Operations Research

Introduction to Graphs

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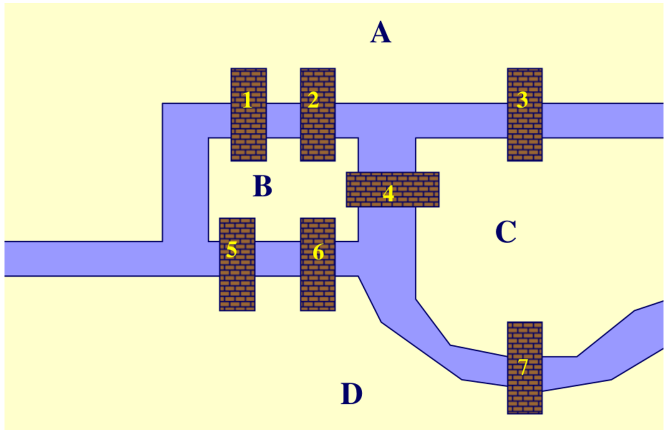
Introduction (1/6)

The premise of Graphs : the bridges of Koenigsberg : Euler 1736

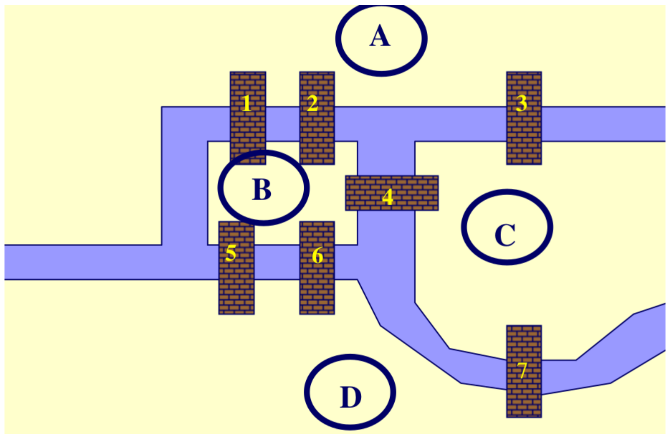
- Leonard Eüler Visited Koenigsberg
- People wondered whether it is possible to take a walk, end up where you started from, and cross each bridge in Koenigsberg exactly once
- Generally it was believed to be impossible

Introduction (2/6)



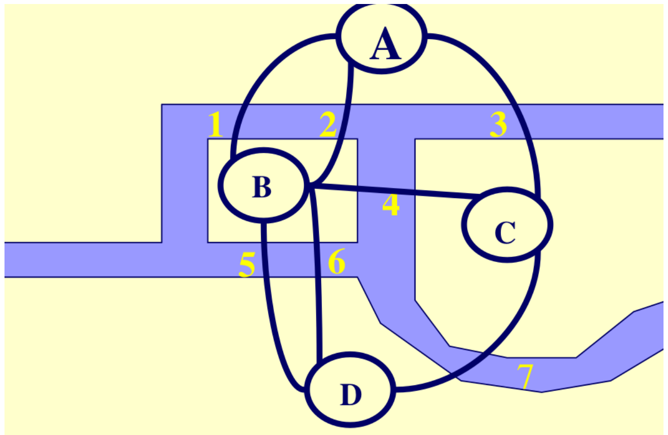
Is it possible to start in A, cross over each bridge exactly once, and end up back in A ?

Introduction (3/6)



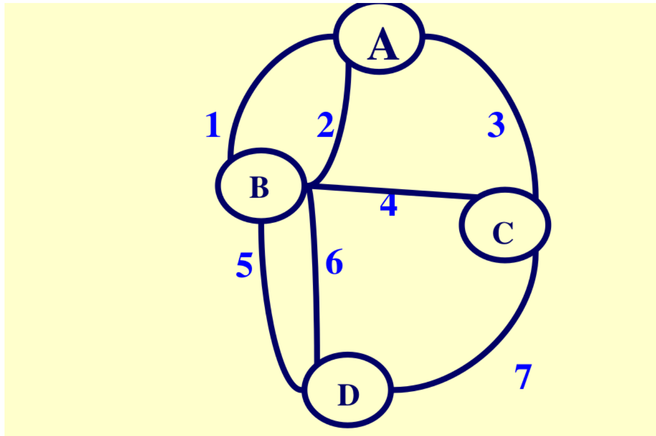
Graph modeling : Land masses are vertices.

