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hosted by Prof. Yutaka Yamamoto
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Hybrid and Switched Systems: Modeling and Analysis

João P. Hespanha

University of California
at Santa Barbara

UCSB

References

The following slides were adapted from the course

ECE594D— *Hybrid Control and Switched systems*

taught at the University of California, Santa Barbara during the Spring of 2002.

A fairly complete list of references can be found in the courses web page:

http://www.ece.ucsb.edu/~hespanha/ece594d-hybrid/

but most of the material taught is covered by the following references:

[1] A. van der Schaft and H. Schumacher. *An Introduction to Hybrid Dynamical Systems*. Lecture Notes in Control and Information Sciences 251, Springer-Verlag, 2000.

[2] J. Hespanha. *Encyclopedia of Life Support Systems*, Chapter Stabilization Through Hybrid Control. Feb. 2001. To appear

[3] J. Hespanha. Tutorial on Supervisory Control. Lecture notes for the tutorial workshop "Control Using Logic and Switching" offered at the 40th Conf. on Decision and Control, Orland, FL, Dec. 2001.

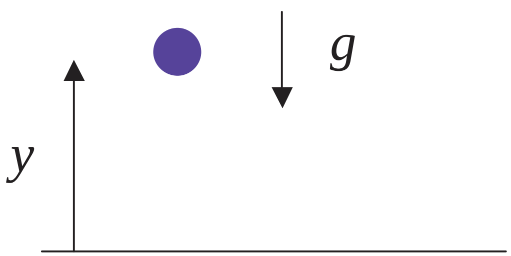
The references [2] and [3] can be found in the publications section of my web page:

http://www.ece.ucsb.edu/~hespanha/

Outline

1. **Examples of hybrid systems**
   - Bouncing ball
   - Thermostat
   - Transmission
   - Multiple-tank
   - Supervisory control
2. **Formal models for hybrid systems**
3. **Switched systems**
   - Motivation
   - Linear switched systems
   - Lyapunov stability, asymptotic stability, exponential stability
4. **Stability of switched systems under arbitrary switching**
   - Instability caused by switching
   - Common Lyapunov function
   - Converse results
   - Algebraic conditions
5. **Controller realizations for stable switching**
6. **Stability under state-dependent switching**
   - State-dependent common Lyapunov function
   - Multiple Lyapunov functions
   - LaSalle’s invariance principle

Example #1: Bouncing ball



Free fall $\equiv \quad \ddot{y} = -g$

Collision $\equiv \quad y^+(t) = y^-(t) = 0$

$$\dot{y}^+(t) = -c\dot{y}^-(t)$$

$c \in [0,1] \equiv$ energy absorbed at impact

Notation: given $x : [0,\infty) \rightarrow \mathbb{R}^n \equiv$ piecewise continuous signal

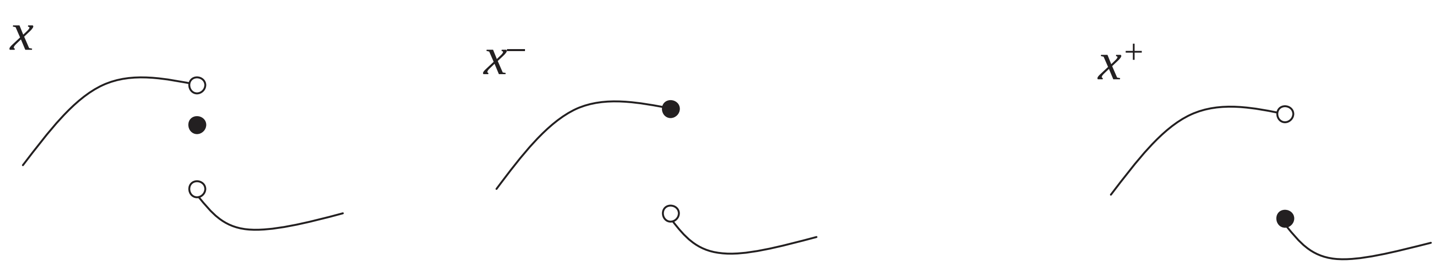
$$x^- : (0,\infty) \rightarrow \mathbb{R}^n \qquad x^-(t) := \lim_{\tau \uparrow t} x(t), \quad \forall t > 0$$

$$x^+ : [0,\infty) \rightarrow \mathbb{R}^n \qquad x^+(t) := \lim_{\tau \downarrow t} x(t), \quad \forall t \geq 0$$

at points t where x is continuous $x(t) = x^-(t) = x^+(t)$

By convention we will generally assume right continuity, i.e.,

$$x(t) = x^+(t) \qquad \forall\, t \geq 0$$



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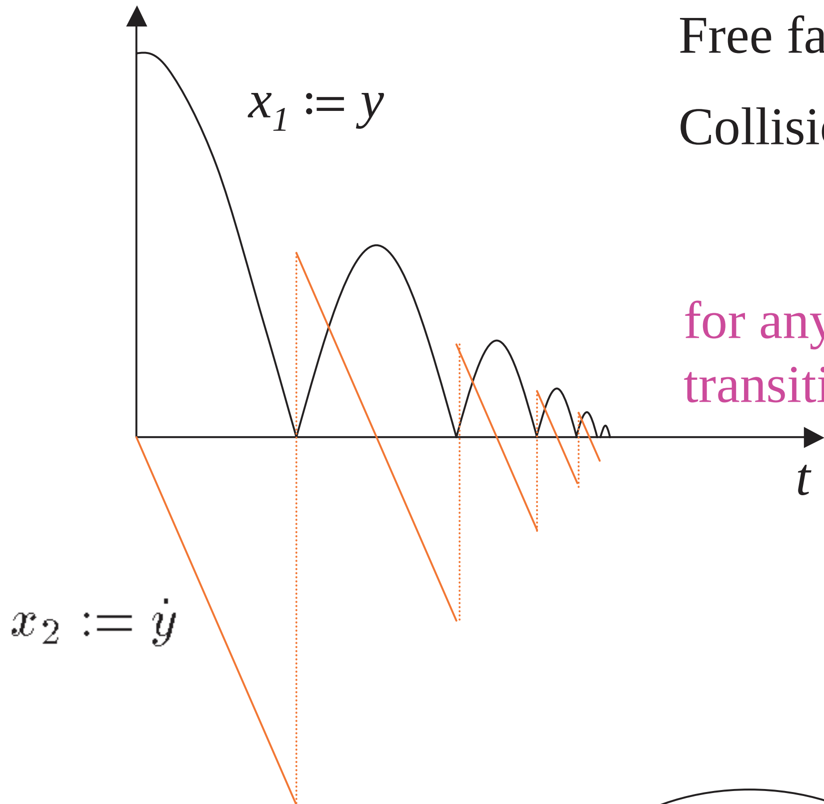
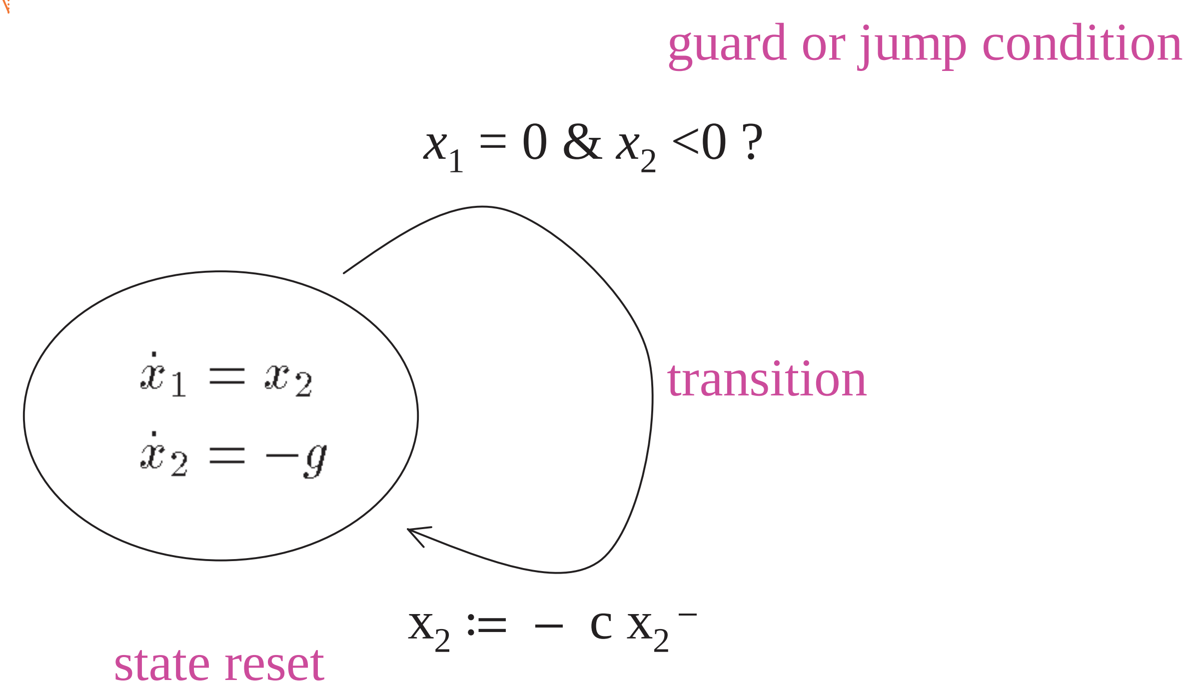


Free fall $\equiv \quad \ddot{y} = -g$

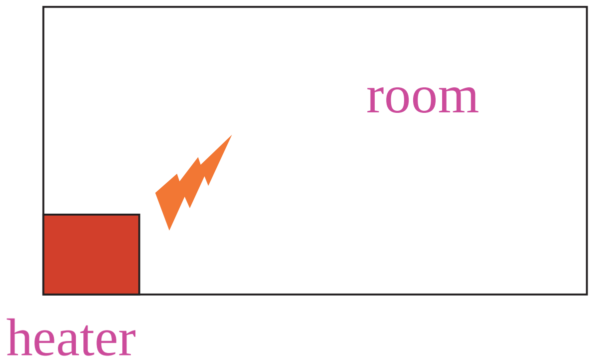
Collision $\equiv \quad y^+(t) = y^-(t) = 0$

$\dot{y}^+(t) = -c\dot{y}^-(t)$

for any c < 1, there are infinitely many transitions in finite time (Zeno phenomena)



Example #2: Thermostat



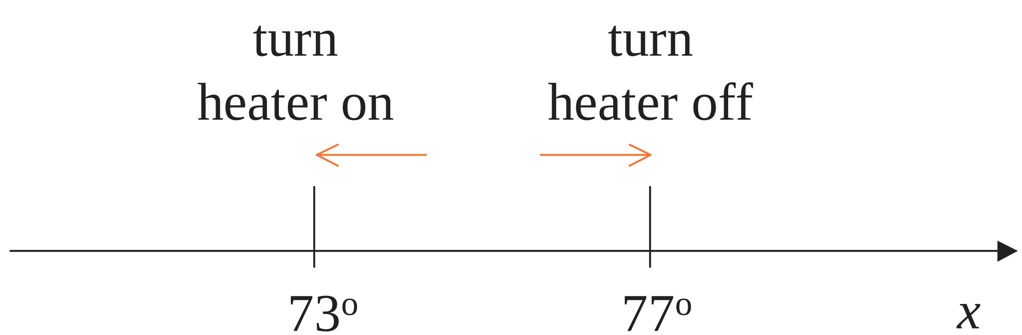
goal $\equiv$ regulate temperature around 75°

$x \equiv$ mean temperature

when heater is off: $\dot{x} \approx -x + 50$ ($x \to 50°$)

when heater is on: $\dot{x} \approx -x + 100$ ($x \to 100°$)

event-based control



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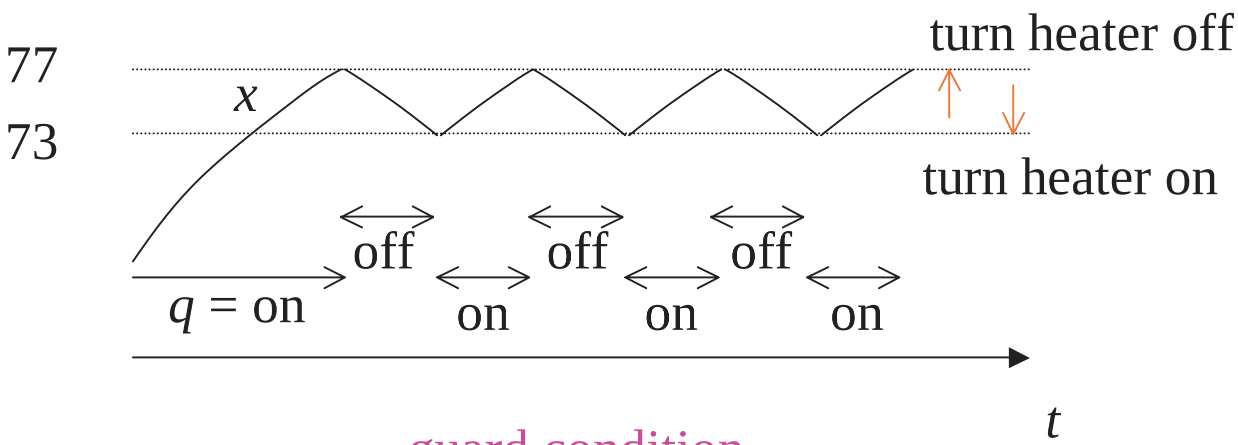
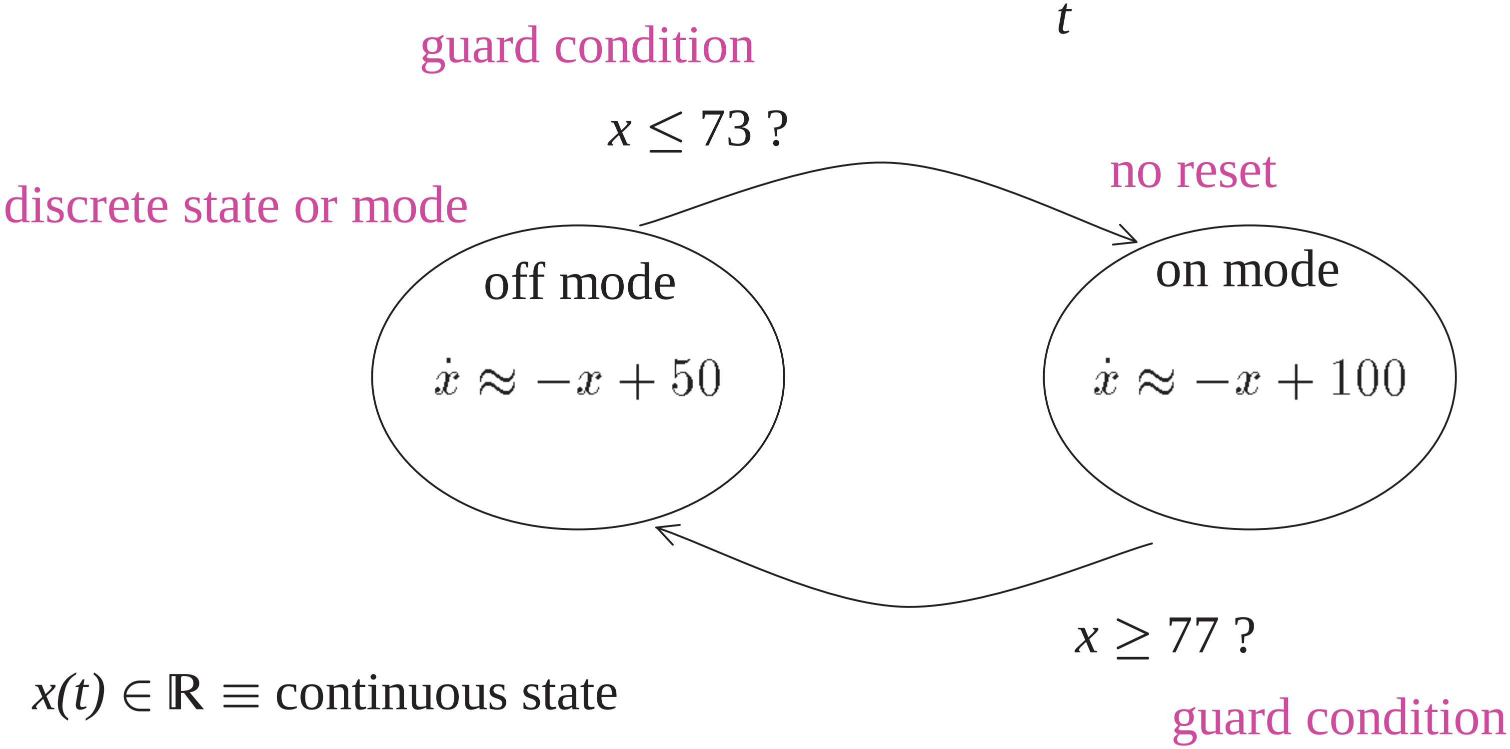




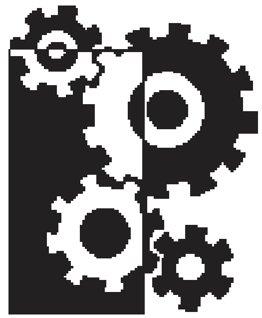
$x(t) \in \mathbb{R} \equiv$ continuous state

$q(t) \in \{$ off, on $\} \equiv$ discrete state

Example #3: Transmission

throttle
$u \in [-1,1]$

$g \in \{1,2,3,4\}$
gear

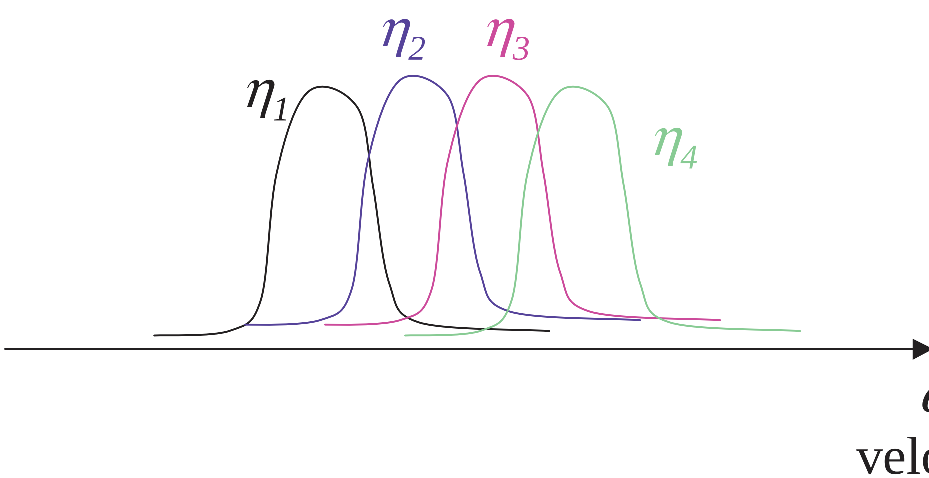


position
θ

ω
velocity

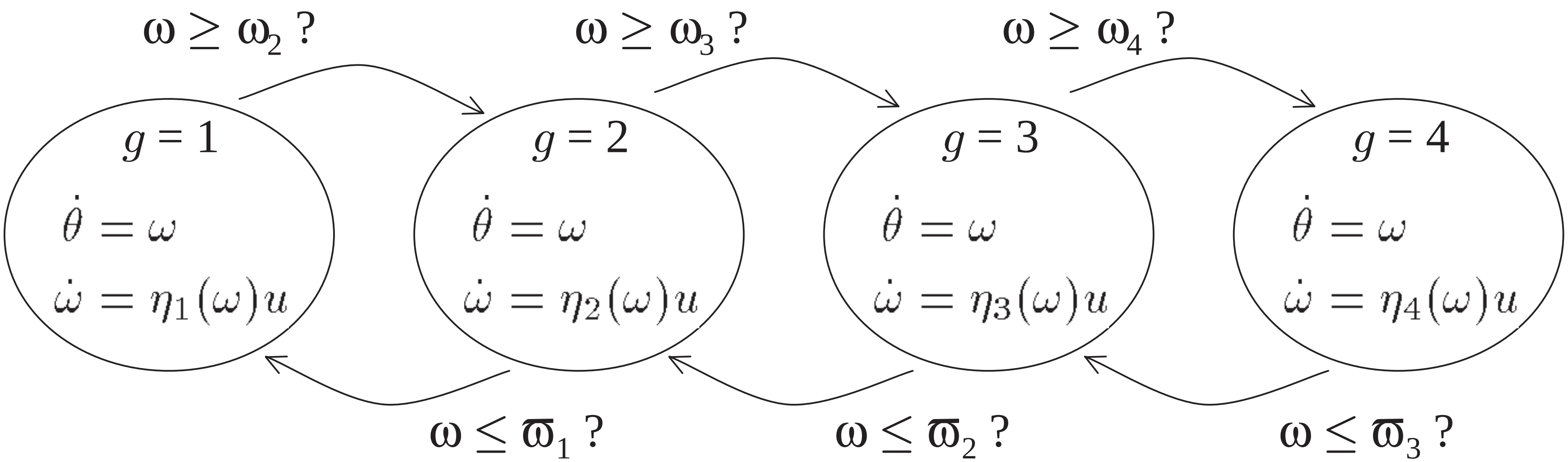
$$\dot{\theta} = \omega$$
$$\dot{\omega} = \eta_g(\omega)u$$

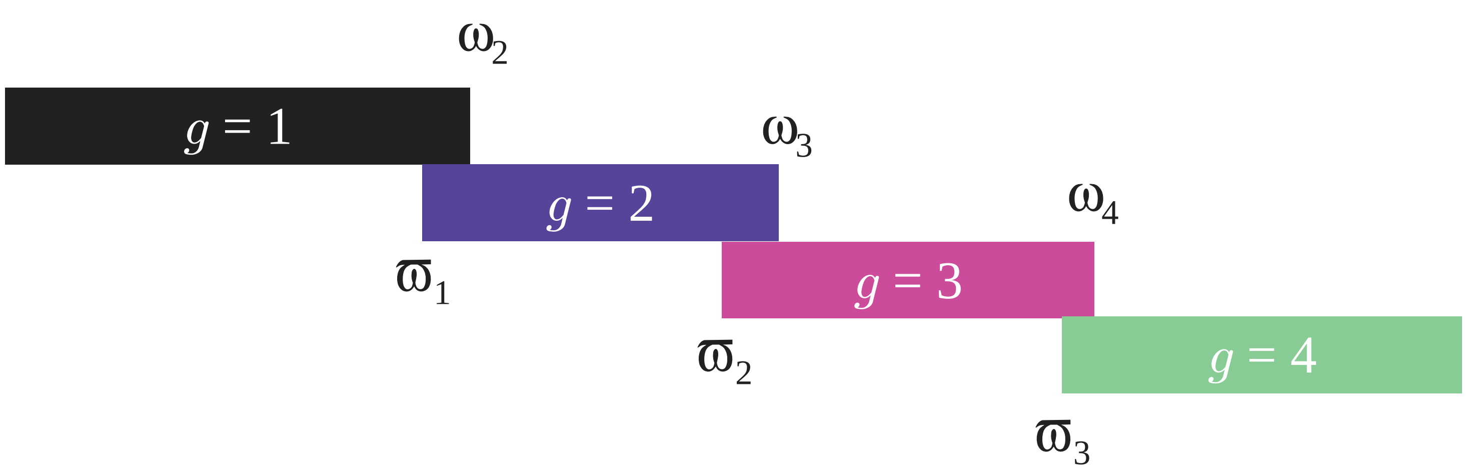
$\eta_k \equiv$ efficiency of the k^{th} gear



