A follow-up on "Measurement of gravity by the voltage difference induced on a solenoid by a free-falling magnet:" The flux of an actual solenoid

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CHAPTER 1

Introduction

This article is a follow-up of the earlier article that I wrote as a part of a submission for the subject ESC612 (From wire to wireless), "*Measurement of gravity by the voltage difference induced on a solenoid by a free-falling magnet.*" In that article, I approximated a solenoid as a simple circular loop that lies on the $x - y$ plane, and let the magnet hover at height h over it. At the end, the gravitational acceleration measured was $9.52\mathrm{m\,s^{-1}}$, which is frankly better than I expected. However, I think that the g value could be better if the geometry of the solenoid is actually used. Therefore, this article is going to address the magnetic flux of an actual solenoid, and compare it with the two other approximations that are commonly used.

1.1 Geometries of the solenoid and its approximation

1.1.1 Geometry of an actual solenoid

The solenoid is modeled as a helix that lies along the z axis and centered at $(x, y) = (0, 0)$ shown in fig. 1.1

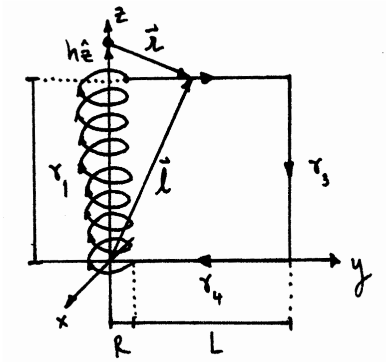

FIG. 1.1 | THE SOLENOID'S GEOMETRY

The solenoid of height $\frac{H}{2\pi}$ that's wounded around itself ω times, where $\omega \in \mathbb{Z}$ is parameterized by the contour γ which is given in four parts: $\gamma_1, \ldots, \gamma_4$. γ_1 represents the actual solenoid, parameterized by

$$\gamma_1 : \left(x = R\sin(\omega\alpha), y = R\cos(\omega\alpha), z = \frac{H}{2\pi} \right) \qquad \alpha \in [0, 2\pi] \tag{1.1}$$

The contour γ_2 to γ_4 represents the wire that connects the bottom of the solenoid to the top:

$$\gamma_2 : \big(x = 0, y = L\alpha + R, z = H \big) \qquad \alpha \in [0, 1] \tag{1.2}$$

$$\gamma_3 : \big(x = 0, y = L + R, z = \alpha H \big) \qquad \alpha \in [1, 0] \tag{1.3}$$

$$\gamma_4 : \big(x = 0, y = L\alpha + R, z = 0 \big) \qquad \alpha \in [1, 0] \tag{1.4}$$

1.1.2 Geometry of the approximations

Before I wrote this article, I decided asked my good friend, *Nattawee Wiwatthanakul* for an opinion on how the magnetic flux may be calculated. He proposed that the solenoid's flux can be approximated with a stack of circular wires shown in fig. 1.2. The wire also has radius R, and is stacked on itself ω times ($\omega \in \mathbb{Z}^+$), reaching the height of H.

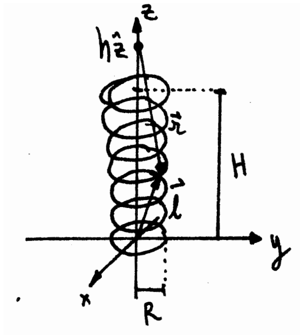

FIG. 1.2 | STACKED CIRCLES APPROXIMATION FOR THE SOLENOID

Each loop contributes its own magnetic flux Φ_{Bi}, and thus the total magnetic flux must be the sum of each loop:

$$\Phi_B = \sum_{i=0}^{\omega} \Phi_{Bi} \tag{1.5}$$

1.2 Magnetic flux

The magnetic flux, Φ_B is given by

$$\Phi_B = \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{a} \tag{1.6}$$

where $\mathbf{B}$ is the magnetic field that's on the wire, and $\mathrm{d}\mathbf{a}$, the area integration element. However, this cannot be done with the geometry of the solenoid because the area enclosed by the wire cannot be parameterized in a simple way. Thus, the area integral must be transformed into a line integral using the Green's theorem. Given that the vector magnetic potential $\mathbf{A}$ where $\mathbf{B} = \nabla \times \mathbf{A}$, eq. (1.6) becomes

$$\Phi_B = \int_S (\nabla \times \mathbf{A}) \cdot \mathrm{d}\mathbf{a} = \oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{l}. \tag{1.7}$$

