

Bài 5: Tìm tất cả ma trận vuông cấp 2.

$$\text{đ/c } X^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} a^2 + bc = 0 \\ ab + bd = 0 \\ ca + dc = 0 \\ cb + d^2 = 0 \end{cases} \Leftrightarrow \begin{cases} ab + bd = 0 \\ ca + dc = 0 \\ a^2 - d^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} ab + bd = 0 \\ ca + dc = 0 \\ a - d = 0 \\ a + d = 0 \end{cases}$$

$$\textcircled{*} a - d = 0 \Leftrightarrow a = d$$

$$\Rightarrow \begin{cases} db + bd = 0 \\ cd + dc = 0 \end{cases} \Leftrightarrow \begin{cases} 2db = 0 \\ 2dc = 0 \end{cases}$$

$$\Rightarrow \begin{cases} d = 0 \\ b = 0 \\ c = 0 \end{cases} \Rightarrow \begin{cases} d = 0 \\ b = c \end{cases} \quad \textcircled{1}$$

$$\textcircled{*} a + d = 0 \Rightarrow \begin{cases} 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow a = \pm \sqrt{-bc}$$
$$\Rightarrow \begin{cases} -db + bd = 0 \\ -dc + dc = 0 \end{cases} \Leftrightarrow \begin{cases} 0 = 0 \\ 0 = 0 \end{cases} \Rightarrow b = 0 \neq bc < 0$$

$$\textcircled{1} \Leftrightarrow b=0 \Rightarrow \begin{cases} a^2 + c^2 = 0 \\ ac + b \cdot 0 = 0 \end{cases}$$

$$\Rightarrow ac + c \cdot 0 = 0$$

$$\Rightarrow \begin{cases} a = 0 \\ c = 0 \\ a = 0 \\ e = 0 \end{cases}$$

$$b; X^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Leftrightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{cases} a^2 + bc = 1 \\ ab + bd = 0 \text{ } \textcircled{1} = \\ cd + dc = 0 \text{ } \textcircled{2} \\ cb + d^2 = 1 \end{cases}$$

$$\textcircled{1} \Rightarrow b(a+d) = 0$$

$$\Rightarrow \begin{cases} b = 0 \\ a+d = 0 \end{cases}$$

$$\textcircled{2} \Rightarrow \begin{cases} a = -d \\ b = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = \pm 1 \\ d = \pm 1 \end{cases}$$

$$\textcircled{1} \quad a = -d$$

$$\Rightarrow a = \pm \sqrt{1 - bc} \quad \forall bc \leq 1.$$

$$\textcircled{2} \Rightarrow c(a+d) = 0$$

$$\Rightarrow \begin{cases} c = 0 \\ d = -a \end{cases}$$

$$\Rightarrow \begin{cases} a = \pm 1 \\ d = \pm 1 \end{cases}$$

$$\begin{cases} a = \pm \sqrt{1 - bc} \quad \forall bc \leq 1. \\ d = -a \end{cases}$$

## Bài 6

$$\text{a; } \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A; x^2 - (a+d)x + ad - bc = 0$$

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$$\Leftrightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 - (a+d) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + ad - bc = 0$$

$$\Leftrightarrow \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} - \begin{bmatrix} a^2 & ba \\ ac & da \end{bmatrix} - \begin{bmatrix} ad & db \\ dc & d^2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} be - bd \\ dc & cb + d^2 - da \end{bmatrix} - \begin{bmatrix} acd & db \\ dc & d^2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} bc - ad & 0 \\ 0 & db - da \end{bmatrix} + ad - bc = 0$$

$$\Leftrightarrow (be - ad) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + ad - bc = 0$$

$$\Leftrightarrow 0 = 0$$

$$b; A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} a^2 + bc = 0 \\ ab + bd = 0 \\ ca + dc = 0 \\ cb + d^2 = 0 \end{cases}$$

$$\textcircled{1} \Rightarrow b(a+d) = 0 \Rightarrow \begin{cases} b = 0 \\ a = -d \end{cases}$$

$$\textcircled{2} \quad b = 0 \Rightarrow a = 0 \Rightarrow d = 0$$

$$\textcircled{3} \quad a = -d \Rightarrow a^2 = -bc \quad (\Rightarrow a = \pm \sqrt{-bc}) \quad \text{und } b \neq 0$$

