MAE 263F MS Comprehensive Exam Problem

Atti W. Lau

I. Introduction

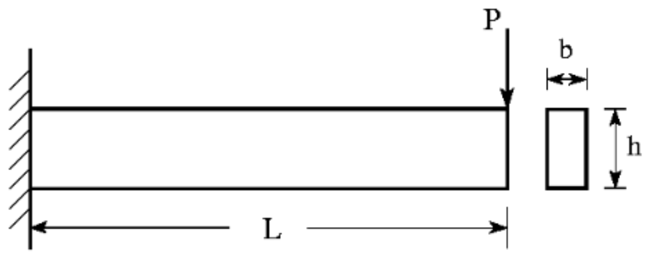

Figure 1: Beam, fixed at left end with a point load at the right end. (Jawed).

This problem calls for both the implicit simulation of a cantilever beam that is fixed at its left and is under a point load P applied vertically at its right end. The beam is 0.1 m long, with a Young's modulus of 200 GPa. It has a solid rectangular cross-section with a width of b = 0.01 m and a height of 0.002 m.

The implicit solver is based on the Newton-Raphson method and employs functions by M. K. Jawed to calculate force terms and their gradients. The solver employs the algorithm to simulate a discrete elastic beam (DEM).

The main value of interest for the applied point load is 10 N, and the vertical deflection of the beam is calculated from the implicit simulation. The deflection in response to a range of loads, from 1 to 100 N, is also simulated. The results of the implicit beam simulation are then compared to that analytically predicted from Euler-Bernoulli beam theory, which states that the deflection of this beam is determined by

$$\delta = \frac{PL^3}{3EI}$$

where E is the Young's modulus and I is the moment of inertia of the beam.

II. Code Explanation

The pseudocode for the solver is as follows:

- Define solver parameters: # of nodes, timestep, total simulation time, implicit calculation tolerance
- Define physical parameters of the beam and the load
- Set the vectors and arrays for:
    - Initial positions and DOFs of beam nodes
    - Diagonal mass matrix with mass of each node
    - Initial velocity
    - External forces; in this case, the point force
    - Free and fixed nodes in the DOF vector
- Time loop:
    - Set the boundary conditions
    - Make an initial guess for the DOF vector: q = q0
    - Implicit solving loop: while error > tolerance,
        - Add the inertial term to f and J
        - Calculate and add the stretching and bending forces to f and J
        - Add the point force load to f
        - Calculate the new positions of the free nodes with q = q – J/f
        - Recalculate error
    - Update q0 and u0
- Record maximum deflection
- Compare to Euler-Bernoulli beam deflection

The implicit solver is set to use N = 50 nodes, a timestep of Δt = .05 seconds, and run for a total simulation time of 0.05 seconds. This simulation time was found to be sufficient for the beam to reach its steady-state deflection for P = 100 N.

Due to the shape of the beam, the beam is discretized into rectangular prism elements whose physical dimensions are b x h x dl, where dl is the length of the beam divided by (N – 1). As the beam will not deflect "out of the page", in the direction along its width, its moment of inertia I is calculated as

$$I = \frac{bh^3}{12}$$

The x- and y-positions of each node are the DOFs for the problem. They are contained in a position vector **q**, where

$$\boldsymbol{q} = \begin{bmatrix} x1 \\ y1 \\ x2 \\ \vdots \end{bmatrix}$$

The governing equation for the motion of these spheres is

$$\boldsymbol{f} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \frac{\partial \boldsymbol{E}^{elastic}}{\partial \boldsymbol{q}} - \boldsymbol{F}_{ext} = 0$$

$$\boldsymbol{q}_{k+1} = \boldsymbol{q}_k - \boldsymbol{J}/\boldsymbol{f}$$

where $\mathbf{M}$ is the mass matrix containing the masses of the spheres, $\mathbf{E}^{\text{elastic}}$ is the elastic energy of the beam (including stretching and bending), $\mathbf{F}_{\text{ext}}$ is a column vector the same size as $\mathbf{q}$ and represents the external force on the beam (Jawed & Lim). In this case, the external force is simply the vertical point load on its end, so the only nonzero element in $\mathbf{F}_{\text{ext}}$ is the last element.

At the beginning of each timestep, the boundary conditions are applied. This beam is fixed at its left end, so the x and y-positions of its first node, x1, and y1, are set to 0. Additionally, the fixed condition makes it so that the beam is not allowed to rotate at that end. Therefore, the y-position of the next node, y2, is also set to 0 so that the beam cannot rotate about its leftmost node. The fixed elements of the DOF vector are elements 1, 2, and 4, while all other elements are free. The Newton-Raphson loop calculates the deflections for the free elements.

