# Multi Notes

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## Contents

# 1 Jan 11. 2023

## 1.1 Syllabus

Learning opportunity is a waste of time. Teach your parents stuff, YHPL is not happy (it is meaningless). Each knowledge check is worth more (20%).

## 1.2 Overview of 1D

We are about to complete calculus. You want to take real analysis or complex analysis after 1D.

| Class | Topic |
|---|---|
| 1A | single variable derivatives |
| 1b | single variable integrals |
| 1C | multi-variable derivatives |
| 1D | multi-variable integra;s |

### 1.2.1 Topics Covered in 1D

Topics covered in 1D:

1. work
2. line integral = work done by force
3. vector fields
4. flux integral = rate of flow thru a thin net
5. Chapter 20: Green's Theorem, curl, divergenence
6. That's the end

If we meet YHPL, she will bake us a cake.

## 1.3 Review from 1C

### 1.3.1 review of quartic surfaces

Review quadric surfaces Review table that YHPL posted on quartic surfaces.

| eqn | name |
|---|---|
| $z = x^2 - y^2$ | Hyperbolic paraboloid |
| $z = y^2$ | Parabolic cylinder |
| $z = \sqrt{4 - x^2 - y^2}$ | half-hemisphere of a sphere |
| $z = \sqrt{x^2 + y^2}$ | elleptic cone |
| $z = x^2 + y^2$ | elliptic paraboloid |
| $z = 1 - \sqrt{x^2 + y^2}$ | elliptic cone |
| $z = 6 - 3x - 2y$ | plane |
| $z = 6 - 2y$ | plane thru (37.0.6) and normal vector is $\vec{n} = \langle 0, 2, 1 \rangle$ |
| $z = 4 - x^2 - y^2$ | Elliptic Paraboloid |

| x y | 1 | 4 | 7 | 10 |
|---|---|---|---|---|
| 4 | 5 | 6 | 9 | 37 |
| 2 | 11 | 14 | 7 | 12 |
| 0 | 16 | 21 | 0 | 15 |

### 1.3.2 review of integrals

Recall that the definition of integral is

$$\int_I f(x)\,dx = \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) partialtax_i$$

where $f(x_i^*)$ is the height of the $i^{th}$ rectangle and $partialtax_i$ is the width of the $i^{th}$ rectangle. We are summing up areas on many small rectangles. Note that area can be negative if the function goes below the x-axis (called signed area). Note that all the $partialtax$s do not all have to be the same width.

**Theorem 1** *Average* $avg = \frac{1}{|I|} \int_I f(x)\,dx$

## 1.4 Multi-Variable integration

Given that $z = f(x, y)$, we find the area by splitting the region into rectangles under the curve. We split the x-axis and we split the y-axis. We integrate over a region $R$. Adding up the volume of all the little rectangular prisms approximates the volume of our original curve.

**Definition 1** *The definition of a* ***multi-variable*** *integral is*

$$\iint_R f(x, y)\,dx\,dy = \int_R f(x, y) dA = \lim_{n\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i^*, y_j^*) partialtax_i partialtay_j$$

In a double integral, you have to integral twice.

### 1.4.1 numerical approximation in multiple variables

Let's use this table as an example of a function

Let $R = [1, 10] \times [[0, 4]]$.

$$\int_R f dA \approx (14 + 6 + 9 + 37 + 12 + 7) \cdot 3 \cdot 2$$

Example: Let $z = e^{-(x^2+y^2)}$. If you sample the function using the bottom left approximation, then this is an overestimate because the rest of the rectangle with have higher $z$-values.

Given a contour map, we can approximate an integral. Divide up the graph and pick a point from each division to represent the whole.

## 1.5 Summary of surfaces

```
var('x y z')
```

# 2 January 12, 2023

no class due to doctor's appointment

# 3 January 15, 2023

No class due to MLK day

# 4 January 18, 2023

If we are integrating over a non-rectangular region, the inner bounds must depend on the outer bounds.

## 4.1 Integrating non-rectangular regions

Example:

A strange region may be given by a triangle an semi-cirle

$$\int_{x=-1}^{x=0}\int_{y=1}^{y=x+2} f dydx + \int_{x=0}^{x=\sqrt{3}}\int_{y=2}^{y=\sqrt{4-x^2}} f dydx$$

But we could also do as y

$$\int_{y=1}^{y=2}\int_{x=-1}^{x=y-2} f dxdy + \int_{y=1}^{y=2}\int_{x=0}^{x=\sqrt{4-y^2}} f dxdy$$

Note that it could be (no need to split)

$$\int_{x=1}^{x=2}\int_{y=1}^{y=\sqrt{4-x^2}} f dxdy$$

Excercise: Given the integral

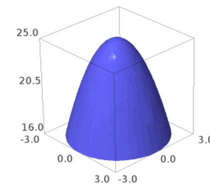
$$\int_{-1}^{0}\int_{1}^{4-x} f dydx$$

reverse the order of the integrals

$$\int_{y=1}^{y=4}\int_{x=-1}^{x=0} f dxdy + \int_{y=4}^{y=5}\int_{x=-1}^{x=4-y} f dxdy$$

## 4.2 Application of Double Integrals

We use double integrals to find volume.




**Example 1** *Exercise: given this weird parabola, find volume.*

$$s_1 : z = 25 - x^2 - y^2, s_2 : z = 16$$

*To find volume*

$$\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} 9 - x^2 - y^2 dydx$$

*Pro =-Tip: always project orthoganlly onto the xy plane*

Another example

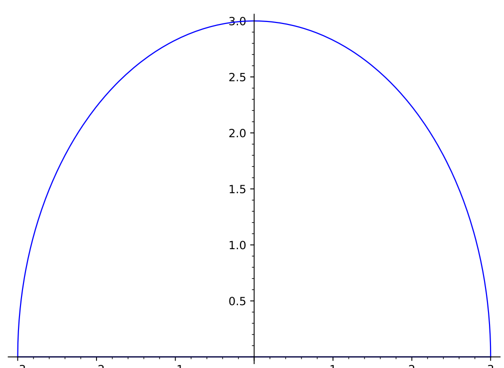
**Example 2** *Given the mass-density of*

$$s(x, y) = \sqrt{x^2 + y^2}$$

*the mass of a triangle is given by*

$$\int_{x=0}^{x=2} \int_{y=0}^{y=4-2x} s(x, y) dydx$$

**Example 3** *Example: Given a city with the shape of a semi-circle. Find the average distance from the city to the ocean.*




*YHPL strongly recommends reading ahead.*
*The average distance from the city to the ocean is given by*

$$\frac{\int_{x=-3}^{x=3} \int_{y=0}^{y=\sqrt{9-x^2}} y \, dydx}{\int_{x=-3}^{x=3} \int_{y=0}^{y=\sqrt{9-x^2}} 1 \, dydx}$$

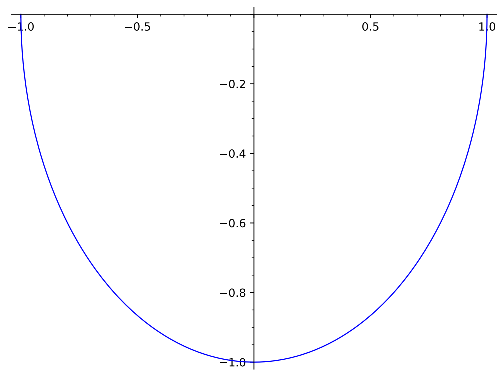
*To get the average distance to the ocean, we get the total distance and then divide by the total area.*

**Example 4** *Without computation, find the sign of*

$$\int_R (y^3 - y) dA$$

*and*

$$\int_R e^y x dA$$




*where the region $R$ is ?*

*Solution: Remember the definition. The first integral will be negative because $y < 0 \implies y^3 > y \implies y^3 - y > 0$. The second integral will be 0, because the negative $x$ perfectly cancels out the positive $x$.*

**Example 5** *What is the sign of*

$$\int_R cos(x) dA$$

*where $R$ is the same region as above? Solution: It is positive because $\cos x > 0$ for $-1 < x < 1$. Cosine only becomes 0 at $\pi/2 \approx 1.5$.*

### 4.3 Triple Integral

**Definition 2** *Given a funciton $w = f(x, y, z)$ and a region $W \subset \mathbb{R}^3$*

$$\iiint_w f(x, y, z) dV$$

There are $3! = 6$ different ways to integrate a triple integral. Where the inner integrals can depend on the outer integrals.

**Example 6** *Redo the parabola volume question using a triple integral.*

$$\int_{x=-3}^{x=3} \int_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} \int_{z=16}^{z=25-x^2-y^2} 1 \, dzdydx$$

## 5 January 19, 2023

### 5.1 Review

Review of what we learned last week

1. find volume by integrating over 1

2. determine sign of integral by using sign of integrand over region
3. You can swap the order of integration (either $dxdy$ or $dydx$), and sometimes one order of integration will be easier than the other
4. When approximating a double integral, you pick a point from each region to approximate the entire region. From a contour map, you can choose the sample point and then multiply that by the area of the region.
5. you can find average function value by taking double integral over region and then divide by the area of that region.

**Example 7** *Here's an application problem: Estimate the average snowfall in Colorado based on this map. Sample based on the midpoint of each rectangular region.*

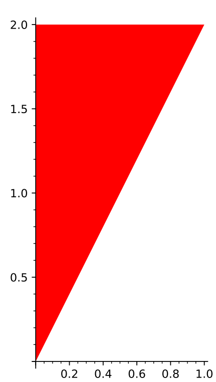
$$\frac{1}{16}(16+16+19+13+8+28+18+13+2+24+17+11+0+16+8+7)=\frac{27}{2}$$

## 5.2 16.4 – Polar Coordinates

In polar coordinates, $(r,\theta)$, a point is represented by its distance from the origin, $r$, and the angle it makes with the positive $x$-axis, $\theta$.

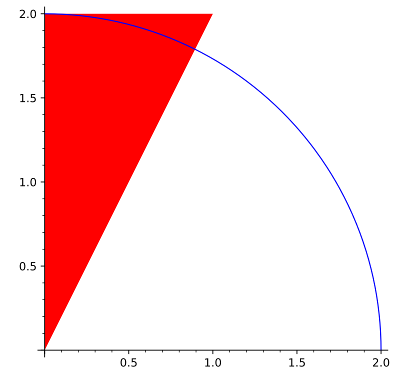
**Example 8** *Let's revisit the problem of the average distance from the city to the ocean that we did last week. The same integral becomes*

$$\begin{aligned}&\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=3} r^2\sin(\theta)drd\theta\\ &=\int_0^{\pi}\sin\theta d\theta\cdot\int_0^3 r^2dr\\ &=(\cos\pi-\cos 0)(\frac{27}{3})\\ &=-18\end{aligned}$$




**Example 9** *Another example: We plot over the region*

$$\begin{aligned}&\int_{x=0}^{x=1}\int_{y=2x}^{y=2} xdydx\\ &=\int_{\theta=\arctan 2}^{\theta=\pi/2}\int_{r=0}^{r=2/\sin\theta} r^2\cos\theta drd\theta\end{aligned}$$



*If you want r to be the outer, then it's a bit harder We split it into 2 regions*

$$\int_{r=0}^{r=2} \int_{\theta=\arctan 2}^{\theta=\pi/2} r^2 \cos\theta d\theta dr + \int_{r=2}^{r=\sqrt{5}} \int_{\theta=\arctan 2}^{\theta=\arcsin 2/r} r^2 \cos\theta d\theta dr$$

### 5.2.1 Polar–Rectangular Conversions

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$x^2 + y^2 = r^2$$

$$\theta = \arctan\frac{y}{x}$$

$$partialtaA \approx rpartialta\theta partialtar$$

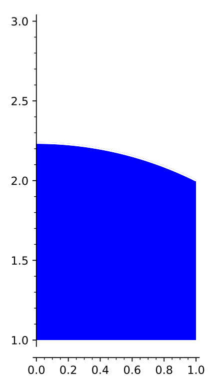
$$dA = r\,d\theta dr = r\,drd\theta$$

$$\int_R f dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) drd\theta$$

## 6 Chapter 16 Knowledge Check Practice

1. Consider the double integral $\int_R partialta(x,y)dA$ where $partialta(x,y)$ is the distance from $(x,y)$ to $(0,0)$ and $R$ is the region bounded by the y-axis, the line $y = 1$, the line $x = 1$ and the semi-cirle $y = \sqrt{5 - x^2}$

   (a) Draw $R$

   Solution:

   (b) find $partialta(x,y)$.
   Solution: $partialta(x,y) = \sqrt{x^2 + y^2}$

(c) What is the practical meaning of the double integral
Solution: the total mass of the region $R$.