By parameterization,

$$\oint_\gamma \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \oint \mathbf{A} \cdot \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\alpha}\, \mathrm{d}\alpha \tag{1.8}$$

CHAPTER 2

Evaluation of the magnetic flux

2.1 Magnetic flux of an actual solenoid

2.1.1 Evaluating the flux on the solenoid's contour

The magnetic dipole is hovering at height h above the middle of the solenoid. The **A** field of the perfect magnetic dipole is given by

$$\mathbf{A}(\boldsymbol{\mathscr{r}}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \boldsymbol{\mathscr{r}}}{\mathscr{r}^3}. \tag{2.1}$$

We denote the vector that points from the origin to the contour as **l**. In the solenoid's contour (γ_1),

$$\mathbf{l} = R\sin(\omega\alpha)\hat{\mathbf{x}} + R\cos(\omega\alpha)\hat{\mathbf{y}} + \frac{H}{2\pi}\alpha\hat{\mathbf{z}}. \tag{2.2}$$

Trivially,

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\alpha} = R\omega\cos(\omega\alpha)\hat{\mathbf{x}} - R\omega\sin(\omega\alpha)\hat{\mathbf{y}} + \frac{H}{2\pi}\hat{\mathbf{z}}. \tag{2.3}$$

Geometrically,

$$h\hat{\mathbf{z}} + \boldsymbol{\mathscr{r}} = \mathbf{l} \tag{2.4}$$

$$
\begin{aligned}
\boldsymbol{\mathscr{r}} &= \mathbf{l} - h\hat{\mathbf{z}} && (2.5)\\
&= R\sin(\omega\alpha)\hat{\mathbf{x}} + R\cos(\omega\alpha)\hat{\mathbf{y}} + \left(\frac{H}{2\pi}\alpha - h\right)\hat{\mathbf{z}}, && (2.6)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathscr{r} &= \sqrt{\left(R\sin(\omega\alpha)\right)^2 + \left(R\cos(\omega\alpha)\right)^2 + \left(\frac{H}{2\pi}\alpha - h\right)^2} && (2.7)\\
&= \sqrt{R^2 + \left(\frac{H}{2\pi}\alpha - h\right)^2}. && (2.8)
\end{aligned}
$$

From eq. (2.1),

$$
\begin{aligned}
\mathbf{m} \times \boldsymbol{\mathscr{r}} &= -m\hat{\mathbf{z}} \times \left(R\sin(\omega\alpha)\hat{\mathbf{x}} + R\cos(\omega\alpha)\hat{\mathbf{y}} + \left(\frac{H}{2\pi}\alpha - h\right)\hat{\mathbf{z}}\right) && (2.9)\\
&= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & -m \\ R\sin(\omega\alpha) & R\cos(\omega\alpha) & \frac{H}{2\pi}\alpha - h \end{vmatrix} && (2.10)\\
&= m\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} \\ R\sin(\omega\alpha) & R\cos(\omega\alpha) \end{vmatrix} && (2.11)\\
&= mR\left(\cos(\omega\alpha)\hat{\mathbf{x}} - \sin(\omega\alpha)\hat{\mathbf{y}}\right). && (2.12)
\end{aligned}
$$

Therefore,

$$
\mathbf{A}(\alpha) = \frac{\mu_0}{4\pi}\frac{mR\left(\cos(\omega\alpha)\hat{\mathbf{x}} - \sin(\omega\alpha)\hat{\mathbf{y}}\right)}{\left(R^2 + \left(\frac{H}{2\pi}\alpha - h\right)^2\right)^{\frac{3}{2}}} \qquad (2.13)
$$