$$\textcircled{4} \quad c = 0 \Rightarrow a \Rightarrow a = 0; d = 0$$

$\Rightarrow A$  ist diagonal

$$\left[ \begin{array}{cc} 0 & 0 \\ c & 0 \end{array} \right], \left[ \begin{array}{cc} 0 & b \\ 0 & 0 \end{array} \right], \left[ \begin{array}{cc} \sqrt{-bc} & b \\ c & -\sqrt{-bc} \end{array} \right]$$

$$\left[ \begin{array}{cc} \cancel{\sqrt{-bc}} & b \\ c & \cancel{\sqrt{-bc}} \end{array} \right]$$

$$\text{und } b \cdot c \leq 0$$

$$\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \xrightarrow{0 \neq c} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} = c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^k = 0 \forall k > 2$$

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} = b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^k = 0 \forall k > 2$$

$$\begin{bmatrix} F_{bc} & b \\ c & -\sqrt{bc} \end{bmatrix} = 0 \forall k > 2.$$

Bài 7:

$$c_1 \begin{vmatrix} a_1 + b_1 x & a_1 - b_1 x & c_1 \\ a_2 + b_2 x & a_2 - b_2 x & c_2 \\ a_3 + b_3 x & a_3 - b_3 x & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1 - b_1 x & c_1 \\ a_2 & a_2 - b_2 x & c_2 \\ a_3 & a_3 - b_3 x & c_3 \end{vmatrix} + b_1 x \begin{vmatrix} a_1 - b_1 x & c_1 \\ a_2 - b_2 x & c_2 \\ a_3 - b_3 x & c_3 \end{vmatrix}$$

Bài 7:

$$\begin{vmatrix} a_1 + b_1x & a_1 - b_1x & c_1 \\ a_2 + b_2x & a_2 - b_2x & c_2 \\ a_3 + b_3x & a_3 - b_3x & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1 - b_1x & c_1 \\ a_2 & a_2 - b_2x & c_2 \\ a_3 & a_3 - b_3x & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & a_1 - b_1x & c_1 \\ b_2x & a_2 - b_2x & c_2 \\ b_3x & a_3 - b_3x & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 - b_1x & c_1 \\ a_2 - b_2x & c_2 \\ a_3 - b_3x & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & -b_1x & c_1 \\ b_2x & -b_2x & c_2 \\ b_3x & -b_3x & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + x \begin{vmatrix} a_1 & -b_1 & c_1 \\ a_2 & -b_2 & c_2 \\ a_3 & -b_3 & c_3 \end{vmatrix} + x^2 \begin{vmatrix} b_1 & -b_1 & c_1 \\ b_2 & -b_2 & c_2 \\ b_3 & -b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + (-x) \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & e_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$+ (-x) \begin{vmatrix} a_1 & b_1 & e_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1x & -b_1x & c_1 \\ b_2x & -b_2x & c_2 \\ b_3x & -b_3x & c_3 \end{vmatrix}$$

$$\tilde{F} = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} + \begin{vmatrix} b_1 x & -b_1 x & c_1 \\ b_2 x & -b_2 x & c_2 \\ b_3 x & -b_3 x & c_3 \end{vmatrix}$$

$$= 2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$V_i = \begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0; \begin{vmatrix} b_1 x & -b_1 x & c_1 \\ b_2 x & -b_2 x & c_2 \\ b_3 x & -b_3 x & c_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a_1 + b_1 x & a_1 - b_1 x & c_1 \\ a_2 + b_2 x & a_2 - b_2 x & c_2 \\ a_3 + b_3 x & a_3 - b_3 x & c_3 \end{vmatrix} = -2x \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$b; \begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & bc \\ 0 & a-b & bc-ac \\ 0 & a-c & bc-ab \end{vmatrix}$$

$$= (a-b)(a-c) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 1 & -b \end{vmatrix}$$

$$= (a-b)(a-c) \begin{vmatrix} 1 & a & bc \\ 0 & 1 & -c \\ 0 & 0 & b-c \end{vmatrix}$$

$$\begin{vmatrix} 1-a & a^2 \\ 1-b & b^2 \\ 1-c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 0 & a-b & a^2-b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$\simeq (a-b)(a-c) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

$$= (a-b)(a-c) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a+b \\ 0 & 0 & b-c \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} 1 & a & bc \\ 1 & b & ac \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(a-c) \cdot (b-c)$$

