ECE 240 Formula Sheet

By Benjamin Kong & Lora Ma

1. Time-domain signals

A continuous-time signal takes the form

$$x(t+nT) = x(t) \quad n \in \mathbb{Z}.$$

A signal $z(t) = \alpha x(t + aT_1) + \beta x(t + bT_2)$ will be periodic if

$$\frac{T_1}{T_2} = \frac{a}{b}$$

for some $a, b \in \mathbb{Z}$.

Let $x(t)$ be some signal.

- The *total energy* is given by
$$E = \lim_{T\to\infty} \int_{-L}^{L} |x(t)|^2 \, \mathrm{d}t.$$
- The *average power* is given by
$$P = \lim_{T\to\infty} \frac{1}{2L} \int_{-L}^{L} |x(t)|^2 \, \mathrm{d}t.$$
- If $x(t)$ is periodic,
$$P = \frac{1}{T} \int_{0}^{T} |x(t)|^2 \, \mathrm{d}t.$$

E finite $\to$ **Energy signal** $\to P = 0$.

E infinite and P finite $\to$ **Power signal**.

Periodic signal $\to$ **Power signal**.

Let $x(t)$ be some signal.

- A *time shift* is represented by
$$x(t - t_0).$$
- A *reflection* is represented by
$$x(-t).$$

The *unit step signal* is defined as

$$u(t) = \begin{cases} 1 & t > 0, \\ 0 & t < 0. \end{cases}$$

A *rectangular pulse* is represented as

$$\operatorname{rect}\left(\tfrac{t}{T}\right) = u\left(t + \tfrac{T}{2}\right) - u\left(t - \tfrac{T}{2}\right).$$

A *ramp signal* is represented as

$$r(t) = tu(t) = \begin{cases} t & t \geq 0, \\ 0 & t < 0. \end{cases}$$

The *unit impulse* $\delta(t)$ (Dirac delta function) is defined as

$$\int_{t_1}^{t_2} x(t)\delta(t) \, \mathrm{d}t = x(0) \quad t_1 < 0 < t_2.$$

It has the following properties:

- $\delta(t) = 0$ for $t \neq 0$,
- $\displaystyle\int_{-\infty}^{\infty} \delta(t) \, \mathrm{d}t = 1$,
- $\delta(-t) = \delta(t)$, and
- $\delta(0) = \infty$.

Some $p(t)$ can be used as a model of a delta function if

- $p(t)$ is even,
- $\displaystyle\lim_{\epsilon\to 0^+} p(t) = +\infty$ for $t = 0$,
- $\displaystyle\lim_{\epsilon\to 0^+} p(t) = 0$ for $t \neq 0$, and
- $\displaystyle\int_{-\infty}^{\infty} p(t) \, \mathrm{d}t = 1$ for all $\epsilon > 0$.

If these conditions are satisfied, then

$$\lim_{\epsilon\to 0^+} p(t) = \delta(t).$$

The *sifting property* is represented as

$$\int_{t_1}^{t_2} x(t)\delta(t - t_0) \, \mathrm{d}t = \begin{cases} x(t_0) & t_1 < t_0 < t_2, \\ 0 & \text{otherwise.} \end{cases}$$

If $x(t)$ is continuous at $t = t_0$, the *sampling property* states that

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$

The *scaling property* states that

$$\delta(at + b) = \frac{1}{|a|}\delta\left(t + \tfrac{b}{a}\right) \quad a \neq 0.$$

The derivative of $\delta(t)$ is defined as

$$\int_{t_1}^{t_2} x(t)\delta'(t - t_0) \, \mathrm{d}t = -x'(t_0) \quad t_1 < t_0 < t_2.$$

It has the following properties:

- $\displaystyle x(t) * \delta'(t) = \int_{-\infty}^{\infty} x(\tau)\delta'(t - \tau) \, \mathrm{d}\tau = x'(t)$,
- $x(t)\delta'(t - t_0) = x(t_0)\delta'(t - t_0) - x'(t_0)\delta(t - t_0)$,
- $\displaystyle\int_{-\infty}^{t} \delta'(\tau - t_0) \, \mathrm{d}\tau = \delta(t - t_0)$, and
- $\displaystyle\delta'(-t) = -\delta'(t) \to \int_{-\infty}^{\infty} \delta'(t) \, \mathrm{d}t = 0.$

2. Continuous-time systems

A system is *linear* if the superposition principle can be applied:

$$\alpha x_1(t) + \beta x_2(t) = \alpha y_1(t) + \beta y_2(t).$$

If we have $x(t) \to y(t)$, the system is *time-invariant* if

$$x(t - t_0) \to y(t - t_0).$$

A system is *memoryless* if the present output only depends on the present input.

- linear time-variant & $y(t) = k(t)x(t) \to$ memoryless.
- linear time-invariant & $y(t) = kx(t) \to$ memoryless.

A system is *causal* if the output at any time t_0 only depends on the values of the input for $t \leq t_0$. Equivalently, if

$$x_1(t) = x_2(t) \quad t \leq t_0$$

implies

$$y_1(t) = y_2(t) \quad t \leq t_0,$$

the system is causal.

A system is *invertible* if the input can be determined from the output alone.

A system is *stable* if some bounded input $|x(t)| \leq \infty$ causes a bounded output $|y(t)| \leq \infty$.

For a linear time-invariant (LTI) system, the response $y(t)$ with impulse response $h(t)$ and input $x(t)$ is given by

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau) \, \mathrm{d}\tau = x(t) * h(t).$$

Some properties of convolution are

- $x(t) * \delta(t) = x(t)$,
- $\displaystyle x(t) * u(t) = \int_{-\infty}^{t} x(\tau) \, \mathrm{d}\tau$,
- $x(t) * \delta'(t) = x'(t)$, and
- $\displaystyle\int_{-\infty}^{\infty} y(t) \, \mathrm{d}t = A_h A_x$ (the product of the areas of the two signals being convoluted).

A LTI system is memoryless if

$$h(t) = k\delta(t).$$

A LTI system is causal if

$$h(t) = 0 \quad t < 0.$$

A LTI system described by $h(t)$ is invertible if there exists an $h_1(t)$ such that

$$h(t) * h_1(t) = \delta(t).$$

A LTI system is BIBO stable if

$$\int_{-\infty}^{\infty} |h(\tau)| \, \mathrm{d}\tau < \infty.$$

3. Fourier series