(d) Write the double integral of the form $dxdy$. Do not evaluate.
Solution:

$$\int_{y=1}^{y=2}\int_{x=0}^{x=1}\sqrt{x^2+y^2}\,dxdy+\int_{y=2}^{y=\sqrt{5}}\int_{x=0}^{x=\sqrt{5-y^2}}\sqrt{x^2+y^2}\,dxdy$$

(e) Write the double integral of the form $dydx$. Do not evaluate.
Solution:

$$\int_{x=0}^{x=1}\int_{y=1}^{y=\sqrt{5-x^2}}\sqrt{x^2+y^2}\,dydx$$

(f) Write the double integral of the form $drd\theta$. Do not evaluate.
Solution:

$$\int_{\theta=\pi/4}^{\theta=\arctan(2)}\int_{r=\sec\theta}^{r=\csc\theta}r^2drd\theta+\int_{\theta=\arctan 2}^{\theta=\pi/2}\int_{r=\sec\theta}^{r=\sqrt{5}}r^2drd\theta$$

(g) Write the double integral; of the form $d\theta dr$. Do not evaluate.
Solution:

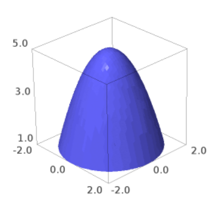
$$\int_{r=1}^{r=\sqrt{2}}\int_{\theta=\arccos(1/r)}^{\theta=\pi/2}r^2d\theta dr+\int_{r=\sqrt{2}}^{r=\sqrt{5}}\int_{\theta=\arcsin(1/r)}^{\theta=\pi/2}r^2d\theta dr$$

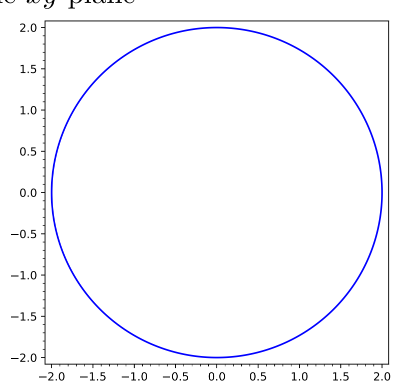
(h) Set up an integral to express the average mass density of $r$
Solution:

$$\frac{\int_{x=0}^{x=1}\int_{y=1}^{y=\sqrt{5-x^2}}\sqrt{x^2+y^2}\,dydx}{\int_{x=0}^{x=1}\int_{y=1}^{y=\sqrt{5-x^2}}1\,dydx}$$

B. A solid region $W$ is bounded above by $z=5-x^2-y^2$ and below by $z=1$.

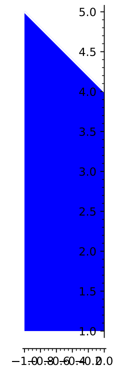
(a) Sketch $W$ in $R^3$, and the projection of $W$ onto the $xy$-plane

Solution: 

and 


C. Reverse the order of the integral

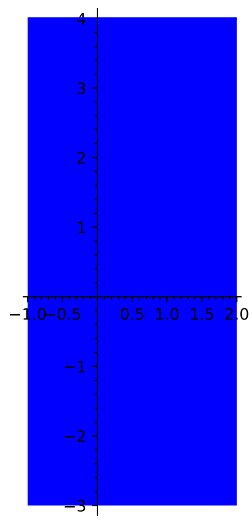
$$\int_{x=-1}^{x=0}\int_{y=1}^{y=4-x}f(x,y)dydx$$




Solution: The first step is always to draw a picture of the region.

$$\int_{y=1}^{y=4} \int_{x=-1}^{x=0} f(x,y)dxdy + \int_{y=4}^{y=5} \int_{x=-1}^{x=4-y} f(x,y)dxdy$$

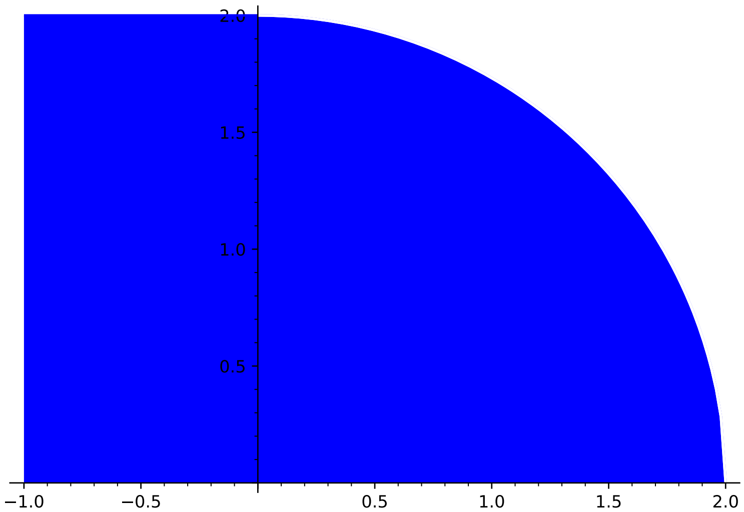
# 7 January 23, 2023

**Example 10** *Consider the region* 

*The limits of the integral*

$$\int_{x=-1}^{x=2} \int_{y=-3}^{y=4} f dydx$$

*are constant because we are using CARTESIAN coordinates. But, by contrast, consider a semicircle. This will have constant limits of integration in POLAR coordaintes*




**Example 11** *Consider the region Do 4 different orders of integration First, dydx*

$$\int_{x=-1}^{x=0}\int_{y=0}^{y=2} x\,dydx + \int_{x=0}^{x=2}\int_{y=0}^{y=\sqrt{4-x^2}} xdydx$$

*Next dxdy*

$$\int_{y=0}^{y=2}\int_{x=-1}^{x=\sqrt{4-y^2}} x\,dxdy$$

*Next drdθ*

$$\int_{\theta=0}^{\theta=\pi/2}\int_{r=0}^{r=2} r^2\cos\theta dr d\theta + \int_{\theta=\pi/2}^{\theta=\pi/2+\arctan(1/2)}\int_{r=0}^{r=2\csc\theta} r^2\cos\theta dr d\theta + \int_{\theta=\pi/2+\arctan(1/2)}^{\theta=\pi}\int_{r=0}^{r=-\sec\theta} r^2\cos\theta dr d\theta$$

*Finally dθdr*

$$\int_{r=0}^{r=1}\int_{\theta=0}^{\theta=\pi} r^2\cos\theta d\theta dr + \int_{r=1}^{r=2}\int_{\theta=0}^{\theta=\arccos(-1/r)} r^2\cos\theta d\theta dr + \int_{r=2}^{r=\sqrt{5}}\int_{\theta=\arcsin(2/r)}^{\theta=\arccos(-1/r)} r^2\cos\theta d\theta dr.$$

By the way, our first knowledge check on chapter 16 is on 01/26

**Example 12** *Imagine a circular dinner plate with radius 10cm where the mass density of the dinner plate is given by*

$$partialta(x,y) = \sqrt{x^2+y^2}$$

*use polar coordinates to find the total mass of the plate.*
*Solution:*

$$\int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=10} r^2 dr d\theta$$

$$= \int_{\theta=0}^{\theta=2\pi} 1000/3 d\theta$$

$$= 2\pi * 1000/3 = \frac{2000\pi}{3}$$

**Example 13** *What is*

$$\int_{-\infty}^{\infty} e^{-x^2} dx?$$

*Solution: Let*

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

*Then*

$$\begin{aligned}
I^2 &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\
&= \int_{x=-\infty}^{x=\infty} \int_{y=-\infty}^{y=\infty} e^{-(x^2+y^2)} dydx \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=\infty} e^{-r^2} rdrd\theta \\
&= \pi
\end{aligned}$$

*So* $I = \sqrt{\pi}$

## 7.1 16.5 - Cylindrical and Spherical Coordinates

### 7.1.1 Cylindrical Coordinates

Cylindrical coordinates are like polar coordinates in 3D. It uses 1 angle and 2 lengths. They are described by $(r, \theta, z)$ The conversion is Rectangular to cylindrical:

$$(x, y, z) \to (\sqrt{x^2 + y^2}, \arctan(y/x), z)$$

cylindrical to rectangular:

$$(r, \theta, z) \to (r\cos\theta, r\sin\theta, z)$$

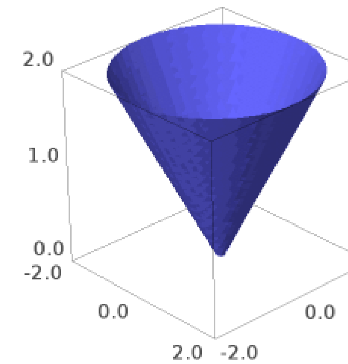
The reason they are called cylindrical coordiantes is because if you are trying to integrate a cylinder, the bounds of integration are constant

### 7.1.2 Spherical Coordaintes

Spherical Coordinates use 2 angles and 1 length. The coordinates are of the form $(\rho, \theta, \phi)$ The conversions are

$$\begin{aligned}
\rho^2 &= x^2 + y^2 + z^2 \\
x &= \rho\sin\phi\cos\theta \\
y &= \rho\sin\phi\sin\theta \\
z &= \rho\cos\phi \\
r &= \rho\sin\phi \\
\phi &= \arccos(\frac{z}{\sqrt{x^2 + y^2 + z^2}}) \\
dV &= \rho^2 \sin\phi d\rho d\phi d\theta
\end{aligned}$$

# 8 January 25, 2023




**Example 14** *Find the volume of this cone using all integration methods.*

*Solution: Note that the equation for this cone is* $x^2 + y^2 = z^2$

$$\int_{x=-2}^{x=2}, \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} 2 - \sqrt{x^2+y^2}dydx$$

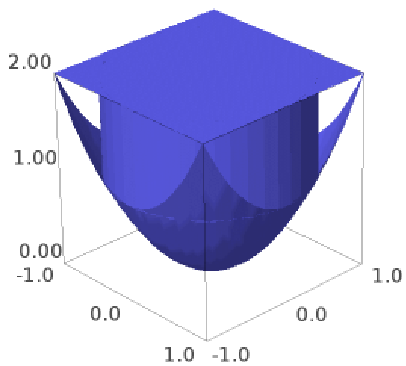
$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r(2-r)drd\theta$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=r}^{z=2} rdzdrd\theta$$

$$\int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=\sqrt{x^2+y^2}}^{z=2} 1dzdydx$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/4} \int_{\rho=0}^{\rho=2\csc\phi} \rho^2 \sin\phi d\rho d\phi d\theta$$

**Example 15** *Consider the 3d solid which is a dome on the bottom and a cylinder on the top.*

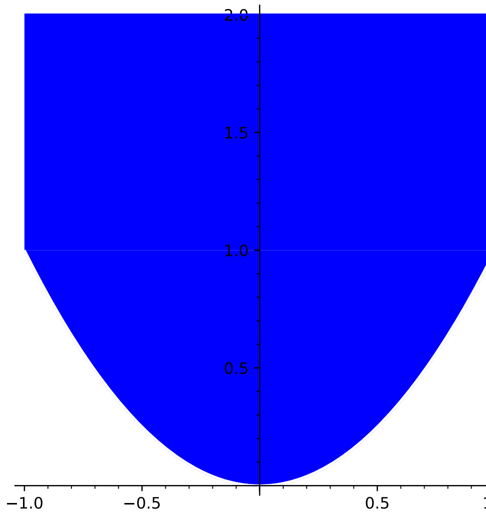



*Find the mass given that the mass density for a point is given by the distance from $z = 2$.*

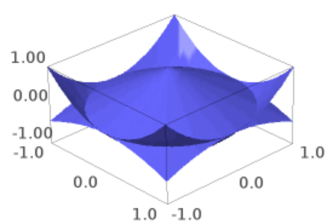
$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=r}^{z=2} (2-z) r dz dr d\theta$$

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\arctan 1/2} \int_{\rho=0}^{\rho=2\sec\phi} \rho^2 \sin\phi d\rho d\phi d\theta$$
$$+ \int_{\theta=0}^{\theta=2\pi} \int_{\phi=2\csc\theta}^{\phi=\pi/2} \int_{\rho=0}^{\rho=\csc\phi} (2-\rho\sin\phi)\rho^2 \sin\phi d\rho d\phi d\theta$$
$$+ \int_{\theta=0}^{\theta=2\pi} \int_{\phi=\arctan 1}^{\phi=\pi/2} \int_{\rho=0}^{\rho=\cos\phi/\sin^2\phi} (2-\rho\sin\phi)\rho^2 \sin\phi d\rho d\phi d\theta$$

$$\int_{x=-1}^{x=1} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \int_{z=x^2+y^2}^{z=2} 2 - z dz dy dx$$