And thus, from eq. (2.3)

$$
\begin{aligned}
\mathbf{A} \cdot \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\alpha} &= \frac{\mu_0 mR}{4\pi}\left(R^2 + \left(\frac{H}{2\pi}\alpha - h\right)^2\right)^{-\frac{3}{2}}\\
&\qquad \left(\cos(\omega\alpha)\hat{\mathbf{x}} - \sin(\omega\alpha)\hat{\mathbf{y}}\right) \cdot \left(R\omega\cos(\omega\alpha)\hat{\mathbf{x}} - R\omega\sin(\omega\alpha)\hat{\mathbf{y}} + \frac{H}{2\pi}\hat{\mathbf{z}}\right)\\
&= \frac{\mu_0 mR}{4\pi}\left(R^2 + \left(\frac{H}{2\pi}\alpha - h\right)^2\right)^{-\frac{3}{2}} R\omega\left(\cos^2(\omega\alpha) + \sin^2(\omega\alpha)\right) && (2.14)\\
&= \frac{\mu_0 mR^2\omega}{4\pi}\left(R^2 + \left(\frac{H}{2\pi}\alpha - h\right)^2\right)^{-\frac{3}{2}} && (2.15)
\end{aligned}
$$

Thus, the magnetic flux that's generated by the contour γ_1 is given by

$$\int_{\gamma_1} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \int_0^{2\pi} \mathbf{A} \cdot \frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\alpha} \,\mathrm{d}\alpha \tag{2.16}$$

$$= \int_0^{2\pi} \frac{\mu_0 m R^2 \omega}{4\pi} \left(R^2 + \left(\frac{H}{2\pi}\alpha - h \right)^2 \right)^{-\frac{3}{2}} \mathrm{d}\alpha \tag{2.17}$$

$$= \frac{\mu_0 m R^2 \omega}{4\pi} \int_0^{2\pi} \left(R^2 + \left(\frac{H}{2\pi}\alpha - h \right)^2 \right)^{-\frac{3}{2}} \mathrm{d}\alpha \tag{2.18}$$

This integral can be easily solved by a simple substitution. Let $u = \dfrac{H}{2\pi}\alpha - h$, then $\mathrm{d}u = \dfrac{H}{2\pi}\,\mathrm{d}\alpha$. The bounds are changed from $\alpha = 0 \rightarrow u = -h$, and $\alpha = 2\pi \rightarrow u = H - h$.

$$\frac{\mu_0 m R^2 \omega}{4\pi} \int_0^{2\pi} \left(R^2 + \left(\frac{H}{2\pi}\alpha - h \right)^2 \right)^{-\frac{3}{2}} \mathrm{d}\alpha$$

$$= \frac{\mu_0 m R^2 \omega}{4\pi} \cdot \frac{2\pi}{H} \int_{-h}^{H-h} \left(R^2 + u^2 \right)^{-\frac{3}{2}} \mathrm{d}u \tag{2.19}$$

$$= \frac{\mu_0 m R^2 \omega}{4\pi} \cdot \left(\frac{u}{R^2 (R^2 + u^2)^{\frac{1}{2}}} \Bigg|_{u=-h}^{u=H-h} \right) \tag{2.20}$$

$$= \frac{\mu_0 m R^2 \omega}{2H} \left(\frac{H-h}{\sqrt{R^2 + (H-h)^2}} + \frac{h}{\sqrt{R^2 + h^2}} \right) \tag{2.21}$$

2.1.2 Evaluating the flux on the solenoid's cable

The contour γ_2 to γ_4 does not contribute to the magnetic flux. This is because the vector $\mathbf{m} \times \boldsymbol{\imath}$ and $\mathbf{l}$ are orthogonal. The first one lies on the x axis, but the other lies along the y axis only. Therefore, the only part that contributes to the magnetic flux is γ_1:

$$\Phi_B = \frac{\mu_0 m R^2 \omega}{2H} \left(\frac{H-h}{\sqrt{R^2 + (H-h)^2}} + \frac{h}{\sqrt{R^2 + h^2}} \right) \tag{2.22}$$

2.2 Magnetic flux of a stacked circular wire

Each of the loop contributes a magnetic flux $\Phi_B(h)$ where h is the height from the center of the wire to the magnet. The contour of a circular wire at radius R that's at $z = 0$ is parameterized by

$$\gamma : (R\cos(\alpha), R\sin(\alpha), 0) \quad \alpha \in [0, 2\pi] \tag{2.23}$$

