2a}\right)^{n}\sqrt{\frac{\pi}{a}} \tag{2.21}$$

As a convention in this study, I shall define

$$\int_{\infty}^{\infty} x^n \exp\left[-ax^2 - bx - c\right] \mathrm{d}x = \Omega_n(a,b,c;x) \equiv \zeta_n(a,b,c)\kappa(a,b,c) \tag{2.22}$$

where

$$\zeta_n(a,b,c) \equiv \left[\sum_{k=0}^{m} \frac{1}{(n-2k)!k!}\left(\frac{\sqrt{a}}{b^2}\right)^{2k}\right] n!\left(\frac{b^2}{2a}\right)^n \sqrt{\frac{\pi}{a}}, \tag{2.23}$$

$$\kappa(a,b,c) \equiv \exp\left[\frac{b^2}{4a} - c\right]. \tag{2.24}$$

I shall refer to ζ as the coefficient of the Gaussian integral, and κ, the resulting exponent. The following section is a table for values of ζ_n from $n = 1$ to 10.

Comment on the generalized Gaussian integral The generalized Gaussian integral can be cast in the form of Meijer's G function. For $n \geq 0$,

$$\int x^n \exp\left[-ax^2 - bx - c\right] \mathrm{d}x$$
$$= \frac{e^{-c}}{2\sqrt{\pi}ab}\left[-\frac{2^{n+1}a}{b^n} G^{1,2}_{2,1}\left(\begin{matrix}\frac{1-n}{2}, -\frac{n}{2} \\ 0\end{matrix}\middle| \frac{4ae^{-2i\pi}}{b^2}\right) + a^{\frac{1-n}{2}} b G^{2,1}_{1,2}\left(\begin{matrix}\frac{1-n}{2} \\ 0, \frac{1}{2}\end{matrix}\middle| \frac{b^2}{4a}\right)\right] \tag{2.25}$$

However, this form isn't that helpful because the Meijer's G function is defined with integral of products.

Originally, I was going to evaluate everything using `SymPy`. However, `SymPy` doesn't bother integrating any Gaussian integrals that have complex arguments. My own implementation is written in chapter 5

2.1 Tables of Gaussian integrals with varying orders

ζ_0: $+\frac{1}{1}\sqrt{\frac{\pi}{a}}$

ζ_1: $-\frac{b}{2}\sqrt{\frac{\pi}{a^3}}$

ζ_2: $+\frac{1}{4}\sqrt{\frac{\pi}{a^5}}(2a + b^2)$

ζ_3: $-\frac{b}{8}\sqrt{\frac{\pi}{a^7}}(6a + b^2)$

ζ_4: $+\frac{1}{16}\sqrt{\frac{\pi}{a^9}}(12a^2 + 12ab^2 + b^4)$

ζ_5: $-\frac{b}{32}\sqrt{\frac{\pi}{a^{11}}}(60a^2 + 20ab^2 + b^2)$

ζ_6: $+\frac{1}{64}\sqrt{\frac{\pi}{a^{13}}}(120a^3 + 180a^2b^2 + 30ab^4 + b^6)$

ζ_7: $-\dfrac{b}{128}\sqrt{\dfrac{\pi}{a^{15}}}(840a^3 + 420a^2b^2 + 42ab^4 + b^6)$

ζ_8: $+\dfrac{1}{256}\sqrt{\dfrac{\pi}{a^{17}}}(1680a^4 + 3360a^3b^2 + 840a^2b^4 + 56ab^6 + b^8)$

ζ_9: $-\dfrac{b}{512}\sqrt{\dfrac{\pi}{a^{19}}}(15120a^4 + 10080a^3b^2 + 1512a^2b^4 + 72ab^6 + b^8)$

CHAPTER 3

Path integrals of bipartite systems in the position basis

3.1 General theory for the propagator of two particles

3.1.1 Non-interacting systems

The propagator for a single particle system is used to find the probability distribution $\tilde{P}(x)$ of a state at anytime, i.e., for a state $|\psi(t_0)\rangle$ of a single particle,

$$\tilde{P}(q_F, t_F) = \langle q_F|\psi(t_F)\rangle = \int K(q_F, q_0; t_F - t_0)\psi(q_0, t_0)\,\mathrm{d}q_0\,. \tag{3.1}$$

The probability of finding the particle at any point (q_F, t_F) in spacetime is defined as

$$P(q_F, t_F) \equiv |\tilde{P}(q_F, t_F)|^2 \tag{3.2}$$

This idea can be easily extended to describe any multiparticle system. But we shall only concern two particles systems in this study. Because the conventional notation is quite cumbersome, I shall introduce my own concise notation. From now on, every unprimed variable belong to particle one, and every unprimed variable, to particle one. E.g., the first particle Hilbert space shall be written with $\mathcal{E}$, and for the second particle, $\mathcal{E}_2$. This extends to all physical variables, e.g., position and momenta. The combined state shall be referred as η, and the separate states as ξ. Note that primed time do not exist; the interaction must happen simultaneously at a point in time; therefore, all the time variables t will not be primed.

Let there be a combined state ket of two subsystems $|\eta\rangle$. If η is separable at any point in time t_0, it follows that

$$|\eta\rangle = |\xi\rangle \otimes |\xi'\rangle\,. \tag{3.3}$$

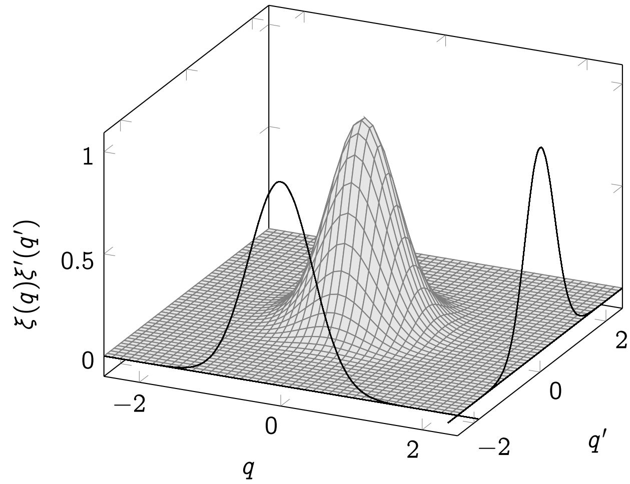

FIG. 3.1 | EXAMPLE OF THE JOINT PROBABILITY AMPLITUDE OF A GAUSSIAN WAVE PACKET.

If there are no interactions between the subsystems following t_0, the Hamiltonian operator must be separable which also implies that the time evolution operator $\hat{U}$ is also separable. From the property of tensor product, the quantity $\langle q, q'|\eta(t_0)\rangle$ can be written as

$$\langle q, q'|\eta(t_0)\rangle = \left[\langle q| \otimes \langle q'|\right] \left[|\xi\rangle \otimes |\xi'\rangle\right] = \langle q|\xi(t_0)\rangle \langle q'|\xi'(t_0)\rangle = \xi(q, t_0)\xi'(q', t_0), \tag{3.4}$$

which represents the product of the probability that ξ is at (q, t_0), and ξ' is at q', t_0. We call $\langle q, q'|\eta(t_0)\rangle$, the **joint probability distribution**. The modulus squared of the joint probability distribution gives the **joint probability amplitude**, $|\langle q, q'|\eta(t_0)\rangle|^2$ which represents the probability to find ξ at (q, t_0) and ξ' at (q', t_0).

In the position basis, the joint probability amplitude is a function that takes in the position of both particles and outputs the probability that both ξ and ξ' will be there. Therefore, the joint probability amplitude encodes a surface, which can be plotted as shown in fig. 3.1. The probability amplitude of each particle can be found by projecting the surface onto the axis of each corresponding position basis.

3.1.2 Interacting systems

Let there be a state ket $|\eta\rangle$ which describes the quantum state of two particles. When the two particles are interacting, the time-evolution operator cannot be factored into tensor products of two operators. Therefore, there can only be one combined time-evolution operator for both of the systems:

$$\hat{U}(t_F, t_0) = \exp\left[-\mathrm{i}\Delta t \left(\frac{\hat{p}^2}{2m} + \frac{\hat{p}'^2}{2m'} + V(\hat{q}, \hat{q}', t)\right)\right] \tag{3.5}$$

The consequence of the combined time-evolution operator is that, we cannot calculate the joint probability distribution between different time of the subsystem. For simplicity' sake, let the index

$j = i + 1$, where $t_j = t_i + \Delta t$ in which $\Delta t \to 0$.

$$\left\langle q_j, q'_j \middle| \eta(t_j) \right\rangle \tag{3.6}$$
$$= \left\langle q_j, q'_j \middle| \hat{U}(t_j, t_i) \middle| \eta(t_i) \right\rangle . \tag{3.7}$$
$$= \iint \mathrm{d}q_i \, \mathrm{d}q'_i \, \left\langle q_j, q'_j \middle| \hat{U}(t_j, t_i) \middle| q_i, q'_i \right\rangle \left\langle q_i, q'_i | \eta(t_i) \right\rangle . \tag{3.8}$$

As seen, the form of transition element is

$$\left\langle q_j, q'_j \middle| \exp\left[-\mathrm{i}\Delta t \left(\frac{\hat{p}^2}{2m} + \frac{\hat{p}'^2}{2m'} + V(\hat{q}, \hat{q}', t) \right) \right] \middle| q_i, q'_i \right\rangle . \tag{3.9}$$