*The spherical integral seems complicated, so you should break up the shape.*




**Example 16** *Write an integral for $f$ for the region which is bounded by $z = 1 - \sqrt{x^2+y^2}$ and $z = -1 + x^2 + y^2$*

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=1} \int_{z=r^2-1}^{1-r} f r \, dz dr d\theta$$

$$\int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} \int_{z=-1+x^2+y^2}^{1-\sqrt{x^2+y^2}} f \, dz dy dx$$

*Remember that* $z = 1 - \sqrt{x^2+y^2} \implies \rho = \frac{1}{\cos\phi + \sin\phi}$

$$\int_{\theta=0}^{\theta=2\pi} \int_{\phi=3\pi/4}^{\phi=\pi} \int_{\rho=0}^{\rho=}$$

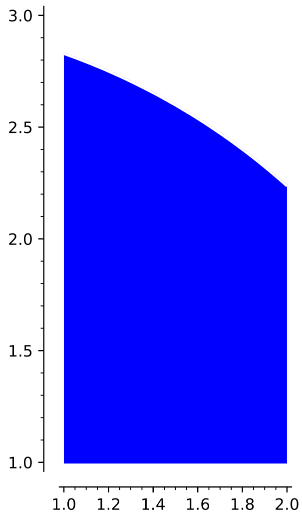
# 9 Knowledge Check Solutions

Consider the double integral

$$\int_1^2 \int_1^{\sqrt{9-x^2}} partialta(x,y)dA$$

where $partialta(x,y)$ is the distance from the y-axis

1. Sketch $R$




2. find $partialta(x,y)$
   Solution: $partialta(x,y) = x$

3. What is the practical meaning of the integral?
   Solution: The total mass

4. dydx

$$\int_1^2 \int_1^{\sqrt{9-x^2}} xdydx$$

5. dx dy

$$\int_1^{\sqrt{5}} \int_1^2 xdxdy + \int_{\sqrt{5}}^{\sqrt{8}} \int_1^{\sqrt{(9-y^2)}} xdxdy$$

6. $drd\theta$

$$\int_{\arctan(1/2)}^{\arctan(1/1)} \int_{1/\sin\theta}^{2/\cos\theta} r^2\cos\theta drd\theta + \int_{\arctan(1)}^{\arctan(\sqrt{5}/2)} \int_{1/\cos\theta}^{2/\cos\theta} r^2\cos\theta drd\theta + \int_{\arctan(\sqrt{5}/2)}^{\arctan(\sqrt{5})} \int_{1/\cos\theta}^{3} r^2\cos\theta drd\theta$$

7. $d\theta dr$

$$\int_{\sqrt{2}}^{\sqrt{5}} \int_{\arcsin(1/r)}^{\arccos(1/r)} r^2\cos\theta d\theta dr + \int_{\sqrt{5}}^{3} \int_{\arccos(2/r)}^{\arccos(1/r)} r^2\cos\theta d\theta dr$$

8. Let $W$ be the solid defined by $z \leq 8 - x^2 - y^2$, $x^2 + y^2 \leq 4, z \geq -3$. Sketch $W$.

9. dydx
   Solution:
$$\int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{4-x^2}}(11-x^2-y^2)$$

10. $drd\theta$
    Solution:
$$\int_0^{2\pi}\int_0^2 r(11-r^2)drd\theta$$

11. dzdydx
    Solution:
$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-3}^{8-x^2-y^2} 1dzdydx$$

12. $dzdrd\theta$
    Solution:
$$\int_0^{2\pi}\int_0^2\int_{-3}^{8-r^2} rdzdrd\theta$$

13. $d\rho d\phi d\theta$
    Solution :
$$\int_0^{2\pi}\int_0^{\arctan(2/4)}\int_0^{\frac{-\cos\phi+\sqrt{\cos^2\phi+32\sin^2}}{2\sin^2\phi}} \rho^2\sin\phi d\rho d\phi d\theta$$
$$+\int_0^{2\pi}\int_{\arctan(2/4)}^{\pi-\arctan(2/3)}\int_0^{2/\sin\phi}\rho^2\sin\phi d\rho d\phi d\theta$$
$$+\int_0^{2\pi}+\int_{\pi-\arctan(2/3)}^{\pi}\int_0^{-3/\cos\phi}\rho^2\sin\phi d\rho d\phi d\theta$$

14. Use cylindrical coordiantes to compute the total mass if the mass density is the distance from the origin
    Solution
$$\int_0^{2\pi}\int_0^2\int_{-3}^{8-x^2-y^2} r(r^2+z^2)dzdrd\theta$$

# 10 January 30, 2023 - Vector Fields

**Example 17** *Remember that you can parameterize the graph of a circle by*

$$\vec{r}(t)=\langle a\cos t, a\sin t\rangle, 0\le t\le 2\pi`$$

*If we add the angular frequency $\omega$, we get a more generalized form*

$$\vec{r}(t)=\langle a\cos\omega t, a\sin\omega t\rangle, 0\le t\le \frac{2\pi}{\omega}`$$

*And the speed of this particle is $a\omega$*

*Remeber that $\vec{v} = \frac{d\vec{r}}{dt}$, so*

$$\vec{v}(t) = \langle -a\omega \sin(\omega t), -a\omega \cos(\omega t)\rangle$$

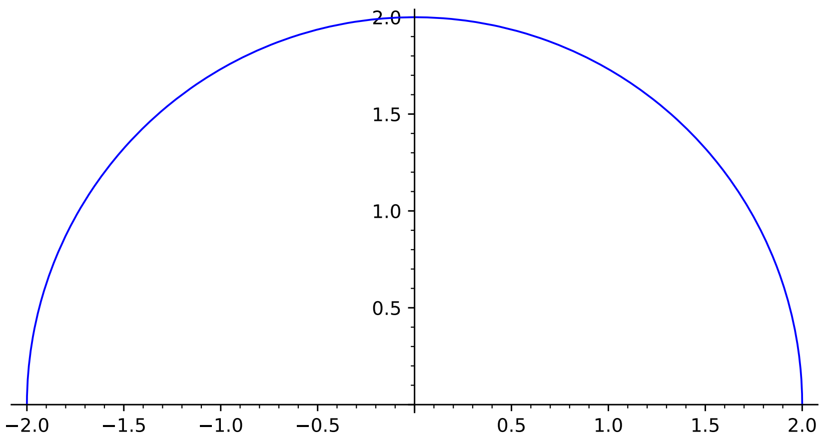
*so $\|\vec{v}\| = a\omega$*

*Let's take the derivative again to get*

$$\vec{a}(t) = \langle -a\omega^2 \cos(\omega t), -a\omega^2 \sin(\omega t)\rangle$$

*Note that $\vec{r}$ is always perpendicular to $\vec{v}$ because $\vec{r}\cdot\vec{v} = 0$. Also note that $\vec{a} = -\omega^2\vec{r}$.*

*Don't overgeneralize: $\vec{v}$ is not always perpendicular to $\vec{r}$, but it is perpendicular in circular motion.*




**Example 18** *Consider the following graph*
*Which is traced out by a particle over 5 seconds. Find the location of the particle at time $t$.*
*Solution: The equation of the particle is*

$$\vec{r}(t) = \langle 2\cos(-\pi t/5), 2\sin(-\pi t/5)\rangle$$

*You can also paramtetrize $(r,\theta)$ by $t$.*

$$r = 2, \qquad \theta = \pi - \pi t/5$$

**Example 19** *A particle moves in a circle of radius 2 at $5m/s$. Find the equation of the particle.*
*Solution:*

$$\vec{r} = \langle 2\sin(5t/2), 2 - \cos(5t/2)\rangle, 0 \le t \le \pi/5$$

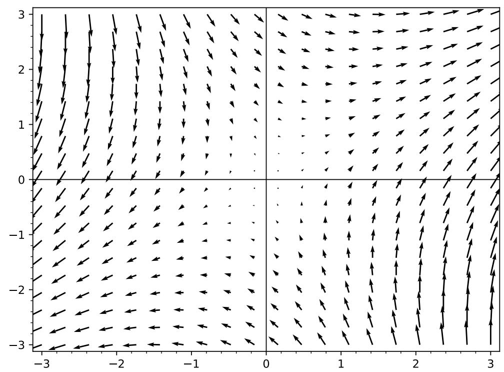
**Definition 3** *A **vector field** is a function from $\mathbb{R}^n \to \mathbb{R}^n$. It takes in a vector and spits out a vector. For example,*

$$F(x,y) = \langle x+y, x-y\rangle$$

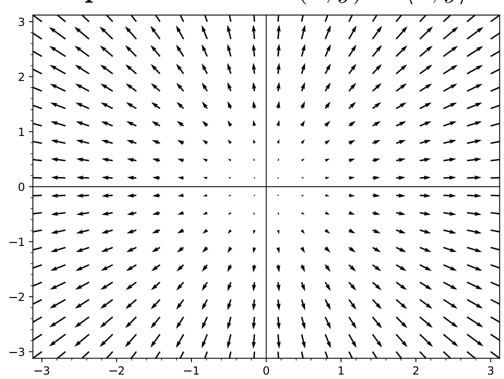
From now on, everything is about vector fields.

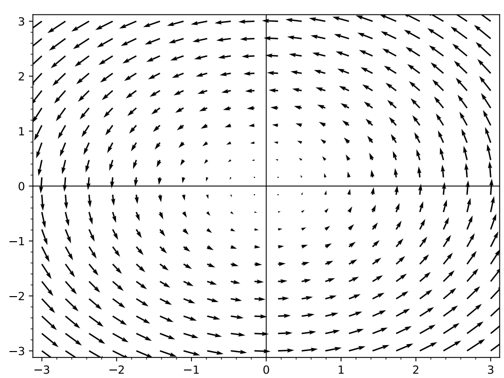
The way to sketch a vector field in $\mathbb{R}^2$ is to draw a little arrow at each point representing the output vector.

**Example 20** *For example, here is a sketch of* $F(x, y) = \langle x + y, 2x - y\rangle$



**Example 21** *sketch* $F(x, y) = \langle x, y\rangle$ *and* $F(x, y) = \langle -y, x\rangle$*? Solution:*





# 11 Februrary 1 2023

## 11.1 17.3 - Flow Line

**Definition 4** *The* ***flow line*** *is how a particle in a vector field would "flow" (imagine the vector field is a force exerted on the particle). A path* $\sigma(t) : \mathbb{R} \to \mathbb{R}^n$ *is a flow-line in a vector field* $\vec{F}(x) : \mathbb{R}^n \to \mathbb{R}^n$ *iff*

$$\sigma'(t) = \vec{F}(\sigma(t))$$

Let's review parametric functions

**Example 22** *Find the flow line defined by $\vec{r}(t)$ if the vector field is $\vec{F} = \langle 1, 3 \rangle$ and $\vec{r}(0) = \langle 2, 4 \rangle$*
*Solution:*

$$\vec{r}(t) = \langle 2 + t, 4 + 3t \rangle$$

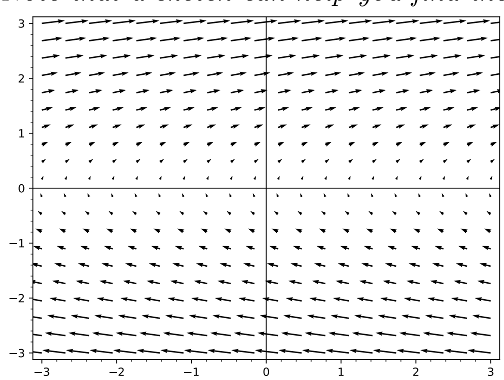
**Example 23** *Find the flow line of $\vec{r}(t)$ if the vector field is $\vec{F}(x, y) = \langle y, 2y \rangle$ and $\vec{r}(1) = \langle 3, 4 \rangle$*
*Solution:*

$$\begin{aligned} r' &= F \circ r \\ \langle x', y' \rangle &= \langle y, 2y \rangle \\ \vec{r}(t) &= \langle 2\, e^{(2\,t)} + 1, 4\, e^{(2\,t)} \rangle \end{aligned}$$

Note that you can find flow lines in sage using

```
# I am going to do this later
print('hello world')
```

