

THE  
LONDON, EDINBURGH, AND DUBLIN  
PHILOSOPHICAL MAGAZINE  
AND  
JOURNAL OF SCIENCE.

---

[SEVENTH SERIES.]

---

SUPPLEMENT, JUNE 1929.

---

CVI. *A Molecular Theory of Friction.* By G. A. TOMLINSON, B.Sc. (*of the National Physical Laboratory*) \*.

1. IT is now generally agreed that friction is, in some way, a consequence of the forces which molecules exert on one another when sufficiently close together. It is known that two molecules can exert a strong attraction on each other, and there is evidence of various kinds that the field of attraction probably extends to a distance of several times the molecular diameter, the force being a rapidly diminishing inverse function of the distance between the molecules.

Since the molecules of a solid are in a state of equilibrium, we have also to recognize the existence of a repulsive force between them, counteracting the cohesive attraction. There is little or no experimental information concerning this repulsive force, but there is certain indirect evidence that the force increases from zero very abruptly when two atoms approach to a critical distance. The highly stable equilibrium which exists amongst the atoms of a metal can only be explained by supposing that the rate of change of the repulsion is much greater than that of the attraction, when the atoms are drawn apart slightly by an elastic strain. The work of Born and Landé† and others on the energy of the crystal lattice led to similar conclusions. A similar conception has been advanced in recent years by various

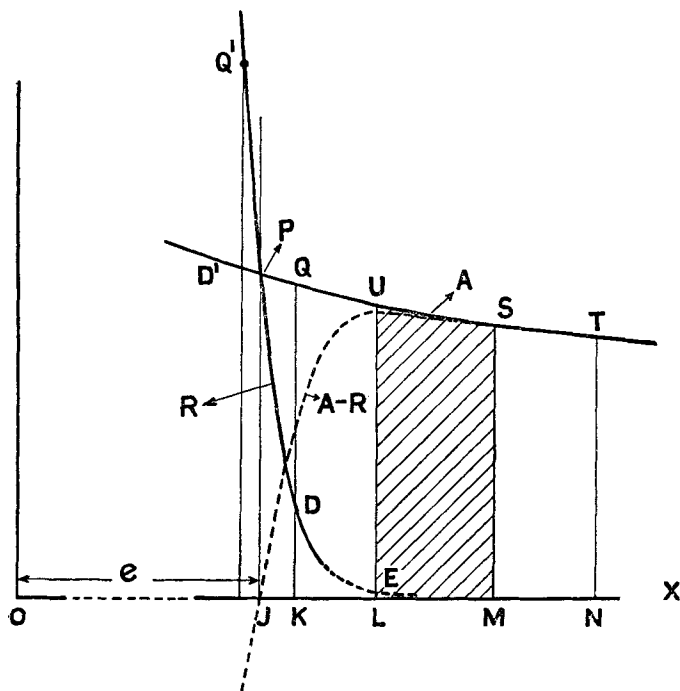
\* Communicated by Sir J. Petavel, K.B.E., F.R.S.

† *Deut. Phys. Gesell.* (1918).

other physicists, including Lennard Jones \* and T. W. Richards †.

The present paper contains a tentative theory of dry solid friction based upon the assumed existence of molecular attractions and repulsions having such characteristics, as shown diagrammatically in fig. 1 by the curves A and R. These curves represent descriptively the supposed attraction and repulsion as functions of the distance between the mole-

Fig. 1.



cules. No assumptions need be made as to the form of the curves, but the slope of the repulsion curve is assumed to be much greater than that of the attraction curve, and the effective range is supposed to be much less; in fact, considerably smaller than the dimensions of the molecules.

Because of the very limited present knowledge concerning interatomic forces, such a theory must necessarily be to a

\* Proc. Roy. Soc. cxii.

† Journ. Amer. Chem. Soc. xlv. to xlviii.

large degree speculative, and its proper place in this paper should be at the end, as a suggested explanation of the experimental work. But although the experiments to be described in the paper may be regarded as having some general interest, they would exhibit no particular connexion with each other, apart from their relation to the tentative theory, the probability of which they were designed to test. It therefore seemed preferable, in order to admit of an adequate seriatim discussion of the results, to give first a preliminary description of the theory, the details of which are further discussed in relation to the several experiments to be described.

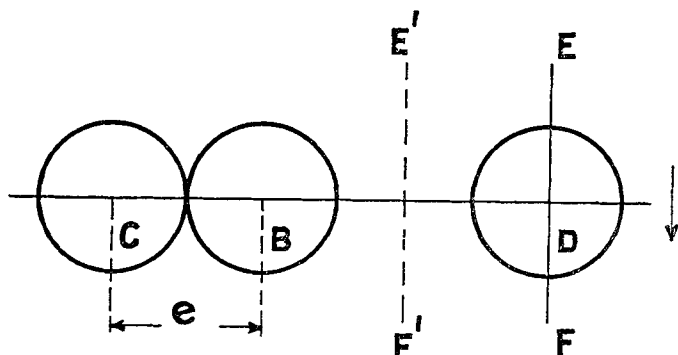
Consider two solid bodies in contact and with relative sliding motion. By our hypothesis, only a certain number of molecules of the one body will approach those of the other body sufficiently closely to suffer repulsion, as any material surface will have quite a coarse structure compared with the range of the repulsive force. Many other molecules of one body may experience some attraction towards the molecules of the other body without entering the repulsion field at all. The applied force between the bodies, together with the sum of all the cohesive attractive forces, is supported by the sum of all molecular repulsions. (Some very anomalous cases of friction are described later in the paper in which the cohesive attraction between the two bodies produces an important effect on the friction. Normally such attractions are negligible by comparison with the applied force.) As the two bodies move relatively, there is a continuous change taking place in the pairs of molecules bearing the load, and the theory assumes that when two molecules come into contact, *i.e.*, come into each other's repulsion field, and then separate, a loss of energy occurs which is manifest as friction. The theory makes no assumptions regarding the cause of the attractive and repulsive forces, but treats the molecules as if they were simple centres of fields of force. Also the ultimate distribution of the lost energy amongst the various possible degrees of freedom is not considered in the theory, which deals solely with the molecular mechanism by which work may become transformed into heat.

It is difficult to understand how friction could be explained as a loss of energy occurring when two atoms approach within each other's attraction field alone. Consider for simplicity a single atom D (fig. 2) forming part of a body which is moving in the direction of EF past another body, of which B and C form two atoms in the state of equilibrium

characteristic of a solid. Referring to the curves A and R of fig. 1, the point P corresponds to the initial undisturbed condition of the atom B, which takes up a position at a distance of O J from C such that the attraction and repulsion on it are equal and opposite and of amount P J. We may observe the very stable condition which follows from the assumed rates of change of the two forces. A small displacement of B in either direction from the point J is accompanied by a strong restoring force.

Let us suppose that the atom D in moving past B along the line E F approaches B to within a distance such as O N in fig. 1; that is, within the attraction field but outside the range of the repulsion. The passage of A will cause a slight disturbance in the position of B, which will move

Fig. 2.



away from C, supposing C to be fixed, to a distance O K in fig. 1 such that the net force Q D now exerted on B by C equals the attraction T N of atom D. The atom D in proceeding further along E F then withdraws from B, which returns to its initial position. It is conceivable that B arrives back to its original position with some appreciable velocity and therefore with some added energy, the aggregate of which might correspond to the loss of energy in friction. If this were the case, however, the energy imparted to B would be reduced as the velocity of relative motion of the two bodies was reduced. But in moving two actual bodies over one another the same total number of atomic approaches would occur whatever the velocity of motion, and therefore the work done, and consequently the coefficient of friction, would be reduced continuously as the velocity was reduced. This is quite contrary to experience. On the other hand, it

known that the characteristic atomic frequency of most of the elements is of the order of  $10^{13}$  per second ; and it seems probable that on account of its extremely small natural period, the displacement of the atom B will follow the increase and decrease of the attraction of D in an almost perfectly dead-beat manner. In other words, the mechanical work done on the atom D by the attraction during its approach does not differ materially from the work done by D against the attraction during its recess.

To explain friction it appears necessary to suppose the existence of some irreversible stage in the passage of one atom past another, in which heat energy is developed at the expense of external work. It is possible to suggest a means by which this may happen in the case of those atoms which come near enough together to experience repulsion.