[Hedlund, Rantzer 1999]

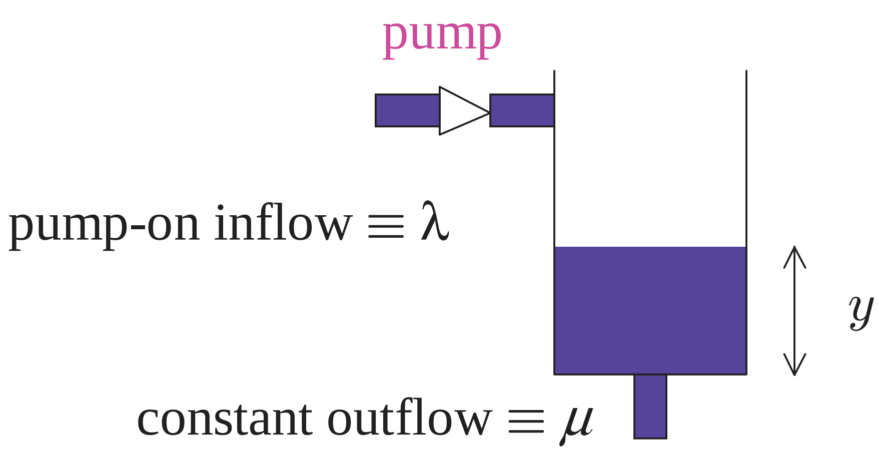
Example #3: Automatic transmission





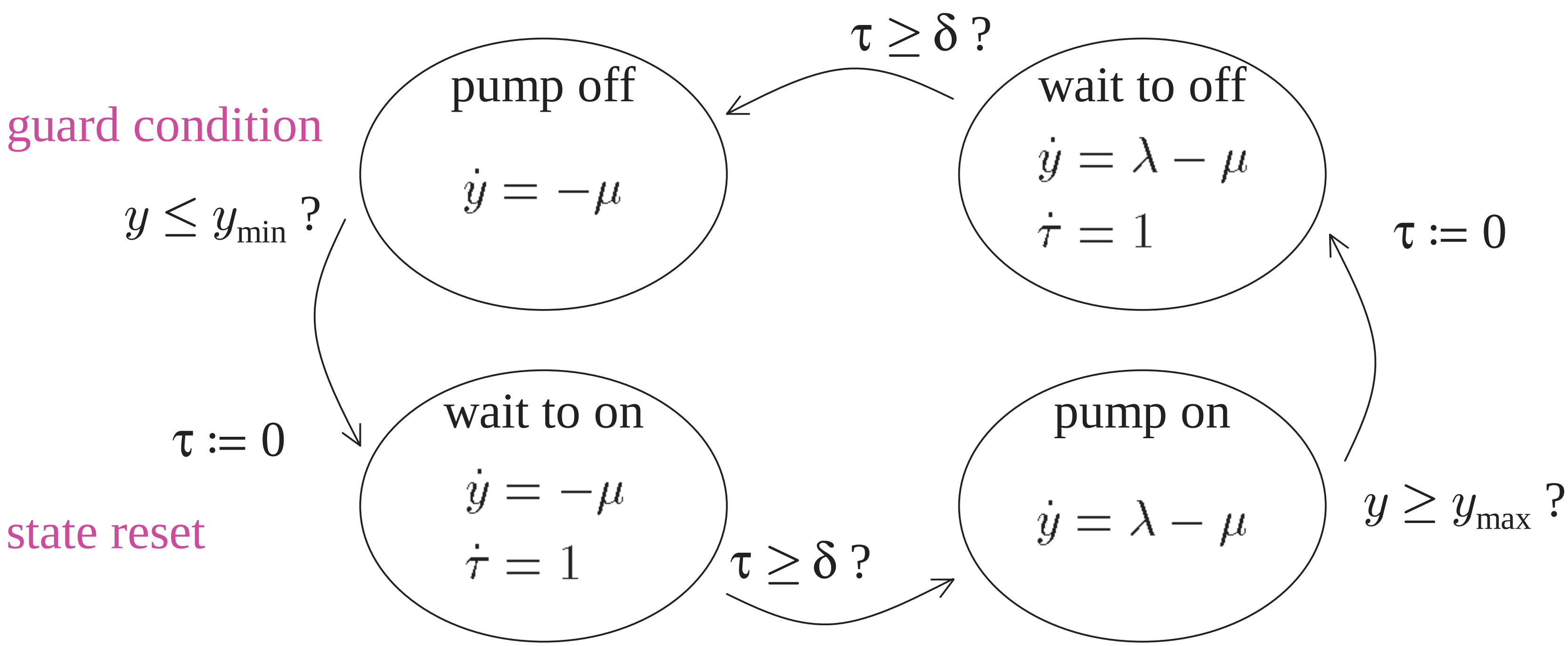
Example #4: Tank system

*



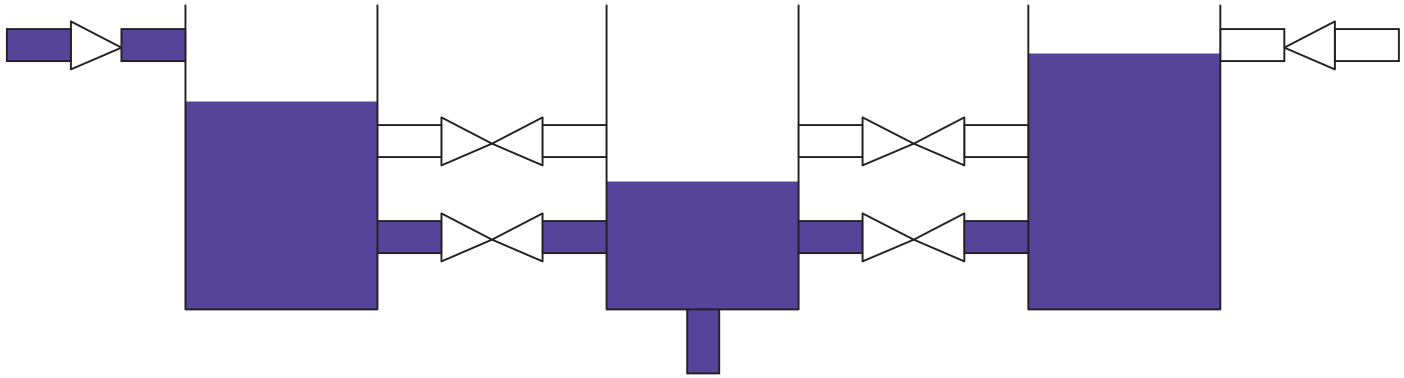
goal $\equiv$ prevent the tank from emptying or filling up

$\delta \equiv$ delay between command is sent to pump and the time it is executed



How to choose $y_{\min}$ and $y_{\max}$ for given μ, λ, δ ?

Example #4: Multiple-tank system



$y_1 \leq y_{1,\min}$ & $y_2 \geq y_{2,\max}$?

$\tau \geq \delta_k$?

$\tau := 0$

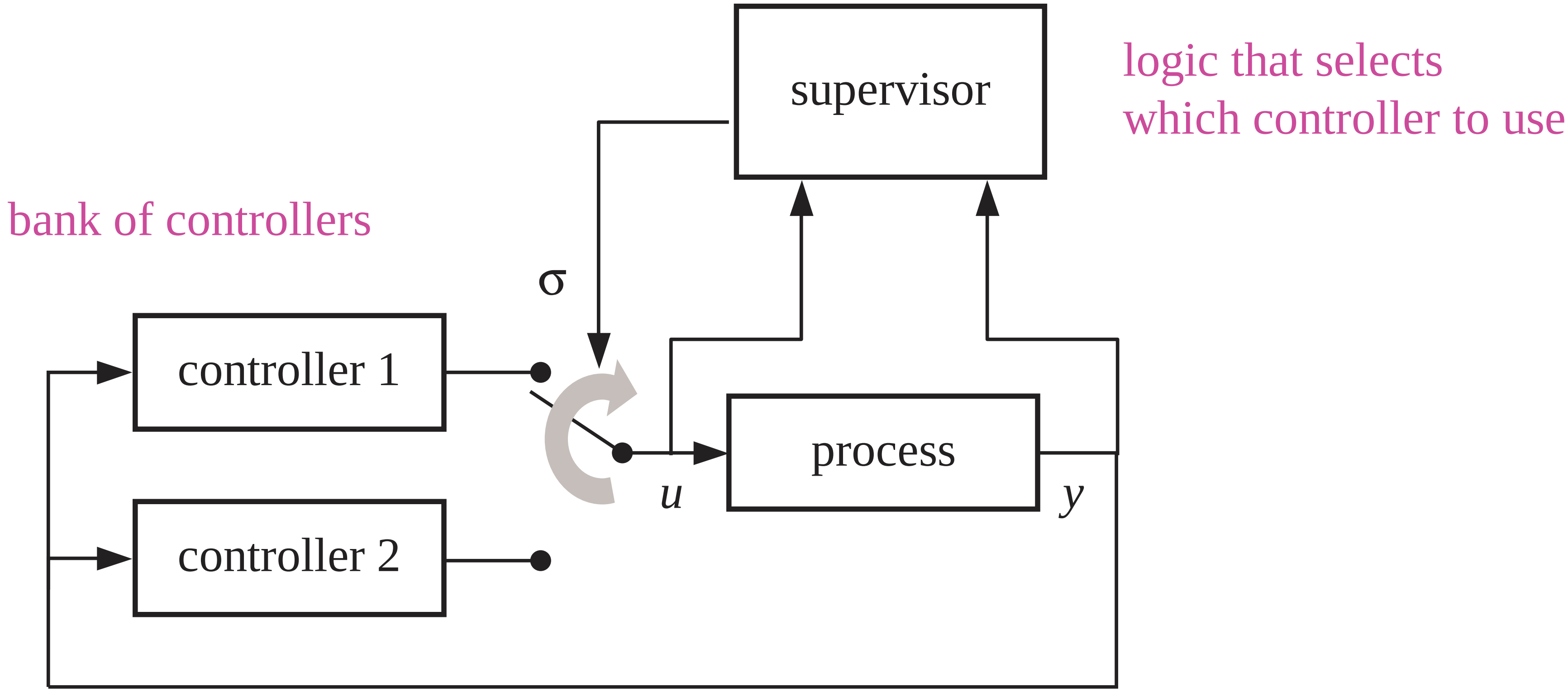
$$\dot{y}_1 = c_1$$
$$\dot{y}_2 = c_2$$
$$\dot{y}_3 = c_3$$
$$\vdots$$
$$\dot{\tau} = 1$$

Initialized rectangular hybrid automata

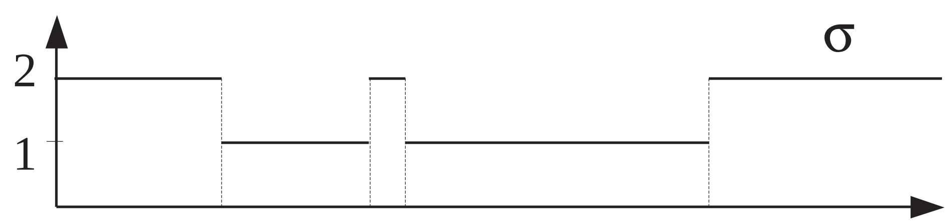
- all differential equations have constant r.h.s.
- all jump cond. are of the form: state var. 1 $\in$ fixed interval 1 & state var. 2 $\in$ fixed interval 2 & etc.
- all resets have constant r.h.s.

Most general class of hybrid systems for which there exist completely automated procedures to compute the set of reachable states

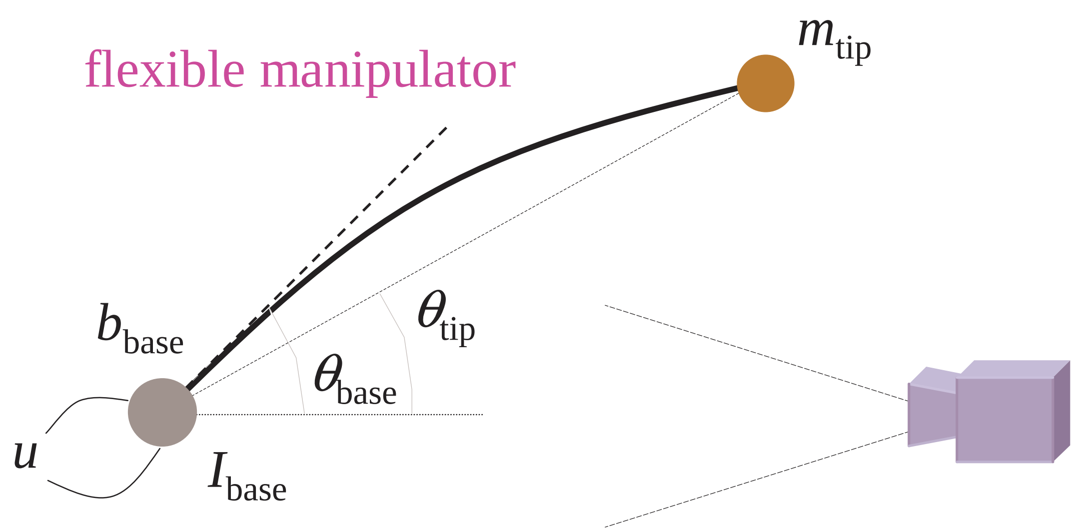
Example #5: Supervisory control



$\sigma \equiv$ switching signal taking values in the set $\{1,2\}$



E.g. #5 a): Vision-based control of a flexible manipulator



goal $\equiv$ drive θ_{tip} to zero, using feedback from

θ_{base} $\rightarrow$ encoder at the base

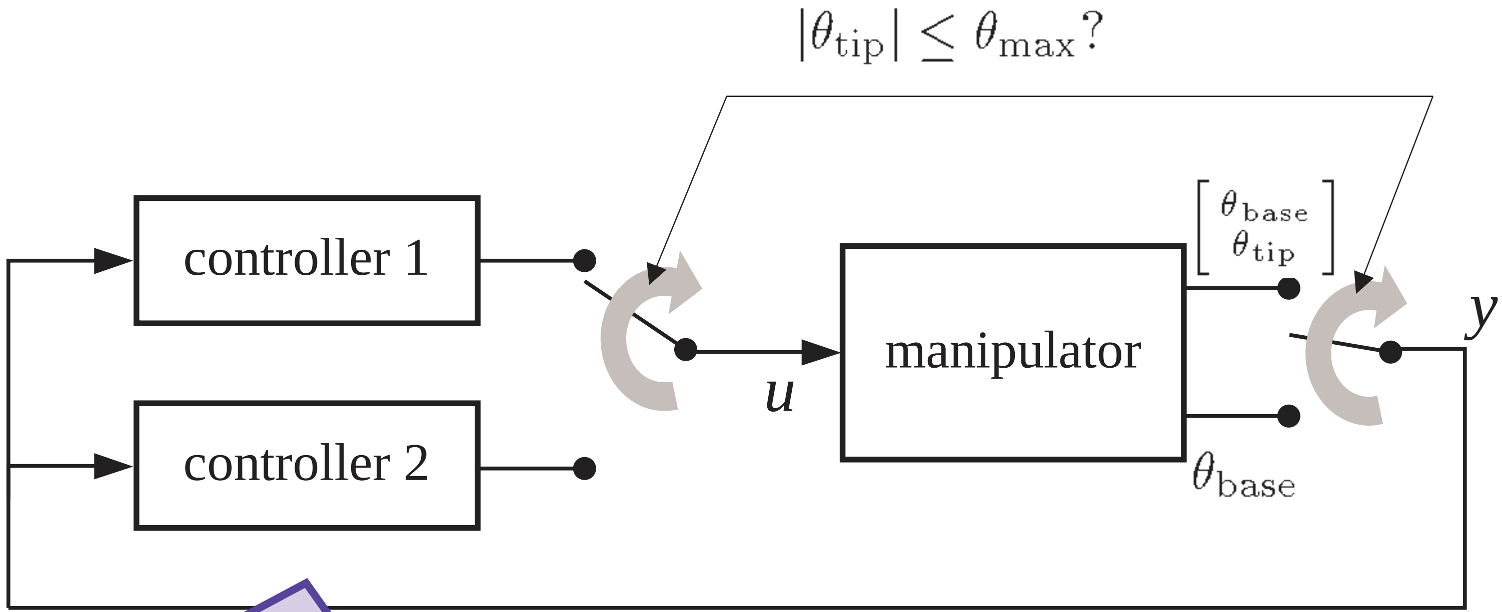
θ_{tip} $\rightarrow$ machine vision (needed to increase the damping of the flexible modes in the presence of noise)

To achieve high accuracy in the measurement of θ_{tip} the camera must have a **small field of view**

output feedback output:

$$y := \begin{cases} \begin{bmatrix} \theta_{\text{base}} \\ \theta_{\text{tip}} \end{bmatrix} & |\theta_{\text{tip}}| \leq \theta_{\max} \\ \theta_{\text{base}} & |\theta_{\text{tip}}| > \theta_{\max} \end{cases}$$

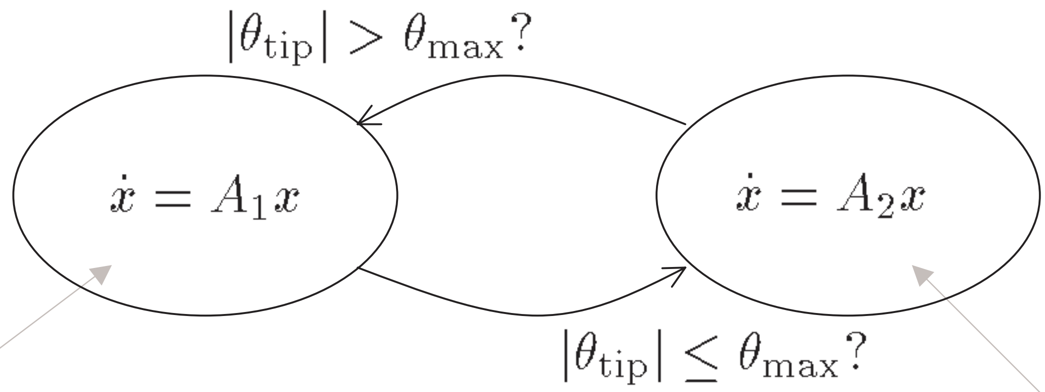
E.g. #5 a): Vision-based control of a flexible manipulator



controller 1 optimized for feedback from θ_{base} and θ_{tip}
and
controller 2 optimized for feedback only from θ_{base}

E.g., LQG controllers that minimize $\lim_{T\to\infty} \frac{1}{T} E\left[\int_0^T \theta_{\text{tip}}^2 + \dot{\theta}_{\text{tip}}^2 + \rho u^2 dt\right]$

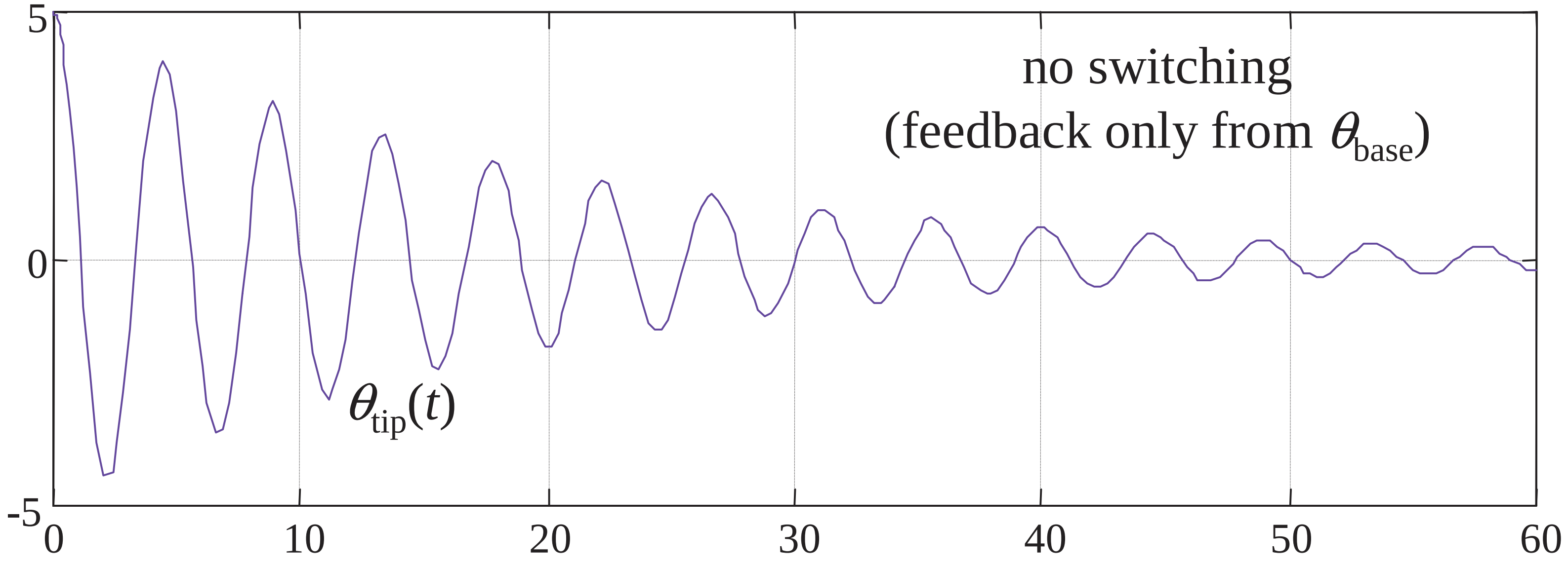
E.g. #5 a): Vision-based control of a flexible manipulator

