Linear Algebra Notes

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Definition 1 *A **linear equation** of the variables $x_1, x_2, \dots x_n$ is any equation of the form*

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $a_1, \dots a_n$ are called the coefficents and b is called the constant.

Definition 2 *A **system of linear equation** is a set of 1 or more linear equations.*

Example 1

$$x + 2y = 3$$

is a linear system and

$$
\begin{aligned}
x - 2y &= 1 \\
2x + 3y &= 4 \\
x + y &= 5
\end{aligned}
$$

is a linear system. But

$$5\sqrt{x} - y = 1$$

is not a linear system.

Given a linear system, we want to find all substitutions which satisfy the linear system. A **solution set** is a set of n-tuples, all of which are valid solutions to a given system of linear equations. Our central problem is: given a linear system, find its solution set. Big problems

1. Existance Question: Do there exist any solutions? If there are, we call the system **consistent**. If there are no solutions, we call the solution **inconsistent**.
2. Uniqueness Question: How many solutions are there? If there is exactly 1 solution, we call that **unique**.

Theorem 1 *If a linear system has solutions, it either has 1 solution or infintely many solutions. A linear system cannot have 12 solutions.*

Example 2

$$x + 2y = 5, x + 4y = 6$$

has solution

$$x = -4, y = 9/2$$

This linear system corresponds to the matrix multiplication

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

We can turn this into an ***augmented matrix*** *to get*

$$\begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$$

We are allowed to perform row operations on the rows to get

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 9/2 \end{bmatrix} \implies \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 9/2 \end{bmatrix}$$

So $(x, y) = (-4, 9/2)$. *The solution is unique.*

Theorem 2 *We are allowed to do these elementary row operations*

Definition 3 *We say two systems of equations are* ***equivalent*** *if their solutions sets are equal.*

1. *replacement: replace one row with the sum of itself and another row*
2. *interchange: swap 2 rows*

3. *scaling: multiply all entries in a row by a constant*

The reason the elementary operations work is because they give an equivalent system, i.e. they do not change the solution set.

Convince yourself that each of the elementary operations do not change the solution set.

Example 3 *Let's solve this system*

$$\begin{aligned} kx + y + 2z &= 1 \\ x + z &= h \\ y - z &= 1 \end{aligned}$$

The solution is

$$x = -3h/(k-3), y = \frac{(h+1)k-3}{k-3}, z = \frac{hk}{k-3}$$

The system would have no solutions (inconsistent) iff $k = 3$ and $h \neq 0$. The system has infinite solutions if $k = 3$ and $h = 0$. The system has 1 solution if $k \neq 3$.

Definition 4 *A matrix is in* ***echelon form*** *iff*

1. *All nonzero rows are above any rows of all zeroes*
2. *each leading entry in a row is in a column to the right of the leading endtry of the row above it*
3. *All entries in a column below a leading entry are zeroes*

Definition 5 *A matrix is in* ***reduce row echelon form*** *iff*

1. *The leading entry in each nonzero row is* 1
2. *each leading* 1 *is the only nonzero entry in its column*

Example 4 *This matrix*

$$\begin{bmatrix} * & * & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & * \end{bmatrix}$$

is in REF, but not in RREF.

Example 5 *This matrix*

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in REF, and it would be in RREF if we changed the 2 *to a* 1.

Theorem 3 *A given matrix has infinitely many equivalent REF, but only 1 RREF.*

Example 6 *Reduce this matrix*

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

Solution:

$$\begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

Definition 6 *If a matrix is in REF, the* ***pivots*** *are leading entries. The* ***pivot columns*** *are the columns which contain a pivot.*

Definition 7 *A* ***basic variable*** *is a variable which corresponds to a pivot column. A* ***free variable*** *is a variable which does not correspond to a pivot column.*

Example 7 *Given the matrix*

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 4 \\ 0 & 1 & 3 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix is in RREF. Variables x_1, x_2, x_4 are basic variables and variable x_3 is a free variable.

If the righmost column of A is a pivot column, then we cannot tell if $\mathbf{A}\vec{x} = \vec{b}$ is consistent or inconsistent.