Therefore,

$$\mathbf{l} = R\cos(\alpha)\hat{\mathbf{x}} + R\sin(\alpha)\hat{\mathbf{y}}, \tag{2.24}$$

and thus,

$$\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\alpha} = -R\sin(\alpha)\hat{\mathbf{x}} + R\cos(\alpha)\hat{\mathbf{y}}. \tag{2.25}$$

Geometrically,

$$h\hat{\mathbf{z}} + \boldsymbol{\mathscr{r}} = \mathbf{l} \tag{2.26}$$

$$\boldsymbol{\mathscr{r}} = -R\sin(\alpha)\hat{\mathbf{x}} + R\cos(\alpha)\hat{\mathbf{y}} - h\hat{\mathbf{z}}, \tag{2.27}$$

and

$$\mathscr{r} = \sqrt{R^2\sin^2(\alpha) + R^2\cos(\alpha) + h^2} = \sqrt{R^2 + h^2}. \tag{2.28}$$

From eq. (2.1),

$$\mathbf{m} \times \boldsymbol{\mathscr{r}} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 0 & 0 & -m \\ R\cos(\alpha) & R\sin(\alpha) & -h \end{vmatrix} \tag{2.29}$$

$$= m \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} \\ R\cos(\alpha) & R\sin(\alpha) \end{vmatrix} \tag{2.30}$$

$$= mR\big(\sin(\alpha)\hat{\mathbf{x}} - \cos(\alpha)\hat{\mathbf{y}}\big); \tag{2.31}$$

therefore,

$$\mathbf{A}(\alpha) = \frac{\mu_0 m R}{4\pi} \frac{\sin(\alpha)\hat{\mathbf{x}} - \cos(\alpha)\hat{\mathbf{y}}}{(R^2 + h^2)^{\frac{3}{2}}}, \tag{2.32}$$

and

$$\mathbf{A}(\alpha)\cdot\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\alpha} = \frac{\mu_0 mR}{4\pi}(R^2+h^2)^{-\frac{3}{2}} \tag{2.33}$$
$$\cdot\left(\sin(\alpha)\hat{\mathbf{x}} - \cos(\alpha)\hat{\mathbf{y}}\right)\cdot\left(-R\sin(\alpha)\hat{\mathbf{x}} + R\cos(\alpha)\hat{\mathbf{y}}\right)$$
$$= \frac{\mu_0 mR^2}{4\pi}(R^2+h^2)^{-\frac{3}{2}}\left(\sin^2(\alpha)+\cos^2(\alpha)\right) \tag{2.34}$$
$$= \frac{\mu_0 mR^2}{4\pi}\frac{1}{(R^2+h^2)^{\frac{3}{2}}}. \tag{2.35}$$

The magnetic flux of a single loop is then

$$\int \mathbf{A}(\alpha)\cdot\frac{\mathrm{d}\mathbf{l}}{\mathrm{d}\alpha}\,\mathrm{d}\alpha = \int_0^{2\pi}\frac{\mu_0 mR^2}{4\pi}\frac{1}{(R^2+h^2)^{\frac{3}{2}}}\,\mathrm{d}\alpha \tag{2.36}$$
$$= \frac{\mu_0 mR^2}{4\pi}\frac{1}{(R^2+h^2)^{\frac{3}{2}}}\int_0^{2\pi}\mathrm{d}\alpha \tag{2.37}$$
$$= \frac{\mu_0 mR^2}{4\pi}(R^2+h^2)^{-\frac{3}{2}}2\pi \tag{2.38}$$
$$= \frac{\mu_0 mR^2}{2}(R^2+h^2)^{-\frac{3}{2}}. \tag{2.39}$$

In fig. 1.2, the top-most loop is at $z = H$. Therefore, the distance between the magnet to each loop from the bottom-most to top-most is

$$h, h-\frac{H}{\omega}, h-\frac{2H}{\omega}, \dots, h-\frac{(\omega-1)H}{\omega}, h-H. \tag{2.40}$$

Therefore, the total magnetic flux given by the stacked circle approximation is

$$\frac{\mu_0 mR^2}{2}\sum_{n=0}^{\omega}\left(h-\frac{nH}{\omega}\right)^{-\frac{3}{2}}. \tag{2.41}$$