Let us now consider the atom D as passing by the elementary body B C along a nearer line  $E'F'$  and approaching so close that D and B come into mutual repulsion. As D approaches, the two atoms will exert an increasing attraction, and under this force B will move away slightly from C as before. This process continues until the atoms are separated by a distance O L in fig. 1, at which

distance  $\frac{dA}{dx} = \frac{dR}{dx}$ , A and R denoting the forces of attrac-

tion and repulsion. At this period A—R has a maximum value and B is equidistant from D and C. As D approaches still nearer, the symmetrical position of B will be maintained,

since for all positions to the left of L,  $\frac{dR}{dx}$  exceeds  $\frac{dA}{dx}$ , and

the symmetrical position is one of stable equilibrium. A small movement of D then brings both pairs of molecules into the position corresponding to the point P, *i. e.*, B occupies its original position relative to C with D added to it in a similar stable relationship. Any approach of D now brings into play a powerful net repulsion such as is shown by  $Q'D'$ , and this is supposed to be the condition of those molecules which are assisting to support the external load applied to the contact surfaces.

Consider next what occurs when the molecule D is withdrawn. Up to the position corresponding to the point L, the molecule B remains in stable equilibrium equidistant from D and C, and everything up to this point is reversible. At the point L the equilibrium of B changes from a stable into an unstable condition. As this transition is smooth and continuous, there will be a certain probability that the

molecule B will persist in its symmetrical position of equilibrium as D recedes, until B is at some distance O M from both C and D, when, owing to its instability, B will suddenly fly back to its position of stable equilibrium at a point between J and L. This point, in fact, is the same that B occupied when D in its approach was at the same point that it is now supposed to occupy in recess.

The molecule B thus returns to its stable position with a store of kinetic energy equal to the area U S M E, this being the work done on B by D during the irreversible portion of the cycle from L to M. We thus have a possible mechanism by which the recession of two molecules is not the reverse of their approach, but contains an irreversible stage in which mechanical work is transformed into molecular kinetic energy or heat.

According to the theory suggested, friction is in a sense a matter of chance, as the displacement L M of the molecules into the unstable region may vary greatly. Actually we are concerned with the statistical average of a very great number of different values, and the uniformity always observed in experiments on friction is due to the large number of molecules in contact. It may be remarked that on this theory the two bodies in contact become virtually united at all the points where atomic repulsion occurs, by the formation of a continuous chain of atoms in stable equilibrium. On severing the atomic bond the plucking action of one atom on the other dissipates a quantity of energy which is independent of the velocity of relative motion, and in this respect the theory is in agreement with experiment. Further, only those atoms which contribute to the elastic reaction, and so help for a short time to support the load, pass through the irreversible stage and so involve friction. This is again in conformity with the fact that the friction is proportional to the load.

In the simple case we have considered in fig. 2 it will be seen to be a matter of chance whether the atom B ultimately attaches itself to C or to D. In the case of actual bodies there seems to be little doubt that the disturbed atom has a strong bias in favour of a return to its original position, and probably nearly all do so. Some, however, definitely do not—a question which is discussed later.

2. We shall next obtain an expression for the coefficient of friction. The force F between the surfaces is supposed to be supported by the sum of the molecular repulsions

$$F = \Sigma (p).$$

Since the force of repulsion varies very rapidly with distance, the individual values of  $p$  must vary widely from one place to another and from one instant to another, but a very large number of molecules are concerned under conditions which justify us in assuming the total number  $n$  to be statistically constant so long as the load is constant. This being so, we can assign a statistical average value  $\bar{p}$  to the force of repulsion of a pair of molecules, and we then have

$$F = n\bar{p}.$$

The 'actual value of  $n$  will depend on the total load and the shape of the contact surfaces.

As one body slides over the other, a continuous exchange of partners will proceed amongst the repelling molecules. Some will pass out of the repulsion range and others will enter it. At each approach and separation a loss of energy occurs, of which the average value will be called  $W$ .

If we can find an expression for the total number  $N$  of separations taking place in a movement  $x$ , we can equate the loss of energy to the external work done :

$$NW = \mu Fx.$$

If, for the moment, we imagine the molecules to have a perfectly definite array, we can represent the circumstances, for example, by fig. 3, where the crosses represent the whole of the molecules of one surface, and the circles the proportion of molecules on the second surface which contribute to the repulsion. If the two patterns are moved relatively to each other, say vertically up the page, through a distance  $x$ , which is large compared with  $e$ , it is evident

that the number of separations which will occur is  $n \times \frac{x}{e}$

$n$  being the total number of circles. It may be noticed that in the theoretically ideal distribution shown in the figure we should get a different number of separations if the relative motion of the two bodies were in a different direction—*e. g.*, if the motion were at  $45^\circ$ , the number would be

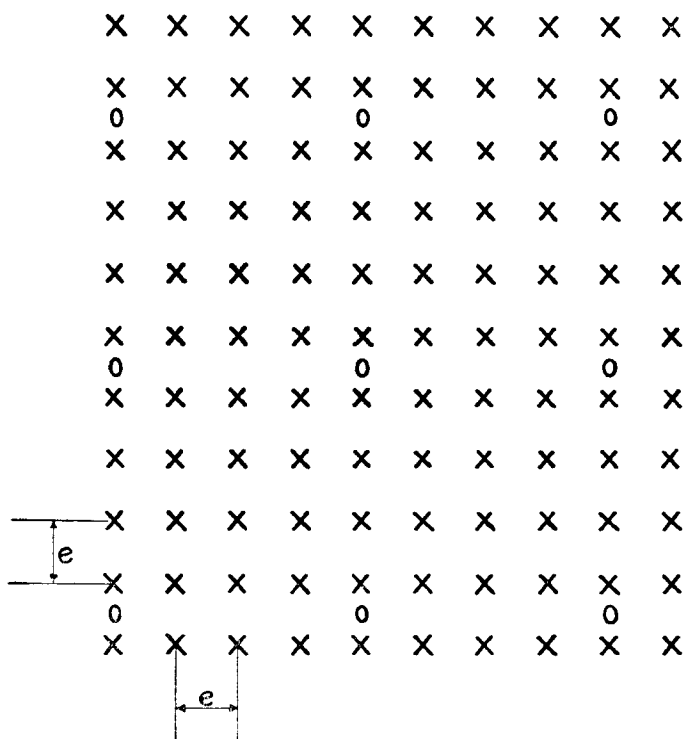
$n \times \frac{x}{\sqrt{2}e}$ . There is some evidence, as a matter of fact, that

friction between crystalline surfaces may depend on the direction of relative motion.

In any ordinary case, however, the general distribution of the molecules in the surfaces and of the individual molecules concerned in the repulsion at any moment will be quite

irregular, and we assume, moreover, that the surfaces will have a coarse structure in comparison with the extent of the repulsive field. There must, however, be a sufficient number of molecules within the repulsive range at any moment to support the load, and the conditions will be statistically

Fig. 3.



constant, so that we may reasonably assume that the number of separations in distance  $x$  will be

$$N = q \times n \times \frac{x}{e},$$

where  $e$  is the *mean* molecular distance and  $q$  a statistics constant whose actual value must remain unknown, but probably not very widely different from unity.

We then have

$$NW = \mu Fx = \mu n \bar{p} x,$$



$$\text{also} \quad NW = q \frac{n^2}{e} W,$$

$$\text{and hence} \quad \mu = \frac{qW}{pe} \dots \dots \dots (1)$$

Remembering our initial assumptions as to the nature of the molecular attraction and repulsion, we can express  $W$  approximately in terms of the normal cohesive attraction and the average displacement  $\bar{l}$  of the molecule into the unstable region of equilibrium (LM of fig. 1). Thus, if we take the force of attraction acting over the small displacement  $\bar{l}$  to be a constant force  $A_0$  slightly smaller than the normal cohesion  $P$  of fig. 1, and neglect the small repulsive force acting between  $L$  and  $M$ , we get as a somewhat free approximation

$$W = A_0 \bar{l}$$

$$\text{and} \quad \mu = q \frac{A_0 \bar{l}}{pe}, \dots \dots \dots (2)$$

a form of expression of more interest than (1).

The result obtained from the theory is so far in agreement with the experimental laws of friction; viz., that the friction is proportional to the load, independent of the area in contact and independent of the velocity of motion.