Theorem 4 *If the rightmost column of $\begin{bmatrix} \mathbf{A} & \vec{b} \end{bmatrix}$ is a pivot column, then $\mathbf{A}\vec{x} = \vec{b}$ is not consistent.*

Theorem 5 *Even if you have a free variable, that doesn't guarantee infintely many solutions.*

If you have a free variable and the system is consistent, then the system has infinitely many solutions.

Example 8 *Given that the RREF is*

$$\begin{bmatrix} 1 & 2 & 3 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

Describe the solution set. It is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 6 \\ 0 \\ 0 \\ 7 \\ 8 \end{bmatrix}$$

This solution set is infinitely big because it has free varaibles. This is called ***parametric vector form*** *of the solution set.*

2 March 22

Let's review what we learned last session about matricies and linear systems.

1. the RREF form of a matrix is unique (but the REF form is not).
2. the elementary row operations do not affect the solution set of a linear system.
3. pivot columns correspond to basic variables, non-pivot columns correpond to free variables.
4. every linear system has either 0, 1, or ∞ solutions.
5. an augmented matrix can be used to represent a linear system. It is the coefficent matrix concatenated with the constant vector.
6. We can express the solution set of a linear system in parametric vector form.
7. If a system is consistent and has a free variable, it has infinitely many variables (but even inconsistent systems may have free variables)
8. row operations do not change pivot columns.

Example 9 *Let*

$$p = (0, 1) \qquad q = (1, 1) \qquad r = (2, 2)$$

Prove there exists no line with all 3 points.

Prove there exists exactly one unique parabola with all 3 points

Solution:

The line has linear system

$$1 = 0a + b \qquad 1 = a + b \qquad 2 = 2a + b$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So inconsistent.

For the parabola, it has the system

$$1 = 0a + 0b + c \qquad 1 = a + b + c \qquad 2 = 4a + 2b + c$$

$$\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 4 & 2 & 1 & 2 \end{bmatrix} \sim$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

i.e.

$$y = \frac{1}{2}x^2 - \frac{1}{2}x + 1$$

solution is unique because no free varaibles.

2.1 1.3 – Vector Equation

Definition 8 ***matrix multiplication by vector:*** *Given*

$$A = [\vec{a_1}, \vec{a_2}, \vec{a_3}, \dots, \vec{a_n}]$$

and

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix}$$

then

$$A\vec{v} = \vec{a_1}v_1 + \vec{a_2}v_2 + \cdots + \vec{a_n}v_n = \sum_{i=1}^{n} \vec{a_i}v_i$$

We call this a ***linear combination*** *of a_i. A linear combination is a sum of scalings of vectors, i.e. scale the vectors and then add them up.*

Example 10

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 4 & 7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2b + 3c \\ 5a + 4b + 7c \end{bmatrix}$$

Definition 9 ***span:*** *Given a set of vectors $S = \{\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}\}$, then $span(S)$ is the set of all linear combinations of the vectors. In other words, it is the set of all vectors reachable by using only the vectors in S. In equational form,*

$$span(S) = \left\{ \sum_{i=1}^{n} c_i\vec{v_i} \;\middle|\; \forall c_i \in \mathbb{R} \right\}$$

Example 11 *In $\mathbb{R}^3$,*

$$span(\{\vec{v}\}) = \{k\vec{v} \quad | \quad k \in \mathbb{R}\}$$

This set is a line going through the origin (unliness $\vec{v} = \vec{0}$, then it's just the origin). The $\vec{0}$ vector is always in the span set.

$$span(\{\vec{u}, \vec{v}\})$$

this is a plane passing through the origin (unless the vectors are linearly dependent, then it's a line). It could be origin, or line thru origin, or plane thru origin.

$$span(\{\vec{u}, \vec{v}, \vec{w}\})$$

is either the origin, or a line thru the origin, or a plane thru the origin, or the entire space.

Example 12

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \qquad v = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Is $Ax = b$ consistent? This is equivalent to asking if b is a linear combination of the columns of A and it is equivalent to asking if b is an element of the spanning set of the columns of A.