The terms in the exponents can be separated due to the vanishing commutator when $\Delta t \to 0$. By separating the terms in the exponents and inserting two complete sets of momentum basis, the equation above turns into

$$\begin{aligned}
&\frac{1}{(2\pi)^2} \iint \mathrm{d}p \, \mathrm{d}p' \, \langle q_j, q'_j | \exp\left[-\mathrm{i}\Delta t \frac{\hat{p}^2}{2m} \right] | p, p' \rangle \, \langle p, p' | \exp\left[-\mathrm{i}\Delta t \frac{\hat{p}'^2}{2m'} \right] \exp\left[-\mathrm{i}\Delta t V(\hat{q}, \hat{q}', t) \right] | q_i, q'_i \rangle \\
&= \frac{1}{(2\pi)^2} \iint \mathrm{d}p \, \mathrm{d}p' \exp\left[-\mathrm{i}\Delta t \left(\frac{p^2}{2m} + \frac{p'^2}{2m'} + V(q_i, q_i, \Delta t) \right) \right] \langle q_j | p \rangle \, \langle q'_j | p' \rangle \, \langle q_i | p \rangle^* \, \langle q'_j | p' \rangle^* \\
&= \frac{1}{(2\pi)^4} \mathrm{e}^{-\mathrm{i}\Delta t V(q_i, q'_i, t)} \int \mathrm{d}p \exp\left[-\mathrm{i}\Delta t \frac{p^2}{2m} + \mathrm{i}p(q_j - q_i) \right] \int \mathrm{d}p' \exp\left[-\mathrm{i}\Delta t \frac{p'^2}{2m'} + \mathrm{i}p'(q'_j - q'_i)^2 \right] \\
&= \frac{(mm')^{\frac{1}{2}}}{8\pi^3 \mathrm{i}\Delta t} \exp\left[-\mathrm{i}\Delta t V(q_i, q'_i, t) + \frac{\mathrm{i}m'}{2\Delta t}(q'_j - q'_i)^2 + \frac{\mathrm{i}m}{2\Delta t}(q_j - q_i)^2 \right].
\end{aligned} \tag{3.10}$$

To find the propagator for an interacting system, we need to perform successive integrals on q and q', i.e.,

$$K_\eta = \int \cdots \int \mathrm{d}q_N \, \mathrm{d}q'_N \ldots \mathrm{d}q_1 \, \mathrm{d}q'_1 \, \langle q_F, q'_F | \hat{U}(t_F, t_N) | q_N, q'_N \rangle \ldots \langle q_1, q'_1 | \hat{U}(t_1, t_0) | q_0, q'_0 \rangle \tag{3.11}$$

Notice that when there is no interaction between the two systems ($V = 0$), the integrals become separable and reduces down to the form of the non-interacting system's propagator but off by a normalization factor.

There are two common forms of interaction, which is the spring interaction and the coulomb interaction. Both of which includes the term $(q_i - q'_i)^2$ in $V(q, q')$, which causes major problems in integration. When expanded, there is a $q_i q'_i$ term that makes the integral inseparable which causes the integral pattern to not repeat; therefore, we resort to perturbation.

3.1.3 Form of the two particle perturbation series

The perturbation series for one particle are already given by Feynman in his path integrals textbook:

$$K_n(F, 0) = (-\mathrm{i})^n \int \cdots \int K_0(F, n) \prod_{i=1}^{n} V(i) K_0(i, i-1) \, \mathrm{d}\tau_i \tag{3.12}$$

where K_n is the n'th order perturbation, $K(j,k) = K(q_j, t_j; q_k, t_k)$, $K = \sum_n K_n$ and $\mathrm{d}\tau_i = \mathrm{d}q_i\,\mathrm{d}q_i\,\mathrm{d}t_i$. To apply it with two particles, we extend it as follows.

The propagator for two particles can be written as

$$
\begin{aligned}
&K(q_F, q_F'; q_0, q_0'; t_F - t_0) \\
&= \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 - V(q,q')\,\mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] &\quad (3.13) \\
&= \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \exp\left[-\mathrm{i} \int_{t_0}^{t_F} V(q,q')\,\mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] &\quad (3.14)
\end{aligned}
$$

The exponential can be expanded using the Taylor series:

$$
\begin{aligned}
\exp\left[-\mathrm{i} \int_{t_0}^{t_F} V(q,q')\,\mathrm{d}t\right] &= 1 + (-\mathrm{i}) \int_{t_0}^{t_F} V(q,q')\,\mathrm{d}t \\
&\quad + \frac{1}{2!}(-\mathrm{i})^2 \left(\int_{t_0}^{t_F} V(q,q')\,\mathrm{d}t\right)^2 + \frac{1}{3!}(-\mathrm{i})^3 \left(\int_{t_0}^{t_F} V(q,q')\,\mathrm{d}t\right)^3 \qquad (3.15)
\end{aligned}
$$

We then insert this into the original integral. To avoid confusion, I've changed the integration variable from t to s.

$$
\begin{aligned}
&K(q_F, q_F'; q_0, q_0'; t_F - t_0) &\quad (3.16) \\
&= \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] &\quad (3.17) \\
&\quad + (-\mathrm{i})^2 \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] \int_{t_0}^{t_F} V(q(s), q'(s))\,\mathrm{d}s \\
&\quad + \frac{1}{2!}(-\mathrm{i})^3 \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] \int_{t_0}^{t_F} \\
&\qquad \times \int_{t_0}^{t_F} V(q(s), q'(s))\,\mathrm{d}s \int_{t_0}^{t_F} V(q(s'), q'(s'))\,\mathrm{d}s' + \ldots
\end{aligned}
$$

In which,

$$
K_0(q_F, q_F'; q_0, q_0'; t_F - t_0) = \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] \qquad (3.18)
$$

$$
\begin{aligned}
&K_1(q_F, q_F'; q_0, q_0'; t_F - t_0) \\
&= (-\mathrm{i})^2 \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] \int_{t_0}^{t_F} V(q(s), q'(s))\,\mathrm{d}s \qquad (3.19)
\end{aligned}
$$

$$
\begin{aligned}
&K_2(q_F, q_F'; q_0, q_0'; t_F - t_0) \\
&= \frac{1}{2!}(-\mathrm{i})^3 \int_{q_0'}^{q_F'} \int_{q_0}^{q_F} \exp\left[\mathrm{i} \int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] \qquad (3.20) \\
&\qquad \times \int_{t_0}^{t_F} V(q(s), q'(s))\,\mathrm{d}s \int_{t_0}^{t_F} V(q(s'), q'(s'))\,\mathrm{d}s'
\end{aligned}
$$

and so on and so forth. To evaluate each of the perturbation terms, we exchange the integral order and put the time integral at the front. E.g., for the first perturbation term,

$$K_1(q_F, q'_F; q_0, q'_0; t_F - t_0) = (-\mathrm{i}) \int_{t_0}^{t_F} F_1(s)\,\mathrm{d}s \tag{3.21}$$

where

$$\begin{aligned} F_1(s) &= \int_{q'_0}^{q'_F}\int_{q_0}^{q_F} \exp\left[\mathrm{i}\int_{t_0}^{t_F} \frac{m}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 + \frac{m}{2}\left(\frac{\mathrm{d}q'}{\mathrm{d}t}\right)^2 \mathrm{d}t\right] \mathcal{D}[q]\mathcal{D}[q'] \times V(q(s), q'(s)) &(3.22)\\ &= K_0(q_F, q'_F; q_0, q_0; t_F - t_0) V(q(s), q'(s)). &(3.23) \end{aligned}$$

I shall name $(q(s), q'(s')) = (q_1, q'_1)$. Using the composition property,

$$K(q_F, q'_F; q_0, q'_0; t_F - t_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(q_F, q'_F; q_1, q'_1; t_F - t_1) K_0(q_1, q'_1; q_0, q_0; t_1 - t_0)\,\mathrm{d}q_1\,\mathrm{d}q'_1 \tag{3.24}$$

and therefore,

$$F_1(q_1, q'_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(q_F, q'_F; q_1, q'_1; t_F - t_1) V(q_1, q'_1) K_0(q_1, q'_1; q_0, q_0; t_1 - t_0), \tag{3.25}$$

$$\begin{aligned} &K_1(q_F, q'_F; q_0, q'_0; t_F - t_0)\\ &= (-\mathrm{i}) \int_{t_0}^{t_F}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(q_F, q'_F; q_1, q'_1; t_F - t_1) V(q_1, q'_1) K_0(q_1, q'_1; q_0, q_0; t_1 - t_0)\,\mathrm{d}s\,. \end{aligned} \tag{3.26}$$

Using similar argument, we can write the n'th perturbation term as

$$K_n(F, 0; F', 0') = (-\mathrm{i})^n \int \cdots \int K_0(F, n) K_0(F', n') \left[\prod_{i=1}^{n} V(i) K_0(i, i-1) K_0(i', i'-1)\,\mathrm{d}q_i\,\mathrm{d}q'_i\,\mathrm{d}t_i\right]. \tag{3.27}$$

where $i = (q_i, t_i)$. We've dropped the $1/n!$ term off due to the bounds of the time integral, similar to how Feynman did it in his book [1]. The joint probability distribution can also be expanded in the form of the perturbation series.

$$\begin{aligned} \eta(q_F, q'_F; t_F) &= \int K(q_F, q'_F; q_0, q'_0; t_F - t_0) \eta(q_0, q'_0; t_0)\,\mathrm{d}q_0\,\mathrm{d}q'_0 &(3.28)\\ &= \int \sum_n K_n(q_F, q'_F; q_0, q'_0; t_F - t_0) \eta(q_0, q'_0; t_0)\,\mathrm{d}q_0\,\mathrm{d}q'_0 &(3.29)\\ &= \sum_n \int K_n(q_F, q'_F; q_0, q'_0; t_F - t_0) \eta(q_0, q'_0; t_0)\,\mathrm{d}q_0\,\mathrm{d}q'_0 &(3.30) \end{aligned}$$

To calculate the joint probability distribution directly, we'd have to use the Born's expansion, but extended to two particles. For the first order Born's approximation,

$$\begin{aligned} \eta(q_F, q'_F; t_F) &= \eta_0(q_F, q'_F; t_F)\\ &- \mathrm{i} \iiint K(q_F, q'_F; q_0, q'_0; t_F - t) K(q_F, q'_F; q_0, q'_0; t_F - t) V(q_0, q'_0) \eta_0(q_0, q'_0; t_0)\,\mathrm{d}q_0\,\mathrm{d}q'_0\,\mathrm{d}t \end{aligned} \tag{3.31}$$