Further, in the expression for  $\mu$ ,  $A_0$  and  $\bar{p}$  may be regarded as molecular elastic constants which must have some association with the ordinary elastic constants of the material. The theory therefore suggests that the coefficient of friction should be related to the elastic constants. This has been investigated experimentally and a definite relation discovered. A discussion of this follows in section 8.

3. The foregoing theory will now be applied to the case of rolling friction. Consider a cylinder of radius  $r$  and length  $b$  rolling through a distance  $x$  on a horizontal plane surface under a load  $F$ . Let  $P$  be the force to be applied at the axis of the cylinder parallel to the plane to overcome the friction. Then the coefficient of rolling friction will be defined as

$$\lambda = \frac{P}{F}.$$

Making the same assumptions as before, we require to find the total number of molecular separations which occur in rolling a distance  $x$ . It is known that the cylinder and

plane come into contact over a parallel strip of width  $2a$  which can be determined. The intensity of normal stress is a maximum at the centre of the strip, and falls off to zero at the edges according to the equation

$$f = \frac{3}{2} f_{\text{mean}} \sqrt{1 - \frac{y^2}{a^2}} \Bigg\}, \dots \dots (3)$$

$$= \frac{3}{2} \frac{F}{2ab} \sqrt{1 - \frac{y^2}{a^2}} \Bigg\}, \dots \dots$$

where  $y$  is the distance from the centre of the strip.

The maximum stress intensity at the centre strip is thus

$$f_{\text{max.}} = \frac{3}{4} \frac{F}{ab} \dots \dots \dots (4)$$

In rolling a distance  $x$  a rectangular area  $bx$  is swept out, and every portion of this area comes in turn under the normal stress  $f_{\text{max.}}$  Small end-effects are rendered negligible if  $b$  and  $x$  are supposed to be large in comparison with  $a$ . Hence the aggregate force which is applied and removed is

$$\frac{3}{4} \frac{F}{ab} bx = \frac{3}{4} \frac{Fx}{a}.$$

If we had again the ideal geometrical arrangement of molecules in rows, the number of separations associated with the application and removal of this load would be given by the equation

$$n \bar{p} = \frac{3}{4} \frac{Fx}{a}.$$

But as the surfaces in contact are continuously changing by the rolling motion just as in the preceding section they were changed by sliding, and the same irregular distribution of the repulsions exists, the actual number of separations will be  $qn$ ,  $q$  being the same statistical factor as before, since both sliding and rolling have the similar effect of bringing a sequence of different but statistically equivalent molecular distributions into contact.

Hence the loss of energy

$$= \frac{1}{2} qn W$$

$$= \frac{3}{4} \frac{Fx}{ap} q W$$

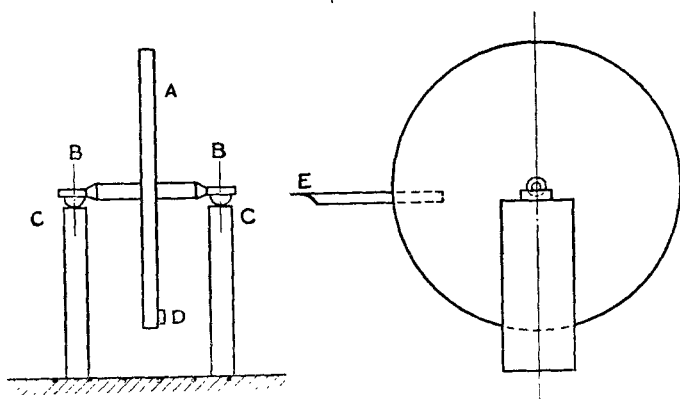
$$= \frac{3}{4} \frac{Fxq}{ap} A_0 \bar{l}.$$



the disk oscillating slowly the successive semi-amplitudes were marked off at the rest positions of the image of the pointer on the screen. In this way the decrement of the amplitude was accurately measured, and from this friction could be found. The resistance of the air was examined by conducting a similar experiment having the disk suspended with its axis vertical on a torsional suspension of piano wire, giving the same frequency of oscillation. It was found that the contribution of the air resistance to the total friction measured in the rolling experiments was negligible. The rolling surfaces were thoroughly cleaned with alcohol-ether-ammonia cleaning mixture before the measurement.

The results obtained are given in Table I.

Fig. 4.



The theoretical relation in equation (6) may be used to calculate  $\lambda$  in terms of  $\mu$ , for comparison with the above experimental result. As regards the value of  $e$ , the distance between the atoms, X-ray measurements give the lattice constant of iron to be  $2.86 \times 10^{-8}$  cm., and this value will be used for  $e$ .

The value of  $2a$  can be found from equation (7) following, and is given in Table I.

$$\text{Hence } \frac{\lambda}{\mu} = \frac{3 \times 2.86 \times 10^{-8}}{4 \times 4.14 \times 10^{-4}} = 0.519 \times 10^{-4}.$$

For clean dry steel Sir William Hardy has found  $\mu = 0.79$ , and the writer has obtained a value 0.39 (see Table VI.): substituting these values,  $\lambda$  is found to be 0.000041 or 0.000020 respectively. If these are compared with the

first three observed values of  $\lambda$  in Table I., a satisfactory order of agreement is found, especially in the case of the higher value of  $\mu$ . It is considered preferable to exclude the last two observed values from this comparison, for reasons discussed later in section 6.

TABLE I.

Diameter of cylinders 0.1000 in.		
Length of cylinders in rolling contact 0.100 in.		
Total weight 600 gm.		
Width of contact area, $2a = 8.28 \times 10^{-4}$ cm.		
Amplitude of oscillation.		Coefficient of friction. $\lambda$ .
cm.	radians.	
0.0090	0.0707	0.000047
0.00506	0.0398	0.000041
0.00135	0.0106	0.000045
0.00038	0.0030	0.000015
0.00024	0.0019	0.000011

5. Some further experimental results on rolling friction will now be examined, and will be shown to be in close agreement with the theoretical expression deduced for the coefficient. The quantity  $a$  which occurs in this expression can be found from the equation given by Hertz in his theory of elastic contact :

$$a^2 = \frac{1}{\pi} Fr(\mathfrak{S}_a + \mathfrak{S}_b) \quad . \quad . \quad . \quad (7)$$

where

$r$  = radius of cylinder,

$F$  = load on the cylinder per unit length,

$$\mathfrak{S} = \frac{3K + 4C}{C(3K + C)},$$

$C$  = modulus of rigidity,

$K$  = modulus of compressibility.

The suffixes  $a$  and  $b$  refer to the two materials forming the roller and the plane.

Where these are both the same we have

$$a^2 = \frac{2}{\pi} FrS. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Hence, using equation (6), which is cleared of all the unknown terms, for the value of  $\lambda$  in terms of  $\mu$ , we have

$$\lambda = \frac{3}{4} \frac{e\mu}{a} = \frac{3}{4} \sqrt{\frac{\pi}{2}} \frac{e\mu}{\sqrt{FrS}}. \quad . \quad . \quad . \quad . \quad (9)$$

This equation indicates that  $\lambda$  is not a constant, but is inversely proportional to  $\sqrt{F}$  and to  $\sqrt{r}$ .

Goodman\* carried out an extensive investigation on

TABLE II.

Load F in lb.	Coefficient of rolling friction.	Coefficient of friction $\times \sqrt{F}$ .
1000.....	0.0040	0.1265
2000.....	0.0028	0.1252
3000.....	0.0023	0.1260
4000.....	0.0020	0.1264
5000.....	0.0018	0.1273
6000.....	0.0016	0.1239
7000.....	0.0015	0.1254
8000.....	0.0017	0.1520
9000.....	0.0017	0.1613
10,000..	0.0018	0.1800

rolling friction, employing commercial ball and roller bearings, and he arrived at the following conclusions:—

- (1) The coefficient of friction decreases as the load increases.
- (2) The coefficient of friction decreases as the radius increases.
- (3) The coefficient of friction is nearly independent of the speed.

These conclusions agree qualitatively with equation (9), but the theory can be tested further as regards the variation of the coefficient of friction with load by comparison with Goodman's results. In Table II. the first two columns are

\* Proc. Inst. Civil Eng. 1911-12.

taken from Goodman's paper, and the third column has been computed by the writer. The measurements were made on a roller bearing having rollers  $\frac{3}{4}$  in. in diameter and 6 in. in length. The speed of the shaft was 80 revolutions per minute and the bearing was run without lubricant.

