Multi Notes

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1 Jan 11. 2023

1.1 Syllabus

Learning opportunity is a waste of time. Teach your parents stuff, YHPL is not happy (it is meaningless). Each knowledge check is worth more (20%).

1.2 Overview of 1D

We are about to complete calculus. You want to take real analysis or complex analysis after 1D.

Class	Topic
1A	single variable derivatives
1b	single variable integrals
1C	multi-variable derivatives
1D	multi-variable integra;s

1.2.1 Topics Covered in 1D

Topics covered in 1D:

1. work
2. line integral = work done by force
3. vector fields
4. flux integral = rate of flow thru a thin net
5. Chapter 20: Green's Theorem, curl, divergenence
6. That's the end

If we meet YHPL, she will bake us a cake.

1.3 Review from 1C

1.3.1 review of quartic surfaces

Review quadric surfaces Review table that YHPL posted on quartic surfaces.

eqn	name
$z = x^2 - y^2$	Hyperbolic paraboloid
$z = y^2$	Parabolic cylinder
$z = \sqrt{4 - x^2 - y^2}$	half-hemisphere of a sphere
$z = \sqrt{x^2 + y^2}$	elleptic cone
$z = x^2 + y^2$	elliptic paraboloid
$z = 1 - \sqrt{x^2 + y^2}$	elliptic cone
$z = 6 - 3x - 2y$	plane
$z = 6 - 2y$	plane thru (37.0.6) and normal vector is $\vec{n} = \langle 0, 2, 1 \rangle$
$z = 4 - x^2 - y^2$	Elliptic Paraboloid

1.3.2 review of integrals

Recall that the definition of integral is

$$\int_I f(x)\,dx = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where $f(x_i^*)$ is the height of the i^{th} rectangle and Δx_i is the width of the i^{th} rectangle. We are summing up areas on many small rectangles. Note that area can be negative if the function goes below the x-axis (called signed area). Note that all the Δxs do not all have to be the same width.

Theorem 1 *Average* $avg = \frac{1}{|I|} \int_I f(x)\,dx$

x y	1	4	7	10
4	5	6	9	37
2	11	14	7	12
0	16	21	0	15

1.4 Multi-Variable integration

Given that $z = f(x, y)$, we find the area by splitting the region into rectangles under the curve. We split the x-axis and we split the y-axis. We integrate over a region R. Adding up the volume of all the little rectangular prisms approximates the volume of our original curve.

Definition 1 *The definition of a **multi-variable** integral is*

$$\iint_R f(x,y)\,dx\,dy = \int_R f(x,y)dA = \lim_{n\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{m} f(x_i^*, y_j^*)\Delta x_i \Delta y_j$$

In a double integral, you have to integral twice.

1.4.1 numerical approximation in multiple variables

Let's use this table as an example of a function

Let $R = [1, 10] \times [[0, 4]]$.

$$\int_R f dA \approx (14 + 6 + 9 + 37 + 12 + 7) \cdot 3 \cdot 2$$

Example: Let $z = e^{-(x^2+y^2)}$. If you sample the function using the bottom left approximation, then this is an overestimate because the rest of the rectangle with have higher z-values.

Given a contour map, we can approximate an integral. Divide up the graph and pick a point from each division to represent the whole.

1.5 Summary of surfaces

```
var('x y z')
```

2 January 12, 2023

no class due to doctor's appointment

3 January 15, 2023

No class due to MLK day

4 January 18, 2023

If we are integrating over a non-rectangular region, the inner bounds must depend on the outer bounds.