Introduction (4/6)



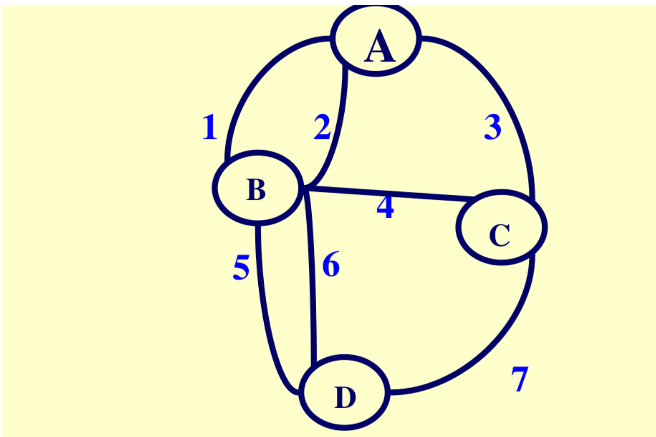
Graph modeling : Bridges are edges.

Introduction (5/6)



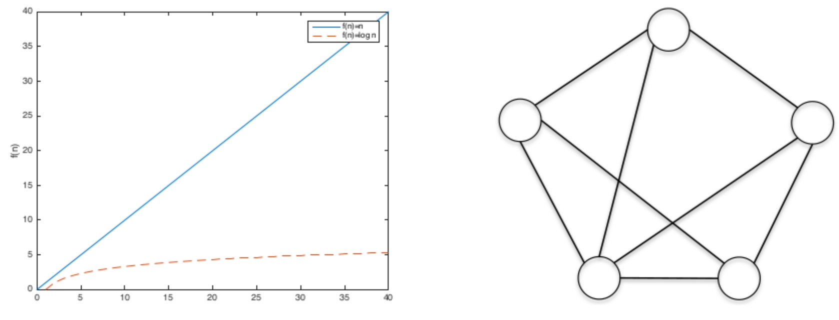
Is there a “walk” starting at A and ending at A and passing through each edge exactly once ?

Introduction (6/6)



Such a walk is called an eulerian cycle. Solving this problem will be discussed later.
Any intuition ?

Graph ?



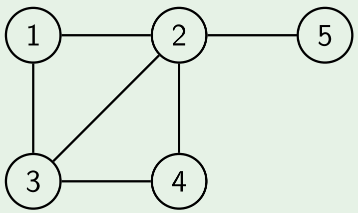
(a) A graph (to most of the world) (b) A graph (in algorithms)

Two flavors of graphs : directed and undirected.

Basic definitions and Terminology

- Undirected graph $G = (V, E)$
- V set of vertices
- E set of edges between unordered pairs of vertices called endpoints
- Captures pairwise relationship between objects
- Graph size parameters : $n = |V|$ (number of vertices), $m = |E|$ (number of edges).

Example



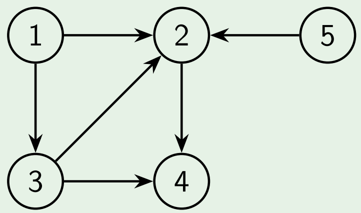
Nodes : $V = \{1, 2, 3, 4, 5\}$, $n = 5$

Edges : $S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{2, 4\}, \{2, 5\}\}$, $m = 6$

Basic definitions and Terminology

- Directed graph $G = (V, E)$
- V set of vertices
- E set of arcs between ordered pairs of vertices
- Captures oriented relationship between objects
- Graph size parameters : $n = |V|$ (number of vertices), $m = |E|$ (number of arcs).