3.2 Density operator and its related quantities in the position basis

In this section, we'll develop the mathematical foundations for analyzing the correlation and entanglement entropy of a bipartite system. We shall focus on the density matrix ($\hat{\rho}$), its trace and partial trace. The trace is used to find the correlation between the two subsystems, and the partial trace, the entanglement entropy. Let us consider a bipartite system under a potential V that doesn't have any internal interactions. The density operator of a bipartite system η can be written as

$$\hat{\rho} = |\eta\rangle\langle\eta| \tag{3.32}$$

If the subsystems are denoted ξ and ξ', each one of them can be expanded into the position basis $\{|q_n\rangle\}$ and $\{|q'_n\rangle\}$:

$$|\xi\rangle = \int \xi(q)\,|q\rangle\,\mathrm{d}q \quad \text{and,} \quad |\xi'\rangle = \int \xi'(q')\,|q'\rangle\,\mathrm{d}q' \tag{3.33}$$

The total state $|\eta\rangle = |\xi\rangle \otimes |\xi'\rangle$ can also be expanded

$$|\eta\rangle = \iint \xi(q)\xi'(q')\,|q,q'\rangle\,\mathrm{d}q\,\mathrm{d}q'\,. \tag{3.34}$$

Analogously,

$$\langle\eta| = \iint \xi^*(q)\xi'(q')\,\langle q,q'|\,\mathrm{d}q\,\mathrm{d}q' \tag{3.35}$$

Therefore, we can write the density operator $\hat{\rho}$ as

$$\hat{\rho} = \iiiint \xi(q)\xi(q')\xi^*(x)\xi^*(x')\,|q,q'\rangle\langle x,x'|\,\mathrm{d}q\,\mathrm{d}q'\,\mathrm{d}x\,\mathrm{d}x'\,, \tag{3.36}$$

where I have changed the integral variable from q, q' to x, x'. The full trace of the density operator is the sum along the diagonal, which for our state

$$\begin{aligned}
\mathcal{T}\mathrm{r}(\hat{\rho}) &= \mathcal{T}\mathrm{r}\big(|\eta\rangle\langle\eta|\big) && (3.37)\\
&= \iint \langle q,q'|\eta\rangle\,\langle\eta|q,q'\rangle\,\mathrm{d}q\,\mathrm{d}q' && (3.38)\\
&= \iint \xi(q)\xi^*(q)\xi'(q')\xi'^*(q')\,\mathrm{d}q\,\mathrm{d}q' && (3.39)\\
&= \iint |\xi(q)\xi'(q')|^2\,\mathrm{d}q\,\mathrm{d}q' = \iint |\eta(q,q')|^2\,\mathrm{d}q\,\mathrm{d}q'\,. && (3.40)
\end{aligned}$$

Or alternatively, we can expand the trace as a sum, which in the position basis becomes an integral, then use the commutation between the integral and the sum. Here, I renamed the integration variable as y, y'.

$$\begin{aligned}
\mathcal{T}\mathrm{r}(\hat{\rho}) &= \mathcal{T}\mathrm{r}\Big(\int\cdots\int \xi(x)\xi'(x')\xi^*(y)\xi'^*(y')\,|x,x'\rangle\langle y,y'|\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'\Big) && (3.41)\\
&= \int\cdots\int \langle q,q'|\Big[\int\cdots\int \xi(x)\xi'(x')\xi^*(y)\xi'^*(y')\,|x,x'\rangle\langle y,y'|\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'\Big]|q,q'\rangle\,\mathrm{d}q\,\mathrm{d}q'\\
&= \int\cdots\int \xi(x)\xi'(x')\xi^*(y)\xi'^*(y')\delta(q-x)\delta(q'-x')\delta(q-y)\delta(q'-y')\,\mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}y\,\mathrm{d}y'\,\mathrm{d}q\,\mathrm{d}q'
\end{aligned}$$

$$= \iint \xi(q)\xi^*(q)\xi'(q')\xi'^*(q')\,\mathrm{d}q\,\mathrm{d}q' \tag{3.42}$$

$$= \iint |\xi(q)\xi'(q')|^2\,\mathrm{d}q\,\mathrm{d}q' = \iint |\eta(q,q')|^2\,\mathrm{d}q\,\mathrm{d}q'\,. \tag{3.43}$$

If the state is pure, we can use the normalization condition to say that the trace of the density operator must be equal to one.

The expectation value of an operator $\hat{A}$ on a state $|\eta\rangle$ is generally defined as

$$\langle \hat{A} \rangle = \frac{\langle \eta|\hat{A}|\eta\rangle}{\langle \eta|\eta\rangle} \tag{3.44}$$

Assuming that the state is normalized, $\langle \eta|\eta\rangle$ is one and hence $\left\langle \hat{A} \right\rangle = \langle \eta|\hat{A}|\eta\rangle$. Since the norm is one, we can also write the expectation value of $\hat{A}$ as

$$\langle \hat{A} \rangle = \langle \eta|\hat{A}|\eta\rangle \langle \eta|\eta\rangle \tag{3.45}$$

$$= \langle \eta| \left[\hat{A}\,|\eta\rangle\langle\eta|\right] |\eta\rangle \tag{3.46}$$

$$= \mathcal{T}\mathrm{r}\big(\hat{A}\,|\eta\rangle\langle\eta|\big) = \mathcal{T}\mathrm{r}\big(\hat{A}\hat{\rho}\big). \tag{3.47}$$

The covariance between two operators: $\hat{A}$ and $\hat{B}$, $\mathrm{Cov}_{\hat{A}\hat{B}}$ is defined as

$$\mathrm{Cov}_{\hat{A}\hat{B}} = \langle \hat{A}\hat{B} \rangle - \langle \hat{A} \rangle \times \langle \hat{B} \rangle = \mathcal{T}\mathrm{r}\big(\hat{A}\hat{B}\hat{\rho}\big) - \mathcal{T}\mathrm{r}\big\{\hat{A}\rho\big\}\,\mathcal{T}\mathrm{r}\big\{\hat{B}\rho\big\}, \tag{3.48}$$

which represents the direction of the relationship between two variables. If the covariance is positive, that means the two operators tend to correlate. If it's negative, then the two operators tend to anti-correlate. And, if the covariance is zero, the two operators are linearly dependent. However, it does not include other forms of dependencies, e.g., quadratic, or cubic. To find the correlation between two operators ($\mathrm{Corr}_{\hat{A}\hat{B}}$), we have to divide the covariance by the product of standard deviation.

$$\mathrm{Corr}_{\hat{A}\hat{B}} = \frac{\mathrm{Cov}_{\hat{A}\hat{B}}}{\sigma_A \sigma_B} \quad \text{and,} \quad \sigma_{\hat{A}} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}, \tag{3.49}$$

However, I shall primarily focus on the covariance: it should be enough to give the big picture of the degree of correlations. This is solely because the tediousness of evaluating the standard deviation.

Let us then consider the partial trace of the density operator. Given a bipartite system, the density operator of the first system (reduced density) can be written as

$$\hat{\rho}_1 = \mathcal{T}\mathrm{r}_2(\hat{\rho}) \tag{3.50}$$

where $\mathcal{T}\mathrm{r}_n$ is the partial trace over the n'th subsystem. The operator $\hat{\rho}_1$ has matrix elements

$$\langle q_n|\hat{\rho}_1|q_{n'}\rangle = \int \langle q_n, q'|\,\hat{\rho}\,|q_{n'}, q'\rangle\,\mathrm{d}q' \tag{3.51}$$

$$= \int \langle q_n, q'|\eta\rangle\,\langle \eta|q_{n'}, q'\rangle\,\mathrm{d}q' \tag{3.52}$$

$$= \int \eta(q_n, q')\eta^*(q_{n'}, q')\,\mathrm{d}q' \tag{3.53}$$

Therefore, the reduced density operator of the first system can be written as

$$\hat{\rho}_1 = \iiint \eta(q_n, q')\eta^*(q_{n'}, q')\,|q_n\rangle\langle q_n'|\,\mathrm{d}q_n\,\mathrm{d}q_{n'}\,\mathrm{d}q'\,. \tag{3.54}$$

Analogously,

$$\langle q_n'|\hat{\rho}_2|q_{n'}'\rangle = \int \langle q, q_n'|\,\hat{\rho}\,|q, q_{n'}'\rangle\,\mathrm{d}q = \int \eta(q, q_n')\eta^*(q, q_{n'}')\,\mathrm{d}q\,, \tag{3.55}$$

$$\hat{\rho}_2 = \iiint \eta(q, q_n')\eta^*(q, q_{n'}')\,|q_n'\rangle\langle q_{n'}'|\,\mathrm{d}q_n'\,\mathrm{d}q_{n'}'\,\mathrm{d}q\,. \tag{3.56}$$

We can then use these reduced density operator to analyze the entanglement entropy of a system.

The bipartite von Neumann entanglement entropy $\mathcal{S}$ of is defined as the von Neumann entropy of either of its reduced states.

$$\mathcal{S}(\hat{\rho}_A) = -\mathcal{T}\mathrm{r}\big(\hat{\rho}_A \ln \hat{\rho}_A\big) \tag{3.57}$$

The result is independent of the reduced state chosen. The base of the logarithm might differ from different authors, but for simplicity, I shall use the natural logarithm: they all differ by just a constant factor. The von Neumann entanglement entropy can then be expanded in an integral form.

3.3 Initial states of choice

From now on, I shall use the letter ξ to refer to a state of a single particle, and the letter η to refer to the combined state of two particles.