The last column shows that over a wide range of loads the theoretical relation between the load and the coefficient of friction is in very good agreement with the experimental results.

6. If the foregoing theory of friction is substantially correct an interesting deduction may be made. If two surfaces have a relative oscillatory motion, either sliding or rolling, and it is possible to reduce the amplitude to such an extent that the displacement of the surfaces or some portion of them becomes comparable with  $\bar{l}$ , we should find that an increasing proportion of the molecules would return through the unstable stage to the stable condition of equilibrium without being released with increased kinetic energy. Or we should expect the friction to become progressively smaller as the amplitude diminishes below a certain value. Such a result has, in fact, been obtained for both sliding and rolling friction in some experiments which will now be described.

A very light oscillator, made from a piece of cork, was pivoted on the point of a fine needle on a polished steel flat surface. A small piece of mirror was attached to the cork and also a piece of magnetized sewing-needle to provide a suitable control couple. The oscillator was contained in an exhausted vessel having a plate-glass window. The oscillations were observed by a reflected image of a cross-line on a remote scale. An oscillation was started by disturbing the magnetic field slightly, and the successive amplitudes were measured at the scale.

It can be shown that the amplitude will decrease uniformly if friction is constant. Thus let  $\theta_1$  and  $\theta_2$  be two successive amplitudes. Then the energy lost in friction is

$$c(\theta_1 + \theta_2),$$

where  $c$  is a constant.

This is equal to the change in potential energy of the system, or

$$c(\theta_1 + \theta_2) = \frac{1}{2}k(\theta_1^2 - \theta_2^2),$$

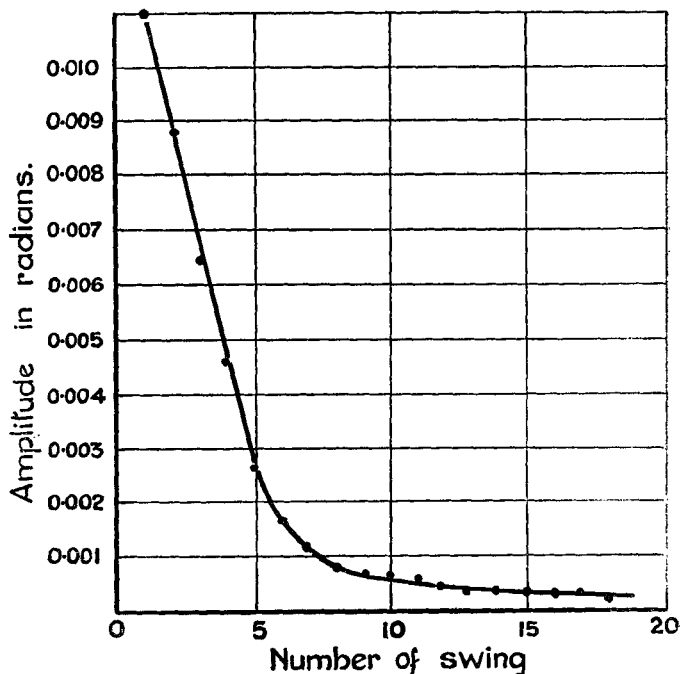
where  $k$  is the stiffness of the controlling couple.

$$\text{Hence } \theta_1 - \theta_2 = \frac{2c}{k} = \text{constant.}$$

The actual result obtained is very different, as shown by the curve in fig. 5, in which there is a well-defined deviation from the straight line at a certain amplitude, followed by a rapidly diminishing friction couple.

It is interesting to make an estimate of the relative linear displacement of the surfaces corresponding to the knee of the curve.

Fig. 5.



In an investigation into the deformation of instrument pivots at the National Physical Laboratory, Stott has found that a small flat is produced, the area of which over a wide range of loads is proportional to the load, or a definite flow-stress exists which is constant for any load. The value of this stress is found by Stott to depend somewhat on the angle of the pivot, and for the present purpose an assumed stress of 500 kilogrammes per square millimetre will be a fairly probable value. The weight of the oscillator was 0.15 gm., and the radius of the small flat, if it is assumed to be circular, is therefore about  $3 \times 10^{-5}$  cm



The bend in the curve corresponds approximately to an amplitude of about  $0.0015$  radian, so that the motion at the edge of the contact flat is about  $4.5 \times 10^{-8}$  cm. This is an amount of nearly the same magnitude as the distance between the atoms as found by X-ray measurements. The linear motion decreases uniformly to zero from the periphery to the centre, and the reduction in friction probably commences at the centre of the compressed area, spreading outwards more and more as the amplitude decays away. Hence the critical amplitude at which the friction begins to diminish is apparently an amount well below atomic dimensions.

TABLE III.

Rolling friction of two crossed cylinders 0.1 in. and 0.4 in. diameter. Total weight 600 gm. Major semi-axis of contact of ellipse, $5.95 \times 10^{-3}$ cm. Minor semi-axis of contact of ellipse, $2.4 \times 10^{-3}$ cm.		
Amplitude of oscillation.		Coefficient of friction. $\lambda$ .
cm.	radians.	
0.00845	0.0665	0.000014
0.00720	0.0567	0.000013
0.00561	0.0442	0.000012
0.00276	0.0217	0.000010
0.000635	0.0050	0.000006
0.000237	0.00187	0.000004

In the case of rolling friction a similar result has already been obtained, as shown in Table I., where the coefficient of rolling friction decreases to about a quarter of its initial value when the amplitude becomes very small. This decrease was always observed in measurements of  $\lambda$ , and two further examples are given in Tables III. and IV. In the first case the 0.1 in. diameter cylinders are rolling upon two other cylinders of 0.4 in. diameter set at right angles to the axis of rolling.

In the second case two  $3\frac{1}{2}$ -mm. steel balls are rolling on flat steel surfaces.

7. The measured coefficients of friction for three different types of rolling, given in Tables I., III., and IV., can be compared with the relative values of the coefficients found from the theoretical expression for  $\lambda$ , as a further test of the probable validity of the theory.

It will first be shown that equation (5) applies equally in all these cases of rolling.

In general, the contact area is an ellipse having semi-axes,  $b$ , parallel to the axis of rolling, and  $a$ , parallel to the

TABLE IV.

Rolling friction of $3\frac{1}{2}$ -mm. steel ball on steel plane. Total weight 600 gm. Radius of contact circle, $3.3 \times 10^{-3}$ cm.		
Amplitude of oscillation.		Coefficient of friction. $\lambda$ .
cms.	radians.	
0.0077	0.0605	0.0000165
0.0067	0.0528	0.0000155
0.00404	0.0318	0.0000125
0.00086	0.00675	0.0000048
0.000442	0.00348	0.0000030
0.000232	0.00183	0.0000017

path of rolling. Let  $a$  and  $b$  correspond to X and Y axes respectively, then the normal contact-stress is given by

$$f = \frac{3}{2} \frac{F}{\pi ab} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}, \dots \dots (10)$$

and the mean stress along the Y axis, parallel to the axis of rotation, is

$$\begin{aligned} \frac{\int_0^b f dy}{\int_0^b dy} &= \frac{\frac{3}{2} \frac{F}{\pi ab} \int_0^b \sqrt{1 - \frac{y^2}{b^2}} dy}{b} \\ &= \frac{3}{8} \frac{F}{ab} \dots \dots \dots (11) \end{aligned}$$

Hence, as in section 3, the total load applied and removed in rolling through a distance  $x$  is

$$2bx \times \frac{3F}{8ab} = \frac{3Fx}{4a}, \quad . \quad . \quad . \quad (12)$$

which is the same result as that previously obtained for the rolling of a cylinder on a plane, that being only a particular case.

According to the theory, therefore, the relative coefficients of friction should be inversely proportional to the values of  $a$ . These have been calculated, and the results are summarized below.

TABLE V.