Bài 8:

$$A = \begin{vmatrix} 1 & 3 & 5 & -1 \\ 2 & -1 & -1 & 4 \\ 5 & 1 & -1 & 7 \\ 7 & 7 & 9 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & -1 \\ 0 & -7 & -11 & 6 \\ 0 & -14 & -26 & 12 \\ 0 & -14 & -26 & 8 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 3 & 5 & -1 \\ 0 & -7 & -11 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & -4 & -4 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & -1 \\ 0 & -7 & -11 & 6 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$$\det A = 1 \cdot (-7) \cdot (-4) \cdot (-4) = 112.$$

$$B = \begin{vmatrix} a+b & ab & a^2+b^2 \\ b+c & bc & b^2+c^2 \\ c+a & ca & a^2+c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a+b & ab & a^2+b^2 \\ b+c & bc & b^2+c^2 \\ c+a & ca & a^2+c^2 \end{vmatrix} \begin{vmatrix} a+b & ab \\ b+c & bc \\ c+a & ca \end{vmatrix}$$

$$(a+b)bc(a^2+c^2) + ab(b^2+c^2)(c+a) + (a^2+b^2)(b+c)ca$$

$$- (c+a).bc(a^2+b^2) - ca(b^2+c^2)(a+b) - (a^2+c^2)(b+c)ab$$

$$= bc((a+b)(a^2+c^2) - (c+a)(a^2+b^2)) + ca(a^2+b^2)(b+c) -$$

$$(b^2+c^2)(a+b) + ab((b^2+c^2)(c+a) - (a^2+c^2)(b+c))$$

$$= bc(c^2a + ba^2 + bc^2 - ac^2 - b^2c - ab^2)$$

$$+ ca(a^2b + a^2c + b^2c - b^2a - c^2a - c^2b)$$

$$+ ab(b^2c + b^2a + c^2a - a^2b - a^2c - c^2b)$$

$$= \cancel{c^3ba} + \cancel{b^2ac} + \cancel{bc^3} - \cancel{bac} - \cancel{b^2c^2} - \cancel{bac} + \cancel{acb} + \cancel{ac^2c}$$

$$+ \cancel{b^2ac^2} - \cancel{b^2a^2c} - \cancel{c^2a^2} - \cancel{cab} + \cancel{abc} + \cancel{a^2b^2} + \cancel{c^2ba}$$

$$- \cancel{a^3b^2} - \cancel{abc} - \cancel{b^2a^2}$$

$$= b^2c^3 - b^3c^2 - b^3ac + a^3c^2 + b^2ac^2 - b^2c^2$$

$$- c^3a^2 + abc + a^2b^3 - a^3b^2.$$

$$\text{Q; } D = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 2-x^2 & 2 & 3 \\ 2 & 3 & 1 & 5 \\ 2 & 3 & 1 & 9-x^2 \end{vmatrix} \stackrel{R1 \leftrightarrow R2}{=} \begin{vmatrix} 1 & 2 & 1 & 3 \\ 1 & 2 & 2-x^2 & 3 \\ 2 & 1 & 3 & 5 \\ 2 & 1 & 3 & 9-x^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 1 & 3 \\ 0 & 0 & x^2 - 1 & 0 \\ 0 & -3 & 1 & -1 \\ 0 & -3 & 1 & 3-x^2 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 \cdot x^2 - 1 & 0 \\ -3 & 1 & -1 \\ -3 & 1 & 3-x^2 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x^2 - 1 & 0 \\ 0 & 0 & x^2 - 2 \\ -3 & 1 & 3-x^2 \end{vmatrix} = (-3) \cdot \begin{vmatrix} x^2 - 1 & 0 \\ 0 & x^2 - 2 \end{vmatrix}$$

$$= (-3)(x^2 - 1)(x^2 - 2)$$

$$= 3(x^2 - 1)(x^2 - 2)$$