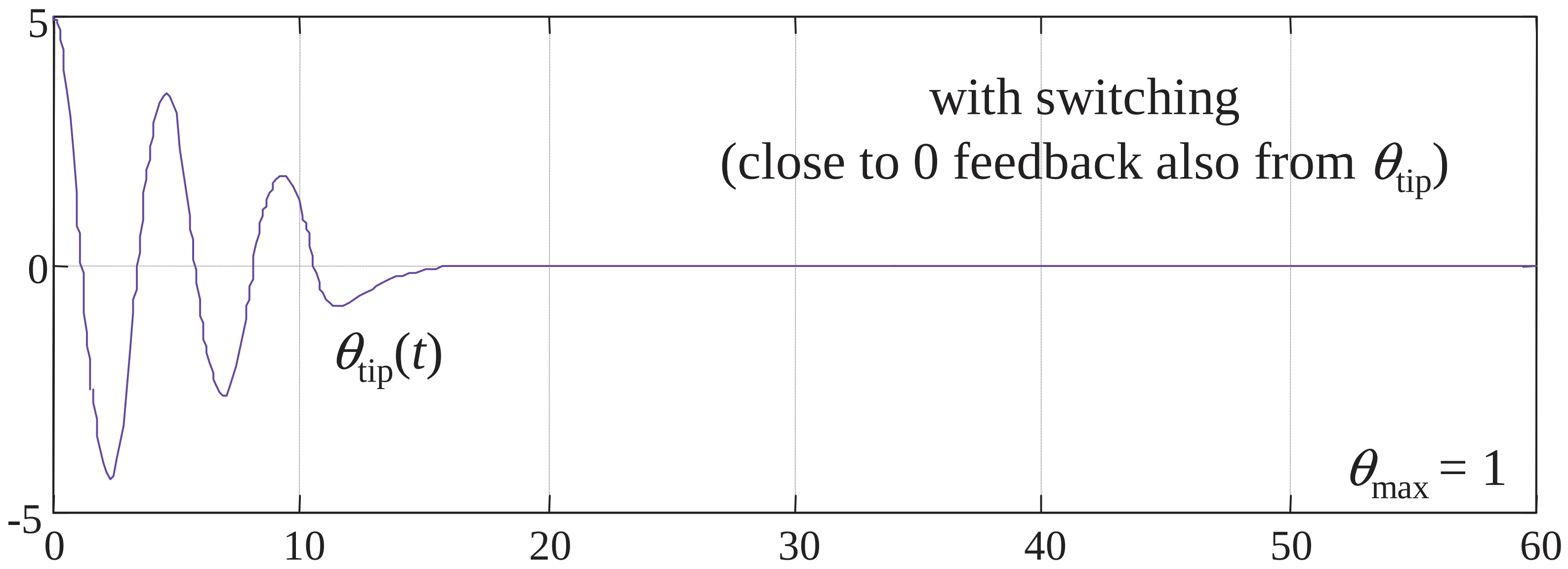


feedback connection with
controller 1
(θ_{base} and θ_{tip} available)

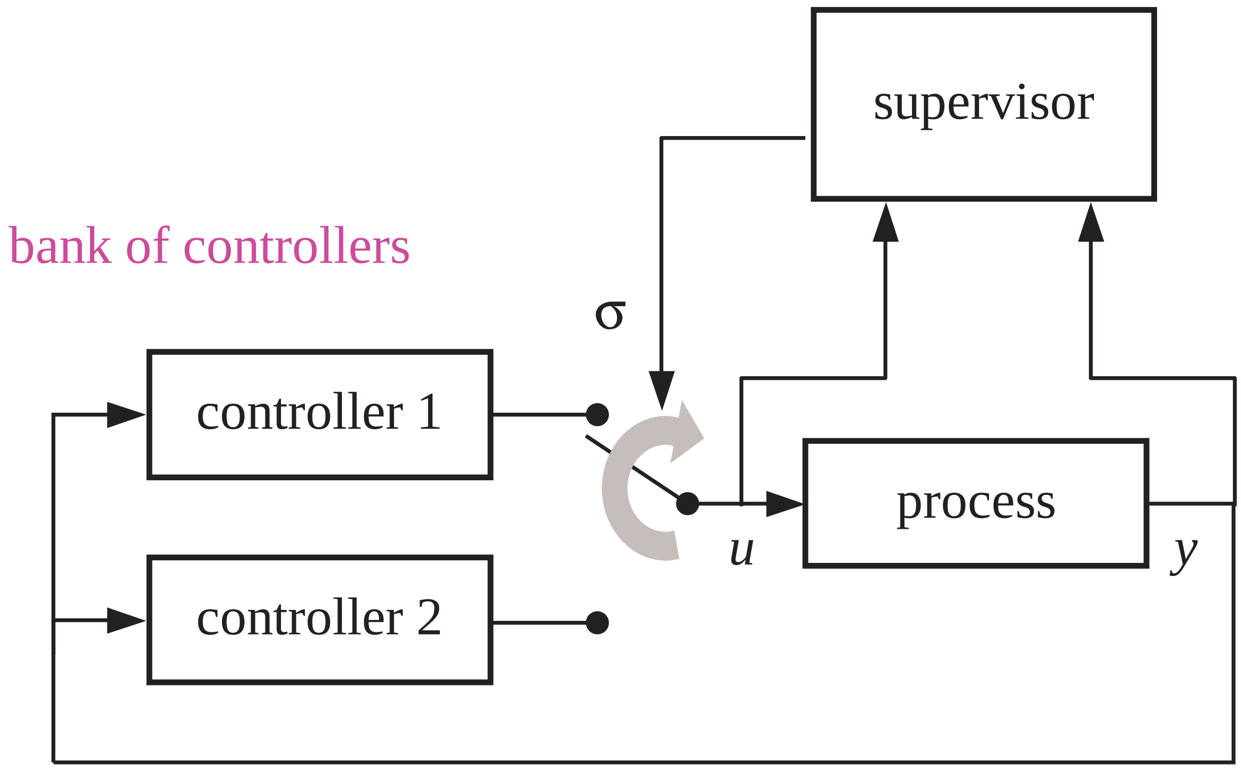
feedback connection
with controller 2
(only θ_{base} available)

E.g. #5 a): Vision-based control of a flexible manipulator





Example #5 b): Adaptive supervisory control



process can either be:

$$\dot{x} = A_1 x + b_1 u$$
$$y = c_1 x$$

or

$$\dot{x} = A_2 x + b_2 u$$
$$y = c_2 x$$

Goal: stabilize process, regardless of which is the actual process model

Supervisor must
- try to determine which is the correct process model by observing *u* and *y*
- select the appropriate controller

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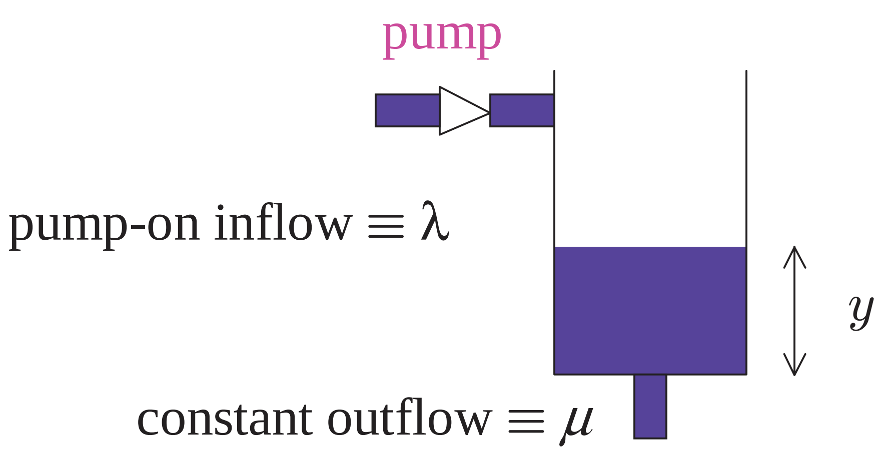
Example #4: Multiple-tank system



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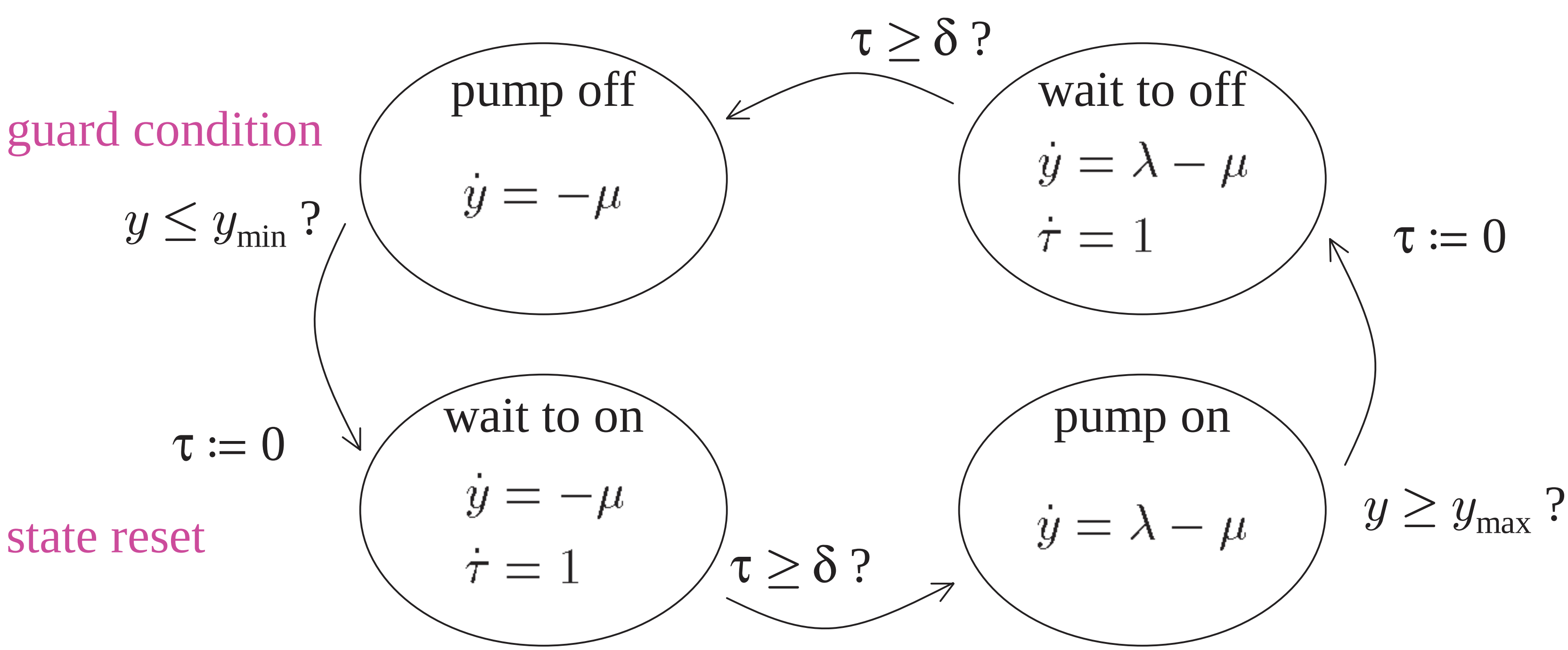
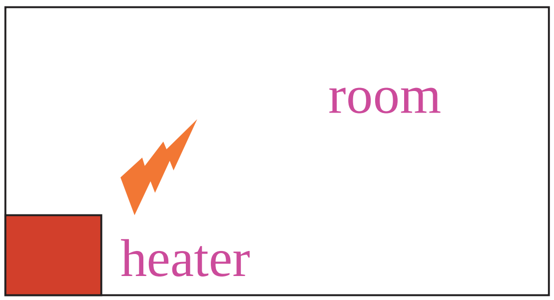
$\delta \equiv$ delay between command is sent to pump and the time it is executed

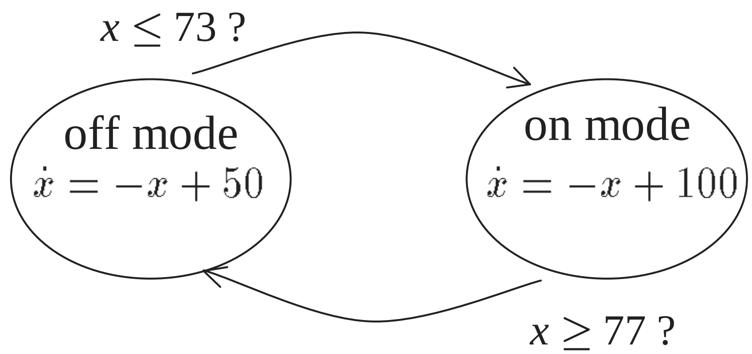


How to formally describe this hybrid system?

Hybrid Automaton
(Example #2: Thermostat)

$x \equiv$ mean temperature





$\mathcal{Q} \equiv$ set of discrete states

$\mathbb{R}^n \equiv$ continuous state-space

$f : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n \equiv$ vector field

$\varphi : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q} \equiv$ discrete transition

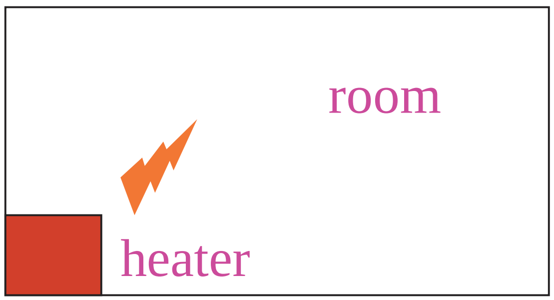
Example: $\mathcal{Q} := \{ \text{off}, \text{on} \}$ $\; n := 1$

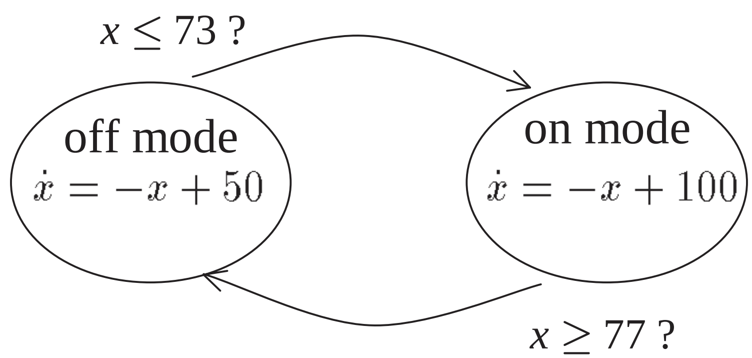
$$f(q,x) := \begin{cases} -x + 50 & q = \text{off} \\ -x + 100 & q = \text{on} \end{cases} \qquad \varphi(q,x) := \begin{cases} \text{on}, & q = \text{off}, x \leq 73 \\ \text{off} & q = \text{off}, x > 73 \\ \text{off}, & q = \text{on}, x \geq 77 \\ \text{on}, & q = \text{on}, x < 77 \end{cases}$$

Hybrid Automaton

(Example #2: Thermostat)

$x \equiv$ mean temperature





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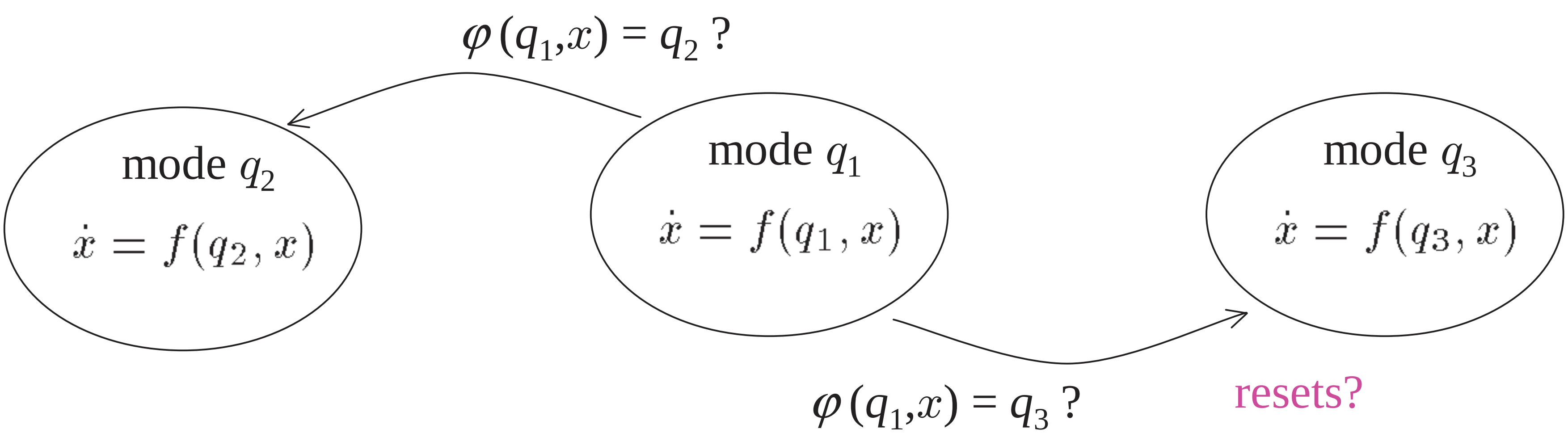
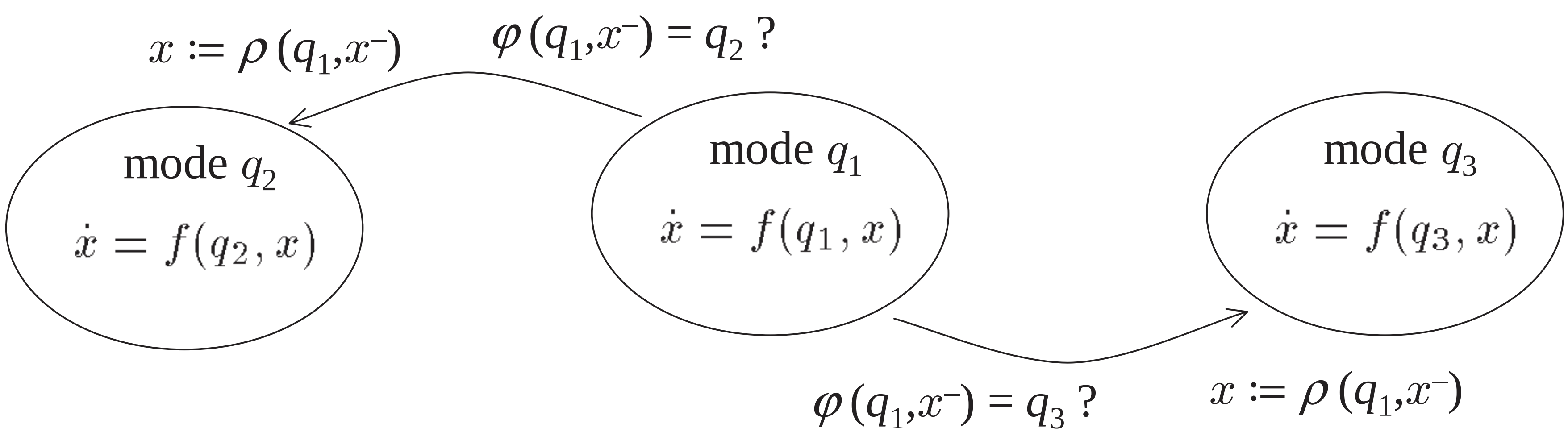


Hybrid Automaton

$$
\begin{aligned}
&\mathcal{Q} && \equiv \text{set of discrete states} \\
&\mathbb{R}^n && \equiv \text{continuous state-space} \\
&f : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n && \equiv \text{vector field} \\
&\varphi : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q} && \equiv \text{discrete transition} \\
&\rho : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n && \equiv \text{reset map}
\end{aligned}
$$



at every transition
we have:

$$
\begin{cases}
q = \varphi(q^-, x^-) \\
x = \rho(q^-, x^-)
\end{cases}
$$

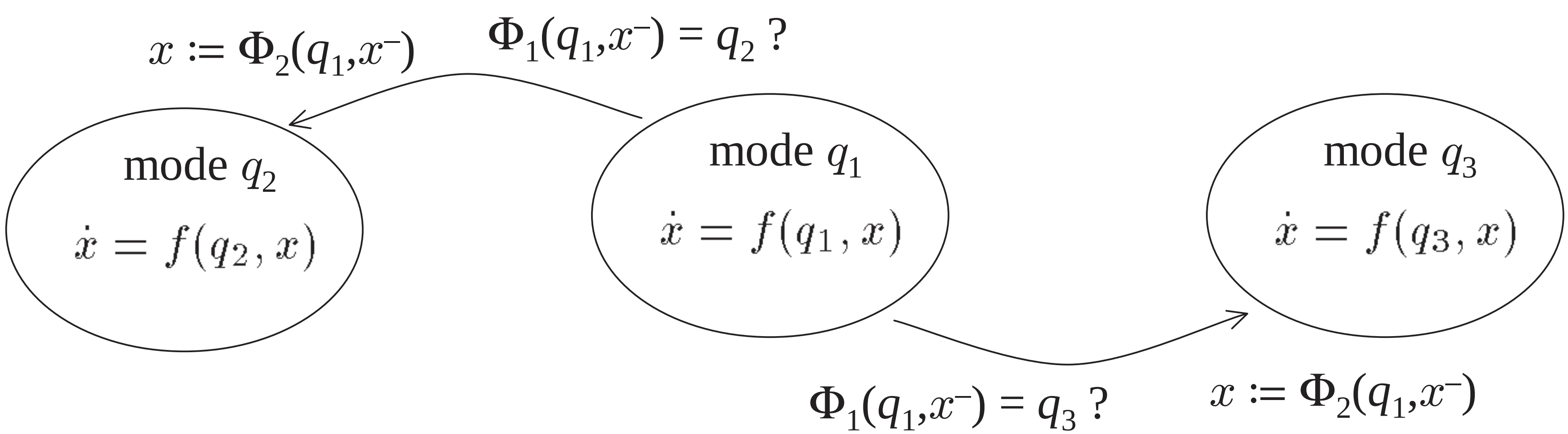
between transitions
we have:

$$\dot{x} = f(q, x)$$

Hybrid Automaton

$\mathcal{Q}$	$\equiv$ set of discrete states
$\mathbb{R}^n$	$\equiv$ continuous state-space
$f : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$	$\equiv$ vector field
$\Phi : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q} \times \mathbb{R}^n$	$\equiv$ discrete transition (& reset map)

$$\Phi(q,x) = \begin{bmatrix} \Phi_1(q,x) \\ \Phi_2(q,x) \end{bmatrix} = \begin{bmatrix} \varphi(q,x) \\ \rho(q,x) \end{bmatrix}$$



at every transition
we have:

$$(q,x) = \Phi(q^-, x^-)$$

between transitions
we have:

$$\dot{x} = f(q,x)$$

Hybrid Automaton

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$\mathbb{R}^n$	$\equiv$ continuous state-space
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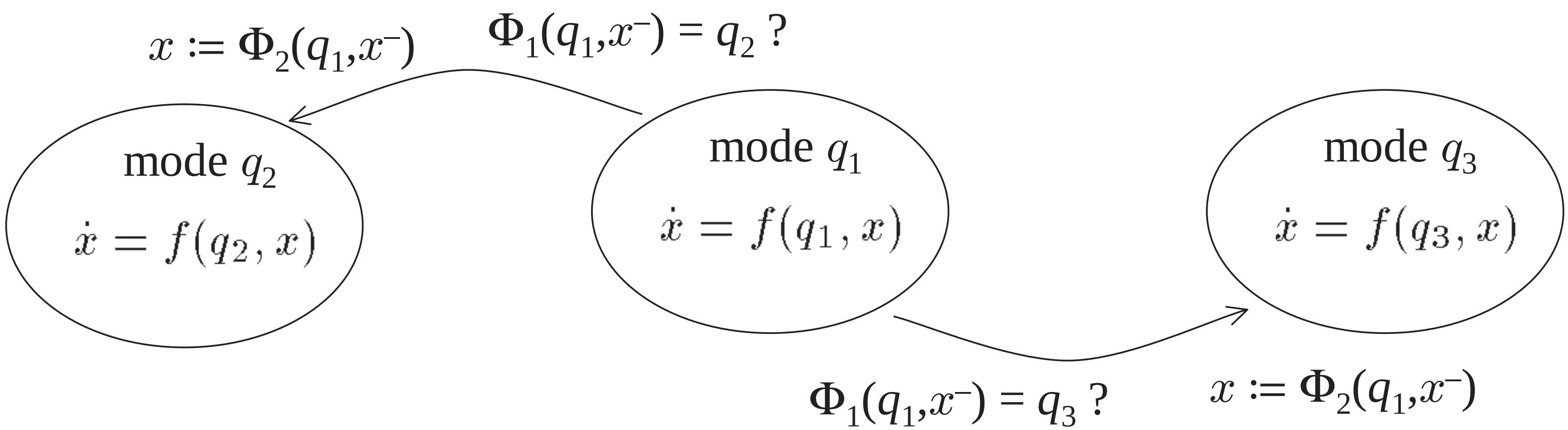
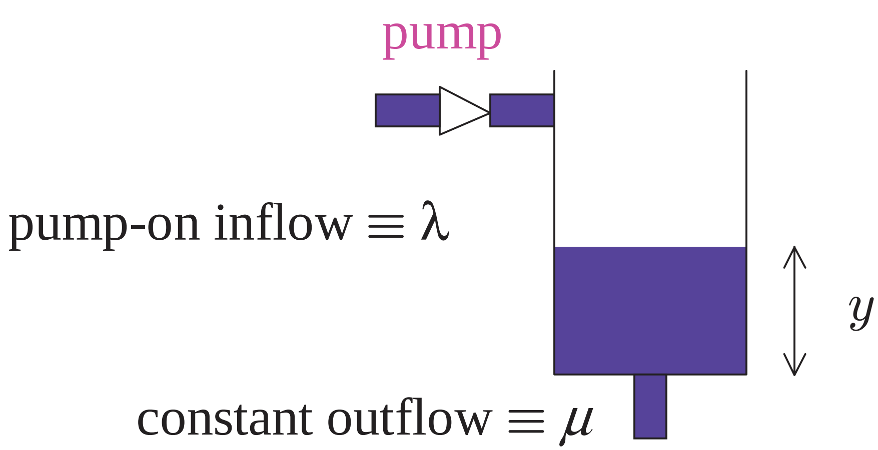


Compact representation of a hybrid automaton

$$\dot{x} = f(q,x) \qquad (q,x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$

Example #5: Multiple-tank system



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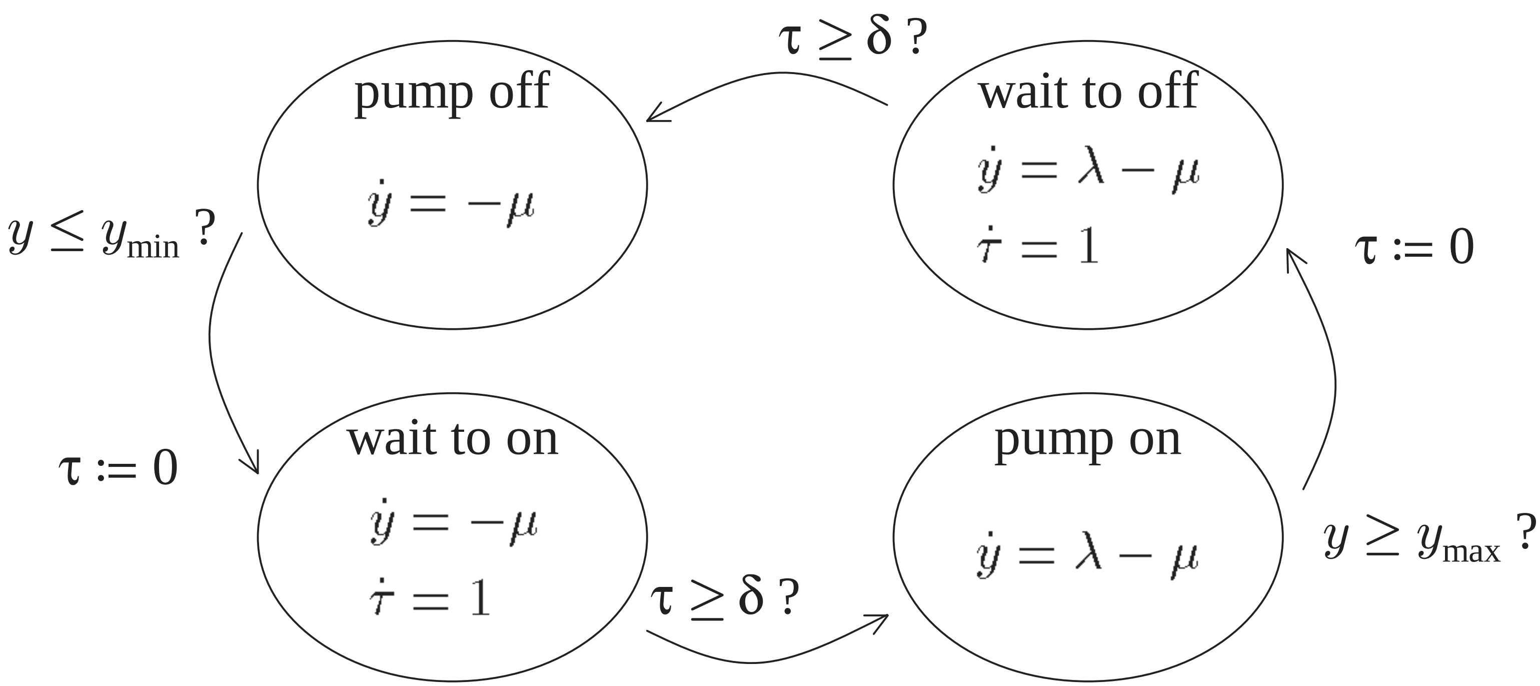

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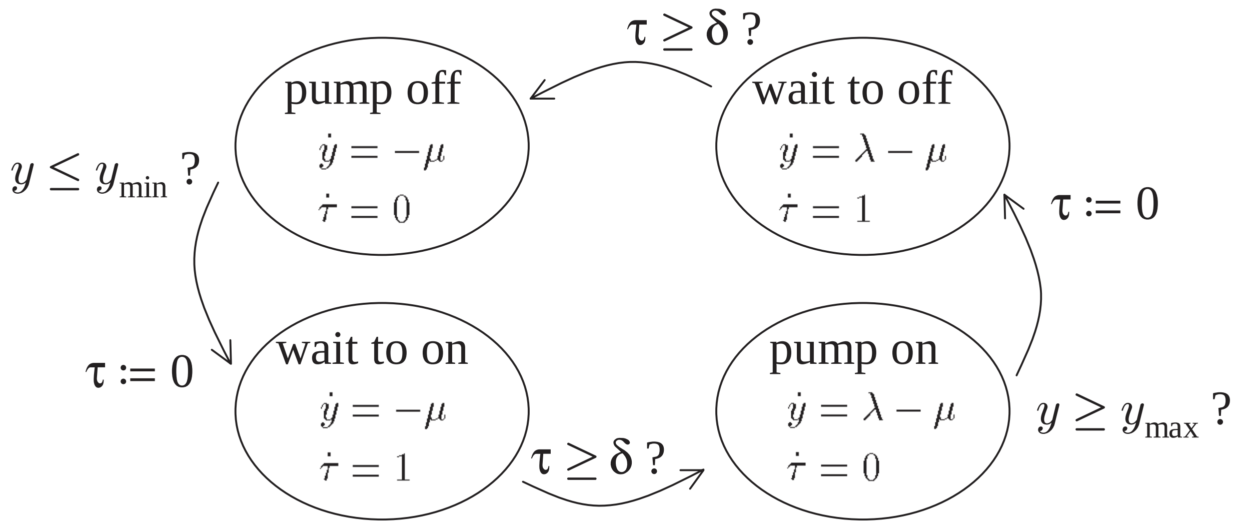


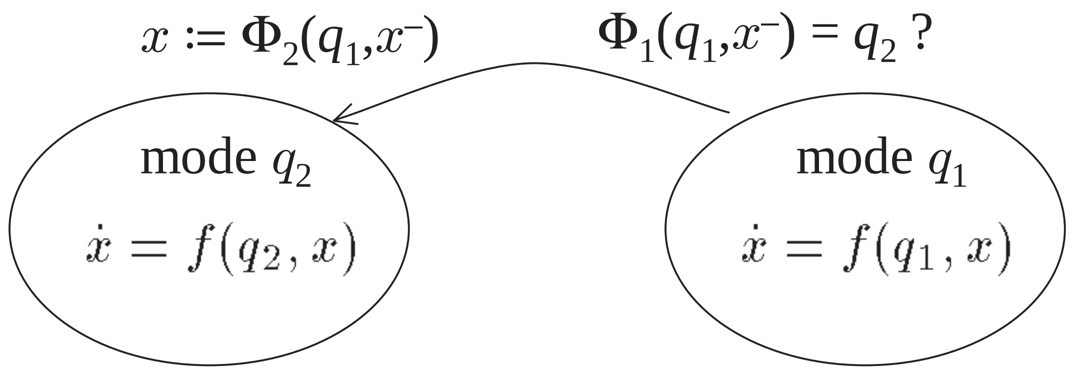
$\mathcal{Q}$:= { off, won, on, woff }
$\mathbb{R}^2 \equiv$ continuous state-space

$$f(q,x) := \begin{cases} \begin{bmatrix} -\mu \\ 0 \end{bmatrix} & q = \text{off} \\ \begin{bmatrix} -\mu \\ 1 \end{bmatrix} & q = \text{won} \\ \begin{bmatrix} \lambda-\mu \\ 0 \end{bmatrix} & q = \text{on} \\ \begin{bmatrix} \lambda-\mu \\ 1 \end{bmatrix} & q = \text{woff} \end{cases}$$

$$\varphi(q,x) := \begin{cases} \text{off} & q = \text{woff}, \tau \geq \delta \\ \text{off} & q = \text{off}, y > y_{\min} \\ \text{won} & q = \text{off}, y \leq y_{\min} \\ \text{won} & q = \text{won}, \tau < \delta \\ \vdots \end{cases} \qquad \rho(q,x) := \begin{cases} x & q = \text{woff}, \tau \geq \delta \\ x & q = \text{off}, y > y_{\min} \\ \begin{bmatrix} y \\ 0 \end{bmatrix} & q = \text{off}, y \leq y_{\min} \\ x & q = \text{won}, \tau < \delta \\ \vdots \end{cases}$$

Solution to a hybrid automaton

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-) \qquad q \in \mathcal{Q},\ x \in \mathbb{R}^n$$



Definition: A ***solution*** to the hybrid automaton is a pair of right-continuous signals

$$x : [0,\infty) \to \mathbb{R}^n \qquad q : [0,\infty) \to \mathcal{Q}$$

such that

1. x is piecewise differentiable & q is piecewise constant
2. on any interval (t_1, t_2) on which q is constant

continuous evolution

$$x(t) = x(t_1) + \int_{t_1}^{t} f\big(q(t_1), x(\tau)\big) d\tau \qquad \forall t \in [t_1, t_2)$$

3. $\big(q(t), x(t)\big) = \Phi\big(q^-(t), x^-(t)\big) \quad \forall t \geq 0$

discrete transitions

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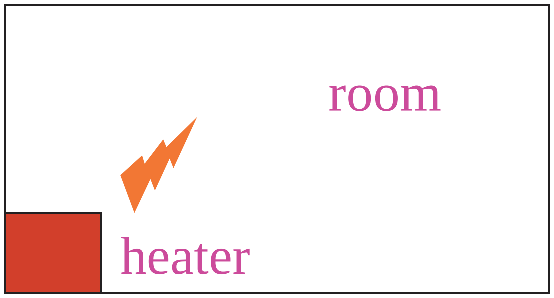
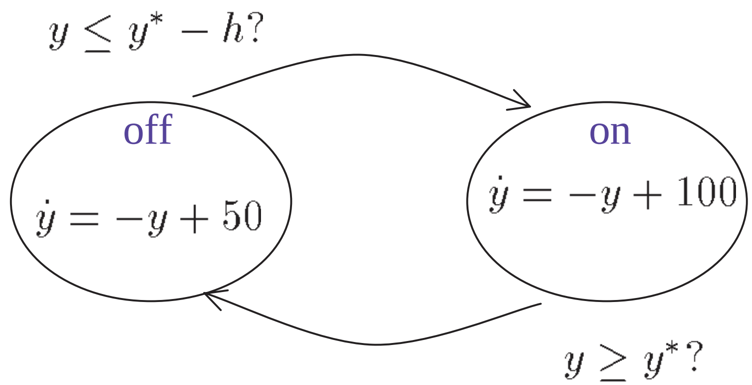
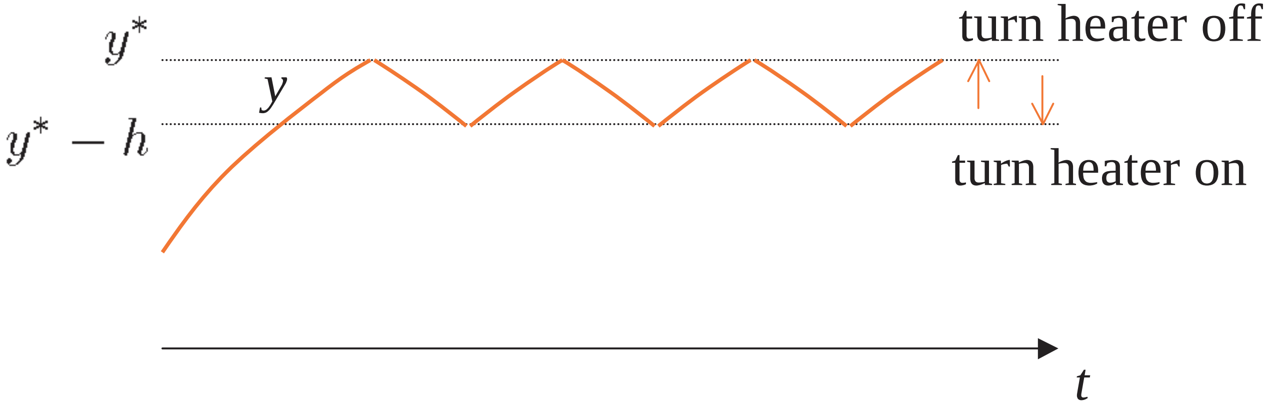
$y \equiv$ mean temperature



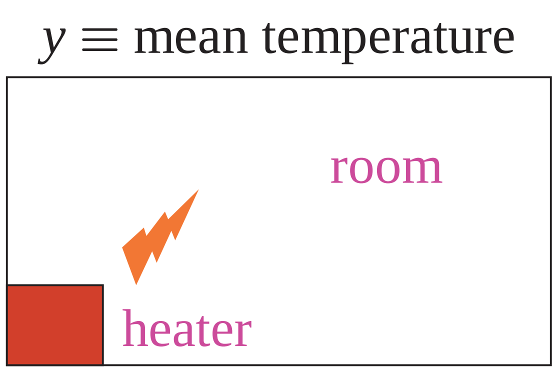


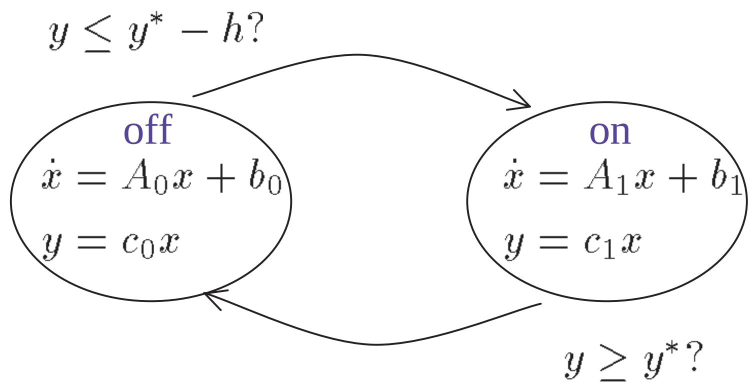


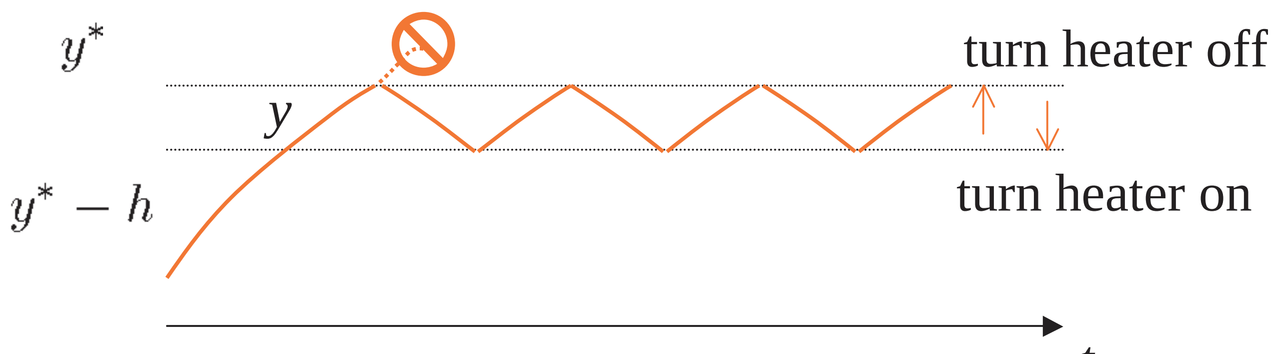
The state of the system remains bounded as $t \to \infty$:

$$\min\{y(0), y^* - h\} \leq y(t) \leq \max\{y(0), y^*\} \qquad \forall t \geq 0$$

Example #2: Thermostat







A_0, A_1 asymptotically stable (all eigenvalues with negative real part)

1. if system would stay in off mode forever then

   eq. state $x_{eq} = A_0^{-1}\, b_0$ is asymptotically stable & $y \to y_{off} := c_0\, A_0^{-1}\, b_0 \leq y^*\text{-}h$

2. if system would stay in on mode forever then

   eq. state $x_{eq} = A_1^{-1}\, b_1$ is asymptotically stable & $y \to y_{on} := c_1\, A_1^{-1}\, b_1 \geq y^*$

With switching, does the overall state x of the system remains bounded as $t \to \infty$?

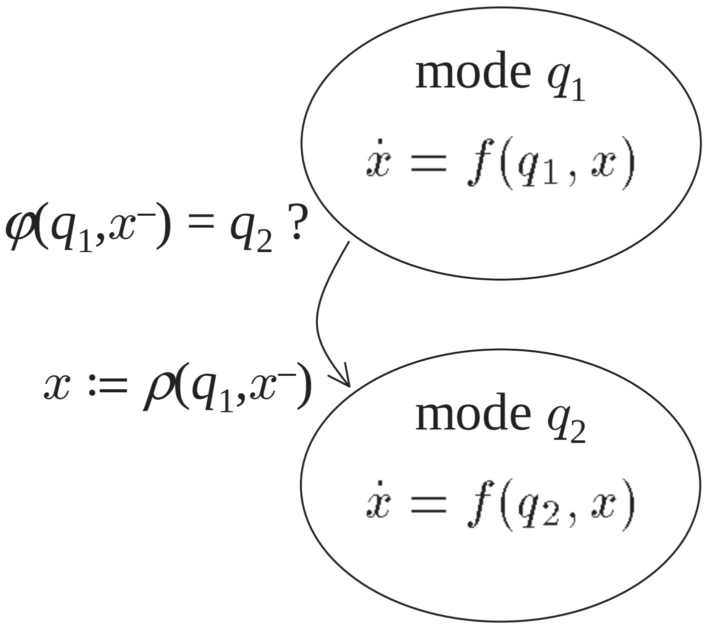
A different view on hybrid systems…

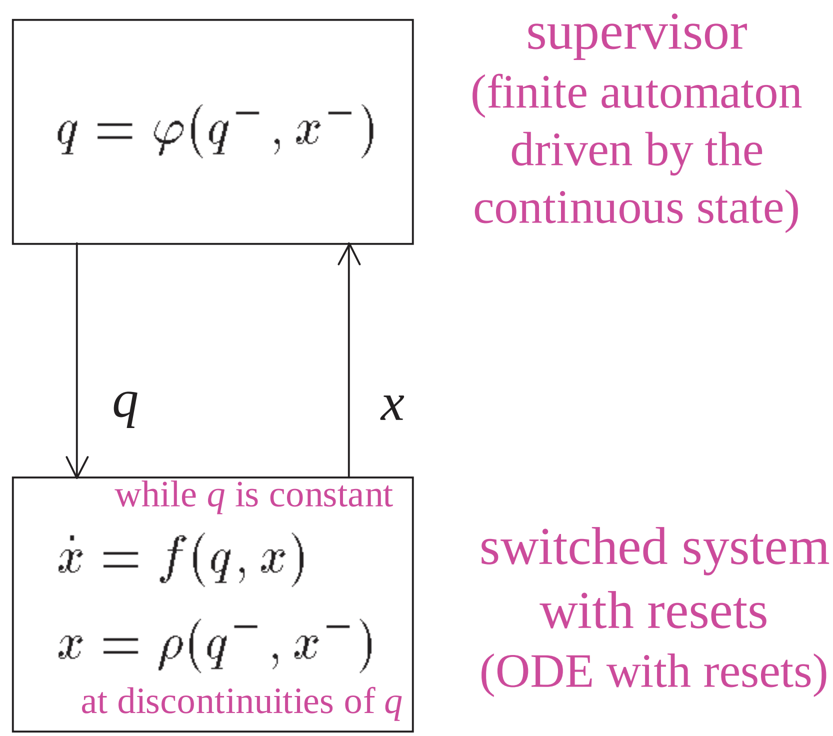
$\mathcal{Q}, \mathbb{R}^n$ $\equiv$ state-spaces

$f : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ $\equiv$ vector field

$\varphi : \mathcal{Q} \times \mathbb{R}^n \to \mathcal{Q}$ $\equiv$ discrete transition

$\rho : \mathcal{Q} \times \mathbb{R}^n \to \mathbb{R}^n$ $\equiv$ reset map



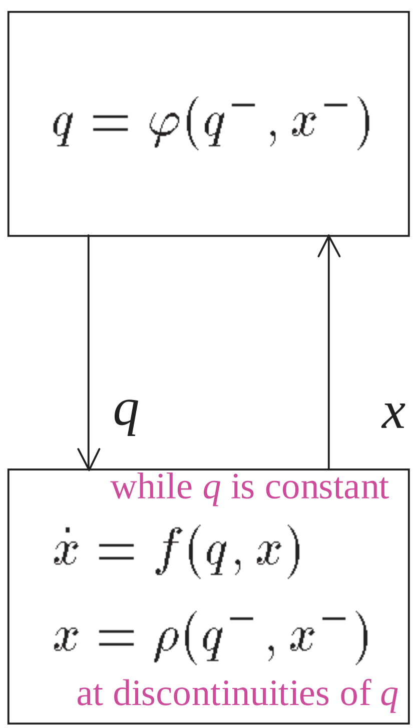


An hybrid system can be viewed as the interconnection of a supervisor and a switched system

(essentially the Simulink/Stateflow model)

Example #2: Thermostat

supervisor
(finite automaton)



switched system
(ODE with resets)

1st On an interval (τ,t) the maximum number of switchings $N(\tau,t)$ is bounded by

$$N(\tau, t) \leq 1 + \frac{c \max\limits_{s \in (\tau,t)} ||x(s)||}{h}(t - \tau)$$

property of the supervisor

+

2nd Assuming that the max. number of switchings $N(\tau,t)$ on (τ,t) is bounded:

$$N(\tau, t) \leq N_0 + \frac{t - \tau}{\tau_D}$$

Then there exist constants α, β, γ s.t.

$$||x(t)|| \leq \alpha ||x(\tau)|| + \beta + \gamma y^*$$

property of the switched system

=

3rd For any choice of τ_D and h such that

$$h \leq c\tau_D \left(\alpha ||x(0)|| + \beta + \gamma y^*\right)$$

x must be bounded for any solution compatible with 1& 2

property of the interconnection

Switched system

parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad p \in \mathcal{Q}$

parameter set

switching signal $\equiv$ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$

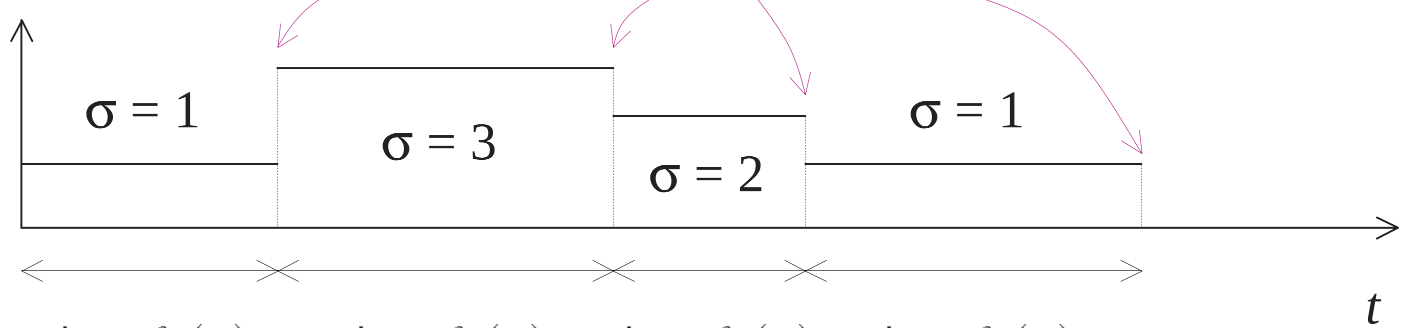
$\mathcal{S} \equiv$ set of admissible switching signals

E.g., $\mathcal{S} := \{ \sigma : N_\sigma(\tau, t) \leq 1 + (t - \tau), \forall\, t > \tau \geq 0 \}$

\# of discontinuities of σ in the interval (τ, t)

$$\dot{x} = f_\sigma(x) \qquad \sigma \in \mathcal{S}$$

switching times (discontinuities of σ)



$\dot{x} = f_1(x) \qquad \dot{x} = f_3(x) \qquad \dot{x} = f_2(x) \qquad \dot{x} = f_1(x)$

A ***solution*** to the switched system is any pair (σ, x) with $\sigma \in \mathcal{S}$ and x a solution to

$$\dot{x} = f_{\sigma(t)}(x)$$

time-varying ODE

Switched system with state-dependent switching

parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad p \in \mathcal{Q}$

parameter set

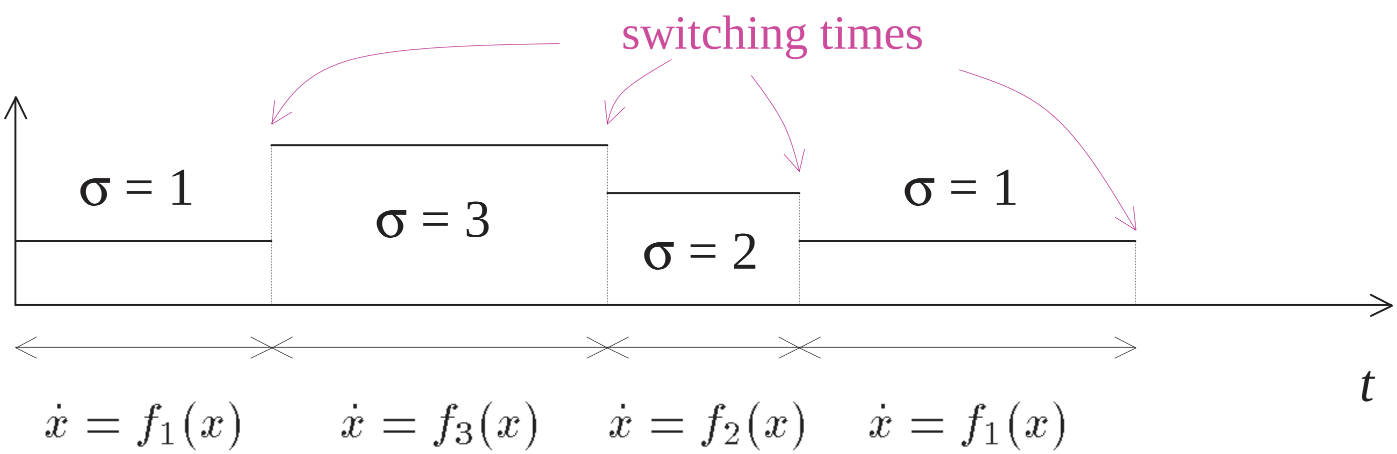
switching signal $\equiv$ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$

$\mathcal{S} \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in $\mathbb{R}^n$

E.g., $\mathcal{S} := \{(\sigma, x) : N_\sigma(\tau, t) \leq 1 + \sup_{s\in(\tau,t)} \|x(s)\| \, (t - \tau), \ \forall\, t > \tau \geq 0 \ \}$

for each x only some σ may be admissible

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$



A ***solution*** to the switched system is a pair $(\sigma, x) \in \mathcal{S}$ for which x is a solution to

$$\dot{x} = f_{\sigma(t)}(x)$$

time-varying ODE

Switched system with resets

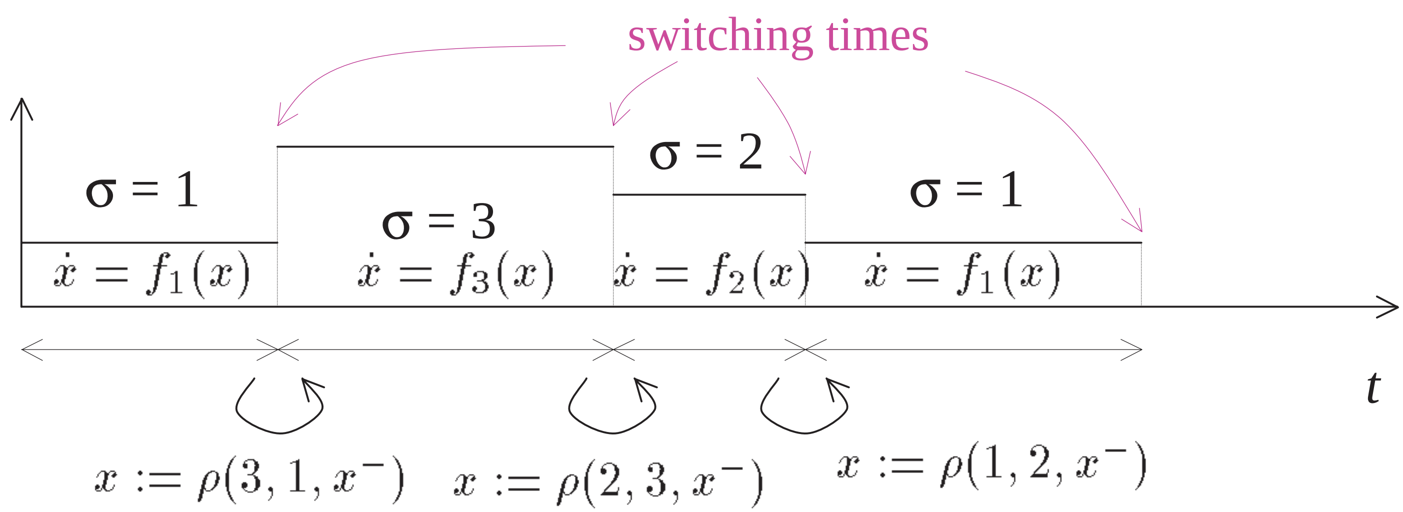
parameterized family of vector fields $\equiv$ $f_p : \mathbb{R}^n \to \mathbb{R}^n$ $\qquad p \in \mathcal{Q}$

parameter set

switching signal $\equiv$ piecewise constant signal $\sigma : [0,\infty) \to \mathcal{Q}$

$\mathcal{S} \equiv$ set of admissible pairs (σ, x) with σ a switching signal and x a signal in $\mathbb{R}^n$

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$



A ***solution*** to the switched system is a pair (σ, x) $\in \mathcal{S}$ for which

1. on every open interval on which σ is constant, x is a solution to

$$\dot{x} = f_{\sigma(t)}(x)$$

time-varying ODE

2. at every switching time t, $x(t) = \rho\big(\sigma(t), \sigma^-(t), x^-(t)\big)$

Time-varying systems vs. Hybrid systems vs. Switched systems

Time-varying system $\equiv$ for each initial condition $x(0)$ there is only one solution

$$\dot{x} = f_{\sigma(t)}(x)$$ (all f_p locally Lipschitz)

Hybrid system $\equiv$ for each initial condition $q(0)$, $x(0)$ there is only one solution

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-)$$

Switched system $\equiv$ for each $x(0)$ there may be several solutions, one for each admissible σ

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$

the notions of stability, convergence, etc.
must address "uniformity" over all solutions

Time-varying systems vs. Hybrid systems vs. Switched systems

Time-varying system $\equiv$ for each initial condition $x(0)$ there is only one solution

$$\dot{x} = f_{\sigma(t)}(x)$$ (all f_p locally Lipschitz)

Hybrid system $\equiv$ for each initial condition $q(0)$, $x(0)$ there is only one solution

$$\dot{x} = f(q, x) \qquad (q, x) = \Phi(q^-, x^-)$$

Switched system $\equiv$ for each $x(0)$ there may be several solutions, one for each admissible σ

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$

every hybrid system can be viewed as a switched system. Indeed, just pick:

$$\mathcal{S} := \big\{(x, \sigma) : (q, x) := \Phi(q^-, x^-)\big\}$$

$$\rho(q_2, q_1, x_1) := \begin{cases} x_2 & \exists x_2 : (q_2, x_2) = \Phi(q_1, x_1) \\ x_1 & \text{otherwise} \end{cases}$$

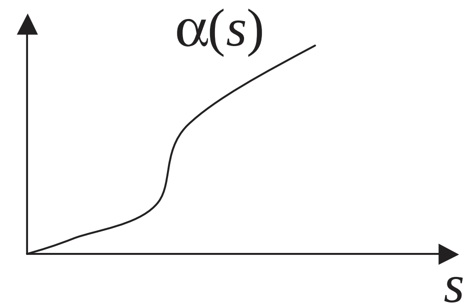
Stability of switched systems

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f_q(x_{eq}) = 0 \ \forall\, q \in \mathcal{Q}$

class $\mathcal{K} \equiv$ set of functions $\alpha : [0,\infty) \to [0,\infty)$ that are

1. continuous
2. strictly increasing
3. $\alpha(0)=0$

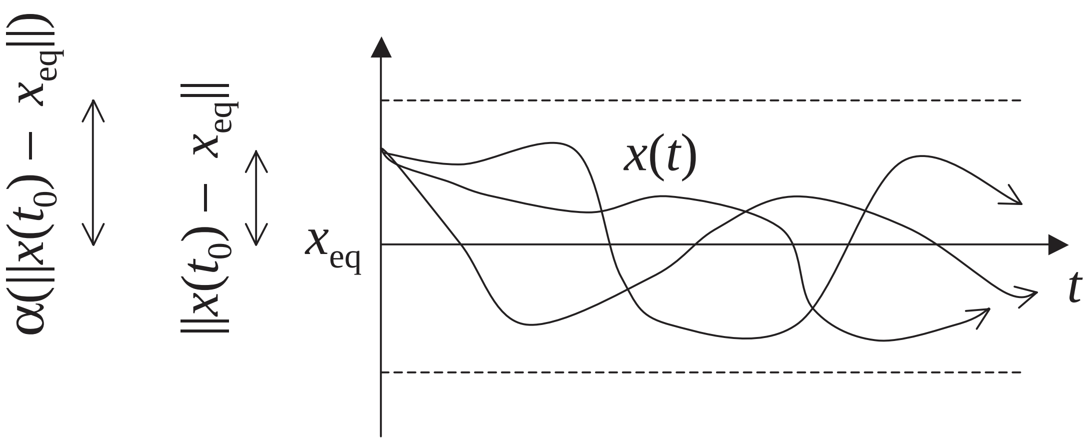


Definition (class $\mathcal{K}$ function definition):
The equilibrium point x_{eq} is ***(globally Lyapunov) stable*** if $\exists\, \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \le \alpha(\|x(t_0) - x_{eq}\|) \qquad \forall\, t \ge t_0 \ge 0$$

along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

α is independent of $x(t_0)$ and σ



Asymptotic stability of switched systems

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$

equilibrium point $\equiv x_{eq} \in \mathbb{R}^n$ for which $f_q(x_{eq}) = 0 \ \forall\, q \in \mathcal{Q}$

class $\mathcal{K} \equiv$ set of functions $\alpha : [0,\infty) \to [0,\infty)$ that are

1. continuous
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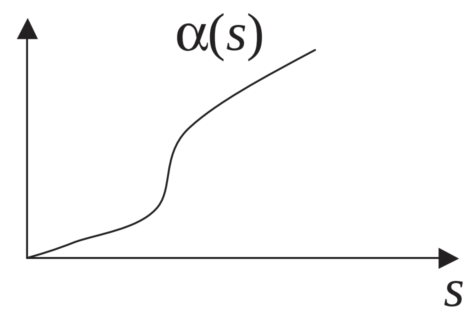
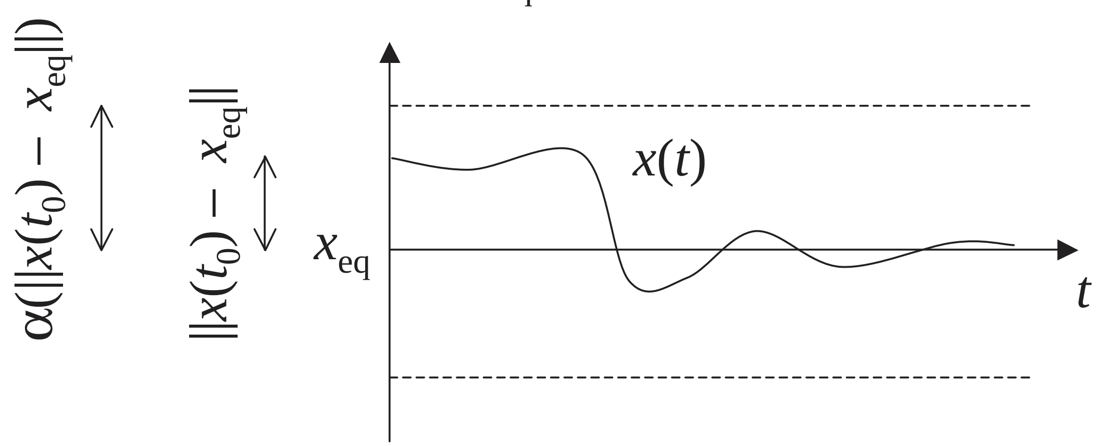


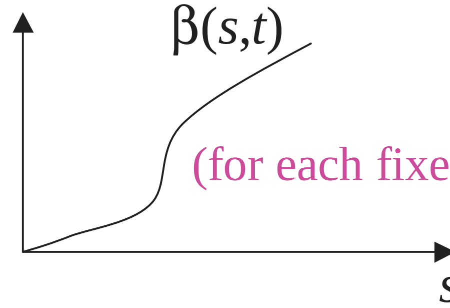
Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***(globally) asymptotically stable*** if it is Lyapunov stable and for every solution that exists on $[0,\infty)$

$$x(t) \to x_{eq} \text{ as } t \to \infty.$$



Uniform asymptotic stability of switched systems

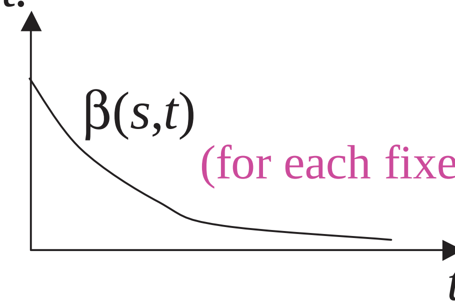
$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}$$



equilibrium point ≡ $x_{eq} \in \mathbb{R}^n$ for which $f(x_{eq}) = 0$

class $\mathcal{KL}$ ≡ set of functions $\beta : [0,\infty)\times[0,\infty)\to[0,\infty)$ s.t.

1. for each fixed t, $\beta(\cdot,t) \in \mathcal{K}$
2. for each fixed s, $\beta(s,\cdot)$ is monotone decreasing and $\beta(s,t) \to 0$ as $t\to\infty$

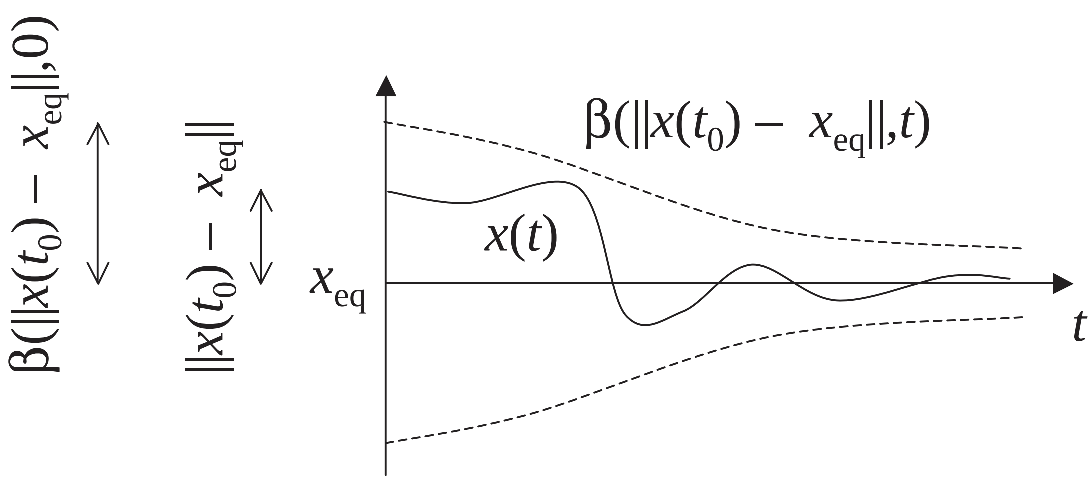


Definition (class $\mathcal{KL}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***uniformly asymptotically stable*** if $\exists\ \beta\in\mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall\ t \geq t_0 \geq 0$$

along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

β is independent of $x(t_0)$ and σ



We have ***exponential stability*** when

$$\beta(s,t) = c\, e^{-\lambda t} s$$

with $c,\lambda > 0$

Three notions of stability

Definition (class $\mathcal{K}$ function definition):
The equilibrium point x_{eq} is ***stable*** if $\exists\, \alpha \in \mathcal{K}$:

$$\|x(t) - x_{eq}\| \leq \alpha(\|x(t_0) - x_{eq}\|) \qquad \forall\, t \geq t_0 \geq 0$$

along any solution $(x, \sigma) \in \mathcal{S}$ to the switched system

α is independent of $x(t_0)$ and σ

Definition:
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***asymptotically stable*** if it is Lyapunov stable and for every solution that exists on $[0,\infty)$

$$x(t) \to x_{eq} \text{ as } t \to \infty.$$

Definition (class $\mathcal{KL}$ function definition):
The equilibrium point $x_{eq} \in \mathbb{R}^n$ is ***uniformly asymptotically stable*** if $\exists\, \beta \in \mathcal{KL}$:

$$\|x(t) - x_{eq}\| \leq \beta(\|x(t_0) - x_{eq}\|, t - t_0) \quad \forall\, t \geq t_0 \geq 0$$

along any solution $(\sigma, x) \in \mathcal{S}$ to the switched system

β is independent of $x(t_0)$ and σ

exponential stability when $\beta(s,t) = c\, e^{-\lambda t} s$ with $c, \lambda > 0$

Example

$$\dot{x} = \sigma x$$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, +1\}$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches and interval between consecutive discontinuities bounded below by 1

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$
with infinitely many switches and interval between consecutive discontinuities below by 1 and above by 2

Example

$$\dot{x} = \sigma x$$

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, +1\}$

unstable

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$

stable but not asympt.

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$ with infinitely many switches

stable but not asympt.

$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$ with infinitely many switches and interval between consecutive discontinuities bounded below by 1

asympt. stable

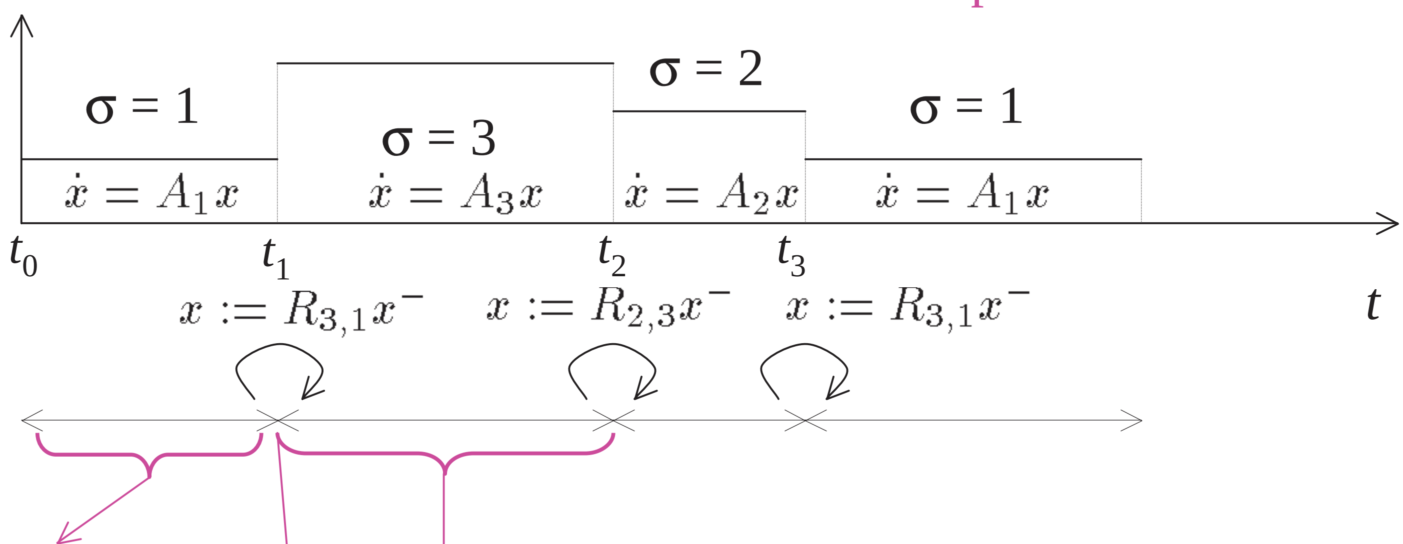
$\mathcal{S} \equiv$ set of piecewise constant switching signals taking values in $\mathcal{Q} := \{-1, 0\}$ with infinitely many switches and interval between consecutive discontinuities below by 1 and above by 2

uniformly asympt. stable

Linear switched systems

$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q,q' \in \mathcal{Q}$

vector fields and reset maps linear on *x*



$$x(t) = e^{A_1(t-t_0)} x(t_0)$$

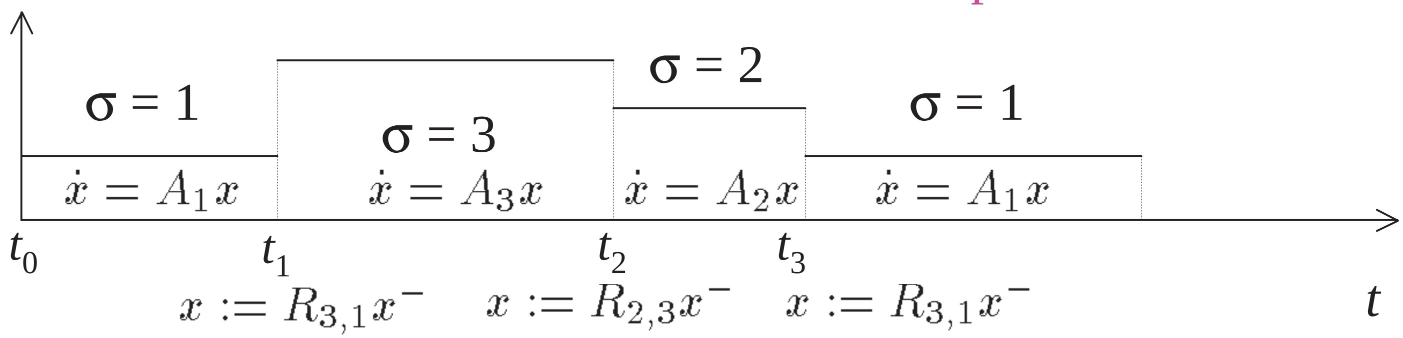
$$x(t_1) = R_{3,1} x^-(t_1) = R_{3,1} e^{A_1(t_1-t_0)} x(t_0)$$

$$x(t) = e^{A_3(t-t_1)} x(t_1) = e^{A_3(t-t_1)} R_{3,1} e^{A_1(t_1-t_0)} x(t_0)$$

Linear switched systems

$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q,q' \in \mathcal{Q}$

vector fields and reset maps linear on *x*



$$x(t) = \Phi_\sigma(t,\tau)\, x(\tau)$$

state-transition matrix for the switched system (σ-dependent)

$$\Phi_\sigma(t,\tau) := e^{A_{\sigma(t_k)}(t-t_k)} R_{\sigma(t_k),\sigma(t_{k-1})} e^{A_{\sigma(t_{k-1})}(t_k - t_{k-1})} \cdots$$
$$\cdots\; R_{\sigma(t_2),\sigma(t_1)} e^{A_{\sigma(\tau)}(t_1-\tau)} \qquad t \geq \tau$$

$t_1, t_2, t_3, \ldots, t_k \equiv$ switching times of σ in the interval $[t,\tau)$

Linear switched systems

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q,q' \in \mathcal{Q}$$

$$x(t) = \Phi_\sigma(t,\tau)x(\tau)$$

state-transition matrix (σ-dependent)

$$\Phi_\sigma(t,\tau) := e^{A_{\sigma(t_k)}(t-t_k)} R_{\sigma(t_k),\sigma(t_{k-1})} e^{A_{\sigma(t_{k-1})}(t_k - t_{k-1})} \cdots$$
$$\cdots R_{\sigma(t_1),\sigma(\tau)} e^{A_{\sigma(\tau)}(t_1-\tau)} \qquad t \geq \tau$$

$t_1, t_2, t_3, \ldots, t_k \equiv$ switching times of σ in the interval $[t,\tau)$

Analogous to what happens for (unswitched) linear systems:

1. $\Phi_\sigma(\tau,\tau) = I \quad \forall \tau$
2. $\Phi_\sigma(t,s)\,\Phi_\sigma(s,\tau) = \Phi_\sigma(t,\tau) \quad \forall\, t \geq s \geq \tau$ (semi-group property)
3. if t is not a switching time, $\Phi_\sigma(t,\tau)$ is differentiable at t and
$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\sigma(t,\tau) = A_{\sigma(t)}\Phi_\sigma(t,\tau)$$
4. if t is a switching time,
$$\Phi_\sigma(t,\tau) = R_{\sigma(t),\sigma^-(t)}\Phi_\sigma^-(t,\tau)$$

for a given σ, Φ_σ is a "solution" to the switched system with resets

but now Φ_σ may not be nonsingular (will be singular if one of the $R_{q\,q'}$ are)

Uniform vs. exponential stability

$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S} \qquad A_q, R_{q,q'} \in \mathbb{R}^{n\times n} \quad q, q' \in \mathcal{Q}$

state-independent switching $\equiv$ $\mathcal{S}$ is such that

$$(\sigma, x) \in \mathcal{S} \Rightarrow (\sigma, z) \in \mathcal{S}$$

for any other piecewise continuous z

only σ determines whether or not (σ, x) is admissible

Theorem:

For switched linear systems with state-independent switching, uniform asymptotic stability implies exponential stability (two notions are equivalent)

Outline

1. **Examples of hybrid systems**
   - Bouncing ball
   - Thermostat
   - Transmission
   - Multiple-tank
   - Supervisory control
2. **Formal models for hybrid systems**
3. **Switched systems**
   - Motivation
   - Linear switched systems
   - Lyapunov stability, asymptotic stability, exponential stability
4. **Stability of switched systems under arbitrary switching**
   - Instability caused by switching
   - Common Lyapunov function
   - Converse results
   - Algebraic conditions
5. **Controller realizations for stable switching**
6. **Stability under state-dependent switching**
   - State-dependent common Lyapunov function
   - Multiple Lyapunov functions
   - LaSalle’s invariance principle

Stability under arbitrary switching

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S}$$

$\mathcal{S}_{\text{all}} \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous

any switching signal is admissible

$\rho(p, q, x) = x \;\forall\, p,q \in \mathcal{Q},\, x \in \mathbb{R}^n$

no resets

If one of the vector fields f_q, $q \in \mathcal{Q}$ is unstable then the switched system is unstable

Why?

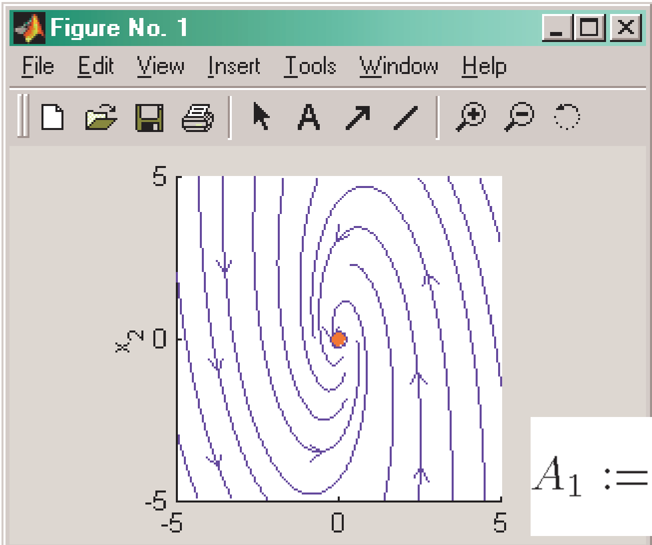
1. because the switching signals $\sigma(t) = q \;\forall\, t$ is admissible
2. for this σ we cannot find $\alpha \in \mathcal{K}$ such that

   $$\|x(t) - x_{\text{eq}}\| \leq \alpha(\|x(t_0) - x_{\text{eq}}\|) \;\; \forall\, t \geq t_0 \geq 0,\ \|x(t_0) - x_{\text{eq}}\| \leq c$$

   (must less for all σ)

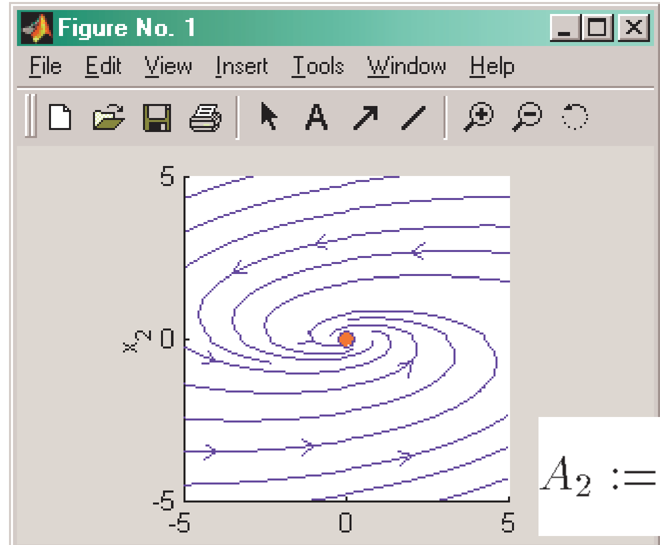
But even if all f_q, $q \in \mathcal{Q}$ are stable the switched system may still be unstable …

Stability under arbitrary switching



$$A_1 := \begin{bmatrix} -.5 & -.4 \\ 3 & -.5 \end{bmatrix}$$

$\dot{x} = A_1 z$ asympot. stable

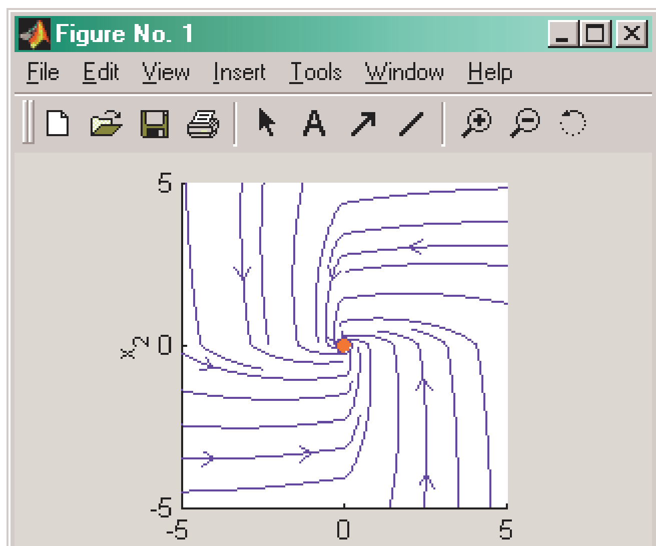


$$A_2 := \begin{bmatrix} -.5 & -3 \\ .4 & -.5 \end{bmatrix}$$

$\dot{x} = A_2 z$ asympot. stable

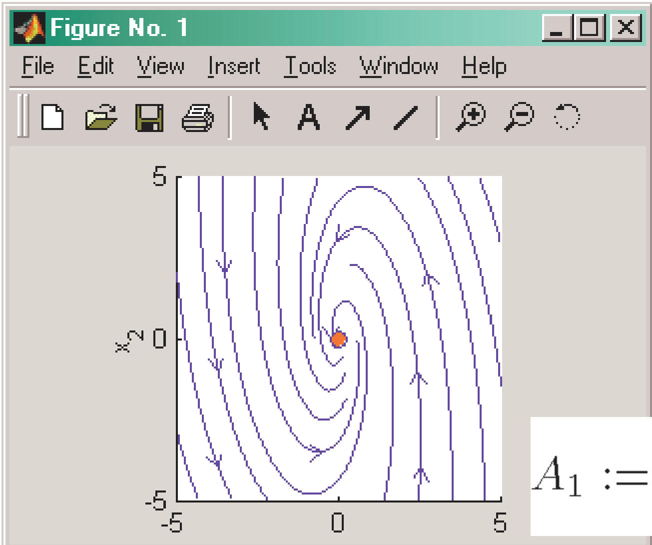
$$\sigma(t) := \begin{cases} 1 & x_1 x_2 \leq 0 \\ 2 & x_1 x_2 > 0 \end{cases}$$

$$\dot{x} = A_\sigma x$$



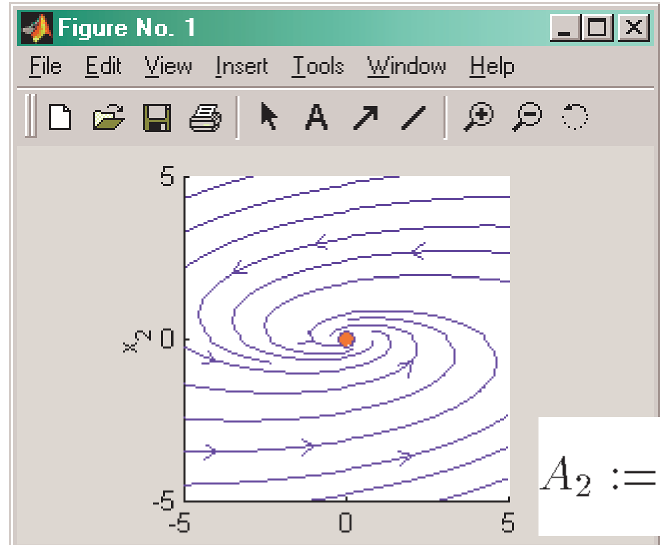
unstable.m

Stability under arbitrary switching



$$A_1 := \begin{bmatrix} -.5 & -.4 \\ 3 & -.5 \end{bmatrix}$$

$\dot{x} = A_1 z$ asympot. stable

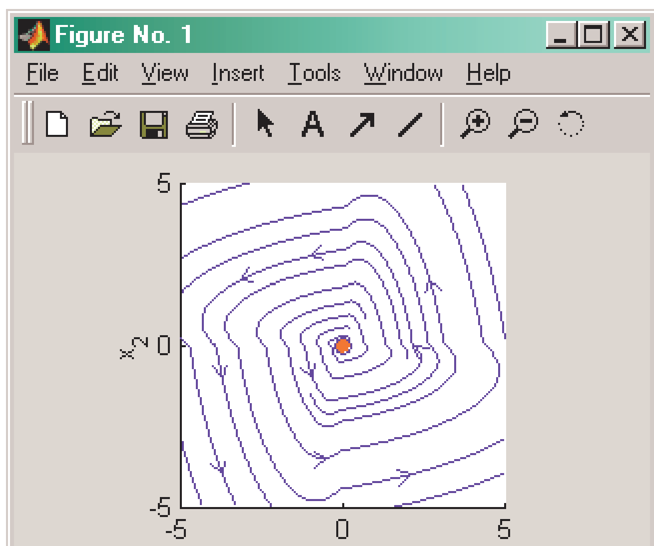


$$A_2 := \begin{bmatrix} -.5 & -3 \\ .4 & -.5 \end{bmatrix}$$

$\dot{x} = A_2 z$ asympot. stable

$$\sigma(t) := \begin{cases} 1 & x_1 x_2 \geq 0 \\ 2 & x_1 x_2 < 0 \end{cases}$$

$$\dot{x} = A_\sigma x$$



for some admissible switching signals the trajectories grow to infinity $\Rightarrow$ switched system is unstable

unstable.m

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then
1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

The same V could be used to prove stability for all the unswitched systems

$$\dot{x} = f_q(x)$$

Common Lyapunov function

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n,\ q \in \mathcal{Q}$$

Then
1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)
1st Take an arbitrary solution (σ, x) and define $v(t) := V(\,x(t)\,)\ \forall\, t \geq 0$

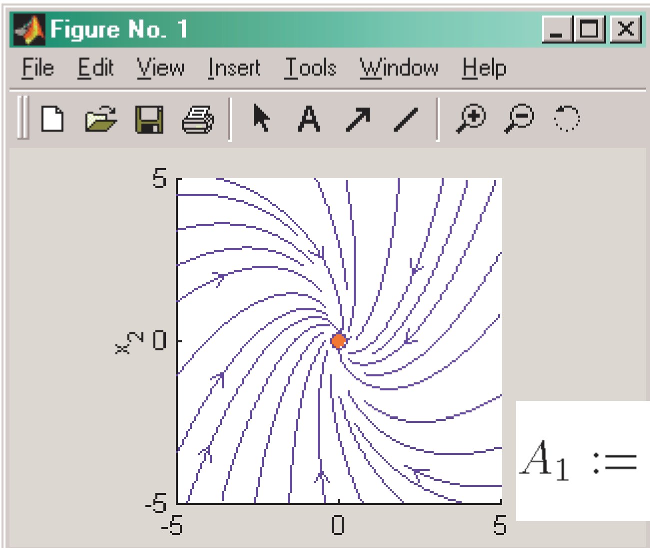
$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq 0$$

2nd Therefore

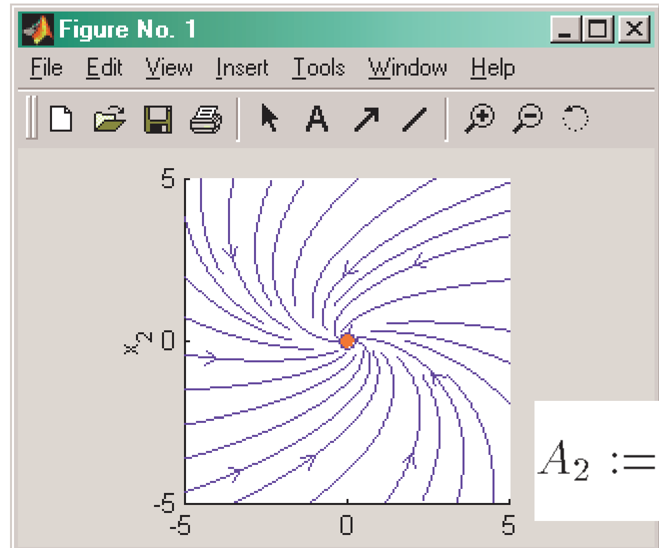
$$v(t) := V\big(x(t)\big) \leq v(0) := V\big(x(0)\big) \qquad \forall t \geq 0$$

$V(\,x(t)\,)$ is always bounded…

Example



$$A_1 := \begin{bmatrix} -1 & .25 \\ -1 & -1 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} -1 & -1 \\ .25 & -1 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

Defining $V(x_1,x_2) := x_1^2 + x_2^2$

common Lyapunov function

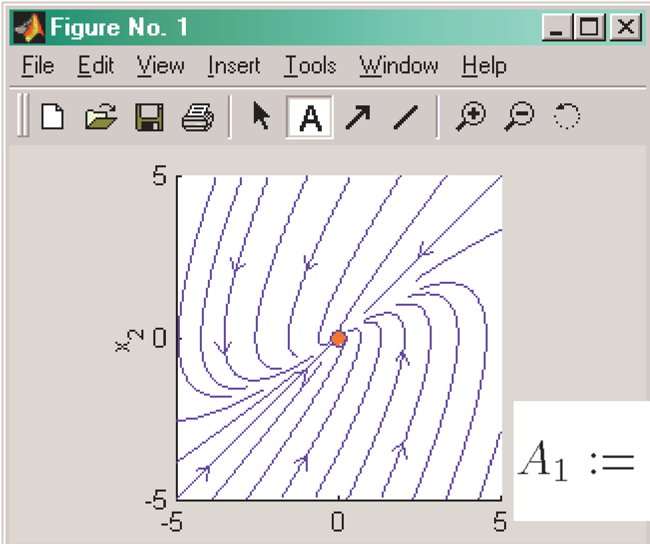
$$\frac{\partial V}{\partial x} A_1 x = -x_1^2 - 1.4375 x_2^2 - (x_1 + .75 x_2)^2 < 0$$

$$\frac{\partial V}{\partial x} A_2 x = -x_1^2 - 1.4375 x_2^2 - (x_1 + .75 x_2)^2 < 0$$

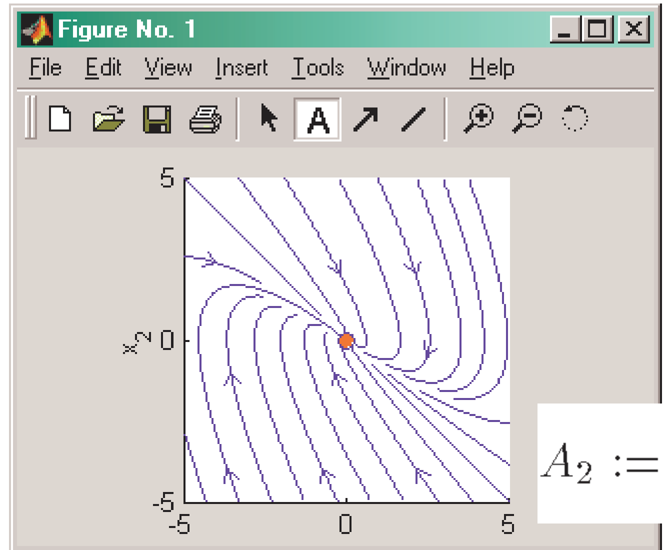
uniform asymptotic stability

stable.m

Example



$$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

Defining $V(x_1,x_2) := x_1^2 + x_2^2$

common Lyapunov function

$$\frac{\partial V}{\partial x} A_1 x = -4x_2^2 \leq 0$$

$$\frac{\partial V}{\partial x} A_2 x = -4x_2^2 \leq 0$$

stability (not asymptotic)

(problems, e.g., close to the x_2=0 axis)

stable.m

Converse result

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

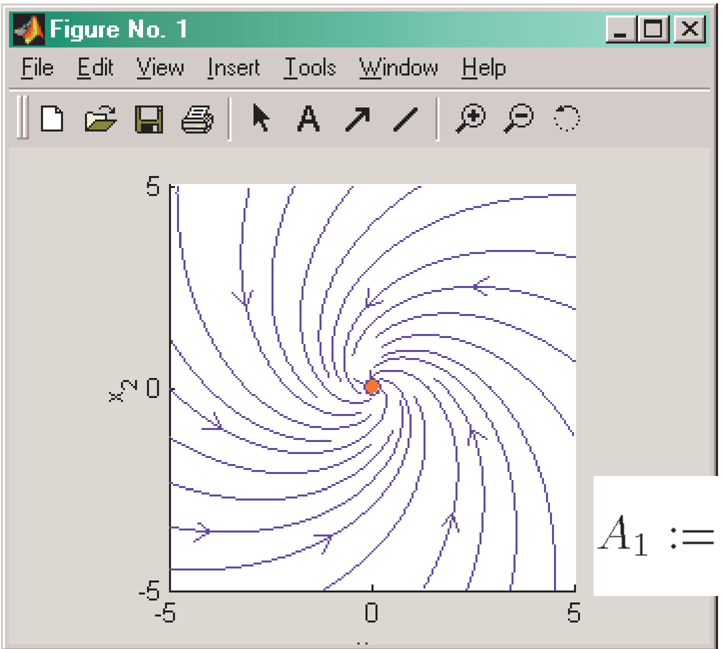
Theorem:
Assume $\mathcal{Q}$ is finite. The switched system is uniformly asymptotically stable (on $\mathcal{S}_{\text{all}}$) if and only if there exists a common Lyapunov function, i.e., continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) < 0 \qquad \forall z \in \mathbb{R}^n \setminus \{0\},\ q \in \mathcal{Q}$$

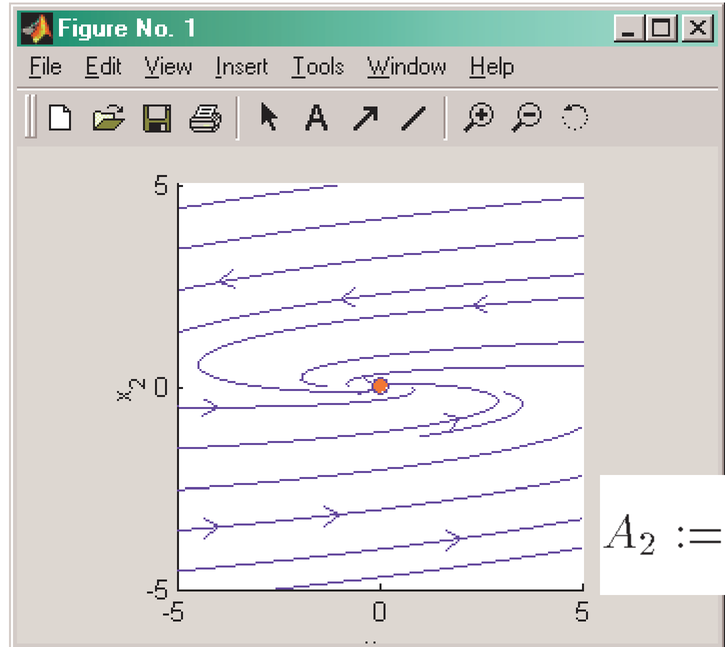
Note that…

1. This result generalized for infinite $\mathcal{Q}$ but one needs extra technical assumptions
2. The sufficiency was already established. It turns out that the existence of a common Lyapunov function is also necessary.
3. Finding a common Lyapunov function may be difficult.
   E.g., even for linear systems V may not be quadratic

Example



$$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$



$$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\dot{x} = A_\sigma x$$

The switched system is uniformly exponentially stable for arbitrary switching but there is no common quadratic Lyapunov function

stable.m

Algebraic conditions for stability under arbitrary switching

$$\dot{x} = A_\sigma x \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

linear switched system

Theorem: If $\mathcal{Q}$ is finite all A_q, $q \in \mathcal{Q}$ are asymptotically stable and

$$A_p A_q = A_q A_p \qquad \forall\, p,q \in \mathcal{Q}$$

then the switched system is uniformly (exponentially) asymptotically stable

Theorem: If all the matrices A_q, $q \in \mathcal{Q}$ are asymptotically stable and upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

(there exists a common Lyapunov function $V(x) = x' P x$ with P diagonal)

Theorem: If there is a nonsingular matrix $T \in \mathbb{R}^{n\times n}$ such that all the matrices

$$B_q = T A_q T^{-1} \qquad (T^{-1} B_q T = A_q)$$

common similarity transformation

are upper triangular or all lower triangular then the switched system is uniformly (exponentially) asymptotically stable

Lie Theorem actually provides the necessary and sufficient condition for the existence of such $T \equiv$ Lie algebra generated by the matrices must be solvable

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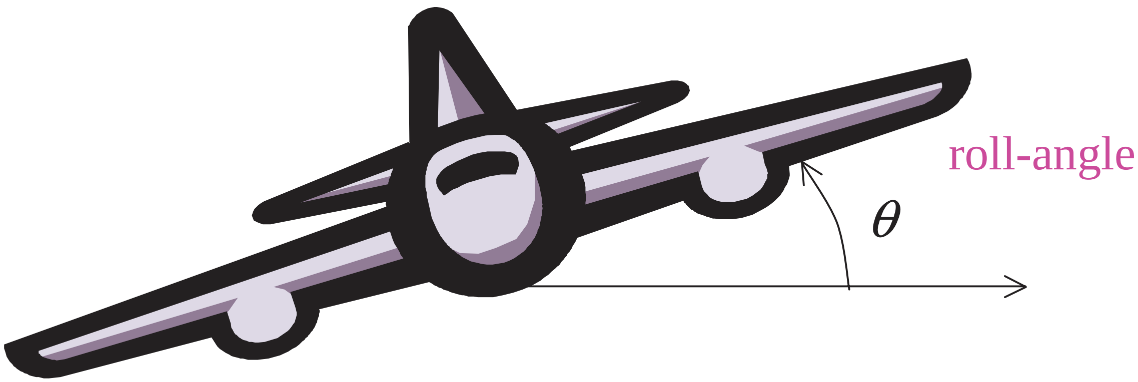
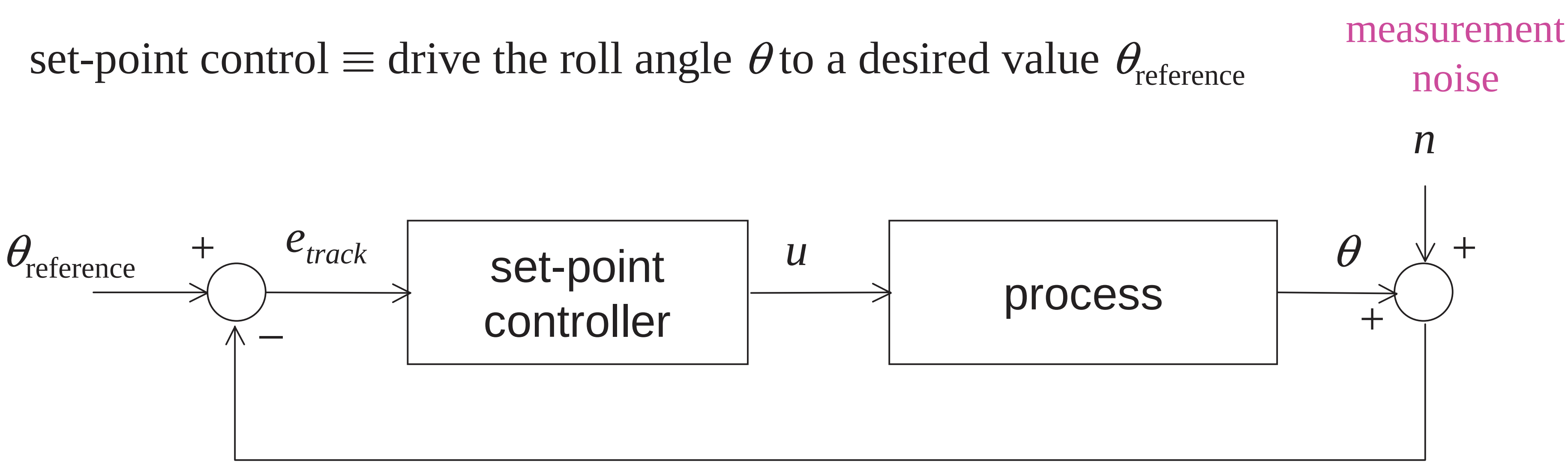
Example #11: Roll-angle control



$$\dddot{\theta} + 50.875\ddot{\theta} + 43.75\dot{\theta} = -1000u$$

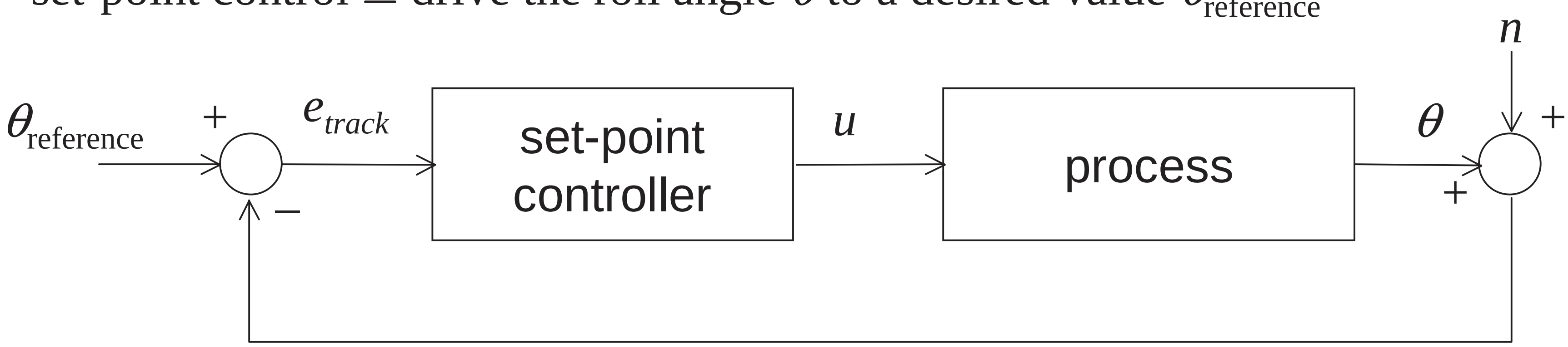
set-point control $\equiv$ drive the roll angle θ to a desired value $\theta_{\text{reference}}$



Example #11: Roll-angle control

set-point control ≡ drive the roll angle θ to a desired value $\theta_{\text{reference}}$

measurement noise



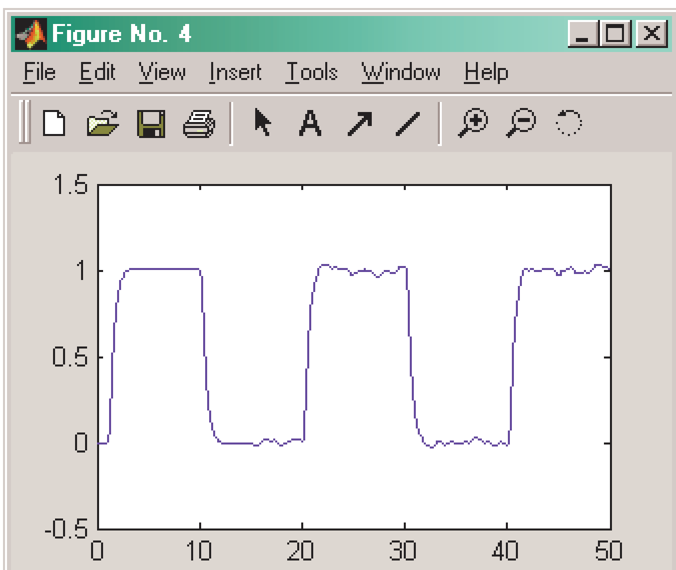
controller 1

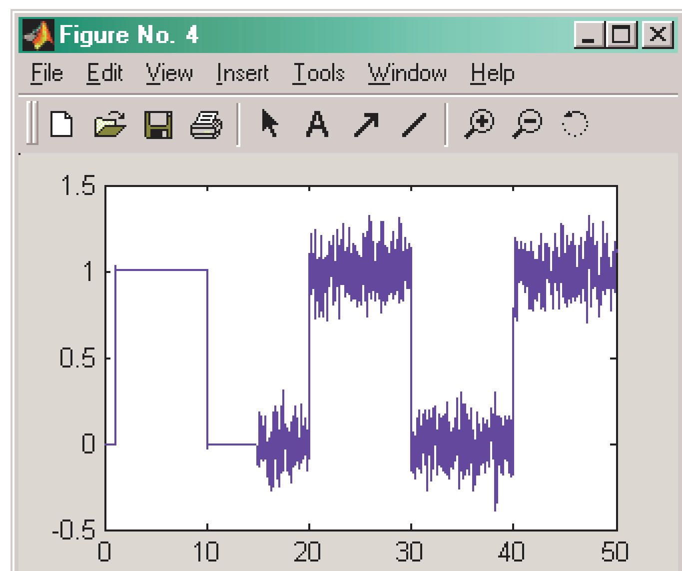
$$\dddot{u} + 63\ddot{u} + 751\dot{u} + 4471u = 6.7\ddot{e} + 340\dot{e} + 316e$$

controller 2

$$\dddot{u} + 974\ddot{u} + 4.7\times 10^5\dot{u} + 1.2\times 10^8 u = 10^6(48\ddot{e} + 322\dot{e} + 316e)$$

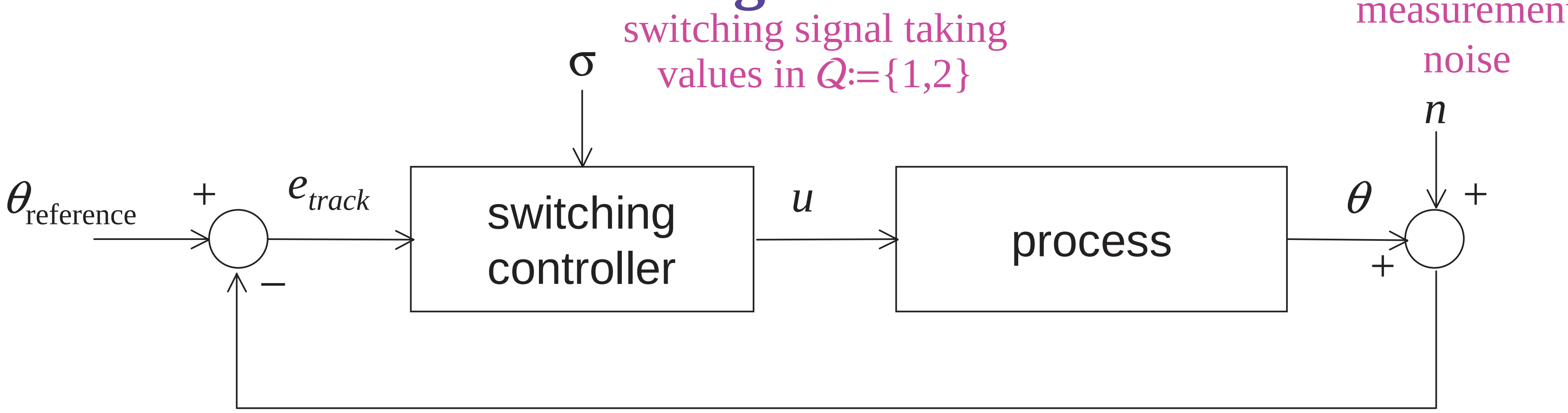
slow but not very sensitive to noise (low-gain)

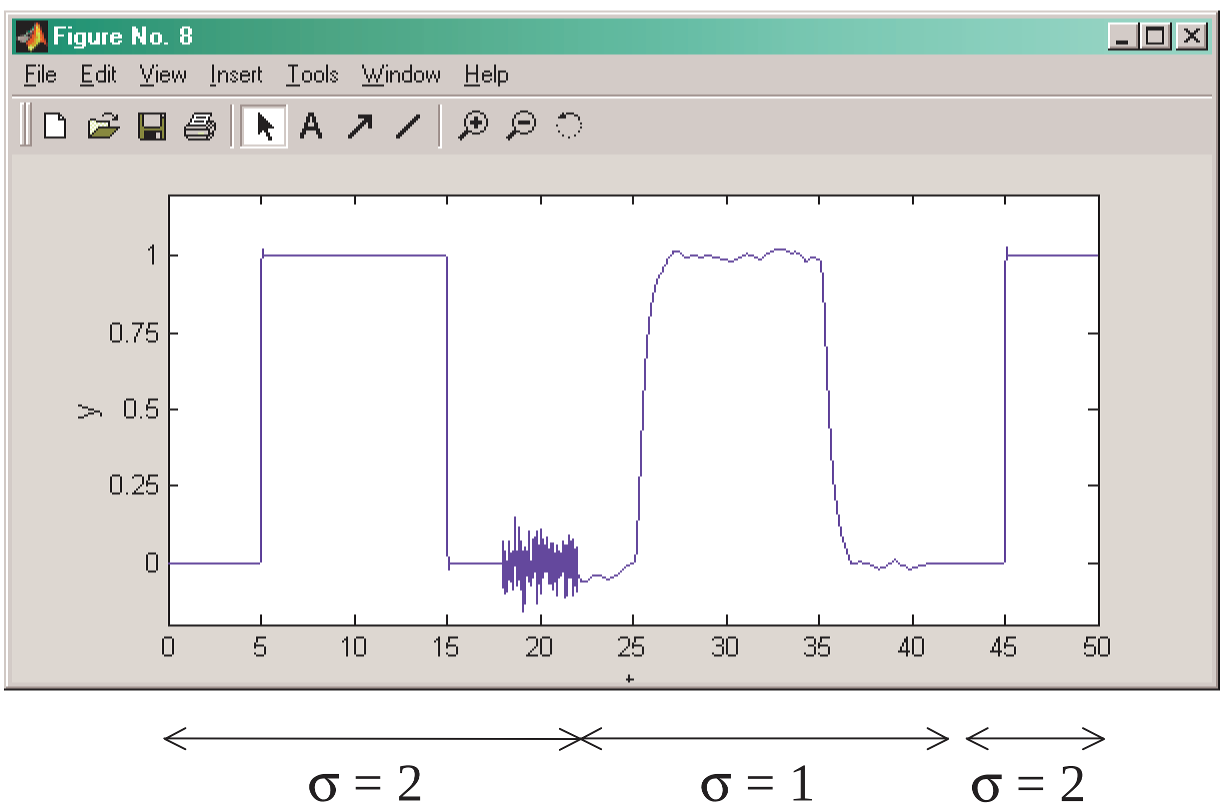




fast but very sensitive to noise (high-gain)

Switching controller





How to build the switching controller to avoid instability ?

Switching controller

controller 1

$$\dddot{u} + 63\ddot{u} + 751\dot{u} + 4471u = 6.7\ddot{e} + 340\dot{e} + 316e$$

realization: $\dot{z} = F_1 z + g_1 e$, $u = h_1 z$

$$F_1 = \begin{bmatrix} -63 & -23 & -17 \\ 32 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \quad g_1 = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

$$h_1 = \begin{bmatrix} 1.7 & 2.7 & .31 \end{bmatrix}$$

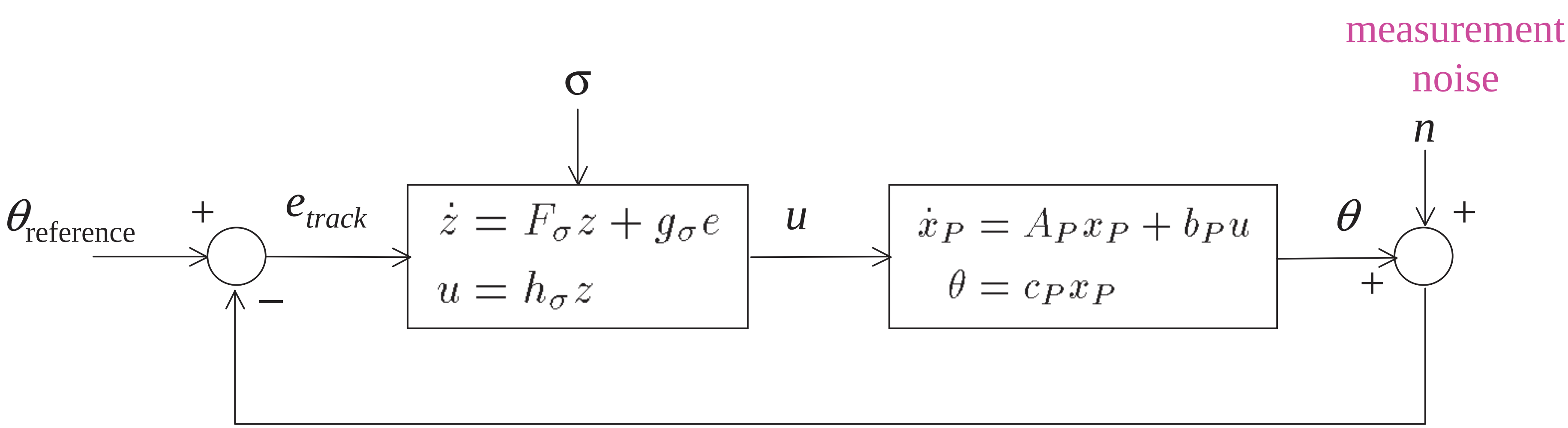
controller 2

$$\dddot{u} + 974\ddot{u} + 4.7 \times 10^5 \dot{u} + 1.2 \times 10^8 u = 10^6 (48\ddot{e} + 322\dot{e} + 316e)$$

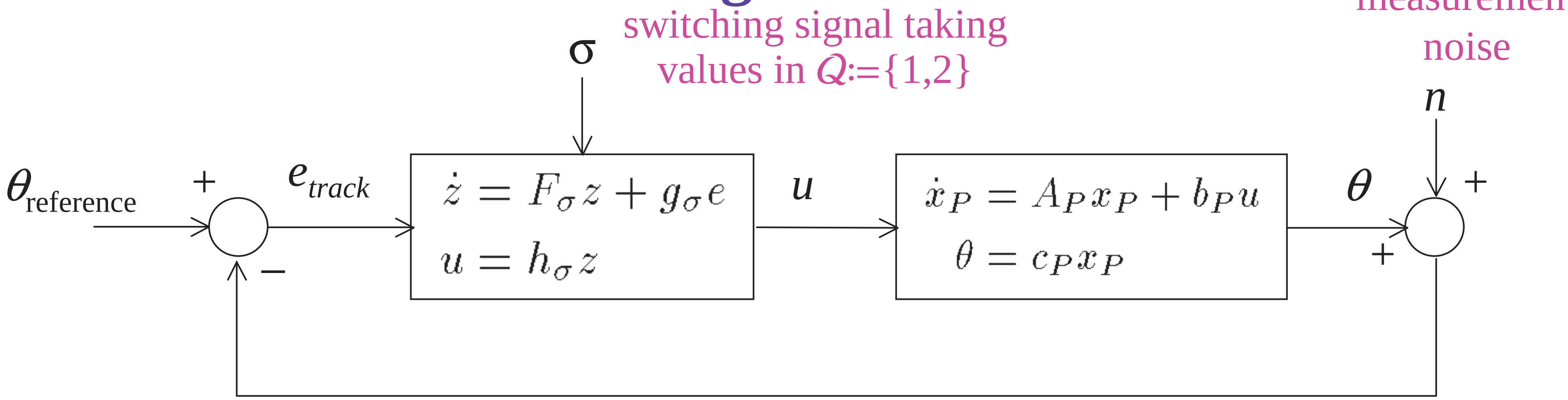
realization: $\dot{z} = F_2 z + g_2 e$, $u = h_2 z$

$$F_2 = \begin{bmatrix} -974 & -459 & -229 \\ 1024 & 0 & 0 \\ 0 & 512 & 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} 8192 \\ 0 \\ 0 \end{bmatrix}$$

$$h_2 = \begin{bmatrix} 5859 & 38 & 0.074 \end{bmatrix}$$



Switching controllers



overall system:

$$\underbrace{\begin{bmatrix} \dot x_P \\ \dot z \end{bmatrix} = \begin{bmatrix} A_P & b_P h_\sigma \\ -g_\sigma c_P & F_\sigma \end{bmatrix} \begin{bmatrix} x_P \\ z \end{bmatrix} + \begin{bmatrix} 0 \\ g_\sigma \end{bmatrix} (\theta_{\text{ref}} - n)}_{\dot x = A_\sigma x + b_\sigma (\theta_{\text{ref}} - n)}$$

$$A_q := \begin{bmatrix} A_P & b_P h_q \\ -g_q c_P & F_q \end{bmatrix} q \in \mathcal{Q} := \{1, 2\}$$

Theorem:
For every family of input-output controller models, there always exist a family a controller realizations such that the switched closed-loop systems is exponentially stable for arbitrary switching.

One can actually show that there exists a common quadratic Lyapunov function for the closed-loop.

In general the realizations are not minimal

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Current-state dependent switching

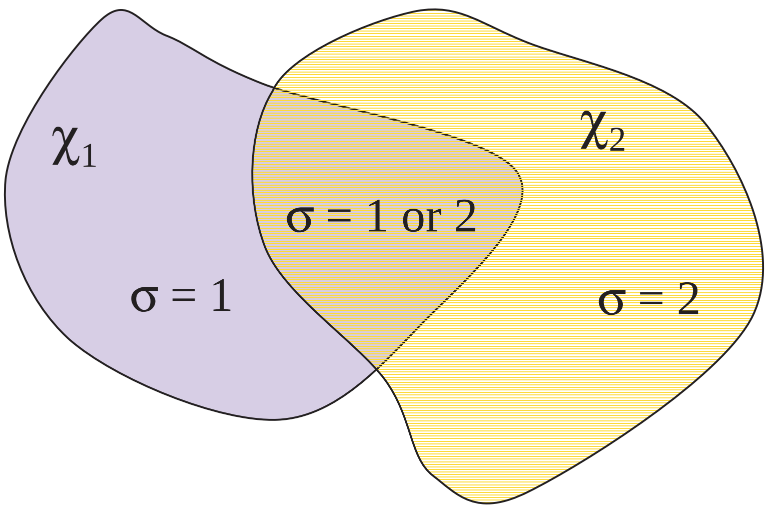
$*$ $\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}$

no resets

$\chi := \{\chi_q \in \mathbb{R}^n : q \in \mathcal{Q}\} \equiv$ (not necessarily disjoint) covering of $\mathbb{R}^n$, i.e., $\cup_{q\in\mathcal{Q}}\, \chi_q = \mathbb{R}^n$

Current-state dependent switching

$\mathcal{S}[\chi] \equiv$ set of all pairs (σ, x) with σ piecewise constant and x piecewise continuous such that $\forall\, t$, $\sigma(t) = q$ is allowed only if $x(t) \in \chi_q$



Thus $(\sigma, x) \in \mathcal{S}[\chi]$ if and only if $x(t) \in \chi_{\sigma(t)}\ \forall\, t$

Common Lyapunov function for arbitrary switching

$$\dot{x} = f_\sigma(x) \qquad (\sigma, x) \in \mathcal{S}_{\text{all}}$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q},\ z \in \mathbb{R}^n$$

Then for arbitrary switching $\mathcal{S}_{\text{all}}$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{eq} = 0$)
1st Take an arbitrary solution (σ, x) and define $v(t) := V(\,x(t)\,)\ \forall\, t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq 0$$

2nd Therefore

$$v(t) := V\big(x(t)\big) \leq v(0) := V\big(x(0)\big) \qquad \forall t \geq 0$$

$V(\,x(t)\,)$ is always bounded…

Common Lyapunov function for current-state dep. switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}[\chi]$$

Theorem:

Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q}, z \in \chi_q$$

Then for current-state dependent switching $\mathcal{S}[\chi]$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{\text{eq}} = 0$)

1st Take an arbitrary solution (σ, x) and define $v(t) := V(x(t))\ \forall t \geq 0$

$$\dot{v} = \frac{\partial V}{\partial x}(x)\dot{x} = \frac{\partial V}{\partial x}(x) f_\sigma(x) \leq W\big(x(t)\big) \leq 0$$

still holds because $x(t) \in \chi_{\sigma(t)}$

2nd Therefore

$$v(t) := V\big(x(t)\big) \leq v(0) := V\big(x(0)\big) \qquad \forall t \geq 0$$

Same conclusions as before …

Common Lyapunov function for current-state dep. switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}[\chi]$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q}, z \in \chi_q$$

Then for current-state dependent switching $\mathcal{S}[\chi]$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Note that:

- Same conclusion would hold for any subset of $\mathcal{S}[\chi]$
- Some (or all) the unswitched systems may not be stable
$$\dot{x} = f_q(x)$$
- This theorem does not guarantee existence of solutions (as opposed to the usual Lyapunov Theorem and the ones for state independent switching)…

Common Lyapunov function for current-state dep. switching

$$\dot{x} = f_\sigma(x) \qquad x = x^- \qquad (\sigma, x) \in \mathcal{S}[\chi]$$

Theorem:
Suppose there exists a continuously differentiable, positive definite, radially unbounded function $V: \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

Then for current-state dependent switching $\mathcal{S}[\chi]$

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{\text{eq}}$ then x_{eq} is (glob) uniformly asymptotically stable.

E.g., $\mathcal{Q} := \{-1, +1\}$, $\chi_{-1} := [0,\infty)$, $\chi_{+1} := (-\infty,0)$

$$\dot{x} = \sigma = \begin{cases} -1 & x \ge 0 \\ +1 & x < 0 \end{cases} \qquad f_{-1}(x) := -1 \quad f_{+1}(x) := +1$$

no solutions exists

For $x_{\text{eq}} = 0$ is an equilibrium point and for $V(z) := z^2$

$$\frac{\partial V}{\partial x}(z - x_{\text{eq}}) f_q(z) = \begin{cases} -2z & q = -1,\ z \ge 0 \\ 2z & q = +1,\ z < 0 \end{cases} \le 0$$

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$V_q : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q} \equiv$ family of Lyapunov functions (cont. dif., pos. def., rad. unb.)

$$\frac{\partial V_q}{\partial x}(z - x_{\mathrm{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q}, z \in \chi_q$$

Given a solution (σ, x) and defining $v(t) := V_{\sigma(t)}(\, x(t)\,)\ \forall\, t \geq 0$

1. On an interval $[\tau, t)$ where $\sigma = q$ (constant)

v decreases

$$\dot{v} = \frac{\partial V_q}{\partial x}(x)\dot{x} = \frac{\partial V_q}{\partial x}(x) f_\sigma(x) = \frac{\partial V_q}{\partial x}(x) f_q(x) \leq W\big(x(t)\big) \leq 0$$

2. But at a switching time t, where $\sigma^-(t) = p \neq \sigma(t) = q$,

$$v^-(t) = V_p\big(x^-(t)\big) \qquad v(t) = V_q\big(x(t)\big)$$

v may be discontinuous

(even without reset)

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

$V_q : \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q} \equiv$ family of Lyapunov functions (cont. dif., pos. def., rad. unb.)

$$\frac{\partial V_q}{\partial x}(z - x_{\mathrm{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

Given a solution (σ, x) and defining $v(t) := V_{\sigma(t)}(\, x(t)\,)\ \forall\, t \ge 0$

1. On an interval $[\tau, t)$ where $\sigma = q$ (constant)

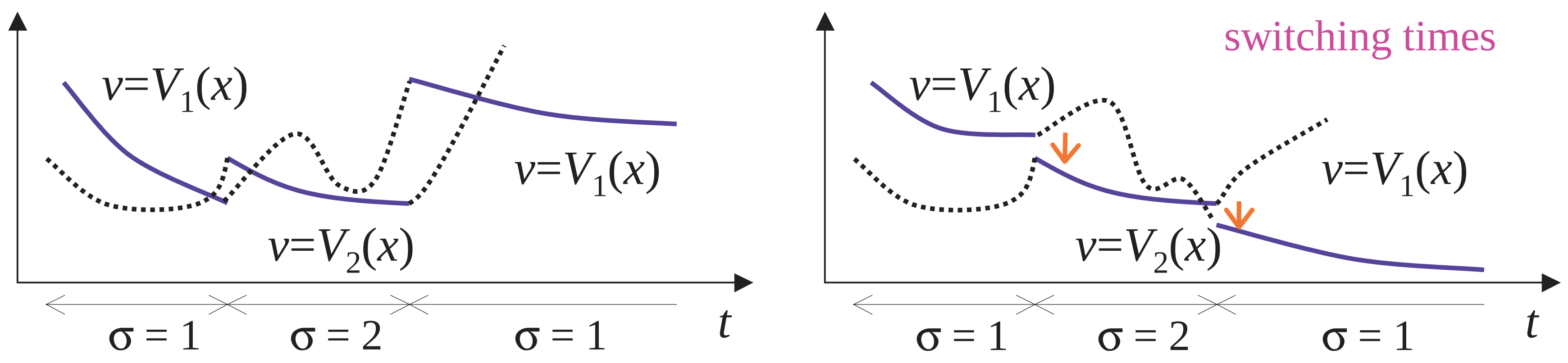
v decreases

$$\dot{v} = \frac{\partial V_q}{\partial x}(x)\dot{x} = \frac{\partial V_q}{\partial x}(x) f_\sigma(x) = \frac{\partial V_q}{\partial x}(x) f_q(x) \le W\big(x(t)\big) \le 0$$

2. But at a switching time t, where $\sigma^-(t) = p \neq \sigma(t) = q$,

$$v^-(t) = V_p\big(x^-(t)\big) \qquad v(t) = V_q\big(x(t)\big)$$

we would be okay if v would not increase at switching times



Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

Theorem: ($\mathcal{Q}$ finite)

Suppose there exists a family of continuously differentiable, positive definite, radially unbounded functions $V_q: \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q}$ such that

$$\frac{\partial V_q}{\partial x}(z - x_{\text{eq}}) f_q(z) \le W(z) \le 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}$ can jump from p to q

$$V_p(z) \ge V_q\big(\rho(q, p, z)\big)$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

Why? (for simplicity consider $x_{eq} = 0$)

1st Take an arbitrary solution (σ, x) and define $v(t) := V_\sigma(\, x(t)\,)\ \forall\, t \ge 0$

while σ is constant: $\dot{v} = \dfrac{\partial V_\sigma}{\partial x}(x)\dot{x} = \dfrac{\partial V_\sigma}{\partial x}(x) f_\sigma(x) \le W\big(x(t)\big) \le 0$

and, at points of discontinuity of σ: $v^-(t) \ge v(t)$ does not increase

from now on same as before …

Multiple Lyapunov functions

$$\dot{x} = f_\sigma(x) \qquad x = \rho(\sigma, \sigma^-, x^-) \qquad (\sigma, x) \in \mathcal{S} \subset \mathcal{S}[\chi]$$

Theorem: ($\mathcal{Q}$ finite)

Suppose there exists a family of continuously differentiable, positive definite, radially unbounded functions $V_q: \mathbb{R}^n \to \mathbb{R}$, $q \in \mathcal{Q}$ such that

$$\frac{\partial V_q}{\partial x}(z - x_{\mathrm{eq}}) f_q(z) \leq W(z) \leq 0 \qquad \forall q \in \mathcal{Q},\ z \in \chi_q$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}$ can jump from p to q

$$V_p(z) \geq V_q\big(\rho(q, p, z)\big)$$

Then

1. the equilibrium point x_{eq} is Lyapunov stable
2. if $W(z) = 0$ only for $z = x_{eq}$ then x_{eq} is (glob) uniformly asymptotically stable.

The V_q's need not be positive definite and radially unbounded "everywhere"

It is enough that $\exists\ \alpha_1, \alpha_2 \in \mathcal{K}_\infty$: $\alpha_1(\|z\|) \leq V_q(z) \leq \alpha_2(\|z\|)$ $\quad \forall\ q \in \mathcal{Q}, z \in \chi_q$

LaSalle's Invariance Principle (ODE)

$$\dot{x} = f(x) \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set $\equiv x(t_0) \in M \Rightarrow x(t) \in M \; \forall \; t \geq t_0$

Theorem (LaSalle Invariance Principle):
Suppose there exists a continuously differentiable, positive definite, radially unbounded function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{\partial V}{\partial x}\big(z - x_{\text{eq}}\big) f(z) \leq W(z) \leq 0 \qquad \forall z \in \mathbb{R}^n$$

Then x_{eq} is a Lyapunov stable equilibrium and the solution always exists globally.
Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{\, z \in \mathbb{R}^n : W(z) = 0 \,\}$$

Note that:

1. When $W(z) = 0$ only for $z = x_{eq}$ then $E = \{x_{eq}\}$.
   Since $M \subset E$, $M = \{x_{eq}\}$ and therefore $x(t) \to x_{eq} \Rightarrow$ asympt. stability
2. Even when E is larger then $\{x_{eq}\}$ we often have $M = \{x_{eq}\}$ and can conclude asymptotic stability.

LaSalle's Invariance Principle (linear system)

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \; \forall\, t \geq 0$

Theorem (LaSalle Invariance Principle–linear system, quadratic V):
Suppose there exists a positive definite matrix P

$$A' P + P A \leq -Q \leq 0$$

Then the system is stable.
Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{\, z \in \mathbb{R}^n : \; Q z = 0 \,\}$$

Note that:

1. Since $Q \geq 0$ we can always write $Q = C' C$ …

LaSalle's Invariance Principle (linear system)

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \; \forall \; t \geq 0$

Theorem (LaSalle Invariance Principle–linear system, quadratic V):
Suppose there exists a positive definite matrix P

$$A'P + PA \leq -C'C \leq 0$$

Then the system is stable.
Moreover, $x(t)$ converges to the largest invariant set M contained in

$$E := \{ z \in \mathbb{R}^n : Cz = 0 \}$$

Why? show that $C'Cz = 0 \Rightarrow Cz = 0$

Note that:

2. When $Q > 0$ then $E = \{0\}$.
   Since $M \subset E$, $M = \{0\}$ and therefore $x(t) \to 0 \Rightarrow$ asympt. stability
3. Even when E is larger then $\{0\}$ we often have $M = \{0\}$ and can conclude asymptotic stability.

When does this happen ?

Asymptotic stability from LaSalle's IP

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \ \forall\, t \geq 0$

$M \equiv$ largest invariant set contained in $E := \{ z \in \mathbb{R}^n : C z = 0 \}$

$$x_0 \in M \text{ if and only if } x(t) := e^{A t} x_0 \in M \subset E \quad \forall\, t \geq 0$$

$$M := \left\{ z \in \mathbb{R}^n : \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} z = 0 \right\}$$

(check that this is indeed an invariant set …)

LaSalle's Invariance Principle (linear system)

$$\dot{x} = Ax \qquad x \in \mathbb{R}^n$$

$M \in \mathbb{R}^n$ is an invariant set if $x(0) \in M \Rightarrow x(t) \in M \ \forall\, t \geq 0$

Theorem (LaSalle Invariance Principle–linear system, quadratic V):
Suppose there exists a positive definite matrix P

$$A' P + P A \leq - C'C \leq 0$$

Then the system is stable. Moreover, $x(t)$ converges to

observability matrix
of the pair (C,A)

$$M := \left\{ z \in \mathbb{R}^n : Oz = 0 \right\} \qquad O := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

When O is nonsingular, we have asymptotic stability
(pair (C,A) is observable)

Back to switched linear systems…

$$\dot{x} = A_\sigma x \qquad x = R_{\sigma,\sigma^-} x^- \qquad (\sigma, x) \in \mathcal{S}$$

Theorem: ($\mathcal{Q}$ finite)

Suppose there exist positive definite matrices $P_q \in \mathbb{R}^{n\times n}$, $q \in \mathcal{Q}$ such that

$$A_q' P_q + P_q A_q \leq - C_q' C_q \leq 0 \qquad \forall\, q \in \mathcal{Q}$$

and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}[\chi]$ can jump from p to q

$$z' P_p z \geq z' R'_{qp} P_q R_{qp} z$$

from general theorem

Then the switched system is stable.

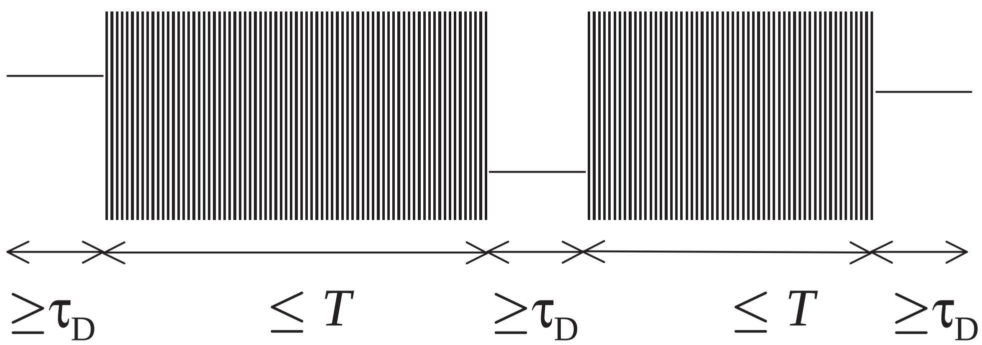
Moreover, if every pair (C_q, A_q), $q \in \mathcal{Q}$ is observable then

1. if $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ then it is asymptotically stable
2. if $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ then it is uniformly asymptotically stable.

Sets of switching signals

$\mathcal{S}_{\text{dwell}}[\tau_D]$ $\equiv$ switching signals with "dwell-time" $\tau_D > 0$, i.e., interval between consecutive discontinuities larger or equal to τ_D

$\mathcal{S}_{\text{p-dwell}}[\tau_D, T] \equiv$ switching signals with "persistent dwell-time" $\tau_D > 0$ and "period of persistency" $T > 0$, i.e., $\exists$ infinitely many intervals of length $\geq \tau_{\mathcal{D}}$ on which sigma is constant & consecutive intervals with this property are separated by no more than T



$$\mathcal{S}_{\text{weak-dwell}} := \cup_{\tau D > 0} \mathcal{S}_{\text{p-dwell}}[\tau_D, +\infty] \equiv \text{each } \sigma \text{ has persistent dwell-time} > 0$$

$$\mathcal{S}_{\text{dwell}}[\tau_D] \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, 0] \subset \mathcal{S}_{\text{weak-dwell}} \subset \mathcal{S}_{\text{all}}$$

LaSalle's IP for switched systems

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and at any $z \in \mathbb{R}^n$ where a switching signal in $\mathcal{S}[\chi]$ can jump from p to q

$$V_p(z) \geq V_q(R_{q\,p}\, z)$$

from general theorem

Then the switched system is stable.

Moreover, if every pair (C_q, A_q), $q \in \mathcal{Q}$ is observable then

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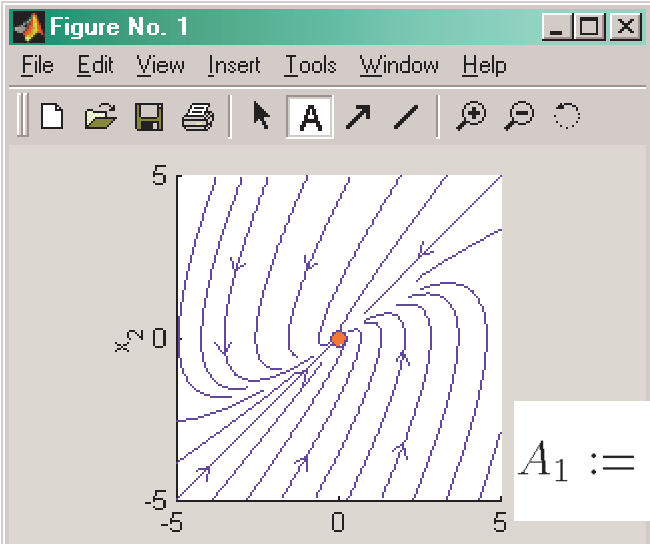
$\mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ $\equiv$ switching signals with "persistent dwell-time" $\tau_D > 0$ and "period of persistency" $T > 0$, i.e., $\exists$ infinitely many intervals of length $\geq \tau_{\mathcal{D}}$ on which sigma is constant & consecutive intervals with this property are separated by no more than T

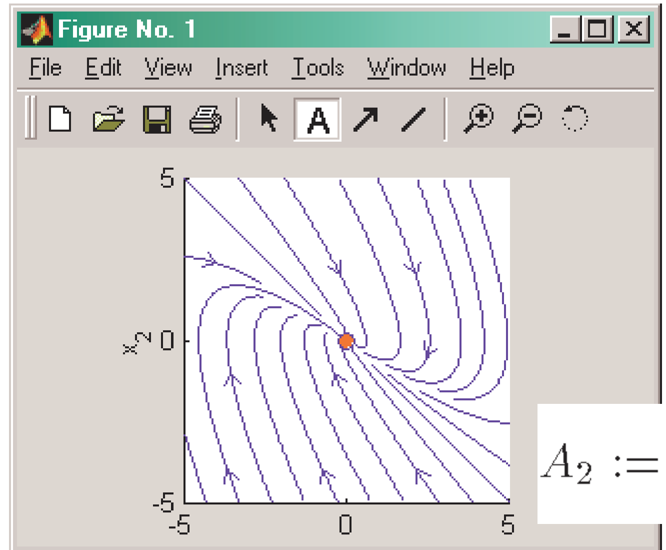
$\mathcal{S}_{\text{weak-dwell}} := \cup_{\tau_D > 0}\, \mathcal{S}_{\text{p-dwell}}[\tau_D, +\infty] \equiv$ each σ has persistent dwell-time > 0

$$\mathcal{S}_{\text{dwell}}[\tau_D] \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, 0] \subset \mathcal{S}_{\text{weak-dwell}} \subset \mathcal{S}_{\text{all}}$$

Example



$A_1 := \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$



$A_2 := \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

$$\dot{x} = A_\sigma x$$

Choosing $P_1 = P_2 = I$ *common Lyapunov function*

$$A_q' P_q + P_q A_q = -c_q' c_q \leq 0 \qquad c_q := \begin{bmatrix} 0 & 2 \end{bmatrix} \qquad \forall q \in \{1, 2\}$$

$$O_q := \begin{bmatrix} c_q \\ c_q A_q \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ \pm 2 & -4 \end{bmatrix} \quad q \in \{1, 2\}$$ *nonsingular (observable)*

1. One can find $\sigma \notin \mathcal{S}_{\text{weak-dwell}}$ for which we do not have asymptotic stability
2. One can find $\sigma \in \mathcal{S}_{\text{weak-dwell}}$, $\sigma \notin \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ for which asymptotic stability is not uniform *(problems, e.g., close to the x_2=0 axis)*

LaSalle's IP for switched systems

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Then the switched system is stable.

from general theorem

Moreover, if every pair (C_q, A_q), $q \in \mathcal{Q}$ is observable then

1. if $\mathcal{S} \subset \mathcal{S}_{\text{weak-dwell}}$ then it is asymptotically stable
2. if $\mathcal{S} \subset \mathcal{S}_{\text{p-dwell}}[\tau_D, T]$ then it is uniformly asymptotically stable.

a) Finiteness of $\mathcal{Q}$ could be replaced by compactness

b) In some cases it is sufficient for all pairs (C_q, A_q), $q \in \mathcal{Q}$ to be detectable (e.g., when $A_q = A + B F_q$)

c) When the pairs (C_q, A_q), $q \in \mathcal{Q}$ are not observable x converges to the smallest subspace $\mathcal{M}$ that is invariant for all unswitched system and contains the kernels of all O_q

d) There are nonlinear versions of this result (no uniformity?)