We are interested in the time evolution of a Gaussian wave packet due to its mathematical simplicity. The normalized state of a Gaussian wave packet is given by

$$\sqrt[4]{\frac{1}{2\pi\sigma}}\exp\left[ipq - \frac{(q-s)^2}{4\sigma^2}\right]. \tag{3.58}$$

For two separable particles, its joint probability is

$$\langle q, q'|\eta(t_0)\rangle = \sqrt[4]{\frac{1}{2\pi\sigma^2}}\sqrt[4]{\frac{1}{2\pi\sigma'^2}}\exp\left[ipq - \frac{(q-s)^2}{4\sigma^2}\right]\exp\left[ip'q' - \frac{(q'-s')^2}{4\sigma'^2}\right] \tag{3.59}$$

$$= \sqrt{\frac{1}{2\pi\sigma\sigma'}}\exp\left[ipq - \frac{(q-s)^2}{4\sigma^2}\right]\exp\left[ip'q' - \frac{(q'-s')^2}{4\sigma'^2}\right] \tag{3.60}$$

As we'll see, the higher order perturbation terms (at least of the spring system) are expressed as the integral of the product between the wave function and some polynomials, which we'll find later.

$$\langle q|\xi(t)\rangle \tag{3.61}$$

$$
\begin{aligned}
&= \int K_0(q_F, q_0; t_F) \langle q_0 | \xi(t) \rangle \, \mathrm{d}q_0 && (3.62)\\
&= \sqrt{\frac{m}{2\pi \mathrm{i} t_F}} \int \exp\left[\frac{\mathrm{i}m}{2t_F}(q_F - q_0)^2\right] \left(\sqrt[4]{\frac{1}{2\pi\sigma}} \exp\left[\mathrm{i}pq_0 - \frac{(q_0 - s)^2}{4\sigma^2}\right]\right) \mathrm{d}q_0 && (3.63)\\
&= \sqrt[4]{-\frac{m^2}{8\pi^3 t_F^2 \sigma^2}} \int \exp\left[\frac{\mathrm{i}m}{2t_F}(q_F - q_0)^2 + \mathrm{i}pq_0 - \frac{(q_0 - s)^2}{4\sigma^2}\right] \mathrm{d}q_0 && (3.64)\\
&= \sqrt[4]{-\frac{m^2}{8\pi^3 t_F^2 \sigma^2}} \int \exp\left[\left(\frac{\mathrm{i}mq_F^2}{2t_F} - \frac{s^2}{4\sigma^2}\right) + q\left(-\frac{\mathrm{i}mq_F}{t_F} + \mathrm{i}p + \frac{s}{2\sigma^2}\right) + q^2\left(\frac{\mathrm{i}m}{2t_F} - \frac{1}{4\sigma^2}\right)\right] \mathrm{d}q_0 && (3.65)\\
&= \sqrt[4]{-\frac{m^2}{8\pi^3 t_F^2 \sigma^2}} \cdot 2\sigma \sqrt{\frac{\pi t_F}{-2\mathrm{i}m\sigma^2 + t_F}} \exp\left[\frac{4\mathrm{i}mpq_F\sigma^2 - mq_F^2 + 2mq_F s - ms^2 - 2\mathrm{i}p^2 t_F \sigma^2 - 2pst_F}{4m\sigma^2 + 2\mathrm{i}t_F}\right]\\
&= \sqrt[4]{\frac{2m^2\sigma^2}{\pi(2m\sigma^2 + \mathrm{i}t_F)^2}} \exp\left[\frac{4\mathrm{i}mpq_F\sigma^2 - mq_F^2 + 2mq_F s - ms^2 - 2\mathrm{i}p^2 t_F \sigma^2 - 2pst_F}{4m\sigma^2 + 2\mathrm{i}t_F}\right]. && (3.66)
\end{aligned}
$$

To plot its probability amplitude, we have to take its modulus squared, which equals to

$$
\frac{m\sigma}{\sqrt{4m^2\sigma^4 + t_F^2}} \exp\left[\frac{-2m^2q_F^2\sigma^2 + 4m^2q_F s\sigma^2 - 2m^2s^2\sigma^2 + 4mpq_F t_F\sigma^2 - 4mpst_F\sigma^2 - 2p^2t_F^2\sigma^2}{4m^2\sigma^4 + t_F^2}\right] \tag{3.67}
$$

3.4 Non-interacting systems: the free particle

3.5 The interacting spring system

The spring potential is given by $V(q, q') = \frac{k}{2}(q - q')^2$. I shall let $\alpha = \frac{k}{2}$ to simplify the notation a bit.

3.5.1 First order perturbation term

From eq. (3.27), set $t_1 = t, t_0 = 0$

$$
\begin{aligned}
&K_1(F, 0; F', 0)\\
&= (-\mathrm{i})^1 \iiint K_0(F, 1) K_0(F', 1') K_0(1, 0) K_0(1, 0') \alpha (q_1 - q_1')^2 \, \mathrm{d}q_1 \, \mathrm{d}q_1' \, \mathrm{d}t\,. && (3.68)\\
&= -\mathrm{i}\alpha \int_0^{t_F} \left[\iint K_0(q_F, q_1; t_F - t) K_0(q_F', q_1'; t_F - t) K_0(q_1, q_0; t) K_0(q_1', q_0'; t)(q_1 - q_1')^2 \, \mathrm{d}q_1 \, \mathrm{d}q_1'\right] \mathrm{d}t\,.
\end{aligned}
$$

Let

$$
I \equiv \iint K_0(q_F, q_1; t_F - t) K_0(q_F', q_1'; t_F - t) K_0(q_1, q_0; t) K_0(q_1', q_0'; t)(q_1 - q_1')^2 \, \mathrm{d}q_1 \, \mathrm{d}q_1'\,; \tag{3.69}
$$

thus, $K_1 = -\mathrm{i}\alpha \int_0^{t_F} I \, \mathrm{d}t$.

We then separate I into three integrals:

$$
I_{P1} = \int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) \, \mathrm{d}q_1 \int K_0(q_F', q_1'; t_F - t) K_0(q_1', q_0'; t) \, \mathrm{d}q_1'\,, \tag{3.70}
$$

$$I_{P2} = \int K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\, \mathrm{d}q_1 \int q_1'^2 K_0(q_F', q_1'; t_F - t) K_0(q_1', q_0'; t)\, \mathrm{d}q_1' \,, \tag{3.71}$$

$$I_{P3} = \int q_1 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\, \mathrm{d}q_1 \int q_1' K_0(q_F', q_1'; t_F - t) K_0(q_1', q_0'; t)\, \mathrm{d}q_1' \,, \tag{3.72}$$

where $I = I_{P1} + I_{P2} + 2I_{P3}$. The integrals without the factor q_1 and q_1^2 can be reduced into the kernel for the free particle:

$$I_{P1} = K_0(q_F', q_0'; t_F) \int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\, \mathrm{d}q_1 \tag{3.73}$$

$$I_{P2} = K_0(q_F, q_0; t_F) \int q_1'^2 K_0(q_F', q_1'; t_F - t) K_0(q_1', q_0'; t)\, \mathrm{d}q_1' \tag{3.74}$$

Since I_{P2} can be obtained by switching all the primed variables with the corresponding unprimed in I_{P1}, we're left with two family of integrals:

$$\int q_1 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\, \mathrm{d}q_1 \quad \text{and,} \quad \int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\, \mathrm{d}q_1 \,. \tag{3.75}$$

To evaluate these, we first simplify the product of kernel under the assumption that $t_F > t$.

$$\begin{aligned}
&K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) \\
&= \sqrt{\frac{m}{2\pi \mathrm{i}(t_F - t)}} \sqrt{\frac{m}{2\pi \mathrm{i} t}} \exp\left[\frac{\mathrm{i}m}{2(t_F - t)}(q_F - q_1)^2 + \frac{\mathrm{i}m}{2t}(q_1 - q_0)^2\right] \qquad (3.76) \\
&= \frac{m}{2\pi} \sqrt{-\frac{1}{t(t_F - t)}} \exp\left[q_1^2 \left(\frac{\mathrm{i}m}{2t} + \frac{\mathrm{i}m}{2(t_F - t)}\right) - q_1 \left(\frac{\mathrm{i}mq_0}{t} + \frac{\mathrm{i}mq_F}{t_F - t}\right) + \left(\frac{\mathrm{i}mq_0^2}{2t} + \frac{\mathrm{i}mq_F^2}{2(t_F - t)}\right)\right] \\
&= \frac{m}{2\pi} \sqrt{-\frac{1}{t(t_F - t)}} \exp\left[-q_1^2 \left(\frac{mt_F}{2\mathrm{i}t(t_F - t)}\right) - q_1(\mathrm{i}m) \left(\frac{q_0}{t} + \frac{q_F}{t_F - t}\right) - \left(\frac{m}{2\mathrm{i}}\right) \left(\frac{q_0^2}{t} + \frac{q_F^2}{t_F - t}\right)\right]
\end{aligned}$$

The normalization factor are pulled out. Both integrals in eq. (3.75) can be evaluated with

$$a = \frac{mt_F}{2\mathrm{i}t(t_F - t)}, \quad b = \mathrm{i}m \left(\frac{q_0}{t} + \frac{q_F}{t_F - t}\right) \quad \text{and,} \quad c = \left(\frac{m}{2\mathrm{i}}\right) \left(\frac{q_0^2}{t} + \frac{q_F^2}{t_F - t}\right) \tag{3.77}$$

in which,

$$\exp\left[\frac{b^2}{4a} - c\right] = \exp\left[\frac{\mathrm{i}m}{2t_F}(q_F - q_0)^2\right] = \sqrt{\frac{2\pi \mathrm{i} t_F}{m}} K_0(q_F, q_0; t_F). \tag{3.78}$$

To summarize,

$$\begin{aligned}
\int q_1 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\, \mathrm{d}q_1 &= -\frac{b}{2}\sqrt{\frac{\pi}{a^3}} \times \frac{m}{2\pi}\sqrt{-\frac{1}{t(t_F - t)}} \times \sqrt{\frac{2\pi \mathrm{i} t_F}{m}} K_0(q_F, q_0; t_F) \\
&= -\frac{b}{2}\sqrt{\frac{\pi}{a^3}} \times \sqrt{\frac{mt_F}{2\pi \mathrm{i} t(t_F - t)}} K_0(q_F, q_0; t_F), \qquad (3.79)
\end{aligned}$$

and

$$\int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\, \mathrm{d}q_1 = \frac{1}{4}\sqrt{\frac{\pi}{a^5}}(2a + b^2) \times \sqrt{\frac{mt_F}{2\pi \mathrm{i} t(t_F - t)}} K_0(q_F, q_0; t_F). \tag{3.80}$$

