

# ECE 240 Formula Sheet

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## 1. Time-domain signals

At a point of discontinuity  $t_0$ , a piecewise function takes the value

$$x(t_0) = \frac{1}{2} \left[ x(t_0^+) + x(t_0^-) \right].$$

A continuous-time signal takes the form

$$x(t + nT) = x(t) \quad n \in \mathbb{Z}.$$

A signal  $z(t) = \alpha x(t + aT_1) + \beta x(t + bT_2)$  will be periodic if

$$\frac{T_1}{T_2} = \frac{a}{b}$$

for some  $a, b \in \mathbb{Z}$ .

Let  $x(t)$  be some signal.

- The *total energy* is given by

$$E = \lim_{L \rightarrow \infty} \int_{-L}^L |x(t)|^2 dt.$$

- The *average power* is given by

$$P = \lim_{L \rightarrow \infty} \frac{1}{2L} \int_{-L}^L |x(t)|^2 dt.$$

- If  $x(t)$  is periodic,

$$P = \frac{1}{T} \int_0^T |x(t)|^2 dt.$$

$E$  finite  $\rightarrow$  **Energy signal**  $\rightarrow P = 0$ .

$E$  infinite and  $P$  finite  $\rightarrow$  **Power signal**.

Periodic signal  $\rightarrow$  **Power signal**.

Let  $x(t)$  be some signal.

- A *time shift* is represented by

$$x(t - t_0).$$

- A *reflection* is represented by

$$x(-t).$$

The *unit step signal* is defined as

$$u(t) = \begin{cases} 1 & t > 0, \\ 0 & t < 0. \end{cases}$$

A *rectangular pulse* is represented as

$$\text{rect}\left(\frac{t}{T}\right) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right).$$

A *ramp signal* is represented as

$$r(t) = tu(t) = \begin{cases} t & t \geq 0, \\ 0 & t < 0. \end{cases}$$

The *unit impulse*  $\delta(t)$  (Dirac delta function) is defined as

$$\int_{t_1}^{t_2} x(t)\delta(t) dt = x(0) \quad t_1 < 0 < t_2.$$

It has the following properties:

- $\delta(t) = 0$  for  $t \neq 0$ ,
- $\int_{-\infty}^{\infty} \delta(t) dt = 1$ ,
- $\delta(-t) = \delta(t)$ , and
- $\delta(0) = \infty$ .

Some  $p(t)$  can be used as a model of a delta function if

- $p(t)$  is even,
- $\lim_{\epsilon \rightarrow 0^+} p(t) = +\infty$  for  $t = 0$ ,
- $\lim_{\epsilon \rightarrow 0^+} p(t) = 0$  for  $t \neq 0$ , and
- $\int_{-\infty}^{\infty} p(t) dt = 1$  for all  $\epsilon > 0$ .

If these conditions are satisfied, then

$$\lim_{\epsilon \rightarrow 0^+} p(t) = \delta(t).$$

The *sifting property* is represented as

$$\int_{t_1}^{t_2} x(t)\delta(t - t_0) dt = \begin{cases} x(t_0) & t_1 < t_0 < t_2, \\ 0 & \text{otherwise.} \end{cases}$$

If  $x(t)$  is continuous at  $t = t_0$ , the *sampling property* states that

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

The *scaling property* states that

$$\delta(at + b) = \frac{1}{|a|} \delta\left(t + \frac{b}{a}\right) \quad a \neq 0.$$

The derivative of  $\delta(t)$  is defined as

$$\int_{t_1}^{t_2} x(t)\delta'(t - t_0) dt = -x'(t_0) \quad t_1 < t_0 < t_2.$$

It has the following properties:

- $x(t) * \delta'(t) = \int_{-\infty}^{\infty} x(\tau)\delta'(t - \tau) d\tau = x'(t)$ ,
- $x(t)\delta'(t - t_0) = x(t_0)\delta'(t - t_0) - x'(t_0)\delta(t - t_0)$ ,
- $\int_{-\infty}^t \delta'(\tau - t_0) d\tau = \delta(t - t_0)$ , and
- $\delta'(-t) = -\delta'(t) \rightarrow \int_{-\infty}^{\infty} \delta'(t) dt = 0$ .

## 2. Continuous-time systems

A system is *linear* if the superposition principle can be applied:

$$\alpha x_1(t) + \beta x_2(t) = \alpha y_1(t) + \beta y_2(t).$$

If we have  $x(t) \rightarrow y(t)$ , the system is *time-invariant* if

$$x(t - t_0) \rightarrow y(t - t_0).$$

A system is *memoryless* if the present output only depends on the present input.

- linear time-variant &  $y(t) = k(t)x(t) \rightarrow$  memoryless.

- linear time-invariant &  $y(t) = kx(t) \rightarrow$  memoryless.

A system is *causal* if the output at any time  $t_0$  only depends on the values of the input for  $t \leq t_0$ . Equivalently, if

$$x_1(t) = x_2(t) \quad t \leq t_0$$

implies

$$y_1(t) = y_2(t) \quad t \leq t_0,$$

the system is causal.

A system is *invertible* if the input can be determined from the output alone.

A system is *stable* if some bounded input  $|x(t)| \leq \infty$  causes a bounded output  $|y(t)| \leq \infty$  for all  $t$ .

Convolution: for a linear time-invariant (LTI) system, the response  $y(t)$  with impulse response  $h(t)$  and input  $x(t)$  is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) d\tau = x(t) * h(t).$$

Some properties of convolution are

- $x(t) * \delta(t) = x(t)$ ,
- $x(t) * u(t) = \int_{-\infty}^t x(\tau) d\tau$ ,
- $x(t) * \delta'(t) = x'(t)$ , and
- $\int_{-\infty}^{\infty} y(t) dt = A_h A_x$  (the product of the areas of the two signals being convoluted).