**Example 24** *Find the flowline if the field is $\vec{F}(x, y) = \langle 2y, 1 \rangle$ and $\vec{r}(0) = \langle 3, 4 \rangle$*
*Solution:*
*Note that a sketch can help you find the solution*




$$\begin{aligned} x' = 2y, y' = 1, x(0) = 3, y(0) = 4 \\ y = t + 4 \\ x = t^2 + 8t + 3 \\ \vec{r}(t) = \langle t^2 + 3, t + 4 \rangle \end{aligned}$$

**Example 25** *Consider the vector field $\vec{F}(x, y) = \langle y, x \rangle$ and $\vec{r}(t) = \langle 3, 4 \rangle$ Solution:*

$$x\,(t) = -\frac{1}{2}\, e^{(-t)} + \frac{7}{2}\, e^{t}, y\,(t) = \frac{1}{2}\, e^{(-t)} + \frac{7}{2}\, e^{t}$$

## 11.2 17.4 – Euler's Method

If we cannot find an exact solution, we can do a numerical approximation using Euler's Method (approximate differential equations with a tangent line). The main idea is that

$$f(a + partialtax) \approx f(a) + partialtax f'(a)$$

# 12 Chapter 17 Practice Knowledge Check

1. A particle moves in the direciton of $\langle 4,3\rangle$ along a straight line at a constant speed of 10 and is located at $(1,2)$ and $(a,0)$ when $t=k$ and $t=5$ respectively. Find $\vec{r}$ Solution:

$$\vec{r}(t) = \langle -31+8t, -22+6t\rangle$$

2. A particle is located at $(0,-4)$ and moves along a full circlular path clockwise centered at the origin at a cosntant speed of $\pi$. Find the particle's veclocity at $t=2$
Solution:

$$4\pi$$

(note that this question was dumb and $\pi$ referred to )

3. let $\vec{F}(x,y) = \langle 2, 4y\rangle$ be ocean current. An ice berg is at $(3,5)$ at $t=0$.

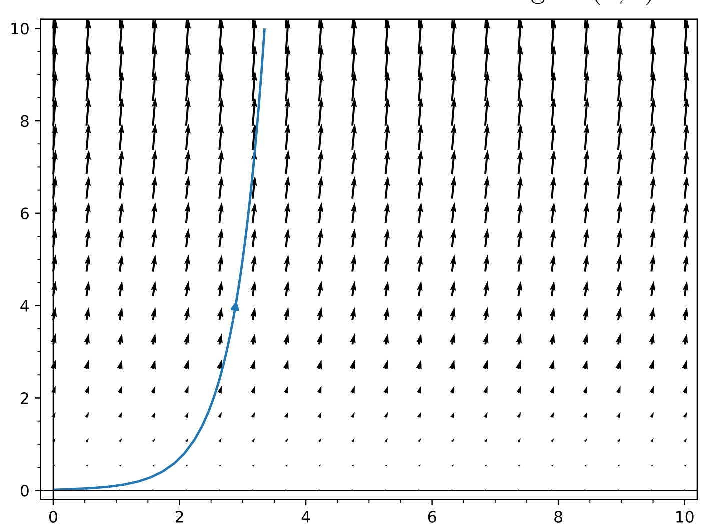
   (a) Use Euler's Method with two steps to approximate the location of the iceberg at t=1.
   Solution:

$$\begin{aligned} r(0) &= \langle 3,5\rangle \\ r(1/2) &\approx \langle 3,r\rangle + 1/2\rangle 2, 4*5\rangle = \langle 4,15\rangle \\ r(1) &\approx \langle 4,15\rangle + 1/2\langle 2, 4*15\rangle = \langle\rangle \end{aligned}$$

   (b) Find the exact location of the iceberg in the ocean current at $t=1$
   Solution:

$$\begin{aligned} x' = 2, y' &= 4y \\ x = 2t+3, y &= 5e^{4t} \\ r(1) &= \langle 5, 5e^4\rangle \end{aligned}$$

   (c) Sketch the vector field and draw the flow line starting at $(3,5)$ on the vector field




   Solution:

# 13 February 2 2023

## 13.1 18 – Line Integral

A line integral can be motivated by considering work from physics. In physics, $W = \vec{F} \cdot partialta\vec{x}$.

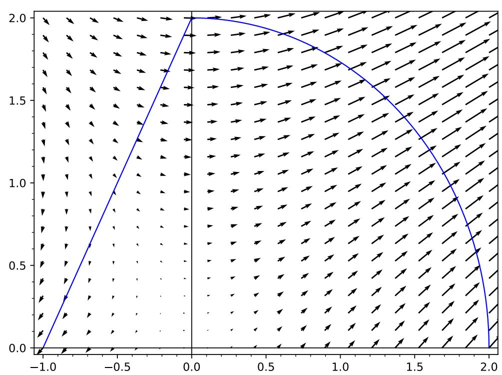
**Definition 5** *Given a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ and a curve $C$ parametrized by the parametric equation $\vec{r}(t) : \mathbb{R} \to \mathbb{R}^n$, then the work done by the field along the curve is given by the* ***line integral***

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

**Example 26** *Given the vector field $\vec{F} = \langle 3, 4 \rangle$ and the path $C$ which is a line from $(0,0)$ to $(0,5)$, find the work done.*
*Solution: Let $\vec{r}(t) = \langle 0, t \rangle, 0 \leq t \leq 5$.*

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^5 \langle 3, 4 \rangle \cdot \langle 0, 1 \rangle dt = 20$$

**Example 27** *Find the work done by the vector field $\vec{F}(x,y) = \langle x + y, x \rangle$ along the curve*



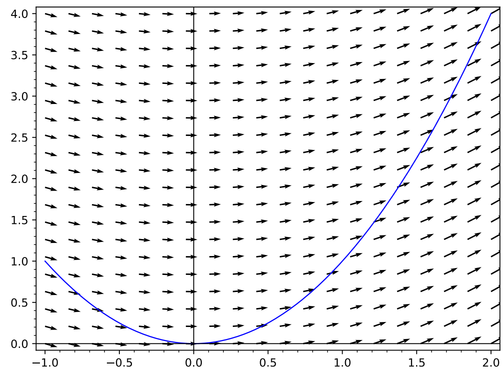

*solution:*
*let $r_1(t) = \langle t, 2t + 2 \rangle, -1 \leq t \leq 0$ and $r_2(t) = \langle 2\sin t, 2\cos t \rangle, 0 \leq t \leq \pi/2$ Then*

$$\int_{-1}^0 \langle 3t + 2, t \rangle \cdot \langle 1, 2 \rangle dt + \int_0^{\pi/2} \langle 2\sin t + 2\cos t, 2\sin t \rangle \cdot \langle 2\cos t, -2\sin t \rangle dt$$

$$\int_{-1}^0 5t + 2dt + \int_0^2 4\sin t\cos t + 4\cos^2 t - 4\sin^2 t dt$$

$$-(\frac{5}{2} + 2) + (1 + 1) = \frac{3}{2}$$

**Example 28** *Given the vector field $\vec{F}(x,y) = \langle 2, x \rangle$ and the curve $C$ which is a parabola given below*



*Solution:*
*Let* $\vec{r}(t) = \langle t, t^2 \rangle, -1 \leq t \leq 2$ *then*

$$\int_{-1}^{2} \langle 2, t \rangle \cdot \langle 1, 2t \rangle dt$$

$$= 12$$

# 14 Februrary 6 2023

Today we are going to talk about line integrals.

## 14.1 Gradient Field

How do we determine if a given vector field is a gradient vector field?

First assume that

**Definition 6** *We say that* $\vec{F}$ *is a gradient vector field iff*

$$\exists \vec{F}, \vec{F} = \nabla f$$

**Example 29** *Let* $\vec{F} = \langle 2xy + 1, x^2 + 3y \rangle$. *Is* $\vec{F}$ *a gradient field? Set up the partial equations.*

$$\frac{\partial f}{\partial x} = 2xy + 1 \qquad \frac{\partial f}{\partial y} = x^2 + 3y$$

*Integrate both sides to get*

$$f(x, y) = x^2 y + x + c(y), \qquad f(x, y) = x^2 y + \frac{3}{2} y^2 + c(x)$$

*Derivate the first wrt to* $y$ *to get*

$$f_y = x^2 + c_y(y) \implies c = 37y + c_2$$

*Then substitute that back in to get*

$$f(x, y) = x^2 y + x + 37y + c_2$$

*so we may conclude that* $f$ *is a gradient field.*

**Example 30** *is $\vec{F} = \langle x, y\rangle$ a gradient field? Yes,,*

$$\vec{F} = \nabla(\frac{1}{2}x^2 + \frac{1}{2}y^2 + C)$$

**Example 31** *Is $\vec{F} = \langle -y, x\rangle$ a gradient field??*

*Assume there is such and $f$ so that*

$$f_x = -y \qquad f_y = x$$

*Integrate to get*

$$f = xy + c(x) \qquad f = -xy + c(y)$$

*Then partially differentiate to get*

$$f_x = y + c'(x) = -y$$

*but this is a contradiction because*

$$y \neq -y$$

*so we conclude that $\vec{F}$ is not a vector field.*

**Theorem 2** ***Fundamental Theorem of Calculus for Line Integrals****: Given a vector field $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$ which is a gradient vector field (which means that $\vec{F} = \nabla f$) and given that $C$ is an oriented curve in $\mathbb{R}^n$ from $p$ to $q$, then*

$$\int_C \vec{F} \cdot d\vec{r} = f(q) - f(p)$$

You should think of this theorem as the analog of the fundamental theorem of calculus from single variable.

**Example 32** *Let $\vec{F} = \langle y^2, 2xy + 1\rangle$.*

*Find the potential function*

$$f_x = y^2 \qquad f_y = 2xy + 1$$

*so*

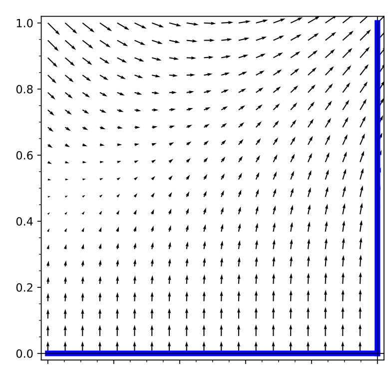
$$f = xy^2 + c(y) \implies f_y = 2xt + c'(y)$$

*This means*

$$c'(y) = 1 \implies c(y) = y$$

*So*

$$\vec{F} = \nabla(xy^2 + y)$$

*Now find the line integral over the curve* 



*We use the FTC from before to find that it is*

$$1 - (-1 \cdot 0^2 - 0) = 1$$

Note that you have to pay attention to the orientation of the curve.

**Definition 7** *A vector field $\vec{F}$ is called* ***path independent*** *(physicst would call it* ***conservative****) iff for all curve $C_1$ and $C_2$*

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{c_2} \vec{F} \cdot d\vec{r}$$

*when $c_1$ and $c_2$ have the same end-points.*

In other words, for a path independent vector field, the work done by the vector field does not depend on the path taken.

**Definition 8** *Given a vector field $\vec{F}$ and a closed curve $C$, the* ***circulation*** *of $\vec{F}$ along $C$ is*

$$\oint \vec{F} \cdot d\vec{r}$$

*which is just a line integral where we end up back where we started.*

Note that for a conservative vector field, the circulation is always 0.

**Definition 9** *we call a vector field $\vec{F}$* ***circulation free*** *iff*

$$\oint_c \vec{F} \cdot d\vec{r} = 0$$

*for all closed curves $c$*

Here is a big theorem

**Theorem 3** *The following statements are all equivalent*

1. *$\vec{F}$ is a gradient field*
2. *$\vec{F}$ is a path independent*
3. *$\vec{F}$ is circulation free*

So if you prove one of these statements, you've proved them all,

## 14.2 Curl

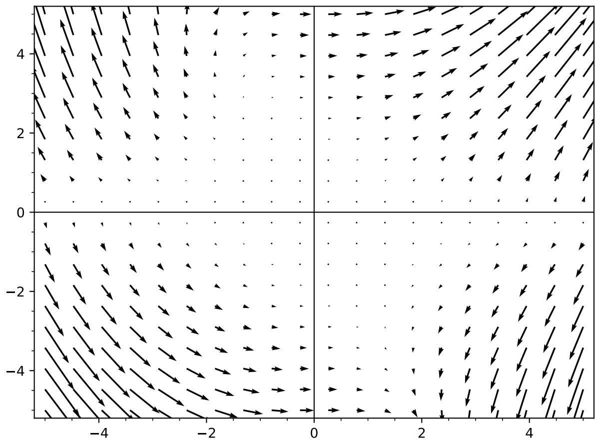
**Definition 10** *Given a vector field $\vec{F}$ and a point $(x, y)$, we defined the* ***curl*** *of $\vec{F}$ at $(x, y)$ to be*

$$\nabla \times \vec{F}(x, y) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

*The physical interpretation is that the curl measures if you placed a windmill at $(x, y)$, the curl measures the how much the vector field will turn the windmill. If the curl is positive, the rotation is counter-clockwise. If the curl is negative, the rotation is clockwise.*

Another name for the curl is the **circulation density**.