On the Gaussian integral with degree one,

$$-\frac{b}{2}\sqrt{\frac{\pi}{a^3}} = \sqrt{\frac{2\pi t}{mt_F^3}} \frac{\sqrt{-\mathrm{i}(t_F - t)^3}}{\mathrm{i}(t_F - t)} \times \left[q_0(t_F - t) + q_F t\right]; \tag{3.81}$$

thus from eq. (3.79),

$$\begin{aligned}\int q_1 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t) &= \sqrt{\frac{2\pi t}{m t_F^3}} \frac{\sqrt{-\mathrm{i}(t_F - t)^3}}{\mathrm{i}(t_F - t)} \times \left[q_0(t_F - t) + q_F t\right] \\ &\qquad \times \sqrt{\frac{m t_F}{2\pi \mathrm{i} t (t_F - t)}} K_0(q_F, q_0; t_F) \quad (3.82) \\ &= -\frac{1}{t_F}\left[q_0(t_F - t) + q_F t\right] K_0(q_F, q_0; t_F). \quad (3.83)\end{aligned}$$

On the Gaussian integral with degree two,

$$\frac{1}{4}\sqrt{\frac{\pi}{a^5}}(2a + b^2) = -\sqrt{\frac{2\pi t}{m^3 t_F^5}} \frac{\sqrt{\mathrm{i}(t_F - t)^5}}{(t_F - t)^2}\left[m\left(q_0(t_F - t) + q_F t\right)^2 + \mathrm{i} t t_F (t_F - t)\right]; \quad (3.84)$$

and thus,

$$\begin{aligned}&\int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\,\mathrm{d}q_1 \quad (3.85) \\ &= \sqrt{\frac{m t_F}{2\pi \mathrm{i} t (t_F - t)}} K_0(q_F, q_0; t_F) \times -\sqrt{\frac{2\pi t}{m^3 t_F^5}} \frac{\sqrt{\mathrm{i}(t_F - t)^5}}{(t_F - t)^2}\left[m\left(q_0(t_F - t) + q_F t\right)^2 + \mathrm{i} t t_F (t_F - t)\right] \\ &= -\frac{1}{m t_F^2}\left[m\left(q_0(t_F - t) + q_F t\right)^2 + \mathrm{i} t t_F (t_F - t)\right] K_0(q_F, q_0; t_F). \quad (3.86)\end{aligned}$$

We're now in the place to finally construct the first order propagator term for the spring system. Recall that

$$K_1 = -\mathrm{i}\alpha \int_0^{t_F} I\,\mathrm{d}t = -\mathrm{i}\alpha\left[\int_0^{t_F} I_{P1}\,\mathrm{d}t + \int_0^{t_F} I_{P2}\,\mathrm{d}t + 2\int_0^{t_F} I_{P3}\,\mathrm{d}t\right] \quad (3.87)$$

From earlier,

$$\begin{aligned}I_{P1} &= K_0(q'_F, q'_0; t_F) \int q_1^2 K_0(q_F, q_1; t_F - t) K_0(q_1, q_0; t)\,\mathrm{d}q_1 \\ &= -\frac{1}{m t_F^2}\left[m\left(q_0(t_F - t) + q_F t\right)^2 + \mathrm{i} t t_F (t_F - t)\right] K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F), \quad (3.88) \\ I_{P2} &= K_0(q_F, q_0; t_F) \int q_1'^2 K_0(q'_F, q'_1; t_F - t) K_0(q'_1, q'_0; t)\,\mathrm{d}q'_1 \\ &= -\frac{1}{m t_F^2}\left[m\left(q'_0(t_F - t) + q'_F t\right)^2 + \mathrm{i} t t_F (t_F - t)\right] K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F), \quad (3.89) \\ I_{P3} &= -\frac{1}{t_F}\left[q_0(t_F - t) + q_F t\right] K_0(q_F, q_0; t_F) \times -\frac{1}{t_F}\left[q'_0(t_F - t) + q'_F t\right] K_0(q'_F, q'_0; t_F) \\ &= \frac{1}{t_F^2}\left[q_0(t_F - t) + q_F t\right]\left[q'_0(t_F - t) + q'_F t\right] K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F). \quad (3.90)\end{aligned}$$

Thus,

$$\begin{aligned}K_1 = -\mathrm{i}\alpha K_0(q_F, q_0; t_F) K_0(q'_F, q'_0; t_F) \frac{1}{t_F^2}\Bigg[&-\frac{1}{m}\int_0^{t_F} m\left(q_0(t_F - t) + q_F t\right)^2 + \mathrm{i} t t_F (t_F - t)\,\mathrm{d}t \\ &-\frac{1}{m}\int_0^{t_F} m\left(q'_0(t_F - t) + q'_F t\right)^2 + \mathrm{i} t t_F (t_F - t)\,\mathrm{d}t \\ &+\int_0^{t_F}\left[q_0(t_F - t) + q_F t\right]\left[q'_0(t_F - t) + q'_F t\right]\mathrm{d}t\Bigg] \quad (3.91)\end{aligned}$$

The integrals are then taken out to be evaluated term by term. Since the integrand of these integrals are all polynomials, we can just plug it into `SymPy.jl`:

$$\int_0^{t_F} m\left(q_0(t_F - t) + q_F t\right)^2 = \frac{t_F^3}{6}\left(2m(q_0+q_F)^2 + \mathrm{i}t_F\right) \tag{3.92}$$

$$\int_0^{t_F} m\left(q_0'(t_F - t) + q_F' t\right)^2 = \frac{t_F^3}{6}\left(2m(q_0'+q_F')^2 + \mathrm{i}t_F\right) \tag{3.93}$$

$$\int_0^{t_F} \left[q_0(t_F - t) + q_F t\right]\left[q_0'(t_F - t) + q_F' t\right] \mathrm{d}t = \frac{t_F^3}{6}\left(2q_0q_0' + q_0q_F' + q_0'q_F + 2q_Fq_F'\right) \tag{3.94}$$

Therefore, the first order perturbation term of the spring system takes the form

$$\begin{aligned} K_1(q_F, q_0; q_F', q_0'; t_F) &= -\mathrm{i}\frac{\alpha t_F}{6} K_0(q_F, q_0; t_F) K_0(q_F', q_0'; t_F) \\ &\times \left[-2\left(m(q_0+q_F)^2 + m(q_0'+q_F')^2 + \mathrm{i}t_F\right) + 2q_0q_0' + q_0q_F' + q_0'q_F + 2q_Fq_F'\right]. \end{aligned} \tag{3.95}$$

Interpretation of the propagator Notice that there are terms

3.5.2 Joint probability contribution from the first order perturbation term

Let us first analyze the time evolution of a separable wave packet with the initial state

$$\eta(q_0, q_0; t_0) \equiv \sqrt{\frac{1}{2\pi\sigma\sigma'}} \exp\left[\mathrm{i}pq_0 + \mathrm{i}p'q_0' - \frac{(q_0 - s)^2}{4\sigma^2} - \frac{(q_0' - s')^2}{4\sigma'^2}\right]. \tag{3.96}$$

For simplicity's sake, let

$$\xi \equiv \exp\left[\mathrm{i}pq_0 - \frac{(q_0-s)^2}{4\sigma^2}\right], \quad \xi' \equiv \exp\left[\mathrm{i}p'q_0' - \frac{(q_0'-s')^2}{4\sigma'^2}\right],$$
$$K \equiv \sqrt{\frac{m}{2\pi\mathrm{i}t_F}} \exp\left[\frac{\mathrm{i}m}{2t_F}(q_F - q_0)^2\right], \quad K' \equiv \sqrt{\frac{m}{2\pi\mathrm{i}t_F}} \exp\left[\frac{\mathrm{i}m}{2t_F}(q_F' - q_0')^2\right].$$

Let us also set t_0 to be 0. Using the first order perturbation term from eq. (3.95), the state at any following time t_F is then

$$\eta(q_F, q_F'; t_F) = \iint \left(K_0(q_F, q_0; q_F', q_0'; t_F - t_0) + K_1(q_F, q_0; q_F', q_0'; t_F - t_0)\right)\eta(q_0, q_0'; t_0)\,\mathrm{d}q\,\mathrm{d}q_0' \tag{3.97}$$

$$\begin{aligned} &= \sqrt{\frac{1}{2\pi\sigma\sigma'}} \iint K_0(q_F, q_0; q_F', q_0'; t_F - t_0)\left\{1 - \mathrm{i}\frac{\alpha t_F}{6}\left[-2\left(m(q_0+q_F)^2\right.\right.\right. \\ &\qquad \left.\left.\left. + m(q_0'+q_F')^2 + \mathrm{i}t_F\right) + 2q_0q_0' + q_0q_F' + q_0'q_F + 2q_Fq_F'\right]\right\} \\ &\qquad \exp\left[\mathrm{i}pq_0 + \mathrm{i}p'q_0' - \frac{(q_0-s)^2}{4\sigma^2} - \frac{(q_0'-s')^2}{4\sigma'^2}\right]\mathrm{d}q_0\,\mathrm{d}q_0' \end{aligned} \tag{3.98}$$

$$\begin{aligned} &= \sqrt{\frac{1}{2\pi\sigma\sigma'}} \iint KK'\xi\xi'\left\{1 - \mathrm{i}\frac{\alpha t_F}{6}\left[-2\left(m(q_0+q_F)^2\right.\right.\right. \\ &\qquad \left.\left.\left. + m(q_0'+q_F')^2 + \mathrm{i}t_F\right) + 2q_0q_0' + q_0q_F' + q_0'q_F + 2q_Fq_F'\right]\right\}\mathrm{d}q_0\,\mathrm{d}q_0' \end{aligned} \tag{3.99}$$