Type of rolling.	$a$ , cm.	Theoretical, $\lambda$ .	Measured, $\lambda$ .
A. Cylinder on plane .....	$4.14 \times 10^{-4}$	0.000041	0.000047
B. Cylinders on cylinders...	$24 \times 10^{-4}$	0.000007	0.000013
C. Spheres on plane .....	$33 \times 10^{-4}$	0.000005	0.000016

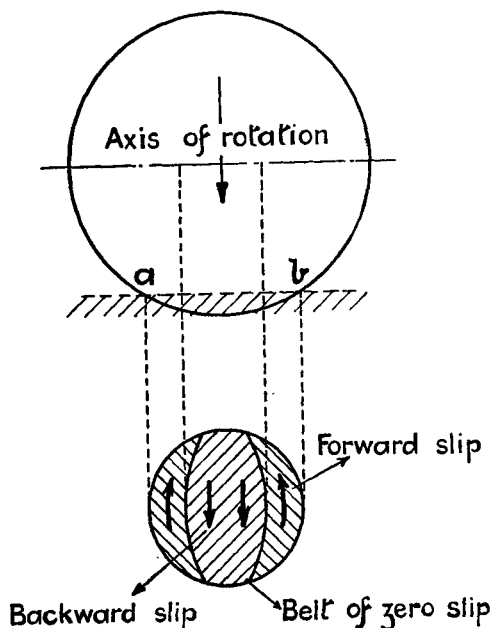
The theory thus shows that the rolling friction in cases B and C should be about  $\frac{1}{6}$  and  $\frac{1}{8}$  respectively of that of case A. Although the measured values are considerably smaller, there is some discrepancy in the relative values as compared with the theoretical values. There is reason for this, however, in the fact that cases B and C are not exactly examples of pure rolling.

Heathcote\* has pointed out in connexion with the friction of ball bearings that some relative sliding motion must take place continuously at the contact. In fig. 6 a sphere rolling on a plane is shown. Different parts of the sphere between  $a$  and  $b$  are at slightly different radii from the axis of rotation, and therefore must have different peripheral velocities, leading to a distribution of relative sliding somewhat as shown in the diagram. There are two symmetrically-placed belts along which no slipping occurs, and inside the belts the surface of the ball slips backwards, while outside the belts the slip is forwards. The form of these belts shown is purely conjectural, but their existence has been demonstrated

\* Proc. Inst. Auto. Eng. xv.

by Heathcote. The rolling action in case C is therefore not pure rolling such as occurs in case A, and there is an additional source of friction present which will tend to increase the measured coefficient of rolling friction above the theoretical value. In case B a similar effect is present, but probably to a less extent when a small cylinder rolls on a large one. Unfortunately this additional friction cannot be computed; but if we recognize its presence, the results of Table V. are in reasonable conformity with those indicated by the theory.

Fig. 6.



8. The expression  $q \frac{A_0}{p} \frac{\bar{l}}{e}$  for the coefficient of friction suggests, as previously remarked, that  $\mu$  may be related to the elastic constants in some way. This possibility will now be examined more closely. Consider, for simplicity, a sphere of one material A sliding on a plane of another material B to which we can apply Hertz's equations for elastic deformation. In the present theory, friction is supposed to be proportional to the number of molecules undergoing repulsion, and at the same time this number is supposed to

be proportional to the load, these suppositions leading to agreement with the well-known fundamental law of friction.

For a load  $F$  and radius  $r$ , Hertz shows that the contact circle has an area

$$\alpha \propto F^{\frac{2}{3}} r^{\frac{1}{3}} (\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{3}} \dots \dots \dots (13)$$

As we are at present only concerned with the nature of the material, we shall in the first place simplify the reasoning by regarding  $F$  and  $r$  as constant, under which conditions

$$\alpha \propto (\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{3}} \dots \dots \dots (14)$$

In the theory it is assumed that

$$F = n \bar{p},$$

and in the case of a soft material the maximum repulsion between two molecules will be small, not because the molecules are incapable of exerting greater repulsion, but rather because the lattice strength will not support any greater force without distortion. Hence for the soft material the mean repulsion  $\bar{p}$  will also be small, and to support the load  $F$  the number of molecules must be correspondingly large. In terms of elasticity the more yielding material has a high value of  $\mathfrak{S}$  and a large contact area  $\alpha$ .

For the given load  $F$  the value of  $n$  is inversely proportional to  $\bar{p}$ . But the coefficient of friction by equation (2) should also be inversely proportional to  $\bar{p}$  and therefore directly proportional to  $n$ , where different materials are concerned, or, substituting in equation (2),

$$\mu = \frac{q A_0 \bar{l} n}{F e}.$$

If, as we may reasonably assume, there is some simple relation for various materials between the molecular quantity  $n$  and the elastic quantity  $\alpha$ , we may write  $n$  as some function of  $\alpha$ , let us say  $f(\alpha)$ .

$$\text{Hence} \quad \mu \propto \frac{q A_0 \bar{l}}{F e} f(\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{3}},$$

and we may therefore expect to find some systematic connexion between  $\mu$  and  $(\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{3}}$ . It has actually been found from the measurement of a large number of coefficients of friction that  $\mu$  is directly proportional to  $(\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{3}}$  with a fair degree of exactness.

Before describing these experiments we shall briefly consider the meaning of this experimental result. It follows

in general that the harder the two materials are the smaller the coefficient of friction will be. Also, since in the present argument  $F$  is a constant and  $q$  may be assumed to have a

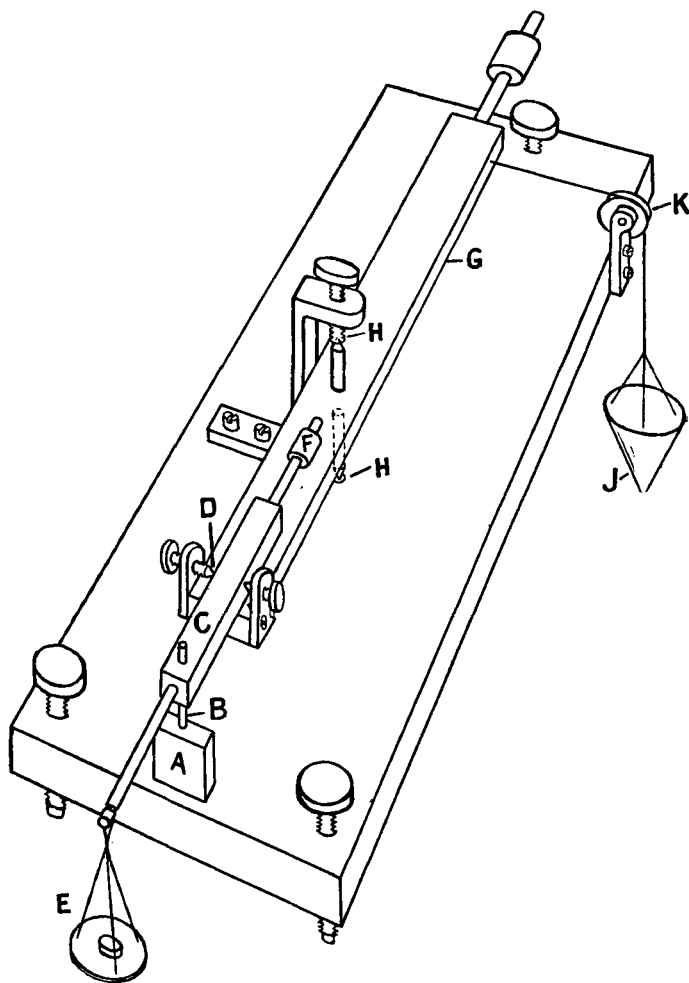
similar value for any material, the quantity  $\frac{A_0 \bar{l}}{e}$  can be regarded as the same for all the different materials used. The atomic distance  $e$  does not vary greatly amongst the different metals; hence  $A_0 \bar{l}$ , which represents the loss of energy in a single molecular separation, is roughly constant. This deduction appears to be not illogical on general grounds, for when the molecular cohesion bond is weak, that is  $A_0$  is small, one would expect the displacement of the molecule beyond its limiting stable position to be greater.