This is not consistent because

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus, $b \notin span(A)$

Example 13

$$x + 2y + 3z = 4 \qquad 5x + 6y + 7z = 8$$

So

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

Theorem 6 *Asking whether the system $Ax = b$ is consistent is the same as asking if $b \in span(A)$*

3 March 23

3.1 1.4 – Matrix Equation

Example 14 *Let*

$$A = \begin{bmatrix} 1 & 4 & 5 \\ -3 & -1 & -14 \\ 2 & 8 & 10 \end{bmatrix}$$

Let $b \in \mathbb{R}^m$. Is $Ax = b$ consistent for any b?
Solution:
Not consistent because if you rref

$$\begin{pmatrix} 1 & 4 & 5 & b_1 \\ -3 & -11 & -14 & b_2 \\ 2 & 8 & 10 & b_3 \end{pmatrix}$$

you get the $-2b_1 + b_3 = 0$.

Now, find the set spanned by the column vectors of A.

(note that the spanning set is NOT $\mathbb{R}^3$, because the columns have a rank of 2). This is

$$span(A) = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \middle| -2b_1 + b_3 = 0 \right\} = \left\{ b_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} | \forall b_2, b_3 \in \mathbb{R} \right\} = span\left(\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\} \right)$$

You can do this is Sage with

```
A =  span(column_matrix([[1,4,5],[3,-11,-14], [2,8,10]]))
```

$$A = \text{RowSpan}_Z \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

Theorem 7 *The following statements are logically equivalent: Let A be an $m \times n$ matrix and let b be an $m \times 1$ vector.*

1. *for all $b \in \mathbb{R}^m$, $Ax = b$ has a solution.*
2. *$span(A) = \mathbb{R}^m$*
3. *The columns of A span $\mathbb{R}^m$*
4. *A has 1 pivot position in every row.*
5. *b can be written as a linear combination of the columns of A.*
6. *$Ax = b$ is consistent for all $b \in \mathbb{R}^m$.*

Make sure you don't mess up the details. We want a pivot in every ROW of A, not every column of A.

Example 15 *Let's apply theorem 7.*

$$span\left(\left\{\begin{bmatrix}1\\2\end{bmatrix}\right\}\right) \neq \mathbb{R}^2$$

because it doesn't have a pivot in the second row.

Example 16

$$span\left(\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}\right) = \mathbb{R}^2$$

because there's a pivot in every row. Note that there is not a pivot in every column, but the theorem requires a pivot in every row.

Another mistake people make is thinking that the theorem applies if there is a pivot in every row of $[A \quad b]$. We are only concerned about a pivot in every row of A.

4 March 27

4.1 Homogenous System

Definition 10 *The linear system $Ax = b$ is called **homogenous** iff $b = \vec{0}$. A homogenous linear system is any system of the form $Ax = \vec{0}$.*

Theorem 8 *A homogenous linear system is always consistent because*

$$\mathbf{A}\vec{0} = \vec{0}$$

Definition 11 *We call $x = \vec{0}$ the **trivial solution** to the linear system $Ax = \vec{0}$.*

Theorem 9 *A homogenous linear system has infinite nontrivial solutions if and only if it contains free variables.*

Example 17 *consider this homogenous linear system*

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution set is

$$span \left\{ \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right\}$$

Example 18 *consider this homogenous linear system*

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y + 4 \\ y \end{bmatrix}$$

This solution set cannot be written as a span because it doesn't contain $\vec{0}$.

The solution set of a homogenous system can always be expressed as the span of some vectors.

Example 19 *Here are 2 systems to consider.*

$$S_1 : x + 2y + 3z = 0$$

$$S_2 : x + 2y + 3z = 4$$

Compare and contrast the solution sets.
Solution:

The solution set for S_1 *is span* $\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$. *The solution set for* S_2 *is*

$$\left\{ \begin{bmatrix} -2y - 3z + 4 \\ y \\ z \end{bmatrix} \middle| \forall y, z \in \mathbb{R} \right\}$$

Theorem 10 *The solution set to* $Ax = b$ *is parallel to the solution set of* $Ax = 0$. *You can think of the solution set of* $Ax = b$ *as being a constant vector translation from the solution set of* $Ax = 0$.

Suppose p *is any solution to the system* $\mathbf{A}\vec{x} = \vec{b}$. *Then, any other solution to* $\mathbf{A}\vec{x} = \vec{b}$ *can be written as* $\vec{p} + \vec{v}_h$, *where* $\vec{v}_h$ *is any solution to* $\mathbf{A}\vec{x} = \vec{0}$.