After the boundary conditions are applied, the Newton-Raphson solving loop calculates positions of each node at the next timestep. The loop calculates and adds each of the terms in the governing equation to do so, and uses the functions gradEb(), hessEb(), gradEs(), and hessEs() written by Dr. Jawed to handle the elastic force terms.

Finally, the vertical deflection at the rightmost node is recorded and compared to the deflection predicted by Euler-Bernoulli beam theory.

III. Results

A. Deflection for Various Point Loads

The following plots show the comparison of end deflection predicted by the implicit simulation after 0.05 seconds and Euler-Bernoulli beam theory

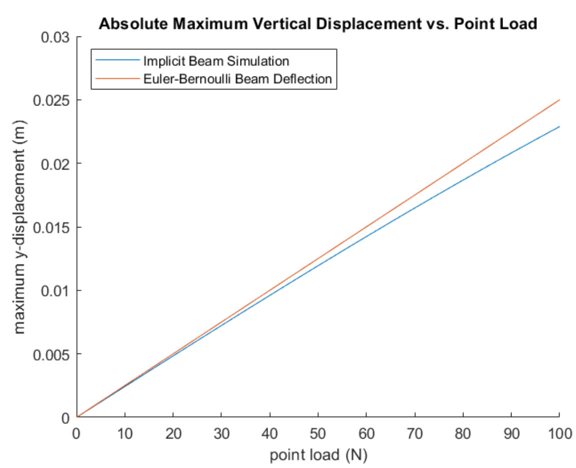

Figure 2: Vertical deflection of the rightmost node after 0.05 seconds as predicted by the simulation and Euler-Bernoulli beam theory.

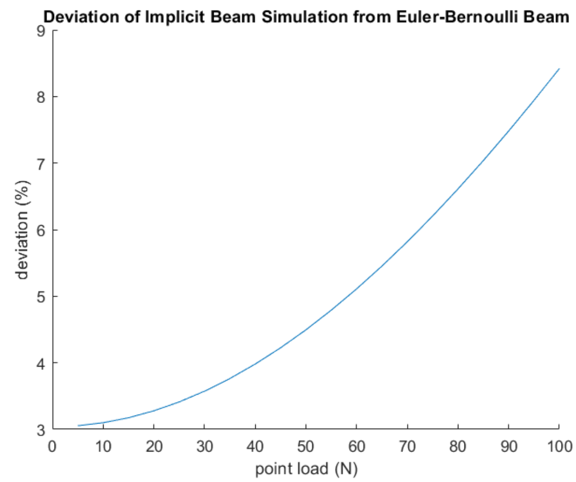

Figure 3: The percentage deviation between the deflections predicted by the simulation and Euler-Bernoulli beam theory.

For a point load of P = 10 N, the implicit simulation predicts an end deflection of 2.422 mm, while the Euler-Bernoulli beam would deflect by 2.5 mm. These deflections are fairly close in value.

At small values of the point load, the deflections predicted by the simulation and Euler-Bernoulli beam theory are nearly the same. However, as the point load magnitude increases, the predicted deflections begin to increasingly deviate in a nonlinear fashion, as can be observed in figures 2 and 3.

B. Prediction Deviation by 10%

By running the code for various point loads it was found that the deflection predictions between the simulation and Euler-Bernoulli beam theory deviate by more than 10% when the point load is at least 116 N. At this load, the implicit simulation gives a displacement of 25.99 mm, while the Euler-Bernoulli beam would deflect by 28.88 mm.

C. Reason for Deviation

At higher point loads, the Euler beam theory's prediction is significantly larger than the one predicted by the solver as the deformation of the beam becomes increasingly nonlinear. The Euler beam theory assumes that the cross-section of the beam stays planar under deformation and only addresses small deformation where linearity may be assumed. The solver is more accurate and general than the Euler beam theory equation as the solver is based on a governing equation that does not make any simplifying assumptions on the deflection and stiffness of the beam, so it is valid for any amount of deformation.

Acknowledgment

I thank Professor M. K. Jawed for his guidance and provided functions for completing this assignment.

References

[1] M. K. Jawed, "Comprehensive Exam Question, Fall 2023". UCLA, 2023.

[2] M. K. Jawed and S. Lim, "Discrete simulation of slender structures" . UCLA, 2019, pp. 16–22.