Note that this method of separation will work for all separable $\eta(t_0)$. Therefore, $\eta(t_0)$ doesn't have to be Gaussian: we just select it to be. The integral above can then be further expanded and broken down into smaller integrals:

$$\begin{aligned}\eta(q_F, q'_F; t_F) = \sqrt{\frac{1}{2\pi\sigma\sigma'}} \Bigg[&\iint KK'\xi\xi' \, \mathrm{d}q_0 \, \mathrm{d}q'_0 \\ &-\mathrm{i}\frac{\alpha t_F}{6} \iint KK'\xi\xi' \Big\{ -2m(q_0^2 + 2q_0 q_F + q_F^2 + q_0'^2 + 2q'_0 q'_F + q_F'^2) \\ &\qquad +2q_0 q'_0 + 2q_F q'_F + q_0 q'_F + q'_0 q_F + \mathrm{i}t_F \Big\} \, \mathrm{d}q_0 \, \mathrm{d}q'_0 \Bigg]\end{aligned} \tag{3.100}$$

I shall let I equals the term in the big square bracket:

$$\begin{aligned}I \equiv \iint KK'\xi\xi' \, \mathrm{d}q_0 \, \mathrm{d}q'_0 - \mathrm{i}\frac{\alpha t_F}{6} \iint KK'\xi\xi' \Big\{ &-2m(q_0^2 + 2q_0 q_F + q_F^2 + q_0'^2 + 2q'_0 q'_F + q_F'^2) \\ &+ 2q_0 q'_0 + 2q_F q'_F + q_0 q'_F + q'_0 q_F + \mathrm{i}t_F \Big\} \, \mathrm{d}q_0 \, \mathrm{d}q'_0\end{aligned} \tag{3.101}$$

and thus,

$$\eta(q_F, q'_F; t_F) \equiv \frac{1}{2\pi\sqrt{\sigma\sigma'}} I \tag{3.102}$$

We then pull the terms that doesn't include q_0 or q'_0 out:

$$\begin{aligned}I = &\left[\iint KK'\xi\xi' \, \mathrm{d}q_0 \, \mathrm{d}q_0 \right] + \left[\iint KK'\xi\xi' \, \mathrm{d}q_0 \, \mathrm{d}q_0 \right] \left(-\mathrm{i}\frac{\alpha t_F}{6} \right) \left(-2m(q_F^2 + q_F'^2) + 2q_F q'_F + \mathrm{i}t_F \right) \\ &- \left(\frac{\mathrm{i}\alpha t_F}{6} \right) \iint KK'\xi\xi' \left\{ -2m(q_0^2 + q_0'^2 + 2q_0 q_F + 2q'_0 q'_F) + 2q_0 q'_0 + q_0 q'_F + q'_0 q_F \right\} \mathrm{d}q_0 \, \mathrm{d}q_0 \\ = &\left[\iint KK'\xi\xi' \, \mathrm{d}q_0 \, \mathrm{d}q_0 \right] \left[1 + \left(\frac{\mathrm{i}\alpha t_F}{6} \right) \left(2m(q_F^2 + q_F'^2) - 2q_F q'_F - \mathrm{i}t_F \right) \right] \\ &- \left(\frac{\mathrm{i}\alpha t_F}{6} \right) \left[\iint KK'\xi\xi' \left\{ -2m(q_0^2 + q_0'^2 + 2q_0 q_F + 2q'_0 q'_F) + 2q_0 q'_0 + q_0 q'_F + q'_0 q_F \right\} \mathrm{d}q_0 \, \mathrm{d}q_0 \right] \\ = &\left[\int K\xi \, \mathrm{d}q_0 \int K'\xi' \, \mathrm{d}q'_0 \right] \left[1 + \left(\frac{\mathrm{i}\alpha t_F}{6} \right) \left(2m(q_F^2 + q_F'^2) - 2q_F q'_F - \mathrm{i}t_F \right) \right] \\ &- \left(\frac{\mathrm{i}\alpha t_F}{6} \right) \Bigg\{ -2m \left[\int q_0^2 K\xi \, \mathrm{d}q_0 \int K\xi' \, \mathrm{d}q'_0 + \int K\xi \, \mathrm{d}q_0 \int q_0'^2 K'\xi' \, \mathrm{d}q'_0 \right] \\ &\quad - 4mq_F \left[\int q_0 K\xi \, \mathrm{d}q_0 \int K\xi' \, \mathrm{d}q'_0 + \int K\xi \, \mathrm{d}q_0 \int q'_0 K'\xi' \, \mathrm{d}q'_0 \right] \\ &\quad + q_F \left[\int K\xi \, \mathrm{d}q_0 \int q'_0 K'\xi' \, \mathrm{d}q'_0 \right] + q'_F \left[\int q_0 K\xi \, \mathrm{d}q_0 \int K\xi' \, \mathrm{d}q'_0 \right] \\ &\quad +2 \left[\int q_0 K\xi \, \mathrm{d}q_0 \int q'_0 K\xi' \, \mathrm{d}q'_0 \right] \Bigg\}\end{aligned} \tag{3.103}$$

For the purpose of legibility, let

$$\begin{aligned}I_0 = \int K\xi \, \mathrm{d}q_0 \,, \quad I_1 = \int q_0 K\xi \, \mathrm{d}q_0 \,, \quad I_2 = \int q_0^2 K\xi \, \mathrm{d}q_0 \,, \\ I'_0 = \int K'\xi' \, \mathrm{d}q'_0 \,, \quad I'_1 = \int q'_0 K'\xi' \, \mathrm{d}q'_0 \quad \text{and,} \quad I'_2 = \int q_0'^2 K'\xi' \, \mathrm{d}q'_0 \,.\end{aligned} \tag{3.104}$$

Thus, I; and hence η, becomes

$$\begin{aligned}\eta(q_F, q'_F; t_F) = \frac{1}{2\pi\sqrt{\sigma\sigma'}} \Bigg\{ &I_0 I'_0 \left[1 + \left(\frac{\mathrm{i}\alpha t_F}{6} \right) \left(2m(q_F^2 + q_F'^2) - 2q_F q'_F - \mathrm{i}t_F \right) \right] \\ &- \left(\frac{\mathrm{i}\alpha t_F}{6} \right) \left[-2m\left(I_2 I'_0 + I_0 I'_2 \right) - 4mq_F \left(I_1 I'_0 + I_0 I'_1 \right) + q_F I_0 I'_1 + q'_F I_1 I'_0 + 2I_1 I'_1 \right] \Bigg\}\end{aligned} \tag{3.105}$$

Note that if you remove the normalization constant $\left(2\pi\sigma\sigma\right)^{-1/2}$, this equation becomes general joint probability distribution for the spring propagator with separable initial state. When $\alpha = 0$, this joint probability distribution reduces down to the joint probability distribution for the free particle.

The integrals listed in eq. (3.104) are then evaluated:

$$I_0 = \int K\xi \, \mathrm{d}q_0 = \sqrt{\frac{m}{2\pi i t_F}} \int \exp\left[ipq_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{im}{2t_F}(q_F - q_0)^2\right] \mathrm{d}q_0 \tag{3.106}$$

$$I_1 = \int q_0 K\xi \, \mathrm{d}q_0 = \sqrt{\frac{m}{2\pi i t_F}} \int q_0 \exp\left[ipq_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{im}{2t_F}(q_F - q_0)^2\right] \mathrm{d}q_0 \tag{3.107}$$

$$I_2 = \int q_0^2 K\xi \, \mathrm{d}q_0 = \sqrt{\frac{m}{2\pi i t_F}} \int q_0^2 \exp\left[ipq_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{im}{2t_F}(q_F - q_0)^2\right] \mathrm{d}q_0 \tag{3.108}$$

All of these integrals are Gaussian integrals with the exponential argument

$$ipq_0 - \frac{(q_0 - s)^2}{4\sigma^2} + \frac{im}{2t_F}(q_F - q_0)^2 \tag{3.109}$$

$$= -\left(\frac{1}{4\sigma^2} - \frac{im}{2t_F}\right) q_0^2 - \left(\frac{imq_F}{t_F} - ip - \frac{s}{2\sigma^2}\right) q_0 - \left(\frac{s^2}{4\sigma^2} - \frac{imq_F^2}{2t_F}\right), \tag{3.110}$$

with

$$a = \frac{1}{4\sigma^2} - \frac{im}{2t_F}, \quad b = \frac{imq_F}{t_F} - ip - \frac{s}{2\sigma^2} \quad \text{and,} \quad \frac{s^2}{4\sigma^2} - \frac{imq_F^2}{2t_F}. \tag{3.111}$$

From section 3.3,

$$\exp\left[\frac{b^2}{4a} - c\right] = \exp\left[\frac{4impq_F\sigma^2 - mq_F^2 + 2mq_F s - ms^2 - 2ip^2 t_F\sigma^2 - 2pst_F}{4m\sigma^2 + 2it_F}\right]. \tag{3.112}$$

I shall let

$$\Xi \equiv \sqrt{\frac{m}{2\pi i t_F}} \exp\left[\frac{4impq_F\sigma^2 - mq_F^2 + 2mq_F s - ms^2 - 2ip^2 t_F\sigma^2 - 2pst_F}{4m\sigma^2 + 2it_F}\right], \tag{3.113}$$

$$\Xi' \equiv \sqrt{\frac{m}{2\pi i t_F}} \exp\left[\frac{4imp'q_F'\sigma'^2 - mq_F'^2 + 2mq_F' s' - ms'^2 - 2ip'^2 t_F\sigma'^2 - 2p's't_F}{4m'\sigma'^2 + 2it_F}\right] \tag{3.114}$$

which represents the unperturbed state of a particle. And,

$$\sqrt{\frac{\pi}{a}} = 2\sigma\sqrt{\frac{\pi t_F}{t_F - 2im\sigma^2}} \equiv G_0, \tag{3.115}$$

$$-\frac{b}{2}\sqrt{\frac{\pi}{a^3}} = 2\sigma\sqrt{\frac{\pi t_F}{(t_F - 2im\sigma^2)^3}}\left(-2iq_F + 2ipt_F\sigma^2 + st_F\right) \equiv G_1, \tag{3.116}$$

$$\frac{1}{4}\sqrt{\frac{\pi}{a^5}} = 2\sigma\sqrt{\frac{\pi t_F}{(t_F - 2im\sigma^2)^5}}\left(-4imt_F\sigma^4 + 2t_F^2\sigma^2 + \left(-2imq_F\sigma^2 + 2ipt_F\sigma^2 + st_F\right)^2\right) \equiv G_2. \tag{3.117}$$