The available data on dry friction were found to be somewhat meagre; and as this relation has a very important bearing on the present theory, a fairly extensive series of coefficients of friction has been measured. In this series ten different materials were used in all the possible combinations, yielding altogether 55 different coefficients of friction. The apparatus used for the purpose is shown by a perspective sketch in fig. 7. A plane surface  $A$  of one of the two materials has a spherical end  $B$  of a rod of the other material in sliding contact with it. The rod is carried by a lever  $C$  pivoted about a horizontal axis on hardened steel points  $D$ . A light scale-pan  $E$  is suspended from the free end of this lever, and the whole can be balanced by a counterpoise weight  $F$ . The supports for the pivot are themselves attached to a second lever  $G$ , which is similarly pivoted about a vertical axis on points  $H$ . A known normal load can be applied by weights in the pan  $E$ , and the tangential friction force is applied by feeding a fine stream of silver sand into a paper hopper  $J$ , which is attached to the lever  $G$  by a thread of cotton passing over a light pulley  $K$ . The effect of the friction of the various pivots on the results was found to be quite negligible, all the pivots being carefully prepared on a special pivot lapping machine in use at the National Physical Laboratory. The apparatus can be balanced and levelled so that the lever  $G$  floats freely in any position.

The contact surfaces, when of metal, were worked up on the finest grade of blue-black emery paper and finally polished on wet chamois leather. The surface was washed with a cleaning fluid and scrubbed with a pad of specially pure cotton-wool until a clinging contact was obtained and the surface emitted a continuous squeaking noise under the

motion of the pad. In the case of lead the surfaces could not be polished and cleaned in the above way, and cleanness

Fig. 7.



was ensured by working with a freshly-cut surface of the metal, prepared by paring it with a clean razor blade. With glass a fused spherical slider and a piece of plate glass were used.

The measured coefficients of friction are given in Table VI., and in Table VII. the computed values of the ratio

$$\frac{(\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{2}}}{\mu} \times 10^8$$

are given. An inspection of Table VII. shows this ratio to be nearly constant. The mean value is  $5.47 \times 10^{-8}$ , and 42 of the 55 values lie within about 10 per cent. of the mean. If the mean value of the ratio is found for each material when paired in turn with all the materials, the constancy is even more striking. These mean values are shown in the last row.

The experimental results thus show that the coefficient of friction of any two surfaces is not a fortuitous constant, but is a function of the elastic moduli  $K$  and  $C$ , and may be expressed with a fair degree of accuracy by an equation,

$$\mu = 0.18 \times 10^8 (\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{2}}.$$

As regards their bearing on the theory of friction, these results certainly appear to support the assumption that the number of molecules in repulsion adjusts itself to be proportional to the load, and that the friction is proportional to this number. It should be made clear that, although the coefficient of friction is found proportional to the contact area in the restricted case of equation (14), it is not to be understood that this applies in the general case of equation (13). In other words, the number of molecules in contact is not proportional to the total area, the number per unit area being a function of the radius of curvature and of the actual load applied, as might be expected, since the intensity of normal stress depends on these variables.

The series of coefficients of friction in Table VI. is in itself of some interest. It will be seen that the values found for any one material taken in turn with all of the materials are in ascending order of magnitude, which is the same for every material. Also, where any two materials  $A$  and  $B$  are concerned, the coefficient  $\mu_{ab}$  is nearly always intermediate between  $\mu_{aa}$  and  $\mu_{bb}$ . These results of course follow at once from the more general relation,

$$\mu \propto (\mathfrak{S}_A + \mathfrak{S}_B)^{\frac{1}{2}}.$$

9. There is no theory of sliding friction which has general acceptance, but in the case of rolling friction a theory was developed by Osborne Reynolds \* which has been considered

\* Phil. Trans. Roy. Soc. 1871,



TABLE VI.

	Hard steel.	Mild steel.	Platinum.	Nickel.	Copper.	Brass.	Aluminium.	Glass.	Tin.	Lead.
Hard steel .....	.393	.410	.398	.428	.548	.535	.649	.605	.785	1.935
Mild steel .....	.410	.411	.427	.429	.533	.506	.605	.721	.786	1.930
Platinum .....	.398	.427	.445	.386	.592	.560	.796	.569	.835	2.070
Nickel .....	.428	.429	.386	.389	.562	.504	.745	.775	.895	2.150
Copper .....	.548	.533	.592	.562	.600	.618	.685	.675	.867	1.945
Brass .....	.535	.508	.560	.504	.618	.634	.706	.873	.752	2.110
Aluminium .....	.649	.605	.796	.745	.935	.706	.937	.845	.941	2.420
Glass .....	.605	.721	.569	.775	.675	.873	.845	.940	1.110	2.250
Tin .....	.786	.786	.855	.895	.867	.752	.905	.941	2.250	3.310
Lead .....	1.935	1.930	2.070	2.150	1.945	2.110	2.000	2.420	2.250	3.310

TABLE VII.

	Hard steel.	Mild steel.	Platinum.	Nickel.	Copper.	Brass.	Aluminium.	Glass.	Tin.	Lead.	O dynes per sq. cm.	K dynes per sq. cm.
Hard steel...	5.55	5.47	5.97	5.32	5.20	5.71	5.38	6.12	5.19	5.23	$89 \times 10^{11}$	$180 \times 10^{11}$
Mild steel ..	5.47	5.64	5.69	5.44	5.43	6.14	5.89	5.20	5.29	5.36	81	16.4
Platinum ..	5.97	5.69	5.70	6.29	5.09	5.73	4.58	5.66	4.94	5.02	61	24.7
Nickel .....	5.32	5.44	6.29	6.03	5.18	6.19	4.81	4.85	4.63	4.81	7.7	17.6
Copper .....	5.20	5.43	5.09	5.18	5.79	5.88	5.80	6.24	5.35	5.44	4.0	13.6
Brass .....	5.71	6.14	5.73	6.19	5.88	6.02	5.98	5.03	6.02	5.10	3.5	10.6
Aluminium ..	5.38	5.89	4.58	4.81	5.80	5.98	4.94	5.65	5.65	5.51	2.7	7.5
Glass .....	6.12	5.20	5.36	4.63	6.24	5.03	5.65	5.24	5.61	4.60	2.9	4.0
Tin .....	4.19	5.29	4.94	4.81	5.35	6.02	5.61	4.60	5.10	5.10	2.05	5.3
Lead .....	5.23	5.36	5.02	4.81	5.44	5.10	5.51	4.60	5.10	4.78	0.34	5.0
Mean values.	5.51	5.56	5.57	5.37	5.54	5.78	5.42	5.52	5.29	5.11		

adequate for many years. In the present paper no distinction is made between rolling and sliding friction, both being supposed to arise in the same molecular plucking action. In thus proposing an alternative theory of rolling friction, we are obliged to inquire whether there are any reasons for considering Reynolds's theory inadequate, especially in view of the established character of this theory and the eminence of its author.

According to Reynolds's theory, rolling friction is caused by slight relative tangential motion between the surfaces in contact, due to unequal elastic surface displacements. There is supposed to be a differential creep motion in continuous operation, and this theory thus reduces rolling friction to a particular case of sliding friction. There can be little doubt that this kind of elastic creep does in general occur, with a contribution to the total loss of energy depending on the amount of load. It is the writer's opinion, however, that the amount of this rubbing motion is much too minute to account for more than a small part of the energy lost in rolling, and that this loss is mainly the result of the molecular disturbances which take place on account of the continual shifting of the loaded area. It may be observed that Reynolds's theory is purely descriptive and has never been applied quantitatively, and no criticism as to the accuracy of Reynolds's deductions is now suggested other than modifying the accepted idea of the magnitude of the creep effect and regarding it as only a small contributory cause of friction.

Several reasons can be adduced showing the elastic creep theory to be inadequate. In the first place, reference must be made to some experiments made by the writer at the National Physical Laboratory in connexion with an investigation on rusting of steel surfaces in contact\*. It is very well known that when two steel surfaces in close contact have a relative sliding motion, the surfaces become pitted and a brown oxidation product is formed. The same result is found when the surfaces are held tightly together if they are subject to vibration. In this case the only motion occurring is the minute amount due to vibration. In the course of this research it was found that a relative motion as small as  $8 \times 10^{-8}$  inch was quite sufficient to cause rusting. On the other hand, it was found that a prolonged rolling motion never led to any formation of rust at all. We may therefore infer that the differential surface motion in rolling is less

\* Proc. Roy. Soc. p. 472 (1927).

than this very small amount. This can be verified by a simple approximate calculation. Taking the case of rolling given in Table I., we can form an estimate of the magnitude of the surface displacement in the contact area. Love\* gives an illustration of the lines of principal stress in the neighbourhood of an elastic contact, which shows that the tangential stress over the surface in contact is a compressive stress which changes to a tensile stress on crossing the border of the contact area. It is therefore zero at the edge of the contact area, and the average will be smaller than the maximum. Let us assume that the average tangential elastic strain is 0.001; then the displacement of the edge of the contact strip relative to the centre is

$$4.14 \times 10^{-4} \times 0.001 \text{ cm.}$$

$$\text{or} \quad 16.3 \times 10^{-8} \text{ inch.}$$

We are only concerned in Reynolds's theory with the differential displacement of the two surfaces, which must be considerably smaller than the above value of the total displacement; hence the calculation roughly bears out the inference from the experiment, that the differential creep does not exceed about  $8 \times 10^{-8}$  inch. Further, the elastic creep is not uniform, but varies between zero and a maximum value at different parts of the contact surface, so that the average creep which is supposed to produce the friction must be a still smaller quantity. It is difficult to understand how such extremely minute relative movement can cause the loss of energy found in rolling, and the difficulty is increased by the fact that the coefficient of sliding friction undergoes a marked diminution at such small amplitudes, as we have shown in section 6.