4.1 Integrating non-rectangular regions

Example:

A strange region may be given by a triangle an semi-cirle

$$\int_{x=-1}^{x=0} \int_{y=1}^{y=x+2} f dy dx + \int_{x=0}^{x=\sqrt{3}} \int_{y=2}^{y=\sqrt{4-x^2}} f dy dx$$

But we could also do as y

$$\int_{y=1}^{y=2} \int_{x=-1}^{x=y-2} f dx dy + \int_{y=1}^{y=2} \int_{x=0}^{x=\sqrt{4-y^2}} f dx dy$$

Note that it could be (no need to split)

$$\int_{x=1}^{x=2} \int_{y=1}^{y=\sqrt{4-x^2}} f dx dy$$

Excercise: Given the integral

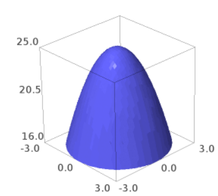
$$\int_{-1}^{0} \int_{1}^{4-x} f dy dx$$

reverse the order of the integrals

$$\int_{y=1}^{y=4} \int_{x=-1}^{x=0} f dx dy + \int_{y=4}^{y=5} \int_{x=-1}^{x=4-y} f dx dy$$

4.2 Application of Double Integrals

We use double integrals to find volume.



Example 1 *Exercise: given this weird parabola, find volume.*

$$s_1 : z = 25 - x^2 - y^2, s_2 : z = 16$$

To find volume

$$\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} 9 - x^2 - y^2 dy dx$$

Pro =-Tip: always project orthoganlly onto the xy plane

Another example

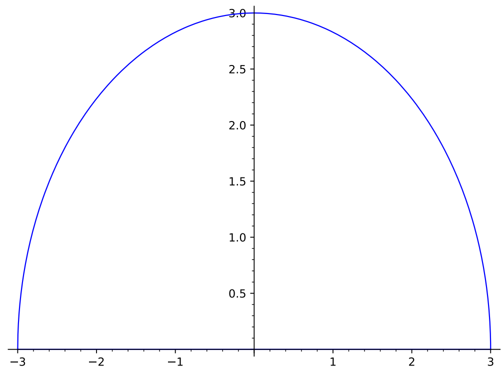
Example 2 *Given the mass-density of*

$$s(x,y) = \sqrt{x^2+y^2}$$

the mass of a triangle is given by

$$\int_{x=0}^{x=2} \int_{y=0}^{y=4-2x} s(x,y)dydx$$

Example 3 *Example: Given a city with the shape of a semi-circle. Find the average distance from the city to the ocean.*



YHPL strongly recommends reading ahead.

The average distance from the city to the ocean is given by

$$\frac{\int_{x=-3}^{x=3} \int_{y=0}^{y=\sqrt{9-x^2}} y\, dydx}{\int_{x=-3}^{x=3} \int_{y=0}^{y=\sqrt{9-x^2}} 1\, dydx}$$

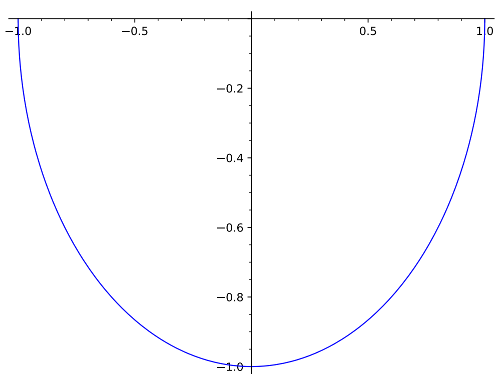
To get the average distance to the ocean, we get the total distance and then divide by the total area.

Example 4 *Without computation, find the sign of*

$$\int_R (y^3 - y)dA$$

and

$$\int_R e^y x dA$$

where the region R is  *?*

Solution: Remember the definition. The first integral will be negative because $y < 0 \implies y^3 > y \implies y^3 - y > 0$*. The second integral will be 0, because the negative* x *perfectly cancels out the positive* x*.*

Example 5 *What is the sign of*

$$\int_R cos(x)dA$$

where R is the same region as above? Solution: It is positive because $\cos x > 0$ *for* $-1 < x < 1$. *Cosine only becomes 0 at* $\pi/2 \approx 1.5$.

4.3 Triple Integral

Definition 2 *Given a funciton* $w = f(x, y, z)$ *and a region* $W \subset \mathbb{R}^3$

$$\iiint_w f(x, y, z)dV$$

There are $3! = 6$ different ways to integrate a triple integral. Where the inner integrals can depend on the outer integrals.