Therefore, with the change of primed and unprimed variables,

$$I_0 = G_0\Xi, \quad I_1 = G_1\Xi, \quad I_2 = G_2\Xi, \quad I_0' = G_0'\Xi', \quad I_1' = G_1'\Xi', \quad \text{and,} \quad I_2' = G_2'\Xi' \tag{3.118}$$

η then becomes

$$\begin{aligned}
\eta &= \frac{1}{\sqrt{2\pi\sigma\sigma'}}\left\{G_0G_0'\Xi\Xi'\left[1+\left(\frac{\mathrm{i}\alpha t_F}{6}\right)\left(2m(q_F^2+q_F'^2)-2q_Fq_F'-\mathrm{i}t_F\right)\right]\right. \qquad (3.119)\\
&\quad-\left(\frac{\mathrm{i}\alpha t_F}{6}\right)\left[-2m\left(G_2G_0'\Xi\Xi'+G_0G_2'\Xi\Xi'\right)-4mq_F\left(G_1G_0'\Xi\Xi'+G_0G_1'\Xi\Xi'\right)\right.\\
&\qquad\left.\left.+q_FG_0G_1'\Xi\Xi'+q_F'G_1G_0'\Xi\Xi'+2G_1G_1'\Xi\Xi'\right]\right\}\\
&= \frac{\Xi\Xi'}{\sqrt{2\pi\sigma\sigma'}}\left\{G_0G_0'\left[1+\left(\frac{\mathrm{i}\alpha t_F}{6}\right)\left(2m(q_F^2+q_F'^2)-2q_Fq_F'-\mathrm{i}t_F\right)\right]\right.\\
&\quad\left.-\left(\frac{\mathrm{i}\alpha t_F}{6}\right)\left[-2m\left(G_2G_0'+G_0G_2'\right)-4mq_F\left(G_1G_0'+G_0G_1'\right)+q_FG_0G_1'+q_F'G_1G_0'+2G_1G_1'\right]\right\}\\
&= \frac{\Xi\Xi'}{\sqrt{2\pi\sigma\sigma'}}\left\{G_0G_0'\left[1+\left(\frac{\mathrm{i}\alpha t_F}{6}\right)\left(2m(q_F^2+q_F'^2)-2q_Fq_F'-\mathrm{i}t_F\right)\right]\right. \qquad (3.120)\\
&\quad\left.-\left(\frac{\mathrm{i}\alpha t_F}{6}\right)\left[G_0(q_FG_1'(1-4m)-2mG_2')+G_0'(q_FG_1(1-4m)-2mG_2)+2G_1G_1'\right]\right\}
\end{aligned}$$

To analyze this statistically, we'd have to normalize the state, i.e.,

$$\iint |\eta(q_F, q_F'; t_F)|^2\,\mathrm{d}q_0\,\mathrm{d}q_0' = 1 \qquad (3.121)$$

where C is the normalization constant. To simplify our notation further, let $q_F = q$, $q_F' = q'$, and $t_F = t$. Thus,

3.5.3 Correlation function of the first order perturbation of interacting spring system

3.5.4 Joint probability contribution from the first order perturbation term on an entangled state

3.5.5 Second order perturbation term

The second order perturbation term, K_2 is

$$\begin{aligned}
K_2 &= \int\cdots\int K_0(F,2)V(2)K_0(2,1)V(1)K_0(1,0)\,\mathrm{d}q_1\,\mathrm{d}q_1'\,\mathrm{d}t_1\,\mathrm{d}q_2\,\mathrm{d}q_2'\,\mathrm{d}t_2 \qquad (3.122)\\
&= \int\cdots\int K_0(q_F,q_2;t_F-t_2)K_0(q_F',q_2';t_F-t_2)K_0(q_2,q_1;t_2-t_1)K_0(q_2,q_1;t_2-t_1) \qquad (3.123)\\
&\quad\times K_0(q_1,q_0;t_1-t_0)K_0(q_1,q_0;t_1-t_0)(q_2-q_2')^2(q_1-q_1')^2\,\mathrm{d}q_1\,\mathrm{d}q_1'\,\mathrm{d}t_1\,\mathrm{d}q_2\,\mathrm{d}q_2'\,\mathrm{d}t_2\\
&= \int_{t_1}^{t_F}\int_{t_0}^{t_F}\left[\int\cdots\int K_0(q_F,q_2;t_F-t_2)K_0(q_F',q_2';t_F-t_2)K_0(q_2,q_1;t_2-t_1)\right. \qquad (3.124)\\
&\quad\times K_0(q_2,q_1;t_2-t_1)K_0(q_1,q_0;t_1-t_0)K_0(q_1,q_0;t_1-t_0)\left(q_1^2q_2^2-2q_1^2q_2q_2'\right.\\
&\qquad+q_1^2\left(q_2'\right)^2-2q_1q_2^2q_1'+4q_1q_2q_1'q_2'-2q_1q_1'\left(q_2'\right)^2+q_2^2\left(q_1'\right)^2\\
&\qquad\left.\left.-2q_2\left(q_1'\right)^2q_2'+\left(q_1'\right)^2\left(q_2'\right)^2\right)\mathrm{d}q_1\,\mathrm{d}q_1'\,\mathrm{d}q_2\,\mathrm{d}q_2'\right]\mathrm{d}t_1\,\mathrm{d}t_2
\end{aligned}$$

The integral once again can be broken into nine integrals that must be integrated w.r.t. time twice later on. All those nine integrals have a product of propagator as a multiplier. We shall evaluate those first, separating the primed and unprimed variables.

$$\begin{aligned}
&K_0(q_F, q_2; t_F - t_2)K_0(q'_F, q'_2; t_F - t_2)K_0(q_2, q_1; t_2 - t_1)K_0(q_2, q_1; t_2 - t_1)K_0(q_1, q_0; t_1 - t_0) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times K_0(q_1, q_0; t_1 - t_0) \\
&= \frac{\mathrm{i}m^3}{8\pi^3(t_F - t_2)(t_2 - t_1)(t_1 - t_0)} \exp\left[\frac{\mathrm{i}m}{2(t_F - t_2)}\left((q_F - q_2)^2 + (q'_F - q'_2)^2\right)\right. \\
&\qquad \left. + \frac{\mathrm{i}m}{2(t_2 - t_1)}\left((q_2 - q_1)^2 + (q'_2 - q'_1)^2\right) + \frac{\mathrm{i}m}{2(t_1 - t_0)}\left((q_1 - q_0)^2 + (q'_1 - q'_0)^2\right)\right]
\end{aligned} \tag{3.125}$$

3.6 The infinite potential well

CHAPTER 4

Path integrals for bipartite system in the momentum basis

4.1 Relations between the position and momentum propagator

CHAPTER 5

Implementation of simulations

5.1 The brute force approach

To represent the joint probability amplitude of a bipartite system, we need three spatial axes. Two for the position basis of each Hilbert space, and one for the probability amplitude. From section 3.1, the joint probability amplitude of a bipartite system represents a surface, and the volume under that surface must be normalized to one. The joint probability function $P(q_F, q_F'; t_F)$ is represented by the modulus squared of the state, projected onto the position basis.

$$
\begin{aligned}
P(q_F, q_F'; t_F) &= \left| \langle q_F, q_F' | \eta(t_F) \rangle \right|^2 && (5.1)\\
&= \left| \iint K(q_F, q_F'; q_0, q_0'; t_F - t_0) \eta(q_0, q_0'; t_0)\, \mathrm{d}q_0\, \mathrm{d}q_0' \right|^2 && (5.2)\\
&= \left| \iint \left\{ \sum_{i=0}^{\infty} K_i(q_F, q_F'; q_0, q_0'; t_F - t_0) \right\} \eta(q_0, q_0', t_0)\, \mathrm{d}q_0\, \mathrm{d}q_0' \right|^2 && (5.3)
\end{aligned}
$$

To implement this equation, we need to discretize it by swapping the integral for the summation sign.

$$
= \left| \sum_{q_0} \sum_{q_0'} \left\{ \sum_{i=0}^{\infty} K_i(q_F, q_F'; q_0, q_0'; t_F - t_0) \right\} \eta(q_0, q_0'; t_0) \right|^2 \qquad (5.4)
$$

The sum of propagators can then be truncated to our liking. And, here is the code in `Julia`.

```julia
using Plots; plotlyjs()
using LinearAlgebra

freePropagator(finPos, startPos, finTime, startTime = 0, m = 1) = sqrt(m / (2 * pi * im * (finTime - startTime))) * exp(im * m / (2 * (finTime - startTime)) * (finPos - startPos)^2)
```

```
freePropagatorC(qf, qfp, q0, q0p, tf, t0) = freePropagator(qf, q0, tf, t0) *
↪  freePropagator(qfp, q0p, tf, t0)
initStateFunction(q0, q0p, σ1, s1, p1, σ2, s2, p2) = (1//2 * pi * σ1)^(1//4) * exp(-(q0 -
↪  s1)^2 / (4 * σ1^2) + im * p1 * q0) * (1//2 * pi * σ2)^(1//4) * exp(-(q0p - s2)^2 / (4 *
↪  σ2^2) + im * p2 * q0p)
springPropagator1(qf, qfp, q0, q0p, tf, t0) = freePropagatorC(qf, qfp, q0, q0p, tf, t0) *
↪  (1 - im * α * tf / 6 * (-2 * (m * (q0 + qf)^2 + m * (q0p + qfp)^2 + im * tf) + 2*q0*q0p
↪  + q0*qfp + q0p*qf + 2*qf*qfp))

# Initialization of the position basis
maxPos = 10
minPos = -10
stepPos = 0.25

pos1Vect = collect(minPos:stepPos:maxPos)
pos2Vect = collect(minPos:stepPos:maxPos)
posVectSize = size(pos1Vect, 1)
posMat = [(i, j) for i in pos1Vect, j in pos2Vect]

posToIndex(pos) = Int32((pos - minPos) / stepPos  + 1)

# Select unentangled and entangled state by commenting/uncommenting
# Unentangled
initState(q) = initStateFunction(q[1], q[2], 0.5, +1, 0, 0.5, -1, 0)
# Entangled
# initState(q) = 1/sqrt(2) * (initStateFunction(q[1], q[2], 0.5, +1, 0, 0.5, -1, 0) -
↪  initStateFunction(q[1], q[2], 0.5, -1, 0, 0.5, +1, 0))
global α
α = 1
m = 1

springPropagator1(qf, qfp, q0, q0p, tf, t0) = freePropagatorC(qf, qfp, q0, q0p, tf, t0) *
↪  (1 - im * α * tf / 6 * (-2 * (m * (q0 + qf)^2 + m * (q0p + qfp)^2 + im * tf) + 2*q0*q0p
↪  + q0*qfp + q0p*qf + 2*qf*qfp))
initMat = initState.(posMat)
initMat = round.(initMat, digits = 7)
```

```

finalMat = Matrix{ComplexF32}(undef, posVectSize, posVectSize)
finalTime = 1

for xf in pos1Vect, xfp in pos2Vect
    sumPos = 0
    for i in 1:posVectSize
        Threads.@threads for j in 1:posVectSize
            x0 = pos1Vect[i]
            x0p = pos2Vect[j]
            sumPos += freePropagatorC(xf, xfp, x0, x0p, finalTime, 0) * initMat[i, j]
        end
    end
    finalMat[posToIndex(xf), posToIndex(xfp)] = sumPos
end
surface(pos1Vect, pos2Vect, abs2.(finalMat))
```

