The Table of Marks for A_5

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Introduction

For a finite group G and a transitive G-set X with stabilizer $H \leq G$, we have a natural isomorphism between the coset space G/H with left multiplication and X with respect to the action of G on X. Since any finite G-set can be decomposed into a disjoint union of transitive G sets, we may represent any action of G on a finite set naturally as an element of the free $\mathbb{Z}$-module generated by

$$\{G/H \mid H \text{ is a representative of a conjugacy class of subgroups of } G\}$$

This module, denoted $\mathfrak{B}(G)$, is called the *Burnside Ring* of G. When considering the action of a finite group on a G-set, it is of course natural to consider the restricted action of its subgroups. This information can be represented by a matrix whose entries entries are given by the order of the set of fixed points of the actions of all conjugacy classes of subgroups of G on the generators of $\mathfrak{B}(G)$. We call this matrix the *Table of Marks* for G and its entries *marks*. Below we construct the Table of Marks for A_5.

Preliminary Results

We begin by listing representatives of all conjugacy classes of subgroups of A_5 along with the number of subgroups in each class [2]:

Conjugacy Class Representative	**Number of Subgroups**
$\langle e \rangle$	1
A_3	10
$\langle (1\ 2\ 3\ 4\ 5) \rangle$	6
V_4	5
$\langle e, (1\ 2)(3\ 4) \rangle$	15
A_4	5
D_{10}	6
$\langle (1\ 2\ 3), (1\ 2)(4\ 5) \rangle$	10
A_5	1

We have a natural partial order by $H \leq K$ if and only if $H \leq gKg^{-1}$ for some $g \in G$, which is extended to a total order on incomparable elements by considering $H \leq K$ if $|H| \leq |K|$. To construct the table of marks, we impose this order on the set of conjugacy classes of A_5 [1], denoting the fixed representatives of the ith conjugacy class as H_i:

$$\begin{aligned}
H_1 &:= \langle e \rangle \\
H_2 &:= \langle (1\ 2)(3\ 4) \rangle \cong \mathbb{Z}/2\mathbb{Z} \\
H_3 &:= A_3 \\
H_4 &:= \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\} \cong V_4 \\
H_5 &:= \langle (1\ 2\ 3\ 4\ 5) \rangle \cong \mathbb{Z}/5\mathbb{Z} \\
H_6 &:= \langle (1\ 2\ 3), (1\ 2)(4\ 5) \rangle \cong S_3 \\
H_7 &:= \langle (1\ 2\ 3\ 4\ 5), (2\ 5)(3\ 4) \rangle \cong D_{10} \\
H_8 &:= A_4 \\
G &:= A_5
\end{aligned} \tag{1}$$

Definition 1. *Let G be a group, let X be a set, and let $\cdot : G \times X \mapsto X$ be an action of G on X which we shall denote using the notation of left multiplication. i.e. for $g \in G,\ x \in X,\ gx := g \cdot x$. We define*

$$\mathrm{Fix}_G(X) := \{x \in X \mid gx = x,\ \forall g \in G\}$$

For clarity, we denote entry (i, j) in the table of Marks, equal to $|\mathrm{Fix}_{H_i}(G/H_j)|$, as $m_{i,j}$ and note several useful results:

Theorem 1. *Let G be a group and let $H_i, H_j \leq G$, then*

$$|\mathrm{Fix}_{H_i}(G/H_j)| = |\left\{gH_jg^{-1} \mid g \in G,\ H_i \leq gH_jg^{-1}\right\}| \cdot [N_G(H_j) : H_j]$$

Proof. See [1]. □

Corollary 1. *Let G be a group and $H \leq G$. Then*

$$|\mathrm{Fix}_H(G/H)| = [N_G(H) : H]$$

Proof. Notice that, for $g \in G$, $h \mapsto ghg^{-1}$ is a bijection from G to itself and so $|gHg^{-1}| = |H|$. So if $H \leq gHg^{-1}$, then $H = gHg^{-1}$ and thus,

$$\begin{aligned} |\left\{gHg^{-1} \mid g \in G, H \leq gHg^{-1}\right\}| &= |\left\{gHg^{-1} \mid g \in G, H = gHg^{-1}\right\}| \\ &= |\{H\}| \\ &= 1 \end{aligned} \tag{2}$$

The result now follows from Theorem 1. □

Employing Theorem 1 to compute each mark $m_{i,j}$, we proceed by finding $[N_G(H_j) \colon H_j]$ for each j. Note that A_5 is simple and therefore the normalizer of only the trivial group and itself.