Example



Nodes : $V = \{1, 2, 3, 4, 5\}$, $n = 5$

Arcs : $S = \{(1, 2), (1, 3), (3, 2), (3, 4), (2, 4), (5, 2)\}$, $m = 6$

Some examples of graphs

- Communication networks : collection of computers connected via a network, vertices $\equiv$ computers, edges $\equiv$ peering relationships
- Wireless networks : vertices $\equiv$ devices in physical location, edges $\equiv$ signal reception
- Information networks : Web graph, vertices $\equiv$ web pages, arcs $\equiv$ hyperlinks
- Social networks : vertices $\equiv$ people, edges $\equiv$ friend relationship
- Transportation networks : map of routes served by an airline carrier, vertices $\equiv$ airports, edges $\equiv$ non-stop flight

Definitions (directed graphs)

- Tails and Heads : A (directed) arc (i, j) has two endpoints i and j : i is the tail of the arc and j is its head.
  The arc (i, j) emanates from node i and terminates at node j. An arc (i, j) is incident to vertices i and j. The arc (i, j) is an outgoing arc of node i and an incoming arc of node j.
- Degrees : The indegree $d^-(i)$ of a node i is the number of incoming arcs of node i and its outdegree $d^+(i)$ is the number of its outgoing arcs. The degree $d(i)$ of a node i is the sum of its indegree and outdegree, $d(i) = d^+(i) + d^-(i)$.

Property

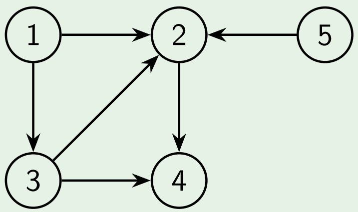
The sum of indegrees of all vertices equals the sum of outdegrees of all vertices and both are equal to the number of arcs m :

$\sum_{i \in V} d^+(i) = \sum_{i \in V} d^-(i) = m$

$\sum_{i \in V} d(i) = 2m$

Illustration

Example



Graph : $n = 5,\ m = 6$

Degrees :

i	1	2	3	4	5	$\sum_i$
d^+	2	1	2	0	1	$6 = m$
d^-	0	3	1	2	0	$6 = m$
d	2	4	3	2	1	$12 = 2m$

Definitions (undirected graphs)

The definitions for directed graphs easily translate into those for undirected networks.

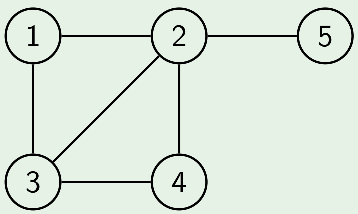
- The vertices i and j are adjacent for an edge $\{i, j\}$.
- Degree : The degree $d(i)$ of a node i is the sum of its adjacent vertices.

Property

$\sum_{i \in V} d(i) = 2m$

Illustration

Example



Graph : $n = 5,\ m = 6$

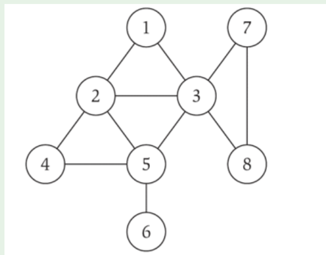
Degrees :

i	1	2	3	4	5	$\sum_i$
d	2	4	3	2	1	$12 = 2m$

Paths and Cycles

- A path in an undirected graph $G = (V, E)$ is a sequence P of vertices $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.
- A path is simple if all vertices are distinct.
- A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first $k - 1$ vertices are all distinct.

Example (Cycle)



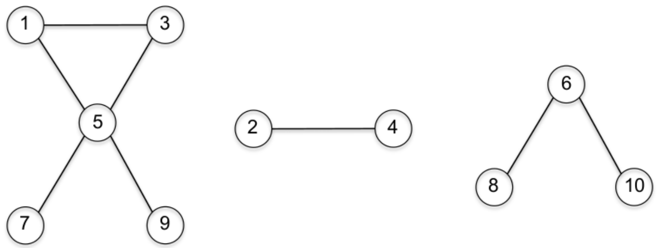
Cycle : $C = (1, 2, 4, 5, 3, 1)$

Graph Connectivity

- An undirected graph $G = (V, E)$ naturally falls into "pieces", which are called connected components.
- More formally, a connected component is a maximal subset $S \subseteq V$ of vertices such that there is a path from any vertex in S to any other vertex in S.