A LTI system is *memoryless* if

$$h(t) = k\delta(t).$$

A LTI system is *causal* if

$$h(t) = 0 \text{ for all } t < 0.$$

A LTI system described by  $h(t)$  is *invertible* if there exists an  $h_1(t)$  such that

$$h(t) * h_1(t) = \delta(t).$$

A LTI system is *BIBO stable* if

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty.$$

### 3. Fourier series

The exponential function

$$e^{j\frac{2\pi nt}{T}} \quad n \in \mathbb{Z}$$

can be used to represent  $x(t)$  via the Fourier series expansion, given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}},$$

where

$$c_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j\frac{2\pi nt}{T}} dt.$$

Note that  $c_n$  can also be expressed as

$$c_n = |c_n| e^{j\angle c_n}.$$

The plot of  $|c_n|$  is called the *amplitude spectrum* of  $x(t)$  while the plot of  $\angle c_n$  is called the *phase spectrum* of  $x(t)$ .

For real valued  $x(t)$ , we have

$$c_n^* = c_{-n}.$$

The Fourier series of  $x(t)$  can also be expressed via the *trigonometric Fourier series* expansion, given by

$$x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos\left(\frac{2\pi nt}{T}\right) + b_n \sin\left(\frac{2\pi nt}{T}\right) \right],$$

where

$$a_0 = c_0 = \frac{1}{T} \int_{\langle T \rangle} x(t) dt,$$

$$a_n = \frac{2}{T} \int_{\langle T \rangle} x(t) \cos\left(\frac{2\pi nt}{T}\right) dt,$$

$$b_n = \frac{2}{T} \int_{\langle T \rangle} x(t) \sin\left(\frac{2\pi nt}{T}\right) dt.$$

Recall that  $\langle T \rangle$  represents any interval of length  $T$ .

Another way to represent  $x(t)$  is via the *amplitude-phase trigonometric Fourier series* expansion, given by

$$x(t) = c_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi nt}{T} + \phi_n\right),$$

where

$$A_n = 2|c_n|, \quad \phi_n = \angle c_n.$$

If  $x(t)$  is *even*,

$$a_0 = \frac{2}{T} \int_0^{\frac{T}{2}} x(t) dt,$$

$$a_n = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \cos\left(\frac{2\pi nt}{T}\right) dt, \\ b_n = 0.$$

If  $x(t)$  is *odd*,

$$a_0 = a_n = 0,$$

$$b_n = \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \sin\left(\frac{2\pi nt}{T}\right) dt.$$

For half-wave odd symmetry where  $x(t+T/2) = -x(t)$ , we have

$$a_n = \begin{cases} \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \cos\left(\frac{2\pi nt}{T}\right) dt, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

$$b_n = \begin{cases} \frac{4}{T} \int_0^{\frac{T}{2}} x(t) \sin\left(\frac{2\pi nt}{T}\right) dt, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Let  $x(t)$  and  $y(t)$  be periodic signals with the same period such that

$$x(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_0 t},$$

$$y(t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{jn\omega_0 t}.$$

Then, via *linearity*,  $z(t) = k_1 x(t) + k_2 y(t)$  can be represented via

$$z(t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{jn\omega_0 t},$$

where  $\alpha_n = k_1 \beta_n + k_2 \gamma_n$ .

Furthermore, the *product* of the previous two signals  $z(t) = x(t)y(t)$  can be represented as

$$z(t) = \sum_{l=-\infty}^{\infty} \alpha_l e^{jn\omega_0 t},$$

where

$$\alpha_l = \frac{1}{T} \int_{\langle T \rangle} x(t) y(t) e^{-jn\omega_0 t} dt \\ = \sum_{m=-\infty}^{\infty} \beta_{l-m} \gamma_m.$$

Using the previous two signals, the *circular convolution* is defined as

$$z(t) = \frac{1}{T} \int_{\langle T \rangle} x(\tau) y(t - \tau) d\tau.$$

The Fourier series coefficients for the circular convolution  $z(t)$  are

$$\alpha_n = \beta_n \gamma_n.$$

Parseval's theorem states that

$$\frac{1}{T} \int_{\langle T \rangle} |x(t)|^2 dt = \sum_{n=-\infty}^{\infty} |\beta_n|^2.$$

If  $x(t)$  has Fourier series coefficients  $c_n$ , then the coefficients representing  $x(t - \tau)$  are

$$d_n = c_n e^{-jn\omega_0 \tau}.$$

The integral of a periodic signal  $x(t)$  is given by

$$\int_{-\infty}^t x(\tau) d\tau = \sum_{n=-\infty}^{\infty} \frac{c_n}{jn\omega_0} e^{jn\omega_0 t}.$$

The *transfer function* of an LTI system is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega \tau} d\tau.$$

Consequently, the response  $y(t)$  of the above LTI system is

$$y(t) = H(\omega) e^{j\omega t}.$$

An LTI system is *distortionless* if

$$y(t) = K x(t - t_d).$$

If  $x(t) = e^{j\omega t}$ , then  $y(t) = H(\omega) e^{j\omega t}$ . Hence, the transfer function of a distortionless LTI system is

$$H(\omega) = K e^{-j\omega t_d}$$

which implies

$$|H(\omega)| = K \quad \text{and} \quad \angle H(\omega) = -\omega t_d.$$