There are two further arguments which show that Reynolds's theory does not fully account for rolling friction. The first is based on the fact that the coefficient of rolling friction is inversely proportional to the width of the contact area which is demonstrated by both Goodman's and the author's experiments. On Reynolds's theory it appears to be quite impossible to account for this result without arriving at some untenable conclusion as to the relation between the slip and the width of the contact strip. Since the coefficient is known to decrease when the load is increased, it follows that the amount of differential creep in rolling any given distance must decrease when the load is increased. The width of the contact increases with the load in a definite manner;

\* 'The Mathematical Theory of Elasticity,' p. 196.

hence we should be forced to the difficult conclusion that the creep decreases as the total arc of contact increases.

The second argument is a simple one. Consider the case of two identical parallel cylinders, one rolling upon the other. In such a case we have perfect symmetry, and the elastic distortions of the two bodies will be exactly alike and the friction in this particular case should on Reynolds's theory be zero, or at least abnormally low, or, if not, some other explanation of the friction must be found. This interesting point was tested by experiment, and the coefficient of friction was determined for parallel cylinders of equal diameter, the lower being stationary with the upper one rolling to and fro upon it. With an amplitude of 0.0188 radian the coefficient was found to be 0.000022, and for direct comparison the friction was then measured after replacing the lower cylinder by a plane surface of the same material and having the same length of contact parallel to the axis of rolling. In the latter experiment with the same amplitude of rolling, the coefficient of friction was found to be 0.000034. The experiment thus shows that the rolling friction of two equal parallel cylinders is not at all exceptionally small, and must arise in some other way than that suggested by Reynolds. It may be remarked that the molecular theory now suggested also offers a simple explanation of the phenomenon described by Reynolds of the oscillation of a roller lying on a plane, if the roller is slightly disturbed. The molecular attractions which the theory supposes are clearly unsymmetrical after a small displacement, and give the resultant restoring couple necessary for an oscillatory rocking motion.

10. One interesting consequence of the molecular theory of friction may be noticed—namely, that by this theory a small loss of energy occurs when two bodies touch with any pressure and separate again in a direction normal to the surfaces; or, in other words, there is some “friction” with purely normal action.

Sir William Hardy has drawn attention to the possibility of this\*, but he states that no friction has been detected with normal motion. The loss of energy involved by the present theory is, in fact, so small as to be imperceptible. It can be easily computed as follows. Let the normal pressure exerted be  $F$ ; then, using the previous notation,

$$F = n\bar{p},$$

\* Proc. Roy. Soc. c. (1922).

and the loss of energy is

$$\begin{aligned} E &= n \times A_0 \bar{l} \\ &= \frac{FA_0 \bar{l}}{p}. \quad . \quad . \quad . \quad . \quad . \quad (15) \end{aligned}$$

Using equation (2), this becomes

$$E = \frac{\mu Fe}{q}.$$

Thus the loss of energy is proportional to the pressure, and is equal to the work done when the applied force is exerted through a distance  $\frac{\mu e}{q}$  an amount comparable with though probably less than the dimensions of the molecules. It is thus not surprising that this loss of energy, if it actually occurs, has not been observed, except as manifested, according to the present theory, in the case of rolling friction, which may be regarded as a continuous process of normal approach and recess.

11. It was remarked in section 1 that when two molecules of actual bodies come into repelling contact and then separate, each molecule has a strong bias to return to its parent body, although momentarily the two are supposed to be in similar equilibrium to that existing within the body of the solid. We shall now discuss this in more detail. The experiments, already referred to, made by the writer on the contact rusting of steel pointed to the conclusion that molecules are torn away by cohesive attraction, and such molecules have a strong affinity for oxygen molecules, the combination giving rise to the brown deposits observed. There is thus definite reason to believe that some of the molecules concerned in contact are detached, and this may be the mechanism of wear.

It can be shown, however, from our knowledge of the ordinary rate of wear of metals, that only a very small proportion of the molecules can be detached.

Each molecule taking part in contact dissipates an amount of energy of average value

$$A_0 \bar{l}.$$

And the mass of the molecule for the present purpose is given closely enough by

$$m = \rho e^3, \quad . \quad . \quad . \quad . \quad . \quad (16)$$

where  $\rho$  is the specific gravity and  $e$  is the distance between the molecules.

In dissipating energy in friction of amount  $E$  ergs, the number of molecular contacts is therefore

$$N = \frac{E}{A_0 \bar{l}}. \quad . \quad . \quad . \quad (17)$$

And the mass of all the molecules involved is

$$M = Nm = \frac{E \rho e^3}{A_0 \bar{l}}. \quad . \quad . \quad . \quad (18)$$

From equation (2) we can substitute

$$A_0 \bar{l} = \frac{\mu}{q} \bar{p} e.$$

Whence

$$M = \frac{E \rho e^2 q}{\mu \bar{p}}. \quad . \quad . \quad . \quad . \quad (19)$$

In this expression  $p$  is unknown and  $q$  is of the order of unity, but is otherwise unknown. If, therefore, we can in any way arrive at some reasonable estimate of  $p$  and take  $q$  as unity, we shall obtain a probable value for the value of  $M$ .

The experiments made by Stott on pivots, already referred to, show that a material has a definite flow-stress and is not capable of supporting any stress exceeding this. Mallock\* and Hankins† have also independently found somewhat similar results. It will be assumed that this state of flow indicates a condition in which all the available molecules are repelling with the maximum possible force, that is, the limiting force the space-lattice can resist.

If the flow-stress is denoted by  $f$ , we then have approximately

$$f = \frac{p_{\max}}{e^2}.$$

In a case of elastic contact  $\bar{p}$  will be smaller than  $p_{\max}$ , and if we assume all the possible values of  $p$  to be equally probably,  $\bar{p}$  will be equal to  $\frac{1}{2} p_{\max}$ .

By taking  $q$  as unity, this leads to the following expression for the value of  $M$ :

$$M = \frac{2E\rho}{\mu f}. \quad . \quad . \quad . \quad . \quad (20)$$

\* 'Nature,' cxvii.

† Proc. Inst. Mech. Eng. (1925).

The value of  $f$  found for unhardened steel by Stott is about 250 kilogrammes per square millimetre, and a concrete example will best show that the calculated value of  $M$  is very large.

If we consider, for example, a brake horse-power test of a 100 KW. motor dissipating this power for 1 hour by means of a brake on a steel flywheel, then the value of  $M$  is of the order  $10^7$  gram. The actual weight of metal which would be worn away is probably only some small fraction of a gram. Thus, although some very free approximations have been necessary in the above computation, we may quite safely conclude that only an extremely small proportion of those atoms taking part in contact are detached from their original position.

This suggests as an interesting possibility that wear may be fundamentally only an accidental accompaniment of friction. For the purpose of providing an illustration, let us assume the actual wear in the above example to be as much as 1 gram. We may anticipate by remarking that quite a large error in this conjecture does not materially affect the conclusions. On this assumption, only 1 in  $10^7$  of the molecules effective in causing friction is detached. The total number of molecules which come into contact is

$$N = \frac{M}{m} = \frac{2E}{\mu f e^3} \quad \dots \dots (21)$$

Taking  $e$  to be  $2.86 \times 10^{-8}$  cm., this has a value of about  $5 \times 10^{28}$ , so that the number detached, although only a very small fraction of this, is still a very large number of the order  $5 \times 10^{21}$ . Hence, as in many molecular phenomena, although the wear may be ultimately traceable to the accidental loss of an occasional molecule, the total number involved is so great that uniformity and consistency are always found in the results perceived by measurement.