**Example 33** *Let $\vec{F} = \langle 2xy + y^2, x^2 + x \rangle$. Find the curl of $\vec{F}$.*




*The curl is*

$$\nabla \times \vec{F} = 1 - 2x - 2y + 2xy$$

**Theorem 4**

$$\vec{F} \text{ is a gradient field } \implies \nabla \times \vec{F} = 0$$

$$\nabla \times \vec{F} \neq 0 \implies \vec{F} \text{ is not a gradient field}$$

Don't misuse the theorem. The inverse of the theorem doesn't hold. If the curl is 0, we cannot conclude that the vector field is a gradient field. Consider the counter-example

$$\vec{F} = \langle \frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \rangle$$

Now find the curl YHPL claims that the curl is 0, but I disagree. The field is not conservative

Here is a useful theorem to compute a line integral.

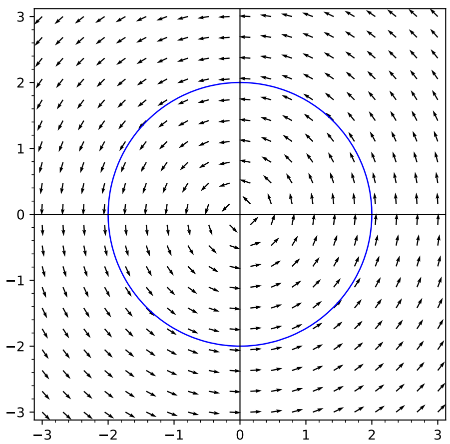
**Theorem 5** *If $\|\vec{F}\|$ is constant along $C$ and $\vec{F}$ is tangent to $C$ everywhere in the same direction, then*

$$\int_C \vec{F} \cdot d\vec{r} = \|\vec{F}\|$$

**Example 34** *To use the previous theoerm, consider the vector field*

$$\vec{F} = \langle \frac{-y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}}$$

*$\|F\| = 1$, and $\vec{F}$ is tangent to any circle centered at $(0, 0)$. So, for example*




*Then*

$$\oint_C \vec{F} \cdot d\vec{r} = 2\pi$$

How to find out if $\vec{F}$ is a gradient field.

1. Find $\nabla \times \vec{F}$. If the curl is not 0, then $\vec{F}$ is not a vector field. If the curl is 0, then inconclusive
2. Solve the differential equations to try and find a potential function

**Theorem 6** ***Curl Test**: If $\nabla \times \vec{F} = 0$ and the domain of $\vec{F}$ has no holes then $\vec{F}$ is a gradient vector field*

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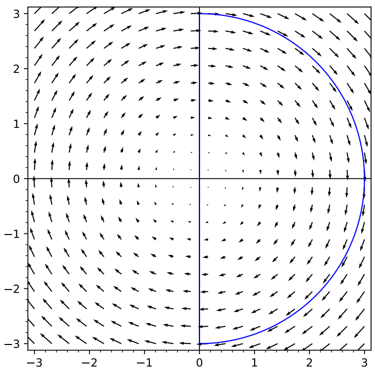
Note that you cannot use the Curl Test to conclude that a given vector field is not a gradient field. The implication goes one way. If you are applying the curl test, the conclusion is always "the field is a gradient field"

**Theorem 7** ***Green's Theorem**: Let $\vec{F}$ be a smooth vector field Let $C$ be a smooth, closed, simple, counter-clockwise curve, and let $R$ be the region enclosed by $C$. Then*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_R \nabla \times \vec{F} dA$$

The proof for Green's Theorem is that the right side of the equality counts up all the little circulations inside a region, and the left side of the equality counts up the total circulation along the boundry of the region.

**Example 35** *Let $\vec{F} = \langle 2y, -2x \rangle$ and let $C$ be the closed curve.*



*Compute*

$$\oint_C \vec{F} \cdot d\vec{r}$$

*Solution:*
*We can solve this integral in 3 different ways.*

1. *Using Green's Theorem*

$$\iint_R \nabla \times \vec{F} dA$$

$$= -\int_{\theta=\pi/2}^{\theta=\pi/2} \int_{r=0}^{r=3} (-4) r dr$$

$$= 18\pi$$

   *Remeber to multiply by $-1$ because this curve is oriented clockwise*

2. *Using parametrization: Let*

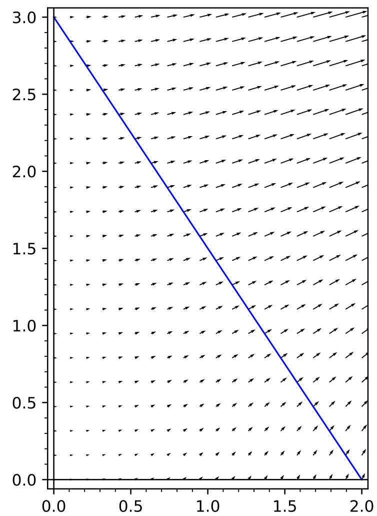
$$r(t) = (0,t), -3 \leq t \leq 3, r(t) = (3\cos(t), 3\sin(t)), -\pi/2 \leq t \leq \pi/2$$

$$\int_{t=0}^{t=3} F(r(t)) \cdot r'(t) dt + \int_{t=-\pi/2}^{t=\pi/2} F(r(t)) \cdot r'(t) dt$$
$$= \int_{t=0}^{t=3} \langle 2t, 0 \rangle \cdot \langle 0, 1 \rangle dt + \int_{t=-\pi/2}^{t=\pi/2} \langle -6\sin(t), 6\cos(t) \rangle \cdot \langle -3\sin(t), 3\cos(t) \rangle dt$$
$$= \int_{t=0}^{t=3} 0 dt + \int_{t=-\pi/2}^{\pi/2} 18 dt$$
$$= 18\pi$$

3. *Geometric intuition: By theorem 5, it's just the path length, which is $3\pi$. $\vec{F}$ is perpendicular to $C$ on the vertical seciton and $\vec{F}$ is parallel to $C$ on the circular section.*

Sometimes we can even use Greens Theorem if the curve $C$ is not closed by drawing in a new line.

**Example 36** *Let $\vec{F} = \langle xy + 1, x\rangle$ and let $C$ be the curve*



*Solution:*
*First, by direct computation: Let*

$$r(t) = \langle t, 3 - 3/2t\rangle, 0 \le t \le 2$$

$$\int_{t=0}^{t=2} \langle 3t - 3/2t^2 - 1, t\rangle \cdot \langle 1, -3/2\rangle dt$$
$$= -(3 - 4 + 2) = -1$$

*Now by Greens Theorem Draw in the other bases of the triangle. Let*

$$r_1(t) = \langle t, 0, \rangle, 0 \le t \le 3$$
$$r_2(t) = \langle 0, t\rangle, 0 \le t \le 2$$

*Then, Green's Theorem gives*

$$\oint_{C - C_2 + C_1} \vec{F} \cdot d\vec{r}$$
$$= \int_{x=0}^{x=2} \int_{y=0}^{y=3-3/2x} (1 - x) dy dx$$
$$= -1$$

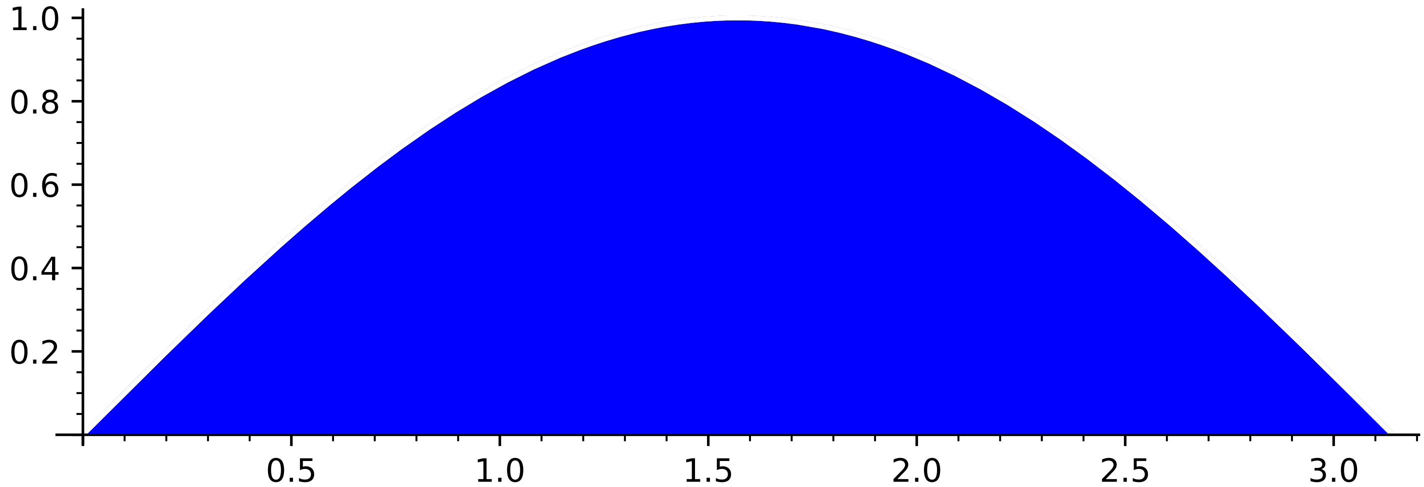
*Now we compute the line integrals of $C_1$ and $C_2$ to subtract them out of the closed curve. We get*

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{t=0}^{t=2} \langle 1, t\rangle \cdot \langle 1, 0\rangle dt = 0$$
$$\int_{C_2} \vec{F} \cdot d\vec{r} \; int_{t=0}^{t=3} \langle 1, 0\rangle \cdot \langle 0, 1\rangle dt = 0$$

*The orientation/sign of curves matters*

We can find the area of any region if we have a function whose curl is 1. For example $\nabla\times\langle 0, x\rangle = 1$.

**Example 37** *Find the area of the region*




*Solution: By Green's Theorem*

$$\int_R \nabla\times\vec{F}dA = \oint_{C_1+c_2} \langle 0, x\rangle\cdot d\vec{r}$$

$$= \int_0^\pi \langle 0, t, \rangle\cdot\langle 1, 0\langle dt + \int_0^\pi \langle 0, t\rangle\cdot\langle 1, \cos t\rangle dt$$

## 16 Feb 9 2023

No Class today due to 12 hour meeting.

## 17 Feb 13 2023

**Example 38** *Let $\vec{F}(x, y) = e^{xy}(y\cos(x)-\sin(x))\vec{i}+xe^{xy}\cos(x)\vec{j}$. $C_2$ is the half unit circle centered at $(1, 0)$ in the first quadrant, traced clockwise from $(0, 0)$ to $(2, 0)$. $C_2$ is the line from $(0, 0)$ to $(2, 0)$*

1. *use the curl test to determine if $\vec{F}$ is a gradient field.*
   *Solution*
   *The curl test has 2 conditions. First*

$$\nabla\times\vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

   *. Next the domain condition. The domain has no holes. By the curl test, $\vec{F}$ is a gradient field*

2. *Find the potential function of $\vec{f}$ Solution:*
   *Set up some differential equations.*

$$f_x = e^{xy})(y\cos(x)-\sin(x)) \qquad f_y = xe^{xy}\cos(x)$$

*So*

$$f = \cos(x)e^{xy} + K(x)$$
$$f_x = -y\sin(x)e^{xy} + K'(x)$$

*so*

$$f = e^{xy}\cos(x)$$

3. *Set up the line integral $\int_{C_1} \vec{F} \cdot d\vec{r}$ using parametrization, do not evaluate.*
   *Solution:*
   *Let*
$$\vec{r}(t) = \langle 1-\cos(t), \sin(t)\rangle, 0 \le t \le \pi$$
$$\int_{t=0}^{t=\pi} \vec{F}(\vec{r}(t))\vec{r}'(t)dt$$

4. *Use the fundamental theorem of line integrals to evaluate $\int_{C_1} \vec{F} \cdot d\vec{r}$*
   *This theorem says we just eval at the beginning and the end*
$$\int_{C_1} \vec{F} \cdot d\vec{r} = f(2,0) - f(0,0) = \cos(2) - 1$$

5. *evaluate $\int_{C_2} \vec{F} \cdot d\vec{r}$ using a parametrisation.*
   *Let*
$$r(t) = \langle t, 0\rangle 0 \le t \le 2$$
   .
$$r'(t) = \langle 1, 0\rangle 0 \le t \le 2$$
$$\int_{t=0}^{t=2} \langle -\sin(t), t\cos(t)\rangle \cdot \langle 1, 0\rangle dt$$

6. *is $\vec{F}$ conservative?*
   *Yes $\vec{F}$ is conservative because $\vec{F}$ is a gradient vector field.*

We already did the chapter 17 KC sample.

