ECE 240 Formula Sheet

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1. Time-domain signals

A continuous-time signal takes the form

$$x(t+nT) = x(t) \quad n \in \mathbb{Z}.$$

A signal $z(t) = \alpha x(t + aT_1) + \beta x(t + bT_2)$ will be periodic if

$$\frac{T_1}{T_2} = \frac{a}{b}$$

for some $a, b \in \mathbb{Z}$.

Let $x(t)$ be some signal.

- The *total energy* is given by
$$E = \lim_{L\to\infty} \int_{-L}^{L} |x(t)|^2 \, \mathrm{d}t.$$
- The *average power* is given by
$$P = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} |x(t)|^2 \, \mathrm{d}t.$$
- If $x(t)$ is periodic,
$$P = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, \mathrm{d}t.$$

E finite $\to$ **Energy signal** $\to P = 0$.

E infinite and P finite $\to$ **Power signal**.

Periodic signal $\to$ **Power signal**.

Let $x(t)$ be some signal.

- A *time shift* is represented by
$$x(t - t_0).$$
- A *reflection* is represented by
$$x(-t).$$

The *unit step signal* is defined as

$$u(t) = \begin{cases} 1 & t > 0, \\ 0 & t < 0. \end{cases}$$

A *rectangular pulse* is represented as

$$\operatorname{rect}\left(\tfrac{t}{T}\right) = u\left(t + \tfrac{T}{2}\right) - u\left(t - \tfrac{T}{2}\right).$$

A *ramp signal* is represented as

$$r(t) = tu(t) = \begin{cases} t & t \geq 0, \\ 0 & t < 0. \end{cases}$$

The *unit impulse* $\delta(t)$ (Dirac delta function) is defined as

$$\int_{t_1}^{t_2} x(t)\delta(t) \, \mathrm{d}t = x(0) \quad t_1 < 0 < t_2.$$

It has the following properties:

- $\delta(t) = 0$ for $t \neq 0$,
- $\displaystyle\int_{-\infty}^{\infty} \delta(t) \, \mathrm{d}t = 1$,
- $\delta(-t) = \delta(t)$, and
- $\delta(0) = \infty$.

Some $p(t)$ can be used as a model of a delta function if

- $p(t)$ is even,
- $\displaystyle\lim_{\epsilon\to 0^+} p(t) = +\infty$ for $t = 0$,
- $\displaystyle\lim_{\epsilon\to 0^+} p(t) = 0$ for $t \neq 0$, and
- $\displaystyle\int_{-\infty}^{\infty} p(t) \, \mathrm{d}t = 1$ for all $\epsilon > 0$.

If these conditions are satisfied, then

$$\lim_{\epsilon\to 0^+} p(t) = \delta(t).$$

The *sifting property* is represented as

$$\int_{t_1}^{t_2} x(t)\delta(t - t_0) \, \mathrm{d}t = \begin{cases} x(t_0) & t_1 < t_0 < t_2, \\ 0 & \text{otherwise}. \end{cases}$$

If $x(t)$ is continuous at $t = t_0$, the *sampling property* states that

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

The *scaling property* states that

$$\delta(at + b) = \frac{1}{|a|}\delta\left(t + \tfrac{b}{a}\right) \quad a \neq 0.$$

The derivative of $\delta(t)$ is defined as

$$\int_{t_1}^{t_2} x(t)\delta'(t - t_0) \, \mathrm{d}t = -x'(t_0) \quad t_1 < t_0 < t_2.$$

It has the following properties:

- $\displaystyle x(t) * \delta'(t) = \int_{-\infty}^{\infty} x(\tau)\delta'(t - \tau) \, \mathrm{d}\tau = x'(t)$,
- $x(t)\delta'(t - t_0) = x(t_0)\delta'(t - t_0) - x'(t_0)\delta(t - t_0)$,
- $\displaystyle\int_{-\infty}^{t} \delta'(\tau - t_0) \, \mathrm{d}\tau = \delta(t - t_0)$, and
- $\displaystyle\delta'(-t) = -\delta'(t) \to \int_{-\infty}^{\infty} \delta'(t) \, \mathrm{d}t = 0$.