12. In the course of the experimental work some unusual cases of friction have been encountered which appear to deserve a brief mention. These are instances in which the cohesive attraction of the surface atoms is not negligibly small by comparison with the external force applied between the surfaces in contact. Consequently the normal reaction between the bodies is considerably greater than the applied force, and extraordinary values are obtained for the coefficient of friction, as usually defined.

In one experiment a light slider was used having 3 feet of lead forming an equilateral triangle. The slider was placed on a clean piece of plate glass which could be tilted to any angle. To obtain clean surfaces of lead, the 3 feet were pared away with a clean razor blade. The glass surface and the blade were cleaned by repeated rubbing with a mixture of alcohol, ether, and ammonia. It was found that the plate could then be tilted completely into the vertical position without any slipping of the slider, giving an infinite coefficient of friction by the ordinary definition. Further, with a little care, it was found possible to turn the glass plate round until it was horizontal with the slider in its original position clinging to the under side, showing that the cohesive attraction could exceed the weight of the slider.

Similar results were obtained with two fibres of quartz or glass crossed at right angles. In this case strong cohesive forces can be observed and measured by the elastic deflexion of the fibre, and tangential frictional forces were measured when the applied normal force was zero or even negative, leading to a series of apparent coefficients of friction passing through infinity to negative values.

These experiments illustrate clearly that the coefficient of friction should be defined in terms of the normal reaction between the bodies, rather than in terms of the applied normal force. Otherwise, under certain circumstances, the friction will be far from proportional to the load, and very abnormal values of the coefficient will be obtained.

13. We shall conclude this paper with a short discussion of the fundamental hypothesis that has been assumed concerning the character of the atomic forces of attraction and repulsion. It has been seen in the course of the paper how the various experimental results support the deductions from the theory, even when the theory has indicated certain relations not hitherto recognized, such as the relation between friction and the elastic constants or the relation between sliding and rolling friction. The validity of the theory, however, depends so much on the simple assumption that the repulsion has a much greater rate of change and a smaller range of action than the attraction, that we shall attempt to show in a simple way how the same hypothesis is consistent with the ordinary phenomena of elasticity. A number of writers have dealt mathematically with the connexion between elasticity and atomic forces, but a



rigorous treatment of the subject is hardly possible with our present limited knowledge of the behaviour of neighbouring atoms in a solid, and we shall therefore only attempt to consider elasticity in an elementary way that seems adequate for the purpose in view.

The characteristic stability of a solid is almost certainly due to the powerful cohesive attraction exerted by the neighbouring atoms acting in equilibrium with an equal and opposite repulsion. Since it is known that each force is some rapidly decreasing inverse function of the distance between the atoms, how does a solid support and recover from a large stress, if its atoms are just in equilibrium under no stress, and an applied tension actually draws them apart and so reduces the cohesive bond? If the attraction and repulsion have characteristics, such as are shown in fig. 1, the tenacity and elasticity of a material can be explained. The point P corresponds to a state of no stress with the atoms under equal repulsive and attractive forces. As a tension is applied causing an external force  $F$  to come into action on the atom, the atoms separate slightly and both  $A$  and  $R$  are diminished, but  $R$  to a considerably greater extent than  $A$ . The rate of change of the *repulsion* is the main factor in determining the elasticity, the variation in the force of attraction being relatively small within the range of elastic strain. The dotted curve in fig. 1 represents the value of  $A - R$  which is equal to the applied force  $F$ . As the tension increases, a point is reached at which  $A - R$  has a maximum value; at this point the atoms will separate with further increase in tension. By our hypothesis the displacement of the atoms at this point is small compared with the field of attraction, and the maximum value of  $A - R$  is not much less than  $PJ$ , the normal cohesion under no stress. Hence the tenacity is almost entirely determined by the cohesive attraction, whereas the elastic modulus is similarly determined by the value of  $\frac{dR}{dx}$ , the attraction being a comparatively flexible bond.

When a compressive stress is applied, the atoms are forced nearer together and we pass upwards along both the curves. The initial rate of change of  $R$  will be the same as before, and therefore the same modulus of elasticity would be expected, which is found by experiment to be the case. Under compression no limiting value of  $A - R$  is indicated as in tension, and if there is a limit to the amount of uniform compression a solid can withstand, it cannot be realized

experimentally. The hypothetical forces thus lead generally to a natural explanation of the elastic behaviour of a solid.

It can be shown more quantitatively from the known data of elasticity that the repulsion field must have a considerably higher gradient and smaller range than the attraction.

Consider any two neighbouring atoms in a body which is subjected to a uniform compression, and let the external pressure  $p$  give rise to a force  $F$  thrusting the atoms closer together. If the distance between the atoms is  $e$ , and an increment  $dF$  in the force produces a decrement  $de$  in this distance, we must have

$$dF = dR - dA$$

$$\text{or} \quad \frac{dF}{de} = \frac{dR}{de} - \frac{dA}{de}.$$

Now,  $\frac{dR}{de}$  and  $\frac{dA}{de}$  are both necessarily negative, and  $\frac{dF}{de}$  is also negative; hence  $\frac{dR}{de}$  must exceed  $\frac{dA}{de}$  for stable equilibrium to exist.

Let it be assumed that the attraction is in the neighbourhood of the point P of fig. 1 can be represented by the simple inverse relation

$$A = \frac{B}{e^n} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

without assuming any value for the exponent  $n$ .

$$\text{Then} \quad \frac{dA}{de} = -\frac{nB}{e^{n+1}} = -\frac{nA}{e} \quad . \quad . \quad . \quad . \quad (23)$$

If  $K$  is the modulus of compressibility,

$$dp = -3K \frac{de}{e};$$

and also  $F = \beta e^2 p$ , where  $\beta$  is a factor not greatly different from unity and dependent on the type of atomic packing.

$$\text{Hence} \quad dF = -3K \beta e de$$

$$\text{or} \quad \frac{dF}{de} = -3K \beta e \quad . \quad . \quad . \quad . \quad (24)$$

We can also write  $A = \beta e^2 f$ , where  $f$  is a stress which corresponds to the limiting cohesion between the atoms.

This stress  $f$  represents the ideal tenacity or the tenacity under a state of triple tension.

$$\text{Hence} \quad \frac{dA}{de} = -\frac{nA}{e} = -\beta nef.$$

$$\text{So that} \quad \frac{dF}{de} = \frac{3K}{nf} \frac{dA}{de}$$

$$\text{and} \quad \frac{dR}{de} = \frac{dF}{de} + \frac{dA}{de} = \left( \frac{3K}{nf} + 1 \right) \times \frac{dA}{de}. \quad . \quad . \quad (25)$$

Considering the quantity within brackets, although there is much uncertainty as to the value of the ideal tenacity  $f$ , there is little doubt that this is a stress considerably smaller than the modulus  $K$ . As regards the exponent  $n$ , there is again a lack of definite knowledge, but values 2 or 4 are frequently suggested. For our present purpose we may at least regard  $n$  as not greatly different from 3, so that we can conclude from equation (26) that the elastic properties of matter qualitatively support our assumption that  $\frac{dR}{de}$  considerably exceeds  $\frac{dA}{de}$ , even if they yield no very definite numerical information as to the ratio of the two force gradients.

That the total range of the repulsion force is small compared with the range of the attraction is also supported by another physical property of solids. It is well known that the total dilatation of a solid from absolute zero to its melting-point is approximately constant for nearly all substances, and amounts to about 0.02 in linear dimensions. Thus the complete range of the solid state corresponds to an expansion of the atomic lattice of only about 2 per cent., beyond which the solid condition of equilibrium vanishes. This quantity, in terms of our hypothesis, must be associated with the effective range of the repulsion. The attraction between atoms, on the other hand, is known to extend for a distance at least exceeding the distance between the atoms.

The author wishes to express his thanks to Sir J. E. Petavel and Mr. J. E. Sears for their continued interest and encouragement, and also to Sir T. E. Stanton and Sir W. B. Hardy for kindly reading and criticizing his first investigation into a subject which owes so much of its advance to them.