Computation of Normalizers

j = 2

$N_G(H_2)$ must be either H_2 itself or a conjugate of $H_4 \cong V_4$, $H_6 \cong S_3$, or $H_7 \cong D_{10}$. If either $H_2 \trianglelefteq H_6$ or $H_2 \trianglelefteq H_7$, then H_2 is the unique Sylow 2-subgroup of either S_3 or D_{10}. However, neither S_3 nor D_{10} contains a unique Sylow 2-subgroup. Since H_4 is abelian, $H_2 \trianglelefteq H_4$. Thus $N_G(H_2) = V_4$, and

$$\begin{aligned} [N_G(H_2) \colon H_2] &= [V_4 \colon H_2] \\ &= \frac{4}{2} \\ &= 2 \end{aligned} \tag{3}$$

j = 3

The only proper subgroups of G containing H_3 are H_3, H_6, and H_8. If $H_3 \trianglelefteq H_8 \cong A_4$, then H_3 is the unique Sylow 3-subgroup of A_4, which is impossible since A_4 has 8 elements of order 3. Now, $[H_6 \colon H_3] = 2 \Rightarrow H_3 \trianglelefteq H_6$. Therefore $N_G(H_3) = H_6$ and

$$\begin{aligned} [N_G(H_3) \colon H_3] &= [H_6 \colon H_3] \\ &= \frac{6}{3} \\ &= 2 \end{aligned} \tag{4}$$

j = 4

Since H_4 is the only subgroup of order 4 contained in $H_8 \cong A_4$, it is the unique Sylow 2-subgroup of A_4, and therefore normal. Hence

$$\begin{aligned}[N_G(H_4)\colon H_4] &= [H_8\colon H_4] \\ &= \frac{12}{4} \\ &= 3\end{aligned} \tag{5}$$

j = 5

The only proper subgroups of G containing H_5 are H_5 itself and H_7. Since $[H_7\colon H_5] = 2$, $H_5 \trianglelefteq H_7$ and

$$\begin{aligned}[N_G(H_5)\colon H_5] &= [H_7\colon H_5] \\ &= 2\end{aligned} \tag{6}$$

j = 6

$N_G(H_6)$ is H_6 itself since H_6 is not properly contained in any proper subgroup of G. Hence

$$\begin{aligned}[N_G(H_6)\colon H_6] &= [H_6\colon H_6] \\ &= 1\end{aligned} \tag{7}$$

j = 7

Observe that the only possibilities for $N_G(H_7)$ are H_7 or H_8. Since H_7 contains an element of order 5 and $H_8 = A_4$ does not, it must be that $H_7 \not\leq H_8$ and so $N_G(H_7) \neq H_8$. Thus $N_G(H_7) = H_7$ and

$$\begin{aligned}[N_G(H_7)\colon H_7] &= [H_7\colon H_7] \\ &= 1\end{aligned} \tag{8}$$

j = 8

$N_G(H_j)$ is clearly H_8 itself, and

$$\begin{aligned}[N_G(H_8)\colon H_8] &= [H_8\colon H_8] \\ &= 1\end{aligned} \tag{9}$$

Computation of Marks

In the interest of the reader, we relegate all trivial computations to the Appendix.

$m_{i,j} \quad i > j$

For all $i > j$, $|H_i| \geq |H_j|$ and so cannot be contained in any conjugate of H_j. Hence the matrix is upper triangular.

$m_{i,j} \quad i = j$

By Corollary 1, the marks on the diagonal are given by the index of H_i in its normalizer.

$\underline{m_{1,j}\ j = 1, 2, \ldots, 9}$

Since $H_1 = \langle e \rangle,\ m_{1,j} = |G/H_j|$.