Example

Connected components : $\{1, 3, 5, 7, 9\}$, $\{2, 4\}$, and $\{6, 8, 10\}$



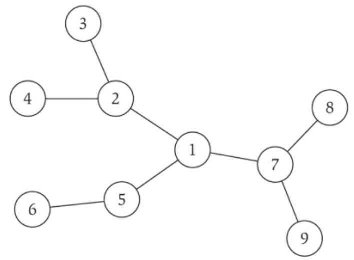
Trees

An undirected graph is a tree if it is connected and does not contain a cycle.

Property

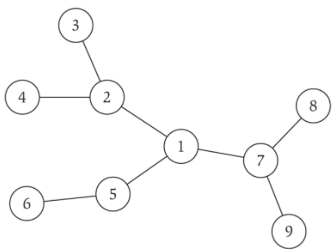
Let G be an undirected graph on n vertices. Any two of the following statements imply the third.

- G is connected.
- G does not contain a cycle.
- G has $n - 1$ edges.

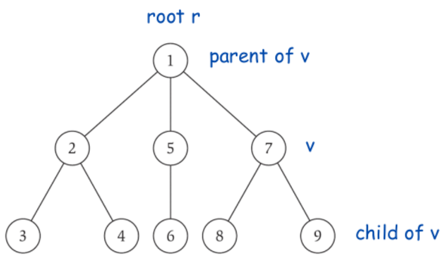


Rooted Trees

Given a tree T, choose a root node r and orient each edge away from r. Models hierarchical structure.



a tree



the same tree, rooted at 1

Measuring the Size of a Graph

- Number of edges in a graph

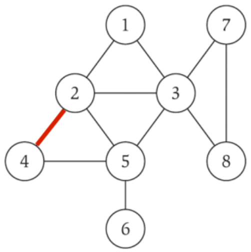
Example (Quizz)

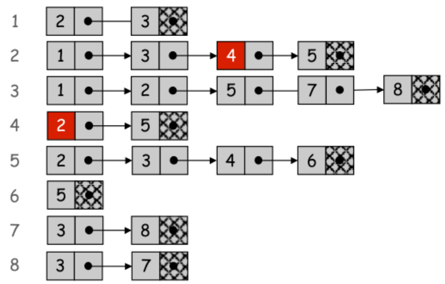
Consider an undirected graph with n vertices and no parallel edges. Assume that the graph is connected. What are the minimum and maximum numbers of edges, respectively, that the graph could have ?

- Sparse vs. Dense Graphs
  A graph is sparse if the number of edges is relatively close to linear in the number of vertices, and dense if this number is closer to quadratic in the number of vertices.

Representing a Graph (1/2)

- Adjacency List
  - An array containing the graph's vertices.
  - For each vertex, a pointer to each of the incident edges.





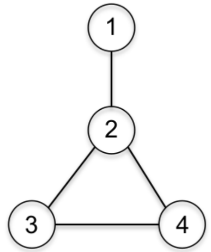
Example (Quizz)

How much space does the adjacency list representation of a graph require, as a function of the number n of vertices and the number m of edges ?

Representing a Graph (2/2)

- Adjacency Matrix
  - square $n \times n$ matrix A (equivalently a two-dimensional array,) with only zeroes and ones as entries. Each entry A_{ij} is defined as :

$$A_{ij} = \begin{cases} 1 \text{ if } (i,j) \text{ belongs to } E, \\ 0 \text{ otherwise.} \end{cases}$$



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Example (Quizz)

How much space does the adjacency matrix representation of a graph require, as a function of the number n of vertices and the number m of edges ?

Comparing the representation

It depends...

- on the density of the graph, i.e. on how the number m of edges compares to the number n of vertices.
- on which operations you want to support.

Example (Quizz)

- How many operations are required to identify if a vertex u is adjacent to a vertex v ?
- How many operations are required to identify the incoming edges of a vertex v ?