**Example 39** *Let $C$ be the curve from $(-1,0)$ to $(1,0)$ on the curve $y = 1-x^2$. Let $\vec{F} = \langle 2, e^{y^{2022}} + x^2 + 3$. Use Greens Theorem to find the work done by $\vec{F}$ on $C$.*
*Solution:*
*We need to close the curve in order to apply Greens Theorem. Note that $\nabla \times \vec{F} = 2x$*

$$\oint_{C_2 - C_1} \vec{F} \cdot d\vec{r} = \iint 2x dA$$

*The area integral is*

$$\int_{x=-1}^{x=1} \int_{y=0}^{y=1-x^2} 2x dy dx = \int_{-1}^{1} 2x(1-x^2)dx = 0$$
$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r}$$

*Let $\vec{r}(t) = \langle t, 0\rangle - 1 \le t \le 1$*

$$\int_{t=-1}^{t=1} \langle 2, t^2 + 4\rangle \cdot \langle 1, 0\rangle dt = 2$$

**Example 40** *Let $\vec{F} = \langle 2, e^{y^{2023}} + x^2 + 3\rangle$ Let $C$ be the clockwise region bounded by $y = 1$, $x = 0$, $y = x$. Find the clockwise circulation of the vector field around the boundry.*

$$\int_0$$

Here we did some problems out of the textbook, and I couldn't type them up because they didn't give the equation.

### 17.1 Chapter 19 - flu

Chapter 19 is about flux. flux is the rate of flow of a vector field through a surface $S$. It is written $\int_S \vec{F} \cdot d\vec{A}$, where $\vec{A}$ is vector normal to $A$ whose length is equal to the area of $A$.

## 18 Chapter 18 Knowledge Check Solutions

1. Consider $F(x, y) = \langle y + 1, x\rangle$ and $G(x, y) = \langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}}$

   (a) Does the curl test apply to $F$?
   Solution:
   Yes, the curl test applies. The $\nabla \times F = 0$ and $F$ has no holes in its domain.

   (b) Is $F$ conservative?
   Solution:
   Yes, $F$ is conservative because of the Curl Test.

   (c) Apply Fundamental Theorem of Calculus to evaluate $\int_C \vec{F} \cdot d\vec{r}$ where $C$ is the curve from $(01, e^{-1})$ to $(1, e)$ along $y = e^x$
   Solution:
   We find that if $\nabla f = \vec{F}$, then $f(x, y) = xy + x + c$ . The FTC says the answer is $f(1, e) - f(-1, e^{-1})$, which is $e + e^{-1} + 2$

   (d) Does the curl test apply to $\vec{G}$?
   Solution:
   No, $\vec{G}$ has a hole at $(0, 0)$

   (e) Find the work done by $\vec{G}$ along the circle of radius 2 at the origin going counterclockwise, starting at the point $(2, 0)$

      i. using explicit parametrization of $C$
      Solution:
      let

$$r(t) = \langle 2\cos t, 2\sin t\rangle \qquad 0 \le t \le 2\pi$$

$$\begin{aligned}
\oint_C \vec{G} \cdot d\vec{r} &= \int_0^{2\pi} \langle -\sin t, \cos t\rangle \cdot \langle -2\sin t, 2\cos t\rangle dt \\
&= \int_0^{2\pi} 2dt \\
&= 4\pi
\end{aligned}$$

ii. using geometric intuition about $\vec{G}$ and $C$
Solution:
$\vec{G}$ is always perpendicular to $C$, so it's just

$$\|\vec{G}\||c| = 14\pi = 4\pi$$

(f) is $\vec{G}$ conservative?
Solution:
No, because in the integral above, the circulation was not 0.

2. Let $\vec{F}(x,y) = \langle 2, e^{y^{2023}} + x^2 + 3\rangle$

(a) Find the clockwise circulation of $\vec{F}$ around the region bounded by $x = 0$, $y = 1, y = x$
Solution:
Just use Greens Theorem. Make sure to multiply by $-1$ because we are going clockwise.

$$\begin{aligned}&\oint_R \vec{F} \cdot d\vec{r}\\&= \iint \nabla \times \vec{F} dA\\&= \int_{x=0}^{x=1} \int_{y=x}^{y=1} 2x dy dx\\&= -1/3\end{aligned}$$

(b) Use Green's theorem to find the work done by the vector field along a curve $C$, where $C$ is from $(-1,0)$ to $(1,0)$ along $y = 1 - x^2$.
Soltion:
You need to close the loop by drawing in an extra line. I suggest defining $C_2$ to be the line from $(-1,0)$ to $(1,0)$. Then, let $R$ be the region enclosed by $C_2 - C$By Green's Theorem

$$\begin{aligned}\oint_R \vec{F} \cdot d\vec{r} &= \iint \nabla \times \vec{F} dA\\ \int_{C_2} \vec{F} \cdot d\vec{r} - \int_C \vec{F} \cdot d\vec{r} &= \int_{-1}^{1} \int_0^{1-x^2} 2x dy dx\\ \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_C \vec{F} \cdot d\vec{r}\end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_{t=-1}^{t=1} \langle 2, t^2 + 3\rangle \cdot \langle 1, 0\rangle dt$$

$$= 4$$

# 19 Feb 16 2023

## 19.1 19.2 – Flux

**Definition 11** *The **flux** of a vector field through a surface is the amount of flow of a vector field through the surface. $S$ is the surface, $\vec{F}$ is the vector field, and $\vec{n}$ is the normal vector to the surface.*

$$\Phi = \iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n} dS$$

*In this notation,* $d\vec{A} = \vec{n}dA$

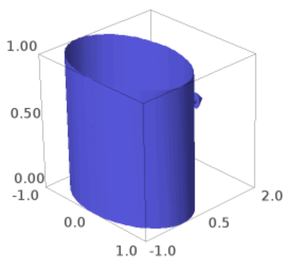
There are several easy cases if we know the surface. You need to memorize these 3 cases:

1. cylinder
2. sphere
3. $z = f(x, y)$

### 19.1.1 Cylinder

**Example 41** *Let $S$ be a cylinder whose axis is the $z$ -axis. There is no top/bottom cap, $S$ is only the sides of the cylinder.*

```
from sage.plot.plot3d.shapes import Cylinder
```




*The unit normal vector $\vec{n}$ for the point with cylindrical coordinates $(r, \theta, z)$ is $\langle \cos\theta, \sin\theta, 0\rangle$.*

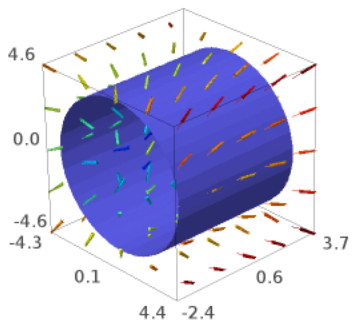
*We can use the unit normal to calculate the flux.*

$$\Phi = \int_{\theta=0}^{\theta=2\pi} \int_{z=a}^{z=b} dz d\theta$$

Here is an example where the vector field is given

**Example 42** *Let $S$ be the surface which is the sides of a cylinder whose axis is the $y$-axis from $y = -2$ to $y = 3$ and whose radius is 4.*

*Let $\vec{F}(x, y) = \langle y, x + 1, z\rangle$. find the flux of $\vec{F}$ through $S$.*




*We can find the flux by*

$$\begin{aligned}
&\Phi \\
&= \int_{y=-2}^{y=3} \int_{\theta=0}^{\theta=2\pi} \langle y4\cos\theta, 4\sin\theta\rangle \cdot \langle -\cos\theta, 0, -\sin\theta\rangle d\theta dy
\end{aligned}$$

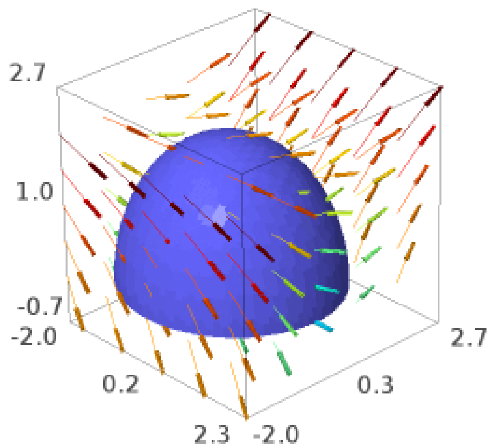
#### 19.1.2 Sphere

For a sphere, use spherical coordinates.

$$\Phi = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=2\pi} \vec{F} \cdot \langle \sin\phi\cos\theta, \sin\phi\cos\theta, \cos\phi\rangle d\phi d\theta$$

Note that a top hemisphere only means the top half of the sphere

**Example 43** *Let $S$ be the top hemisphere and let $\vec{F} = \langle 1, z, y\rangle$. Find the flux.*




$$\Phi$$

$$= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/2} \langle 1, 2\cos\phi, 2\sin\phi\cos\theta\rangle \cdot \langle \sin\phi\cos\theta, \sin\phi\cos\theta, \cos\phi\rangle d\phi d\theta$$

Note that we often set up integrals, but never evaluate them.

### 19.1.3 Implicit Surfaces

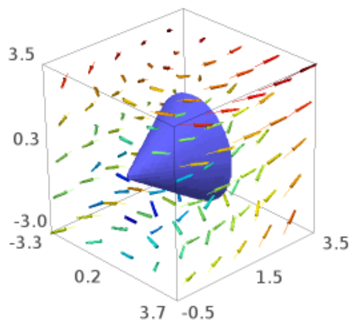
If we are given a surface as $S$ defined by $z = f(x, y)$, then

$$\vec{n} = \langle -f_x, -f_y, 1\rangle$$

Where 1 means that $S$ is oriented upwards and $-1$ means $S$ is oriented downwards.

In general, we project the surface onto a plane.

**Example 44** *Let $S$ be the surface $y = \sqrt{x^2 + y^2}$ for $0 \leq y \leq 2$ where the normal vector is oriented outwards. Let $\vec{F} = \langle 1 + z, x, y\rangle$. Find the flux of $\vec{F}$ through $S$. Note that $S$ is a cone.*




*We have that $\vec{n} = \langle f_x, -1, f_z \rangle$. Note that we use $-1$ because when we project onto $x - z$ plane, the normal vector is pointing down.*

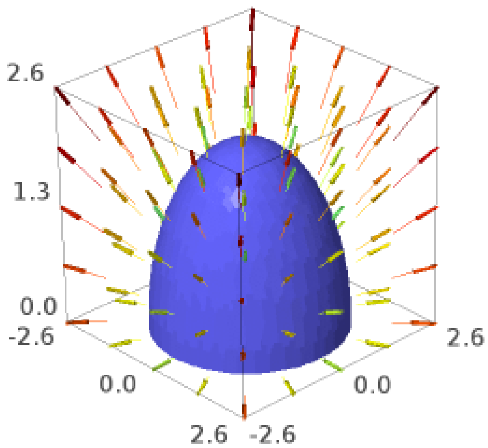
$$\begin{aligned}
\Phi &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \langle 1+z, x, \sqrt{x^2+z^2} \rangle \cdot \langle f_x, -1, f_z \rangle dxdz \\
&= \int \langle 1+z, x, \sqrt{x^2+z^2} \rangle \cdot \langle \frac{1}{2\sqrt{x^2+z^2}}, -1, \frac{1}{2\sqrt{x^2+z^2}} \rangle dxdz \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \langle 1 + r\sin\theta, r\cos\theta, r \rangle \cdot \langle \frac{r\cos\theta}{r}, 1\frac{r\sin\theta}{r} \rangle rdrd\theta
\end{aligned}$$

Here is a useful theorem to compute flux integral.