2. Continuous-time systems

A system is *linear* if the superposition principle can be applied:

$$\alpha x_1(t) + \beta x_2(t) = \alpha y_1(t) + \beta y_2(t).$$

If we have $x(t) \to y(t)$, the system is *time-invariant* if

$$x(t - t_0) \to y(t - t_0).$$

A system is *memoryless* if the present output only depends on the present input.

- linear time-variant & $y(t) = k(t)x(t) \to$ memoryless.
- linear time-invariant & $y(t) = kx(t) \to$ memoryless.

A system is *causal* if the output at any time t_0 only depends on the values of the input for $t \leq t_0$. Equivalently, if

$$x_1(t) = x_2(t) \quad t \leq t_0$$

implies

$$y_1(t) = y_2(t) \quad t \leq t_0,$$

the system is causal.

A system is *invertible* if the input can be determined from the output alone.

A system is *stable* if some bounded input $|x(t)| \leq \infty$ causes a bounded output $|y(t)| \leq \infty$ for all t.

Convolution: for a linear time-invariant (LTI) system, the response $y(t)$ with impulse response $h(t)$ and input $x(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) \, \mathrm{d}\tau = x(t) * h(t).$$

Some properties of convolution are

- $x(t) * \delta(t) = x(t)$,
- $\displaystyle x(t) * u(t) = \int_{-\infty}^{t} x(\tau) \, \mathrm{d}\tau$,
- $x(t) * \delta'(t) = x'(t)$, and
- $\displaystyle\int_{-\infty}^{\infty} y(t) \, \mathrm{d}t = A_h A_x$ (the product of the areas of the two signals being convoluted).

A LTI system is *memoryless* if

$$h(t) = k\delta(t).$$

A LTI system is *causal* if

$$h(t) = 0 \text{ for all } t < 0.$$

A LTI system described by $h(t)$ is *invertible* if there exists an $h_1(t)$ such that

$$h(t) * h_1(t) = \delta(t).$$

A LTI system is *BIBO stable* if

$$\int_{-\infty}^{\infty} |h(\tau)| \, \mathrm{d}\tau < \infty.$$

3. Fourier series

The exponential function

$$e^{j\frac{2\pi nt}{T}} \quad n \in \mathbb{Z}$$

can be used to represent $x(t)$ via the Fourier series expansion, given by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi nt}{T}},$$

where

$$c_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-j\frac{2\pi nt}{T}}\,\mathrm{d}t.$$

Note that c_n can also be expressed as

$$c_n = |c_n|e^{j(\sphericalangle c_n)}.$$

The plot of $|c_n|$ is called the *amplitude spectrum* of $x(t)$ while the plot of $\sphericalangle c_n$ is called the *phase spectrum* of $x(t)$.

For real valued $x(t)$, we have

$$c_n^* = c_{-n}.$$

The Fourier series of $x(t)$ can also be expressed via the *trigonometric Fourier series* expansion, given by

$$x(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\tfrac{2\pi nt}{T}\right) + b_n \sin\left(\tfrac{2\pi nt}{T}\right)\right],$$

where

$$a_0 = c_0 = \frac{1}{T}\int_{<T>} x(t)\,\mathrm{d}t,$$

$$a_n = \frac{2}{T}\int_{<T>} x(t)\cos\left(\tfrac{2\pi nt}{T}\right)\mathrm{d}t,$$

$$b_n = \frac{2}{T}\int_{<T>} x(t)\sin\left(\tfrac{2\pi nt}{T}\right)\mathrm{d}t.$$

Recall that $<T>$ represents any interval of length T.

