Linear Algebra Notes

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Definition 1 *A **linear equation** of the variables $x_1, x_2, \dots x_n$ is any equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, \dots a_n$ are called the coefficents and b is called the constant.

Definition 2 *A **system of linear equation** is a set of 1 or more linear equations.*

Example 1

$$x + 2y = 3$$

is a linear system and

$$\begin{aligned} x - 2y &= 1 \\ 2x + 3y &= 4 \\ x + y &= 5 \end{aligned}$$

is a linear system. But

$$5\sqrt{x} - y = 1$$

is not a linear system.

Given a linear system, we want to find all substitutions which satisfy the linear system. A **solution set** is a set of n-tuples, all of which are valid solutions to a given system of linear equations. Our central problem is: given a linear system, find its solution set. Big problems

1. Existance Question: Do there exist any solutions? If there are, we call the system **consistent**. If there are no solutions, we call the solution **inconsistent**.

2. Uniqueness Question: How many solutions are there? If there is exactly 1 solution, we call that **unique**.

Theorem 1 *If a linear system has solutions, it either has 1 solution or infintely many solutions. A linear system cannot have 12 solutions.*

Example 2

$$x + 2y = 5, x + 4y = 6$$

has solution

$$x = -4, y = 9/2$$

This linear system corresponds to the matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

We can turn this into an ***augmented matrix*** *to get*

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

We are allowed to perform row operations on the rows to get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 9/2 \end{bmatrix}$$

So $(x, y) = (-4, 9/2)$. *The solution is unique.*

Theorem 2 *We are allowed to do these elementary row operations*

Definition 3 *We say two systems of equations are* **equivalent** *if their solutions sets are equal.*

1. *replacement: replace one row with the sum of itself and another row*

2. *interchange: swap 2 rows*

3. *scaling: multiply all entries in a row by a constant*

The reason the elementary operations work is because they give an equivalent system, i.e. they do not change the solution set.

Convince yourself that each of the elementary operations do not change the solution set.

Example 3 *Let's solve this system*

$$\begin{aligned} kx + y + 2z &= 1 \\ x + z &= h \\ y - z &= 1 \end{aligned}$$

The solution is

$$x = -3h/(k-3), y = \frac{(h+1)k-3}{k-3}, z = \frac{hk}{k-3}$$

The system would have no solutions (inconsistent) iff $k = 3$ *and* $h \neq 0$. *The system has infinite solutions if* $k = 3$ *and* $h = 0$. *The system has 1 solution if* $k \neq 3$.

Definition 4 *A matrix is in* ***echelon form*** *iff*

1. *All nonzero rows are above any rows of all zeroes*
2. *each leading entry in a row is in a column to the right of the leading endtry of the row above it*
3. *All entries in a column below a leading entry are zeroes*

Definition 5 *A matrix is in* ***reduce row echelon form*** *iff*

1. *The leading entry in each nonzero row is* 1
2. *each leading* 1 *is the only nonzero entry in its column*

Example 4 *This matrix*

$$\begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & * \end{bmatrix}$$

is in REF, but not in RREF.

Example 5 *This matrix*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in REF, and it would be in RREF if we changed the 2 *to a* 1.

Theorem 3 *A given matrix has infinitely many equivalent REF, but only 1 RREF.*

Example 6 *Reduce this matrix*

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

Solution:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 6 *If a matrix is in REF, the* ***pivots*** *are leading entries. The* ***pivot columns*** *are the columns which contain a pivot.*

Definition 7 *A* ***basic variable*** *is a variable which corresponds to a pivot column. A* ***free variable*** *is a variable which does not correspond to a pivot column.*

Example 7 *Given the matrix*

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in RREF. Variables x_1, x_2, x_4 are basic variables and variable x_3 is a free variable.

If the righmost column of A is a pivot column, then we cannot tell if $\mathbf{A}\vec{x} = \vec{b}$ is consistent or inconsistent.

Theorem 4 *If the rightmost column of* $\begin{bmatrix} \mathbf{A} & \vec{b} \end{bmatrix}$ *is a pivot column, then* $\mathbf{A}\vec{x} = \vec{b}$ *is not consistent.*

Theorem 5 *Even if you have a free variable, that doesn't guarantee infintely many solutions.*

If you have a free variable and the system is consistent, then the system has infinitely many solutions.

Example 8 *Given that the RREF is*

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

Describe the solution set. It is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 8 \end{bmatrix}$$

This solution set is infinitely big because it has free varaibles. This is called ***parametric vector form*** *of the solution set.*

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Let's review what we learned last session about matricies and linear systems.

1. the RREF form of a matrix is unique (but the REF form is not).
2. the elementary row operations do not affect the solution set of a linear system.
3. pivot columns correspond to basic variables, non-pivot columns correpond to free variables.
4. every linear system has either 0, 1, or ∞ solutions.
5. an augmented matrix can be used to represent a linear system. It is the coefficent matrix concatenated with the constant vector.
6. We can express the solution set of a linear system in parametric vector form.
7. If a system is consistent and has a free variable, it has infinitely many variables (but even inconsistent systems may have free variables)
8. row operations do not change pivot columns.