5.2 Parallelization approach

The method in the section above is quite computationally expensive due to the amount of for loops in there, with the time complexity of $O(n^4)$. To reduce this computation time, we must parallelize it on the GPU. First, let's contain $\eta(q_0, q_0'; t_0)$ in a matrix

$$\hat{\eta} = \begin{bmatrix} \eta(x_1, x_1') & \cdots & \eta(x_1, x_n') \\ \vdots & \ddots & \vdots \\ \eta(x_n, x_1') & \cdots & \eta(x_n, x_n') \end{bmatrix}. \tag{5.5}$$

Notice that in eq. (5.4), (q_F, q_F') and t_F is independent of the sum. Therefore, we can create a matrix that's filled with propagators, taking the position index of the matrix, and $(q_F, q_F'; t_F)$ as the universal input:

$$\hat{K}(q_F, q_F'; t_F) = \begin{bmatrix} K(q_F, q_F'; x_1, x_1'; t_F - t_0) & \cdots & K(q_F, q_F'; x_1, x_n'; t_F - t_0) \\ \vdots & \ddots & \vdots \\ K(q_F, q_F'; x_n, x_1'; t_F - t_0) & \cdots & K(q_F, q_F'; x_n, x_n'; t_F - t_0) \end{bmatrix}. \tag{5.6}$$

The Hadamard product (element-wise product) of $\hat{K}(q_F, q_F'; t_F)$ and $\hat{\eta}$, denoted $\hat{K}(q_F, q_F'; t_F) \circ \hat{\eta}$, is

$$\begin{bmatrix} K(q_F, q_F'; x_1, x_1'; t_F - t_0)\eta(x_1, x_1') & \cdots & K(q_F, q_F'; x_1, x_n'; t_F - t_0)\eta(x_1, x_n') \\ \vdots & \ddots & \vdots \\ K(q_F, q_F'; x_n, x_1'; t_F - t_0)\eta(x_n, x_1') & \cdots & K(q_F, q_F'; x_n, x_n'; t_F - t_0)\eta(x_n, x_n') \end{bmatrix}. \tag{5.7}$$

We can then take broadcast the absolute squared function into the matrix, then find the sum of all elements, resulting in

$$\left| \sum_{i=1}^{n} \sum_{j=1}^{n} K(q_F, q'_F; x_i, x'_j; t_F - t_0) \eta(x_i, x'_j) \right|^2, \tag{5.8}$$

which directly equates to the sum in eq. (5.4). Using this method, all the computation can be parallelized on the GPU: vectorized broadcasting for the generation of $\hat{K}$, multiplication broadcasting for the Hadamard product, and array reduction using addition for the sum of all elements. We then implement it as follows.

```julia
using Plots; plotlyjs()
using LinearAlgebra
using CUDA
CUDA.allowscalar(false)

# Initialization of the position basis
maxPos = 5
minPos = -5
stepPos = 0.25

# Gaussian state function
initStateFunction(q0, q0p, σ1, s1, p1, σ2, s2, p2) = (1//2 * pi * σ1)^(1//4) * exp(-(q0 -
↪  s1)^2 / (4 * σ1^2) + im * p1 * q0) * (1//2 * pi * σ2)^(1//4) * exp(-(q0p - s2)^2 / (4 *
↪  σ2^2) + im * p2 * q0p)
posToIndex(pos) = Int32((pos - minPos) / stepPos + 1)

pos1Vect = collect(minPos:stepPos:maxPos)
pos2Vect = collect(minPos:stepPos:maxPos)
posVectSize = size(pos1Vect, 1)
posMat = [(i, j) for i in pos1Vect, j in pos2Vect]
xMesh = pos1Vect * ones(posVectSize)'
yMesh = xMesh'
xMeshCu = CuArray(xMesh)
yMeshCu = CuArray(yMesh)

# The initial state can be uncommented for an entangled state
initState(q) = initStateFunction(q[1], q[2], 0.5, +1, 0, 0.5, -1, 0)# +
↪  initStateFunction(q[1], q[2], 0.5, -3, 0, 0.5, +3, 0)
```

```julia
initMat = initState.(posMat)
initMat = round.(initMat, digits = 6)
initMatCu = CuArray(initMat)

global α, m, finalTime
α = 1
m = 1
finalTime = 0.01

springProp(qf, qfp, q0, q0p, tf, t0) = @. sqrt(m / (2 * pi * im * (tf - t0))) * exp(im * m / (2 * (tf - t0)) * (qf - q0)^2) * sqrt(m / (2 * pi * im * (tf - t0))) * exp(im * m / (2 * (tf - t0)) * (qfp - q0p)^2) * (1 - im * α * tf / 6 * (-2 * (m * (q0 + qf)^2 + m * (q0p + qfp)^2 + im * tf) + 2*q0*q0p + q0*qfp + q0p*qf + 2*qf*qfp))
freeProp(qf, qfp, q0, q0p, tf, t0) = @. sqrt(m / (2 * pi * im * (tf - t0))) * exp(im * m / (2 * (tf - t0)) * (qf - q0)^2) * sqrt(m / (2 * pi * im * (tf - t0))) * exp(im * m / (2 * (tf - t0)) * (qfp - q0p)^2)

finalMat = Matrix{ComplexF32}(undef, posVectSize, posVectSize)
for i in pos1Vect, j in pos2Vect
    # Generation of the propagator matrix
    propMat = springProp(i, j, xMeshCu, yMeshCu, finalTime, 0)
    # Componentwise multiplication
    propMat .*= initMatCu
    # Array reduction using addition
    finalMat[posToIndex(i), posToIndex(j)] = reduce(+, propMat)
end
# Plotting the final surface
surface(pos1Vect, pos2Vect, normalize(abs2.(finalMat)))
```

5.3 Numerical computation of the entanglement entropy and covariance

CHAPTER 6

The Schrödinger's equation for two particles

In this chapter, we'd try to formulate the same interaction problems in terms of the Schrödinger's equation in the non-moving frame of reference. Consider the Hamiltonian

$$\mathcal{H}(q,q') = -\frac{1}{2m}\frac{\partial^2}{\partial q^2} - \frac{1}{2m'}\frac{\partial^2}{\partial (q')^2} + V(q,q') \tag{6.1}$$

The Schrödinger's equation then becomes

$$\mathrm{i}\frac{\partial}{\partial t}\eta(q,q';t) = -\frac{1}{2m}\frac{\partial^2\eta(q,q';t)}{\partial q^2} - \frac{1}{2m'}\frac{\partial^2\eta(q,q';t)}{\partial (q')^2} + V(q,q') \tag{6.2}$$

CHAPTER 7

Path integrals for bipartite system in the momentum basis

7.1 Relations between the position and momentum propagator

APPENDIX A

Integrals evaluation code

All of these codes that I've written are in the `Julia` language, which I've imported three packages: `Sympy`, `OffsetArrays`, and `Plots; plotlyjs()`

A.1 Polynomial extraction

Since we're going to be doing a lot of polynomials rearranging, I've implemented the polynomial extraction function as follows:

```
function extractPolynomial(expr, arg)
    expr isa Sym ? nothing : expr = sympify(1)
    expr = expand(expr)
    polyDegree = Int(degree(expr, arg))
    extractOneSet = []
    for i in 0:polyDegree
        temp = expr.coeff(arg, i)
        push!(extractOneSet, temp)
    end
    polyExt = OffsetVector(extractOneSet, 0:polyDegree)
    return polyExt
end
```

This function accepts two arguments: `expr`, which is the expression you want to extract, and `arg`, the variable that you extract with respect to. For example, inputting $ax^2 + bx + c,\ x$ would give out

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \tag{A.1}$$

which is a zero index matrix.

A.2 Code for the spring problem

A.2.1 First order perturbation

The first order perturbation uses the following symbols:

```
@syms q0 q1 q2
@syms m::(real, positive) t::(real, positive)
q0′, q1′, q2′, qf′, qf = symbols("q_{0}^{\\prime}, q_{1}^{\\prime}, q_{2}^{\\prime},
↪  q_{F}^{\\prime}, q_F")
t0, t1, t2, tf = symbols("t_0, t_1, t_2, t_F", real = true, positive = true)
```

The following code is used to aid the evaluation of integrals.

```
# Simplifying the product of propagators in eq. 2.25
idenFunc = IM*m/(2 * (tf - t)) * (qf - q1)^2 + IM*m/(2*t) * (q1 - q0)^2

idenPoly = extractPolynomial(idenFunc, q1)
idenPolyA = -idenPoly[2]
idenPolyB = -idenPoly[1]
idenPolyC = -idenPoly[0]

display(idenPolyA)
display(idenPolyB)
display(idenPolyC)

```