Warning: you can only think of the solution set of $\mathbf{A}\vec{x} = \vec{b}$ as a translation of the solution set of $\mathbb{A}\vec{x} = \vec{0}$ is the linear system is consistent. Otherwise, it's solution set is the empty set.

4.2 1.6 – Applications

There are a bunch of applications of matricies in the textbook.

Example 20 *Find the general traffic pattern in the freeway network shown in the figure*
Describe the general traffic pattern when the road x_1 is closed.
When $x_4 = 0$ what is the minimum value of x_1?
Solution:
For the solution, we use the general principle that (flow in) - (flow out) = 0.

$$A = \begin{bmatrix} 1 & 0 & -1 & -1 & 0 & 40 \\ -1 & -1 & 0 & 0 & 0 & -200 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \end{bmatrix}$$

$$\begin{aligned} A : x_1 &= x_3 + x_4 + 40 \\ B : 200 &= x_1 + x_2 \\ C : x_2 + x_3 &= 100 + x_5 \\ D : x_4 + x_5 &= 60 \end{aligned}$$

This reduces to

$$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & 100 \\ 0 & 1 & 1 & 0 & -1 & 100 \\ 0 & 0 & 0 & 1 & 1 & 60 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

So x_1, x_2, x_4 are basic variables and x_3, x_5 are free variables. The solution set is

$$\left\{ \begin{bmatrix} x_3 - x_5 + 100 \\ -x_3 + x_5 + 100 \\ x_3 \\ -x_5 + 60 \\ x_5 \end{bmatrix} | \forall x_3, x_5 \in \mathbb{R} \right\}$$

If $x_4 = 0$, then $x_5 = 60$, so we get the solution set

$$\left\{ \begin{bmatrix} x_3 + 40 \\ -x_3 + 160 \\ x_3 \\ 0 \\ 60 \end{bmatrix} | \forall x_3 \in [0, 160] \right\}$$

The minimum value of x_1 is 40.

4.3 1.7 – Linear Independence

Definition 12 *Given a set of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3, \ldots$, we say that the set is* ***linearly independent*** *if and only if the only solution to*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots = \vec{0} \qquad c_i \in \mathbb{R}$$

is the trivial solution.

We say the set is ***linearly dependent*** *if and only if there is a solution to the equation*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \cdots = \vec{0} \qquad c_i \in \mathbb{R}$$

which is not the trivial solution.

Theorem 11 *The columns of* $\mathbb{A}$ *are linearly independent if and only if the only solution to*

$$\mathbf{A}\vec{x} = \vec{0}$$

is the trivial solution $\vec{x} = \vec{0}$.

If one vector lies in the plane spanned by the other two, then the vectors are linearly dependent.

Theorem 12 *To find out if a set of vectors is linearly dependent, RREF their matrix.*

5 March 29

No class due to doctor's appointment

6 March 30

6.1 1.7 – Linearly Indepedence

Definition 13 *A **linearly independent set** is a set of vectors* $\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ *such that*

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_3\vec{v}_3 = 0 \iff c_1 = c_2 = c_3 = \cdots = 0$$

In other words, we say the columns of $\mathbf{A}$ *are linearly independent iff*

$$\mathbf{A}\vec{x} = \vec{0} \iff \vec{x} = \vec{0}$$

There are some special cases to easily determine if a set is linearly independent or dependent.

1. The empty set $\{\}$ is linearly independent.
2. The singleton set $\{\vec{v}\}$ is linearly independent if and only if $\vec{v} \neq \vec{0}$
3. The two-element set $\{\vec{v}_1, \vec{v}_2\}$ is linearly independent iff $\nexists k \in \mathbb{R}, \vec{v}_1 = k\vec{v}_2$.
4. Given a set of p vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_p\}$ in $\mathbb{R}^n$ and $p > n$, then the set is linearly dependent.
5. If a set contains the $\vec{0}$ vector, then it is linearly dependent.

Practically, to find out if vectors are linearly independent, RREF the matrix. If the RREF has a pivot in every row, then the vectors are linearly independent. Otherwise, the vectors are linearly dependent.