**Theorem 8** ***Geometric Intuition Theorem:*** *If $\vec{F}$ is always perpendicular to $S$ and $\|\vec{F}\|$ is constant on $S$, then*

$$\Phi_S = \|\vec{F}\| A$$

**Example 45** *Let $\vec{F}(x, y, z) = \langle x, y, z \rangle$ and let $S$ be the surface which is the upper hemisphere of a sphere with radius 2.*




*By the geometric intuition, the answer is* $\|\vec{F}\|A = 2\pi 2^2 2 = 16\pi$

You can parametrize a surface by a function

$$r(s,t) = \langle x(s,t), y(s,t), z(s,t) \rangle$$

# 20 Feb 20 2023

No class due to holiday.

# 21 Feb 22 2023

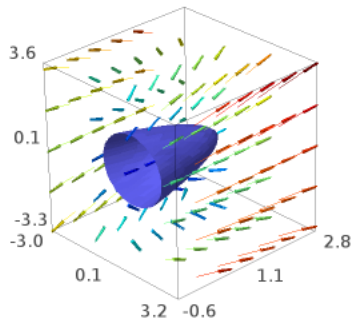
## 21.1 Review

First, let's review what we learned about the flux from last time.

1. the flux is defined as $\Phi = \oint_S \vec{F} \cdot d\vec{A}$
2. If $S$ is a surface defined by $z = f(x,y)$, then $d\vec{A} = \langle -f_x, -f_y, 1 \rangle$
3. if the magnitude of $\vec{F}$ is constant, and $\vec{F}$ is always perpendicular to $S$, then $\Phi = \|\vec{F}\||S|$
4. depending on the surface, it may be more convient to use cylindrical or spherical coordinates
5. pay attention to the orientation of the surface (you may need to multiply by $-1$)

**Example 46** *Set up flux integral for* $\vec{F} = \langle y, 2x+1, y+z \rangle$ *and* $S$ *is the following surfaces. Here are* 6 *examples*

1. *S is bounded by* $y = 2 - (x^2 + z^2)$ *and* $y \geq 0$. *S is oriented inwards.*




*Solution:*

$$d\vec{A} = \langle 2x, -1, 2z\rangle$$

$$\Phi = \int_{x=-\sqrt{2}}^{x=\sqrt{2}} \int_{z=\sqrt{2-x^2}}^{z=\sqrt{2-x^2}} \langle 2-(x^2+z^2), 2x+1, 2-(x^2+z^2)+z\rangle \cdot \langle -2x, -1, -2z\rangle dz dx$$

2. $S : x^2 + y^2 = 9 \; -1 \leq y \leq 2$, *oriented outwards*

   *Solution:*
   *This is just a cylinder*

$$\Phi = \int_{\theta=0}^{\theta=2\pi} \int_{y=-1}^{y=2} \langle y, 2(3\cos\theta + 1), y + 3\sin\theta\rangle \cdot \langle \cos\theta, 0, \sin\theta\rangle 3 dy d\theta$$

3. $S : x = \sqrt{9 - x^2 - z^2}$, *oriented inward*
   *Solution:*
   *This is a half-hemisphere of radius* 3.

$$\Phi = \int_0^{2\pi} \int_0^{\pi} \langle 3\cos\theta\sin\phi, 2(3\sin\phi\cos\theta) + 1, 3\sin\phi\sin\theta + 3\cos\theta\rangle$$
$$\cdot \langle -\sin\phi\cos\theta, -\sin\phi\cos\theta, -\cos\phi\rangle 9\sin\phi d\phi\theta$$

4. $S : y = 1$ *over* $[-1, 2] \times [3, 4]$ *oriented towards positive* $y$.

$$\Phi = \int_{x=-1}^{x=2} \int_{z=3}^{z=4} \langle 1, 2x+1, 1+z\rangle \cdot \langle 0, 1, 0\rangle dz dx$$

5. $S$ : *equilateral triangle with vertices at* $(1,0,0),(0,2,0),(0,0,3)$ *oriented towards the origin. The equation of the surface is* $\frac{x}{1}+\frac{y}{2}+\frac{z}{3}=1$. *The normal vector is constantly* $\vec{n}=\langle -6,-3,-2\rangle$

$$\Phi = \int_{x=0}^{x=1}\int_{y=0}^{y=2-2x} \langle y2x+1, y+3-2x-3/2y\rangle \cdot \langle -3-3/2,-1\langle dydx$$

6. $S : y^2+z^2 \leq 16$ *on* $x=5$ *oriented downward.* $S$ *is a disk.*

$$\Phi = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=4} \langle r\cos\theta, 2(5)+1, r\cos\theta + r\sin\theta\rangle \cdot \langle -1,0,0\rangle r dr d\theta$$

### 21.2 19.3 – Divergence

**Definition 12** *The **divergence** is the flux density. If the divergence is positive, new fluid springs into existance at that point. If the divergence is negative, then fluid disappears at that point. It can be defined as the flux through a very small surface divided by the volume of the region enclosed which encloses that point. We might write*

$$div\vec{F} = \lim_{V(W)\to 0} \frac{\oint_S \vec{F}\cdot d\vec{A}}{V(W)}$$

**Example 47** *Let* $\vec{F} = z, 1+2y, x+3\rangle$ *find divergence of* $\vec{F}$ *at the origin. We can use any surface to find the diverence.* $S_1 : z=\sqrt{x^2+y^2}$ *or* $S_2$ : *cylinder from* $a<y<b$ *or* $S_3$ :
*We will do 2 examples*

**Theorem 9** *The divergence does not depend on what surface you choose. It only depends on the point* $p$ *and field* $\vec{F}$

## 22 Feb 23 2023

**Theorem 10** ***Divergence Theorem*** *If* $S$ *is a closed surface with interior region* $E$*, then* $\Phi$*, the flux through* $S$ *can be calculated as*

$$\Phi = \oint_S \vec{F}\cdot d\vec{A} = \iiint_E \nabla\cdot\vec{F}dV$$

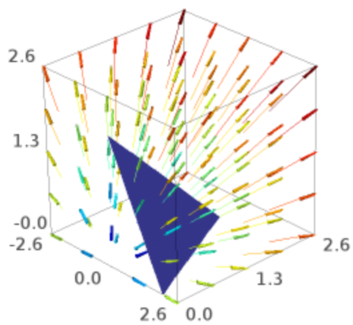
Let's start off with some examples to review divergence.

**Example 48** $\vec{F} = \langle e^{\cos(yz^{2023})}+x^2, 3y+\sin(e^{x^{2023}}), xyz\rangle$. *Let* $S_1 : z=x^2+y^2$ *and* $S_2 : z=4$. *Let* $S=S_1+S_2$ *where* $S$ *is oriented inwards. Find the flux.*
*Solution:*
*By the divergence theorem It's easier in polar coordinates*

$$\begin{aligned}\Phi &= \iiint_E \nabla\cdot\vec{F}dV\\ &= -\int_0^{2\pi}\int_{r=0}^{r=2}\int_{z=r^2}^{z=4}(2r\cos\theta+3)rdzdrd\theta\\ &= -\int_0^{2\pi}\int_0^{2}(4-r^2)(2r\cos\theta+3)rdzdrd\theta\\ &= -24\pi\end{aligned}$$

Now a simple example

**Example 49** *Let $S$ be the equilateral triangle in the 1st octant with veritices at $(2,0,0),(0,2,0),(0,0,2)$ oriented outwards. Let $\vec{F} = 3\langle x,y,z\rangle$. Note that the surface is NOT closed, so in order to use the divergence theorem, we must close the surface.*
*Solution:*




*We can use either direct computation of the flux, or use the divergence theorem if we close the solid ourselves.*

*First, direct computation.*
$\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 1 \implies z = 2 - x - y$

$$\begin{aligned}\Phi &= \int_{x=0}^{x=2}\int_{y=0}^{y=2-x} 3\langle x,y,2-x-y\rangle \cdot \langle 1,1,1\rangle dydx \\ &= \int_{x=0}^{x=2}\int_{y=0}^{y=1-x} 3(x+y+2-x-y)dydx \\ &= 12\end{aligned}$$

*Now, the divergence theorem. Close the solid as a triangular pyramid.*

$$\begin{aligned}\Phi_S &= \int_{x=0}^{x=2}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} 9dzdydx \\ &= 9V \\ &= 12\end{aligned}$$

*Now subtract away the other 3 surfaces (we aren't done yet!)*

$$\Phi_{S_1+S_2+s_3} = 0$$

*You could have evaluated this using geometric intuition because $\vec{F}\cdot d\vec{A}$ is constant, but I won't do that.*

**Example 50** *A greenhouse is in the shape of the graph $z = 9 - x^2 - y^2$ with the floor at $z = 0$. Suppose the temperatutre around the greenhouse is given by $T(x, y, z) = 2x^2 + 2y^2 + (z - 3)^2$. Let $\vec{H} = -\nabla T$ be the heat flux density field.*

*Use the divergence theorem to calculate the total heat flux outward across the boundary wall of the greenhouse?*

*Solution:*
$-\nabla T = -\langle 4x, 4y, 2z - 6\rangle$. $\nabla \cdot \vec{H} = -10$

$$\begin{aligned}\Phi &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=3} \int_{z=0}^{z=9-r^2} -10rdzdrd\theta \\ &= -405\pi\end{aligned}$$

*We now have to subtract the floor (we only want the wall).*

$$\begin{aligned}\Phi_{floor} &= \int_0^{2\pi} \int_0^3 \langle -4r\cos\theta, -4r\sin\theta, -2(0-3)\rangle \cdot \langle 0, 0, -1\rangle rdrd\theta \\ &= -54\pi\end{aligned}$$

# 23 Feb 27 2023

First, a mea culpa from last class. The integral for the half hemisphere really should be

$$\int_0^{\pi/2} \int_0^{2\pi} \langle 3\sin\phi\cos\theta, 2{\cdot}3\cos\phi{+}1, 3\sin\phi\cos\theta{+}3\sin\phi\cos\theta\rangle{\cdot}\langle -\cos\phi, -\sin\phi\cos\theta, -\sin\phi\cos\theta\rangle d\phi d\theta$$

## 23.1 Chapter 20 – Curl in $\mathbb{R}^3$

This chapter is just the 3D curl. Remember that

**Definition 13** ***Geometric Def of Circulation Density:***

$$circ_{\vec{n}}\vec{F}(x, y, z) = \lim_{Area \to 0} \frac{circulation\ around\ C}{Area\ inside\ C} = \frac{\int_C \vec{F} \cdot d\vec{r}}{A(C)}$$

*Whatever curve you choose doesn't matter, so choose the simplest one*

**Definition 14** ***Geometric Definition of Curl:*** *The curl at a point is a vector whose magnitude is equal to the circulation density at that point and whose direction is the direction of the normal vector $\vec{n}$ that maximizes the circulation density.*

**Definition 15** ***Algebraic Defininition of 3D Curl***

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

**Example 51** *Let $\vec{F} = \langle x, 1, x\rangle$. Find $circ_{\langle 0,0,1\rangle}\vec{F}(0, 0, 2)$.*
*Solution:*
*The curve doesn't matter, so let $C$ be a circle with center on the $z$ axis.*

$$circ_{\langle 0,0,1\rangle}\vec{F}(0, 0, 2) = \lim_{r\to 0} \frac{1}{\pi r^2} \int_0^{2\pi} \langle r\cos t, 1, r\cos t\rangle \cdot \langle -r\sin t, t\cos t, 0\rangle dt$$

Let's do an example using the algebaric defnintion

**Definition 16** *let $\vec{F} = \langle xyz, xy, y^2z \rangle$. Find $\nabla \times \vec{F}$.*
*Solution:*

$$\nabla \times \vec{F} = \langle 2yz - 0, xy - 0, y - xz \rangle$$

### 23.2 Stokes Theorem

Here is an interesting fact

**Theorem 11**

$$curl\vec{F} \cdot \vec{n} = circ_{\vec{n}}\vec{F}$$

Stokes Theorem is just the 3d version of Greens Theorem.

**Theorem 12 Stok*es Theorem*:** *Let $S$ be an oriented smooth surface bounded by a simple, closed, boundry curve $C$ with positive (CCW) orientation. Then,*

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot d\vec{A}$$

*it can be interpreted as*

$$\textit{circulation around boundary} = \textit{sum of circulation densities inside interior}$$

*Stokes theorem converts a line integral of $\vec{F}$ to a flux integral of $\nabla \times \vec{F}$.*

The right hand rule is very important for Stokes Theorem. The direction of the boundry must match the direction of the right hand rule for the normal vector $\vec{n}$. In other words, if you walk along $C$, the surface $S$ must always be on your left.

