On Stereographic Projections

Name: Keirn Munro
Email: keirn@math.toronto.edu

Consider $S^n \subset \mathbb{R}^{n+1}$ and let $N = (0, \dots, 0, 1),\ S = (0, \dots, 0, -1),\ U_N = S^n \setminus \{S\},\ U_S = S^n \setminus \{N\}$. Now define the projection from the **south** pole onto the plane

$$\varphi_N \colon U_N \to \mathbb{R}^2$$

by

$$\varphi_N(x_1, \dots, x_n, x_{n+1}) = (1 + x_{n+1})^{-1}(x_1, \dots, x_n)$$

as well as the projection from the **north** pole

$$\varphi_S \colon U_S \to \mathbb{R}^2$$

by

$$\varphi_S(x_1, \dots, x_n, x_{n+1}) = (1 - x_{n+1})^{-1}(x_1, \dots, x_n)$$

Observe that φ_S and φ_N are invertible:

$$\begin{aligned}
\varphi_S^{-1}(x_1, \dots, x_n) &= \frac{1}{1 + \sum\limits_{i=1}^n x_i^2}\left(2x_1, \dots, 2x_n, -1 + \sum_{i=1}^n x_i^2\right) \\
\varphi_N^{-1}(x_1, \dots, x_n) &= \frac{1}{1 + \sum\limits_{i=1}^n x_i^2}\left(2x_1, \dots, 2x_n, 1 - \sum_{i=1}^n x_i^2\right)
\end{aligned} \tag{1}$$

One may then compute transition maps $\varphi_N\varphi_S^{-1},\ \varphi_S\varphi_N^{-1}$ on $\varphi_S(U_N \cap U_S)$ and $\varphi_N(U_N \cap U_S)$ respectively as follows:

$$\begin{aligned}
\varphi_N\varphi_S^{-1}(x_1, \dots, x_n) &= \varphi_N\left(\frac{1}{1 + \sum\limits_{i=1}^n x_i^2}\left(2x_1, \dots, 2x_n, -1 + \sum_{i=1}^n x_i^2\right)\right) \\
&= \left(\frac{1}{1 + \frac{-1 + \sum_{i=1}^n x_i^2}{\frac{1}{1 + \sum_{i=1}^n x_i^2}}}\right)\left(\frac{1}{1 + \sum\limits_{i=1}^n x_i^2}(2x_1, \dots, 2x_n)\right) \\
&= \left(\frac{1 + \sum\limits_{i=1}^n x_i^2}{2\sum\limits_{i=1}^n x_i^2}\right)\left(\frac{1}{1 + \sum\limits_{i=1}^n x_i^2}(2x_1, \dots, 2x_n)\right) \\
&= \frac{1}{\sum\limits_{i=1}^n x_i^2}(x_1, \dots, x_n)
\end{aligned} \tag{2}$$

and by a similar expansion,

$$\varphi_S\varphi_N^{-1} = \frac{1}{\sum\limits_{i=1}^n x_i^2}(x_1, \dots, x_n) \tag{3}$$