Example 21 *Are the column vectors in this matrix linearly independent? Just RREF it.*

6.2 1.8 – Linear Transformations

Definition 14 *A **transformation** is a function* $T : \mathbb{R}^n \to \mathbb{R}^m$. *We call* $\mathbb{R}^n$ *the domain and we call* $\mathbb{R}^m$ *the codomain.*

Definition 15 *Given an* $m \times n$ *matrix* $\mathbf{A}$*, a **Matrix Transformation** is a linear transformation*

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

$$T(\vec{x}) = \mathbf{A}\vec{x}$$

We call $\mathbf{A}$ *the **standard matrix** of* T.

Example 22 *For example, given the matrix*

$$\mathbf{A} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$$

The matrix transformation of **A** *is*

$$T : \mathbb{R}^2 \to \mathbb{R}^3$$
$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

Definition 16 *Given a transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, we say a transformation is a **linear transformation** iff it satisfies 2 axioms:*

1. *Vector addition:*
$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \qquad \forall \vec{u}, \vec{v} \in \mathbb{R}^n$$

2. *Scaling:*
$$T(c\vec{v}) = cT(\vec{v}) \qquad \forall c \in \mathbb{R} \quad \forall \vec{v} \in \mathbb{R}^n$$

Theorem 13 *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then*

$$T(\vec{0}_n) = \vec{0}_m$$

because of the scaling rule.

By the contrapositive, if $T(\vec{0}) \neq \vec{0}$, then T is not a linear transformation.

Theorem 14 *All matrix transformations are also linear transformations. Given a linear transformation and a basis, there is a unique matrix transformation corresponding to the linear transformation.*

In other words, a matrix transformation is basically the same thing as a linear transformation

Theorem 15 *The derivative $\frac{d}{dx}$ from the set of differentiable functions to functions. To see why, it satisfies the axioms*

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$
$$\frac{d}{dx}(cf(x)) = c\frac{d}{dx}(f(x))$$

7 April 12

Theorem 16 *If $T : \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation, then there exists a matrix A called the **standard matrix** defined by*

$$A = [T(e_1), T(e_2), T(e_3), T(e_4), \ldots, T(e_n)]$$

*Where $e_1, e_2, \ldots, e_n$ are the **basis vectors**.*

The way to see this is that any arbitrary vector $\vec{v} \in \mathbb{R}^n$ can be written as a linear combination of the basis vectors as $\vec{v} = c_1\vec{e}_1 + \cdots c_n\vec{e}_n$.

Example 23 *The matrix of a vertical reflection in* $\mathbb{R}^2$ *is*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The matrix of a horizontal contraction (or expansion) by a factor of k *is*

$$A = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix of the projection onto the y *axis is*

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The matrix of the horizonal shear by a factor of k *is*

$$A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

The matrix of a θ *counter-clockwise about the origin is*

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Definition 17 *Let* $f : X \to Y$. f *is called* ***surjective*** *iff every element of* Y *is hit.*

f *is called* ***injective*** *iff* $f(x_1) = f(x_2) \implies x_1 = x_2$.

Surjective functions are also called "onto functions". Injective functions are also called "one-to-one functions"

Definition 18 *We say* $f : X \to Y$ *is* ***bijective*** *iff it is both surjective and injective.*

Theorem 17 *Let* $T : \mathbb{R}^n \to \mathbb{R}^p$ *be a linear transformation, and let* T *have standard matrix* A.

T *is injective/one-to-one iff the equation* $\boldsymbol{A}\vec{x} = \vec{0}$ *only has the solution* $\vec{x} = \vec{0}$. *This means that* A *has a pivot in every column.*

In other words, $T(\vec{x}) = \vec{0} \implies \vec{x} = 0$.

The following are equivalent

1. T *is an injective/one-to-one function*
2. A *has a pivot in every column*
3. A *has no free variables*
4. $T(\vec{x}) = \vec{0} \implies \vec{x} = \vec{0}$
5. *The columns of* $\mathbf{A}$ *are linearly independent.*

Theorem 18 *Let* $T : \mathbb{R}^n \to \mathbb{R}^p$ *be a linear transformation with matrix* $\mathbf{A}$. *The follow are equivalent*

1. T *is a surjection / onto function*
2. A *has a pivot in every row*

3. *The system* $\mathbf{A}\vec{x} = \vec{b}$ *is consistent for every* $\vec{b} \in \mathbb{R}^p$

4. *span*$(A) = \mathbb{R}^p$.