Example 9 *Let*

$$p = (0,1) \qquad q = (1,1) \qquad r = (2,2)$$

Prove there exists no line with all 3 points.

Prove there exists exactly one unique parabola with all 3 points
Solution:

The line has linear system

$$1 = 0a + b \qquad 1 = a + b \qquad 2 = 2a + b$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So inconsistent.

For the parabola, it has the system

$$1 = 0a + 0b + c \qquad 1 = a + b + c \qquad 2 = 4a + 2b + c$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 2 \end{bmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

i.e.

$$y = \frac{1}{2}x^2 - \frac{1}{2}x + 1$$

solution is unique because no free varaibles.

2.1 1.3 – Vector Equation

Definition 8 ***matrix multiplication by vector****: Given*

$$A = [\vec{a_1}, \vec{a_2}, \vec{a_3}, \ldots, \vec{a_n}]$$

and

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$$

then

$$A\vec{v} = \vec{a_1}v_1 + \vec{a_2}v_2 + \cdots + \vec{a_n}v_n = \sum_{i=1}^{n} \vec{a_i}v_i$$

We call this a ***linear combination*** *of a_i. A linear combination is a sum of scalings of vectors, i.e. scale the vectors and then add them up.*

Example 10

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2b + 3c \\ 5a + 4b + 7c \end{bmatrix}$$

Definition 9 ***span***: *Given a set of vectors $S = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$, then span($S$) is the set of all linear combinations of the vectors. In other words, it is the set of all vectors reachable by using only the vectors in S. In equational form,*

$$span(S) = \left\{ \sum_{i=1}^{n} c_i \vec{v}_i \;\middle|\; \forall c_i \in \mathbb{R} \right\}$$

Example 11 *In $\mathbb{R}^3$,*

$$span(\{\vec{v}\}) = \{k\vec{v} \quad | \quad k \in \mathbb{R}\}$$

This set is a line going through the origin (unliness $\vec{v} = \vec{0}$, then it's just the origin). The $\vec{0}$ vector is always in the span set.

$$span(\{\vec{u}, \vec{v}\})$$

this is a plane passing through the origin (unless the vectors are linearly dependent, then it's a line). It could be origin, or line thru origin, or plane thru origin.

$$span(\{\vec{u}, \vec{v}, \vec{w}\})$$

is either the origin, or a line thru the origin, or a plane thru the origin, or the entire space.

Example 12

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Is $Ax = b$ consistent? This is equivalent to asking if b is a linear combination of the columns of A and it is equivalent to asking if b is an element of the spanning set of the columns of A.

This is not consistent because

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $b \notin span(A)$

Example 13

$$x + 2y + 3z = 4 \qquad 5x + 6y + 7z = 8$$

So

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Theorem 6 *Asking whether the system $Ax = b$ is consistent is the same as asking if $b \in span(A)$*

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3.1 1.4 – Matrix Equation

Example 14 *Let*

$$A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -1 & -14 \\ 2 & 8 & 10 \end{bmatrix}$$

Let $b \in \mathbb{R}^m$. Is $Ax = b$ consistent for any b?
Solution:
Not consistent because if you rref

$$\begin{pmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{pmatrix}$$

you get the $-2b_1 + b_3 = 0$.

Now, find the set spanned by the column vectors of A.

(note that the spanning set is NOT $\mathbb{R}^3$, because the columns have a rank of 2). This is

$$span(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \middle| -2b_1 + b_3 = 0 \right\} = \left\{ b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} |\forall b_2, b_3 \in \mathbb{R} \right\} = span\left(\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \right)$$

You can do this is Sage with

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A =  span(column_matrix([[1,4,5],[3,-11,-14], [2,8,10]]))
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$$A = \mathrm{RowSpan}_Z \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem 7 *The following statements are logically equivalent: Let A be an $m \times n$ matrix and let b be an $m \times 1$ vector.*

1. *for all $b \in \mathbb{R}^m$, $Ax = b$ has a solution.*
2. *$span(A) = \mathbb{R}^m$*
3. *The columns of A span $\mathbb{R}^m$*
4. *A has 1 pivot position in every row.*
5. *b can be written as a linear combination of the columns of A.*
6. *$Ax = b$ is consistent for all $b \in \mathbb{R}^m$.*

Make sure you don't mess up the details. We want a pivot in every ROW of A, not every column of A.

Example 15 *Let's apply theorem 7.*

$$span\left(\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} \right) \neq \mathbb{R}^2$$

because it doesn't have a pivot in the second row.

Example 16

$$\mathit{span}\left(\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}2\\1\end{bmatrix}\right\}\right)=\mathbb{R}^2$$

because there's a pivot in every row. Note that there is not a pivot in every column, but the theorem requires a pivot in every row.

Another mistake people make is thinking that the theorem applies if there is a pivot in every row of $[A \quad b]$. We are only concerned about a pivot in every row of A.