$\underline{m_{2,4}}$

Recall that all congujates of $H_4 \cong V_4$ are uniquely determined by their fixed point under the natural action of A_5 on the set $S := \{1, 2, 3, 4, 5\}$. Since a fixed nontrivial element, τ, of any conjugate of V_4 is a 2×2-cycle, τ acts non-trivially on exactly 4 elements of S. Thus, τ is contained in one conjugate of H_4. In the case of $\tau = (1\ 2)(3\ 4)$, the above proves that τ is contained only in H_4 and no other conjugates. So

$$\begin{aligned} &H_2 = \{e, (1\ 2)(3\ 4)\} \leq H_4 \\ &\Rightarrow |\left\{gH_4g^{-1} \mid g \in G,\ H_2 \leq gH_4g^{-1}\right\}| \\ &= |\{H_4\}| \\ &= 1 \end{aligned} \tag{10}$$

and so

$$\begin{aligned} m_{2,4} &= |\left\{gH_4g^{-1} \mid g \in G,\ H_2 \leq gH_4g^{-1}\right\}| \cdot [N_G(H_4) : H_4] \\ &= 3 \end{aligned} \tag{11}$$

$\underline{m_{2,6}}$

H_6 is generated by a 3-cycle, σ, and a 2×2-cycle, τ, such that its action on $S = \{1, 2, 3, 4, 5\}$ has no fixed points, and the elements of S fixed by $\langle \sigma \rangle$ form a transposition in the cycle decomposition of τ. Since the only non-trivial element of H_2 is a 2×2-cycle, it is contained in exactly 2 conjugates of H_6. Hence, $m_{2,6} = 2$.

$\underline{m_{3,6}}$

Since every conjugate of H_6 is uniquely determined by its subgroup of 3-cycles, H_3 is contained in exactly 1 conjugate of H_6. Therefore, $m_{3,6} = 1$.

$\underline{m_{4,6}}$

Recall that every conjugate of $H_4 \cong V_4$ is uniquely determined by it's fixed point and that the collection of $2{\times}2$ cycles in twisted $S_3 \cong H_6$ have no fixed points. So H_4 is not contained in any conjugate of H_6 and thus, $m_{4,6} = 0$.

$\underline{m_{2,7}}$

First we prove the following:

Proposition 1. *If a 2×2-cycle appears in exactly k conjugates of H_7, then every distinct 2×2-cycle appears in exactly k conjugates of H_7.*

Proof. Suppose $\tau \in A_5$ is a 2×2-cycle and appears in exactly k conjugates of H_7.
Fix any of the 15 2×2-cycles $\phi \in A_5$ and note that H_2 has $[A_5 \colon N_G(H_2)] = [A_5 \colon H_4] = 15$ conjugates, implying that all 2×2-cycles are conjugate.

So $\exists \psi \in A_5$ such that $\tau = \psi \phi \psi^{-1}$ and further, if $\tau \in gH_7g^{-1}$, then $\phi \in \psi^{-1}gH_7g^{-1}\psi$. Obviously for $g, h \in A_5$,

$$\begin{aligned} &gH_7g^{-1} \neq hH_7h^{-1} \\ &\Rightarrow \psi^{-1}gH_7g^{-1}\psi \neq \psi^{-1}hH_7h^{-1}\psi \end{aligned} \tag{12}$$

and so ϕ appears in exactly k conjugates of H_7. □

Now, there are exactly 15 2×2-cycles in A_5 and 5 2×2-cycles for every conjugate of H_7, leaving $6\times5=30$ (non distinct) $2\times$-cycles over all conjugates. By the proposition, it must be that each 2×2-cycle, namely $(1\ 2)(3\ 4)\in H_2$, appears exactly $\frac{30}{15}=2$ times over all conjugates of H_7. So, writing this result more suggestively and noting that H_2 is generated by $(1\ 2)(3\ 4)$,

$$|\left\{gH_7g^{-1} \mid g\in G,\ H_2\leq gH_7g^{-1}\right\}|=2 \tag{13}$$

Thus, $m_{2,7}=2$.

