The function $\cos^2 x - \sin x$ models alternating current (AC) circuits. Use the Taylor expansion to approximate its leading behavior for small angles $a = 0$, write the answers in terms of x upto the cubic term.

ans: Given: $f(x) = \cos^2 x - \sin x$

$a = 0$

To find: Taylor series upto cubic term

Solution:

$f(x) = \cos^2 x - \sin x$

$f'(x) = -2\cos x \sin x - \cos x = -\sin 2x - \cos x$

$f''(x) = -2\cos 2x + \sin x$

$f'''(x) = -4\sin 2x + \cos x$

$a = 0$

$f(a) = 1 - 0 = 1$

$f'(a) = 0 - 1 = -1$

$f''(a) = 2 + 0 = -2$

$f'''(a) = 0 + 1 = 1$

Taylor series is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$$

at $a = 0$ & $f(x) = \cos^2 x - \sin x$

we get

$$f(x) = 1 + (-1)x + \frac{2x^2}{2} + \frac{x^3}{3!}$$

$$\Rightarrow f(x) = 1 - x - x^2 + \frac{x^3}{6}$$

Thus the equation to approximate the leading behaviour of AC current for the given function is

$$f(x) = 1 - x - x^2 + \frac{x^3}{6}$$

The function $e^{\sin x}$ describes the spread of disease in epidemiological models where infection rates vary sinusoidally. Approximate to a 3rd degree function using Taylor series to predict short-term disease spread at small deviation of $a = \frac{\pi}{2}$.

ans: Given: $f(x) = e^{\sin x}$

$a = \frac{\pi}{2}$

To find: Taylor polynomial upto 3rd degree

Solution:

$f(x) = e^{\sin x}$

$f'(x) = -\cos x \, e^{\sin x}$

$f''(x) = -\sin x \, e^{\sin x} + \cos^2 x \, e^{\sin x}$

$f'''(x) = -\cos x \, e^{\sin x} - \cos x \sin x \, e^{\sin x}$

$- 2\cos x \sin x \, e^{\sin x} + e^{\sin x} \cos^3 x$

$= e^{\sin x} \cdot \cos x \left[-1 - 3\sin x + \cos x \right]$

$a = \pi/2$ & $f(x) = e^{\sin x}$

$f(a) = e$

$f'(a) = 0$

$f''(a) = -e + 0 = -e$

$f'''(a) = 0$

$\therefore$ the approximate 3rd degree function of taylor series is

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$$

$$\Rightarrow f(x) = e + 0 - \frac{e}{2}\left(x - \frac{\pi}{2}\right)^2 + 0$$

$$f(x) = e - \frac{e}{2}\left(x - \frac{\pi}{2}\right)^2$$

Your professors are tea addicts, to please them you decide to make a "smart tea machine". To control the temperature, the PID is modeled proportionally to $e^x \cos x$, after some trail and error, you decide to linearize the function. The PID manual shows the best value at $a = \frac{\pi}{4}$. Write the final function obtained after linearization.

ans: Given: $e^x \cos x = f(x)$

$$a = \frac{\pi}{4}$$

To find: linear approximation of $f(x)$ at $a = \frac{\pi}{4}$

Solution: $f(x) = e^x \cos x$

$$f'(x) = e^x \cos x - \sin x \, e^x$$

$$f(a) = \frac{e^{\pi/4}}{\sqrt{2}}$$

$$f'(a) = \frac{e^{\pi/4}}{\sqrt{2}} - \frac{e^{\pi/4}}{\sqrt{2}} = 0$$

The linear approximation of the function is given by

$$L(x) = f(a) + f'(a)(x - a)$$

$\therefore$

$$L(x) = \frac{e^{\pi/4}}{\sqrt{2}} + 0$$

$$L(x) = \frac{e^{\pi/4}}{\sqrt{2}}$$

Hence the function can be linearized as

$$L(x) = \frac{e^{\pi/4}}{\sqrt{2}}$$

Find the linearization of $f(x) = 2^x$ at $a = 0$.

ans Given: $f(x) = 2^x$

$a = 0$

To find: Linear approximation of 2^x

Soln: $f(x) = 2^x = y$ (say)

$\ln y = x \ln 2$

$f(a) = 1$

$f'(x) = y'$

$f'(a) = \ln 2$

$\frac{1}{y} y' = \ln 2 \Rightarrow y' = 2^x \ln 2$

$L(x) = f(a) + f'(a)(x - a)$

at $a = 0$

$L(x) = 1 + x \ln 2$

Determine the first three non zero values of taylor series of the function $f(x) = \cos(2x + \frac{\pi}{2})$ at $a = \frac{\pi}{4}$

ans: Given: $f(x) = \cos\left(2x + \frac{\pi}{2}\right)$

$a = \frac{\pi}{4}$

To find: first three non zero terms.

Soln:

$f(x) = \cos\left(2x + \frac{\pi}{2}\right)$ $\quad a = \frac{\pi}{4}$

$f(x) = -\sin 2x$ $\quad f(a) = -1$

$f'(x) = -2\cos 2x$ $\quad\quad f'(a) = 0$

$f''(x) = 4\sin 2x$ $\quad\quad f''(a) = 4$

$f'''(x) = 8\cos 2x$ $\quad\quad f'''(a) = 0$

$f^{IV}(x) = -16\sin 2x$ $\quad\quad f^{IV}(a) = -16$

Taylor series is given by

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$$
$$+ \frac{f^{IV}(a)(x-a)^4}{4!}$$

$$f(x) = -1 + 0 + \frac{4}{2}\left(x-\frac{\pi}{4}\right)^2 + 0 - \frac{16}{4!}\left(x-\frac{\pi}{4}\right)^4$$

$$f(x) = -1 + 2\left(x-\frac{\pi}{4}\right)^2 - \frac{2}{3}\left(x-\frac{\pi}{4}\right)^4$$