Example 24 *Let* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *defined by*

$$T(\vec{x}) = \begin{bmatrix} 1 & -1 & 3 \\ -2 & 2 & -6 \end{bmatrix} \vec{x}$$

Find Range(T)*?*
Solution:
Range$(T) = span(\begin{bmatrix} 1 \\ -2 \end{bmatrix})$
Find the set of all pre-images of $\begin{bmatrix} 2 \\ -4 \end{bmatrix}$*?*
Solution:

$$S = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + l \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} | \forall k, l \in \mathbb{R}^3 \right\} = span \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right) + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Is T *onto/surjective?*
Solution:
no.
Is T *one-to-one/injective?*
Solution:
no.

8 April 13

Definition 19 *Given a matrix* A*, we call* A^{-1} *the* ***inverse*** *of* A *iff* $AA^{-1} = I$ *If* A *has an inverse, then* A *is called* ***invertable****.*

If A is invertiable, then A^{-1} is its unique inverse.
We only talk about inverses in the context of square matricies.
For a 1×1 matrix, the inverse is

$$a^{-1} = \frac{1}{a}$$

For a 2×2 matrix, the inverse is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Theorem 19 *If* A *is invertable, then the system*

$$\mathbf{A}\vec{x} = \vec{b}$$

has the unique solution

$$\vec{x} = \mathbf{A}^{-1}\vec{b}$$

Example 25 *Solve this system*

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Theorem 20

$$(A^{-1})^{-1} = A$$
$$(AB)^{-1} = B^{-1}A^{-1}$$
$$(A^T)^{-1} = (A^{-1})^T$$

Theorem 21 *A matrix A is invertible if and only if it is row-equivalent (by elementary row operatins) to the identity matrix I.*

Theorem 22 ***Finding Inverse**: To find the inverse on the $n \times n$ matrix A, perform row operations $[A \quad I_n]$ to get to $[I_n \quad A]$ In other words*

$$[A \quad I_n] \sim [I_n \quad A]$$

Example 26

$$\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{3}{2} & 1 & 0 \end{pmatrix}$$

Theorem 23 ***Invertible Matrix Theorem**: The following statements are all logically equivalent*

1. *A is an invertiable $n \times n$ matrix*
2. *$A \sim I_n$*
3. *A has n pivot positions*
4. *The only solution $\mathbf{A}\vec{x} = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$*
5. *The columns of A form a linearly inpendent set/basis of $\mathbb{R}^n$*
6. *The linear transformation $T(\vec{x}) = \mathbf{A}\vec{x}$ is one-to-one/injective*
7. *The linear transformation $T(\vec{x}) = \mathbf{A}\vec{x}$ is onto/surjective*
8. *The equation $\mathbf{A}\vec{x} = \vec{b}$ has exactly one solution $\vec{x} = \mathbf{A}^{-1}\vec{b}$ for all $\vec{b} \in \mathbb{R}^n$*

8.1 Chapter 3 – Determinants

Definition 20 *Given a matrix A, the **sub-matrix** A_{ij} is the matrix obtained by deleting row i and column j*

Theorem 24 ***Laplace Expansion**: The determinant can be calculated by*

$$|A| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} |A_{ij}|$$

*Some people call $(-1)^{i+j}|A_{ij}| = c_{ij}$ the **co-factor**.*

Note that Laplace expansion is very slow, it has time complexity $O(n!)$. If you want to calculate the determinant quickly, elementary row operations can calculate the determinant in $O(n^3)$.

Theorem 25 *If A is a triangular matrix, then $|A|$ is just the product of the diagonal entries.*

$$|A| = \prod_{i=1}^{n} a_{ii}$$

It is a common mistake that $A \sim B \implies |A| = |B|$, but this is false.

Theorem 26 *Each of the elementary row operations has an effect on the determinant*
***scaling**: multiplying a row by k multiplies the determinant by k*
***interchange**: swapping two rows multiplies the determinant by -1*
***replacement**: adding a row to another row doesn't change the determinant*

Theorem 27 ***Properties of determinant***

$$|AB| = |A||B|$$

$$|A^{-1}| = |A|^{-1}$$

$$|A^T| = |A|$$