**Example 52** *let $C$ be the CCW circulation on the triangle with verticies at $(0, 0, 2)$, $(0, 2, 0)$ , and $(0, 0, 2)$. Let $\vec{F} = \langle xy, yz, zx \rangle$. Find $\oint_S \vec{F} \cdot d\vec{r}$.*
*Solution:*
*Let $S$ be the triangular region inside $C$. By Stoke's Theorem,*

$$\oint_S \vec{F} \cdot d\vec{r} =$$

## 24 Chapter 19 KC Solutions

Let $\vec{F}(x, y, z) = \langle x + 1, yz, y \rangle$. For each question find the flux of $\vec{F}$ thru $S$

1. let $S : y^2 + z^2 = 4, 1 \leq x \leq 3$, oriented inwards. Find flux of $\vec{F}$ thru $S$.
   Solution:
   This is a cylinder, use cylindrical coordinates

$$\Phi = \int_{\theta=0}^{\theta=2\pi} \int_{x=1}^{x=3} \langle x + 1, 4\sin\theta\cos\theta, 4\sin\theta \rangle \cdot \langle 0, -\cos\theta, -\sin\theta \rangle 2dxd\theta$$

2. Let $S: x^2+y^2+z^2=4, 1 \leq z \leq 2$, oriented downwards.
Solution:
This is a sphere (or a cut of a sphere). Use spherical coordinates

$$\Phi_S = \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi/3} \langle 2\sin\phi 2\sin\phi\sin\theta, 2\sin\phi\cos\theta\rangle \cdot -\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle 4\sin\phi d\phi d\theta$$

3. Let $S: y = x^2+z^2, y \leq 4$ oriented outwards
Solution :
This is a parabola. Project onto the $y=0$ plane. Use $-1$ because "outwards" means down on the $y=0$ plane.

$$\Phi_S = \int_{x=-2}^{x=2} \int_{z=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \langle x, x^2+z^2, z\rangle \cdot \langle 2x, -1, 2z\rangle dzdx$$

4. Compute flux density of $\vec{F}$ at $(0,0,0)$
Solution:

$$\nabla \cdot \vec{F} = 1 + z + 0 = 1$$

5. Compute the flux density of $\vec{F}$ at $(0,0,0)$ using the geometric definition with a closed surface which is composed of 2 surfaces. $S_1: x = y^2+z^2, x \leq a^2$ and $S_2: y^2+z^2 \leq a^2 where x = a^2$.
Solution:

$$\begin{aligned}
\Phi &= \int_{y=-a}^{y=a} \int_{z=-\sqrt{a^2-y^2}}^{z=\sqrt{a^2-y^2}} \langle y^2+z^1+1, yz, y\rangle \cdot \langle -1, 2y, 2z\rangle dzdy \\
&+ \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \langle a^2+1, r\cos\theta r\sin\theta, r\cos\theta\rangle \cdot \langle 1,0,0\rangle rdrd\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} \langle r^2+1, r\cos\theta r\sin\theta, r\cos\theta\rangle \cdot \langle -1, 2r\cos\theta, 2r\sin\theta\rangle rdrd\theta \\
&+ \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} (a^2+1) rdrd\theta \\
&= \frac{\pi}{2} a^2
\end{aligned}$$

Thus, the divergence is

$$V(a) = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=a} (a^2 - r^2) rdrd\theta$$

$$= \frac{\pi}{2} a^4$$

So

$$\nabla \cdot \vec{F} = \lim_{a \to 0} 1 = 1$$

6. Let $S^*$ be the surface defined by $z = \sqrt{x^2+y^2}$ where $0 \leq z \leq 2$ oriented outward.

7. without using the divergence theorem, set up an integral in polar coordinates to find the flux of $\vec{F}$ thru $S^*$.
   Solution:
   $S^*$ is just a cone.

$$\Phi_{S^*} = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} r\cos\theta + 1, r\sin\theta r, r\sin\theta\rangle \cdot \langle \cos\theta, \sin\theta, -1\rangle r dr d\theta$$

8. Compute the flux of $\vec{F}$ thru $S^*$ using the divergence theorem
   Solution :
   note that $S^*$ is not closed, so we need to close it ourselves. Define the disk $S_2 : 0 \leq r \leq 2, z = 2$ where $S_2$ is oriented upwards

$$\begin{aligned}\Phi_{S^*+S_2} &= \iiint_E \nabla \cdot \vec{F} d\vec{V} \\ &= \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \int_{z=r}^{z=2} (1+z) r dz dr d\theta \\ &= 20\pi/3\end{aligned}$$

   Now subtract $\Phi_{S_2}$

$$\begin{aligned}\Phi_{S_2} &= \int_0^{2\pi} \int_0^2 \langle r\cos\theta + 1, r\sin\theta 2, r\sin\theta\rangle \cdot \langle 0, 0, 1\rangle r dr d\theta \\ &= 0\end{aligned}$$

   Our answer is $20\pi/3$

## 25 Chapter 20 KC solutios

Let $\vec{F} = \langle 2x, e^{cosy}, yz\rangle$ and let $P = (0, 0, 1)$.

1. Is $\vec{F}$ a curl field?
   Solution:
   $\nabla \cdot \vec{F} = 2 - \sin y e^{\cos y} + y \neq 0$. $\vec{F}$ is not a curl field because every curl field has 0 divergence. (never justify that it is not a curl field using the divergence test).

2. What direction has the maximum circulation at $P$?
   Solution :
   This is simply the direction of the curl.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & e^{\cos y} & yz \end{vmatrix}$$

   So $\langle 1, 0, 0\rangle$

3. What is the maximum circulation density at $P$?
   Solution:
   This is just the norm of the curl, which is 1

4. Find the circulation density at $P$ in the direction of $\langle 1, 2, 3, \rangle$.
   Solution:
   Just take the dot product of the curl with the unit vector in this direction.

$$\nabla \times \vec{F} \cdot \langle 1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14} \rangle = \frac{1}{\sqrt{14}}$$

   Sanity check: this should be less than 1.

5. Let $C$ be a curve which is a triangle with veritcies at $(1,0,0), (0,1,0), (0,0,1)$ oriented counterclockwise. Use Stoke's theorem to find the work done by the vector field around $C$.
   Solution :
   By the right-hand rule, the normal vector should point towards the origin. The equation of the curve is $S : x + y + z = 1$, so $z = 1 - x - y$. We project the surface onto the $z = 0$ plane.

$$\oint_C \vec{F} \cdot d\vec{r}$$

$$\begin{aligned} &= \iint_S \nabla \times \vec{F} \cdot d\vec{A} \\ &= \int_{x=0}^{x=1} \int_{z=0}^{z=1-x} \langle 1 - x - y, 0, 0 \rangle \cdot \langle -1, -1, -1 \rangle dz dx \\ &= -\frac{1}{6} \end{aligned}$$

6. Assume that $\nabla \cdot \vec{F}(x, y, z) = 0$ and $\nabla \times \vec{F}(x, y, z) \neq 0$ and $D(\vec{F}) = \mathbb{R}^3 \setminus \{(1, 2, 3)\}$.

7. Does the curl test apply to $\vec{F}$?
   Solution:
   No, the curl of $\vec{F}$ is not 0. The domain condition is valid.

8. Does the divergence test apply to $\vec{F}$?
   Solution:
   No, although the divergence is 0, the domain condition is not satisfied.

9. is $\vec{F}$ a gradient vector field?
   No, the curl of every gradient vector field is 0, and this is not 0. (do NOT justify this by the curl test).

10. is $\vec{F}$ a curl field?
    Solution:
    We have no way of knowing. We cannot apply the divergence test, and $\nabla \cdot \vec{F} = 0$ so there is no way of knowing.

## 26 March 1 2023

Let's review that traingle problem from Monday.

We have the triangle with verticies at $(2,0,0),(0,2,0),(0,0,2)$. Let us choose the surface as the 3 triangluar surfaces on $z=0,x=0,y=0$. Then, we use Stoke's Theorem to get

$$\begin{aligned}
\oint_C \vec{F}\cdot d\vec{r} &= \int_{S_1+S_2+S_3} \nabla\times\vec{F}d\vec{A} \\
&= \int_0^2\int_0^{2-x} -\langle y,0,x\rangle\cdot\langle 0,0,1\rangle dydx \\
&+ \int_0^2\int_0^{2-y} -\langle y,z,0\rangle\cdot\langle 1,0,0\rangle dzdy \\
&+ \int_0^2\int_0^{2-x} -\langle 0,z,x\rangle\cdot\langle 0,1,0\rangle dzdx \\
&= \int_0^2\int_0^{2-x} -xdydx + \int_0^2\int_0^{2-y} -ydzdy + \int_0^2\int_0^{2-x} -zdzdx \\
&= \boxed{-4}
\end{aligned}$$

Remember that due to the right-hand-rule, the surface has to be oriented inward.

## 26.1 Curl Field

**Definition 17** ***Curl Field***: *We say a vector field $\vec{F}$ is a Curl Field iff*

$$\exists\vec{G}:\mathbb{R}^n\to\mathbb{R}^n \qquad \nabla\times\vec{G}=\vec{F}$$

*We call $\vec{G}$ the potential function of $\vec{F}$*

You should notice that curl field is very similar to a gradient field. Note that the potential function is not unique: there are infinitely many potential functions for each curl field. Note that you should never try to find the potential of a curl field because it will be very painful to solve all the differential equations.

**Example 53** *Let $\vec{F}=\langle 0,-1,-x\rangle$. Is $\vec{F}$ a curl field?*
*Solution: yes*

Here is a useful theorem for determining if a vector field is a curl field.

**Theorem 13**

$$\vec{F} \textit{ is a curl field} \implies \nabla\cdot\vec{F}=0$$

By contrapositive

**Theorem 14**

$$\nabla\cdot\vec{F}\neq 0 \implies \vec{F} \textit{ is not a curl field}$$

The converse of this theorem is false, so only use it 1 way. Here is a counter-example to caution against using the theorem in reverse.

**Example 54** *Let $\vec{F}=\frac{\vec{r}}{\|\vec{r}\|^3}$. $\nabla\times\vec{F}=0$. But, $\vec{F}$ is not a curl field. This means that $\vec{F}$ is a coutner-example to the inverse of the theorem.*

*Use a geometric relation to see that $\vec{F}$ is not a curl field. Let $S$ be the surface of a sphere with center at the origin. $\Phi_S=4\pi\neq 0$*

Note that the divergene theorem is not applicable in this case because $\vec{F}$ is not defined at $\langle 0,0,0\rangle$

A common technique if a vector field has a limited domain is to define 2 surfaces and do the calculation in between the 2 surfaces to avoid the undefined area. Here is the updated theorem

**Theorem 15** ***Divergence Test***: *If*

1. $\nabla \cdot \vec{F} = 0$
2. *domain condition: Every closed surface in $D(\vec{F})$ has its interior entirely in $D(\vec{F})$.*

*then $\vec{F}$ is a curl field.*

**Example 55** *Let $D(\vec{F}) = \mathbb{R}^3 \backslash \{(1,2,3),(5,6,7)\}$. We cannot conclude that $\vec{F}$ is a curl field because it has holes in its domain.*

**Example 56** *Let $D(\vec{F}) = \mathbb{R}^3 \setminus \{(0,k,0)|k \in \mathbb{R}\}$. Despite having holes, $\vec{F}$ does satisfy the domain condition, so we can apply the divergence test.*

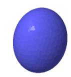
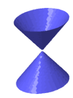
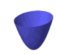
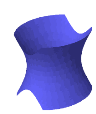
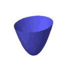
The curl test also works in $\vec{R}^3$

**Theorem 16** ***Curl Test***: *If*

1. $\nabla \times \vec{F} = \vec{0}$
2. *domain condition: Every smooth curl in $D(\vec{F})$ can be contracted to a point, staying within $D(\vec{F})$ at all times.*

*then $\vec{F}$ is a gradient vector field.*

**Example 57** *Let $D(\vec{F}) = \mathbb{R}^2 \setminus \{(x,0)|x \in \mathbb{R}\}$. Here, the domain condition is valid.*

| Name | Equation | Graph |
|---|---|---|
| Ellipsoid | $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2 2} = 1$ |  |
| Cone | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$ |  |
| Elliptic Paraboloid | $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$ |  |
| Hyperboloid of One Sheet | $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ |  |
| Hyperboloid of Two Sheets | $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{z}{c}$ |  |
| Hyperbolic Paraboloid | $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ | |