

ON STEREOGRAPHIC PROJECTIONS

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Consider $S^n \subset \mathbb{R}^{n+1}$ and let $N = (0, \dots, 0, 1)$, $S = (0, \dots, 0, -1)$, $U_N = S^n \setminus \{S\}$, $U_S = S^n \setminus \{N\}$. Now define the projection from the **south** pole onto the plane

$$\varphi_N: U_N \rightarrow \mathbb{R}^n$$

by

$$\varphi_N(x_1, \dots, x_n, x_{n+1}) = (1 + x_{n+1})^{-1}(x_1, \dots, x_n)$$

as well as the projection from the **north** pole

$$\varphi_S: U_S \rightarrow \mathbb{R}^n$$

by

$$\varphi_S(x_1, \dots, x_n, x_{n+1}) = (1 - x_{n+1})^{-1}(x_1, \dots, x_n)$$

Observe that φ_S and φ_N are invertible:

$$\begin{aligned} \varphi_S^{-1}(x_1, \dots, x_n) &= \frac{1}{1 + \sum_{i=1}^n x_i^2} \left(2x_1, \dots, 2x_n, -1 + \sum_{i=1}^n x_i^2 \right) \\ \varphi_N^{-1}(x_1, \dots, x_n) &= \frac{1}{1 + \sum_{i=1}^n x_i^2} \left(2x_1, \dots, 2x_n, 1 - \sum_{i=1}^n x_i^2 \right) \end{aligned} \tag{1}$$

One may then compute transition maps $\varphi_N \varphi_S^{-1}$, $\varphi_S \varphi_N^{-1}$ on $\varphi_S(U_N \cap U_S)$ and $\varphi_N(U_N \cap U_S)$ respectively as follows:

$$\begin{aligned} \varphi_N \circ \varphi_S^{-1}(x_1, \dots, x_n) &= \varphi_N \left(\frac{1}{1 + \sum_{i=1}^n x_i^2} \left(2x_1, \dots, 2x_n, -1 + \sum_{i=1}^n x_i^2 \right) \right) \\ &= \left(\frac{1}{1 + \frac{-1 + \sum_{i=1}^n x_i^2}{1 + \sum_{i=1}^n x_i^2}} \right) \left(\frac{1}{1 + \sum_{i=1}^n x_i^2} (2x_1, \dots, 2x_n) \right) \\ &= \left(\frac{1 + \sum_{i=1}^n x_i^2}{2 \sum_{i=1}^n x_i^2} \right) \left(\frac{1}{1 + \sum_{i=1}^n x_i^2} (2x_1, \dots, 2x_n) \right) \\ &= \frac{1}{\sum_{i=1}^n x_i^2} (x_1, \dots, x_n) \end{aligned} \tag{2}$$

and by a similar expansion,

$$\varphi_S \circ \varphi_N^{-1} = \frac{1}{\sum_{i=1}^n x_i^2} (x_1, \dots, x_n) \tag{3}$$