Example 6 *Redo the parabola volume question using a triple integral.*

$$\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \int_{z=16}^{z=25-x^2-y^2} 1\, dzdydx$$

5 January 19, 2023

5.1 Review

Review of what we learned last week

1. find volume by integrating over 1
2. determine sign of integral by using sign of integrand over region
3. You can swap the order of integration (either $dxdy$ or $dydx$), and sometimes one order of integration will be easier than the other
4. When approximating a double integral, you pick a point from each region to approximate the entire region. From a contour map, you can choose the sample point and then multiply that by the area of the region.
5. you can find average function value by taking double integral over region and then divide by the area of that region.

Example 7 *Here's an application problem: Estimate the average snowfall in Colorado based on this map. Sample based on the midpoint of each rectangular region.*

$$\frac{1}{16}(16 + 16 + 19 + 13 + 8 + 28 + 18 + 13 + 2 + 24 + 17 + 11 + 0 + 16 + 8 + 7) = \frac{27}{2}$$

5.2 16.4 – Polar Coordinates

In polar coordinates, (r, θ), a point is represented by its distance from the origin, r, and the angle it makes with the positive x-axis, θ.

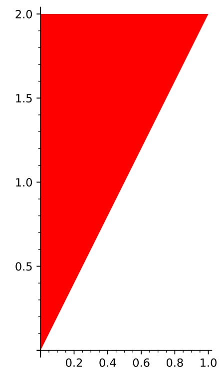
Example 8 *Let's revisit the problem of the average distance from the city to the ocean that we did last week. The same integral becomes*

$$\int_{\theta=0}^{\theta=\pi} \int_{r=0}^{r=3} r^2 \sin(\theta) dr d\theta$$

$$= \int_0^{\pi} \sin\theta d\theta \cdot \int_0^3 r^2 dr$$

$$= (\cos\pi - \cos 0)(\frac{27}{3})$$

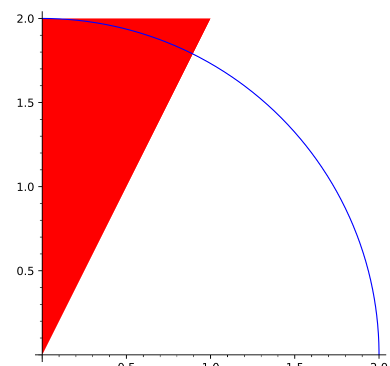
$$= -18$$



Example 9 *Another example: We plot over the region*

$$\int_{x=0}^{x=1} \int_{y=2x}^{y=2} x dy dx$$

$$= \int_{\theta=\arctan 2}^{\theta=\pi/2} \int_{r=0}^{r=2/\sin\theta} r^2 \cos\theta dr d\theta$$



If you want r to be the outer, then it's a bit harder We split it into 2 regions

$$\int_{r=0}^{r=2} \int_{\theta=\arctan 2}^{\theta=\pi/2} r^2 \cos\theta d\theta dr + \int_{r=2}^{r=\sqrt{5}} \int_{\theta=\arctan 2}^{\theta=\arcsin 2/r} r^2 \cos\theta d\theta dr$$

5.2.1 Polar–Rectangular Conversions

$$x = r\cos\theta$$

$$y = r\sin\theta$$

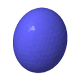
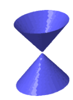
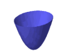
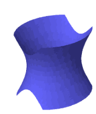
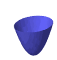
$$x^2 + y^2 = r^2$$

$$\theta = \arctan\frac{y}{x}$$

$$\Delta A \approx r\Delta\theta\Delta r$$

$$dA = r\,d\theta dr = r\,drd\theta$$

$$\int_R f dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)drd\theta$$

Name	Equation	Graph
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2 2} = 1$	
Cone	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$	
Elliptic Paraboloid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$	
Hyperboloid of One Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	
Hyperboloid of Two Sheets	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$	
Hyperbolic Paraboloid	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2}=1$	