Another way to represent $x(t)$ is via the *amplitude-phase trigonometric Fourier series* expansion, given by

$$x(t) = c_0 + \sum_{n=1}^{\infty} A_n \cos\left(\tfrac{2\pi nt}{T} + \phi_n\right),$$

where

$$A_n = 2|c_n|, \qquad \phi_n = \sphericalangle c_n.$$

If $x(t)$ is *even*,

$$a_0 = \frac{2}{T}\int_0^{\frac{T}{2}} x(t)\,\mathrm{d}t,$$

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos\left(\tfrac{2\pi nt}{T}\right)\mathrm{d}t,$$

$$b_n = 0.$$

If $x(t)$ is *odd*,

$$a_0 = a_n = 0,$$

$$b_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\sin\left(\tfrac{2\pi nt}{T}\right)\mathrm{d}t.$$

For half-wave odd symmetry where $x(t+T/2) = -x(t)$, we have

$$a_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{\frac{T}{2}} x(t)\cos\left(\tfrac{2\pi nt}{T}\right)\mathrm{d}t, n \text{ odd},\\ 0, n \text{ even}.\end{cases}$$

$$b_n = \begin{cases} \dfrac{4}{T}\displaystyle\int_0^{\frac{T}{2}} x(t)\sin\left(\tfrac{2\pi nt}{T}\right)\mathrm{d}t, n \text{ odd},\\ 0, n \text{ even}.\end{cases}$$

Let $x(t)$ and $y(t)$ be periodic signals with the same period such that

$$x(t) = \sum_{n=-\infty}^{\infty} \beta_n e^{jn\omega_0 t},$$

$$y(t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{jn\omega_0 t}.$$

Then, via *linearity*, $z(t) = k_1 x(t) + k_2 y(t)$ can be represented via

$$z(t) = \sum_{n=-\infty}^{\infty} \alpha e^{jn\omega_0 t},$$

where $\alpha = k_1\beta_n + k_2\gamma_n$.

Furthermore, the *product* of the previous two signals $z(t) = x(t)y(t)$ can be represented as

$$z(t) = \sum_{l=-\infty}^{\infty} \alpha_l e^{jn\omega_0 t},$$

where

$$\alpha_l = \frac{1}{T}\int_{<T>} x(t)y(t)e^{-jn\omega_0 t}\,\mathrm{d}t$$

$$= \sum_{m=-\infty}^{\infty} \beta_{l-m}\gamma_m.$$

Using the previous two signals, the *circular convolution* is defined as

$$z(t) = \frac{1}{T}\int_{<T>} x(\tau)y(t-\tau)\,\mathrm{d}\tau.$$

The Fourier series coefficients for the circular convolution $z(t)$ are

$$\alpha_n = \beta_n\gamma_n.$$

Parseval's theorem states that

$$\frac{1}{T}\int_{<T>} |x(t)|^2\,\mathrm{d}t = \sum_{n=-\infty}^{\infty} |\beta_n|^2.$$

If $x(t)$ has Fourier series coefficients c_n, then the coefficients representing $x(t-\tau)$ are

$$d_n = c_n e^{-jn\omega_0\tau}.$$

The integral of a periodic signal $x(t)$ is given by

$$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\tau = \sum_{n=-\infty}^{\infty} \frac{c_n}{jn\omega_0}e^{jn\omega_0 t}.$$

The *transfer function* of an LTI system is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,\mathrm{d}\tau.$$

Consequently, the response $y(t)$ of the above LTI system is

$$y(t) = H(\omega)e^{j\omega t}.$$

An LTI system is *distortionless* if

$$y(t) = Kx(t-t_d).$$

If $x(t) = e^{j\omega t}$, then $y(t) = H(\omega)e^{j\omega t}$. Hence, the transfer function of a distortionless LTI system is

$$H(\omega) = Ke^{-j\omega t_d}$$

which implies

$$|H(\omega)| = K \quad \text{and} \quad \sphericalangle H(\omega) = -\omega t_d.$$