$\underline{m_{2,8}}$

The conjugates of $H_8\cong A_4$ are uniquely defined by their fixed points. Since H_2 acts non-trivially on exactly 4 elements, H_2 is therefore only a subgroup of eH_8e^{-1} and so $m_{2,8}=1$.

$\underline{m_{3,8}}$

$A_3\leq gH_8g^{-1}$ for any conjugate of H_8 that fixes 4 or 5, so $m_{3,8}=2$.

$\underline{m_{4,8}}$

Both H_4 and H_8 are completely determined by their respective unique fixed points. Thus, $m_{4,8}=1$.

$\underline{m_{i,9}\quad i=1,\ldots,9}$

Since $G/G\cong\langle e\rangle,\ m_{i,9}=1$

Table of Marks

$H\backslash\mathfrak{B}(G)$	G/H_1	G/H_2	G/H_3	G/H_4	G/H_5	G/H_6	G/H_7	G/H_8	G/G
H_1	60	30	20	15	12	10	6	5	1
H_2		2	0	3	0	2	2	1	1
H_3			2	0	0	1	0	2	1
H_4				3	0	0	0	1	1
H_5					2	0	1	0	1
H_6						1	0	0	1
H_7							1	0	1
H_8								1	1
G									1

Appendix

$\underline{m_{2,3}}$

Since H_2 contains a product of disjoint 2-cycles and H_3 does not, Theorem 1 yields that $m_{2,3} = 0$.

$\underline{m_{3,4}}$

Note that $H_3 \not\leq gH_4g^{-1}$ for any $g \in G$ since $H_3 = A_3 \cong \mathbb{Z}/3\mathbb{Z}$ and no conjugate of $H_4 \cong V_4$ contains a 3-cycle. Hence $m_{3,4} = 0$.

$\underline{m_{2,5}}$

Since no conjugate of $H_5 = \langle(1\ 2\ 3\ 4\ 5)\rangle \cong \mathbb{Z}/5\mathbb{Z}$ contains an involution, $m_{2,5} = 0$.

$\underline{m_{3,5}}$

Similar to the case above, no conjugate of H_5 contains a 3-cycle. Therefore, $m_{3,5} = 0$.

$\underline{m_{4,5}}$

Again, since conjugate of H_5 contains an involution $H_4 \cong V_4$ is not contained in any conjugate of H_5. Therefore $m_{4,5} = 0$.

$\underline{m_{5,6}}$

H_6 contains no elements of order 5, so H_5 cannot be contained in any conjugate of H_6. Thus, $m_{5,6} = 0$.

$\underline{m_{3,7}}$

H_7 contains no elements of order 3, so H_3 cannot be contained in any conjugate of H_7. Thus, $m_{3,7} = 0$.

$\underline{m_{4,7}}$

Since no 2×2-cycles in H_7 share a fixed point, no conjugate of H_7 can contain H_4, and $m_{4,7} = 0$.

$\underline{m_{5,7}}$

Each copy of H_7 is uniquely determined by any of the 5-cycles it contains and so H_5 is only a subgroup of eH_7e^{-1} thus, $m_{5,7} = 1$.

$\underline{m_{6,7}}$

Note that H_6 contains a 3-cycle and H_7 contains no elements of order 3. Thus, $m_{6,7} = 0$.

$\underline{m_{5,8}}$

Observe that H_5 contains a 5-cycle and H_8 does not, so $m_{5,8} = 0$.

$\underline{m_{6,8}}$

Since H_6 acts non-trivially on 5 elements and H_8 has a fixed point, $m_{6,8} = 0$.

$\underline{m_{7,8}}$

Similarly, H_7 contains no fixed points, so $m_{7,8} = 0$.

References

[1] Götz Pfeiffer. *The Burnside Ring and the Table of Marks.* URL: http://schmidt.ucg.ie/~goetz/pub/marks/node1.html.

[2] *Subgroup Structure of Alternating Group: A5.* URL: https://groupprops.subwiki.org/wiki/Subgroup_structure_of_alternating_group:A5.