The Shapes of Multi-Dimensional Arrays

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Introduction

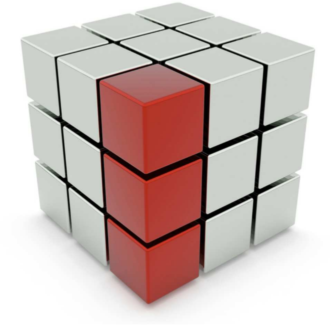
Multidimensional arrays and linear algebra

$$\begin{pmatrix} 0.5 \\ 1.2 \\ 4.1 \end{pmatrix}$$

1D

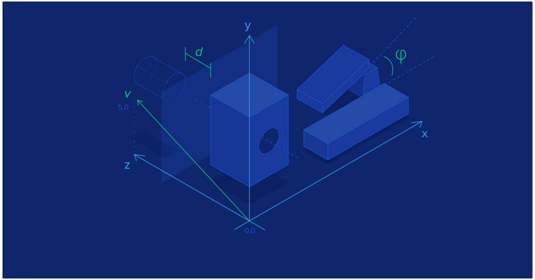
$$\begin{pmatrix} 0.5 & 4.5 & -0.3 \\ 1.2 & -0.9 & 1.4 \\ 4.1 & -3.4 & 0.1 \end{pmatrix}$$

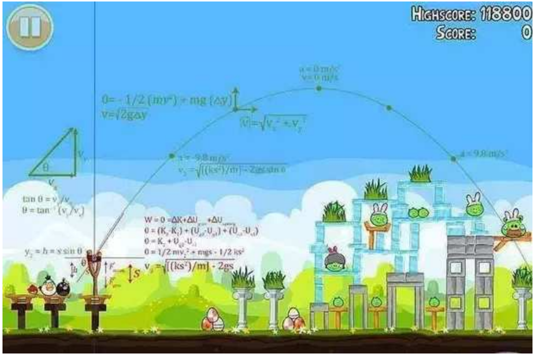
2D



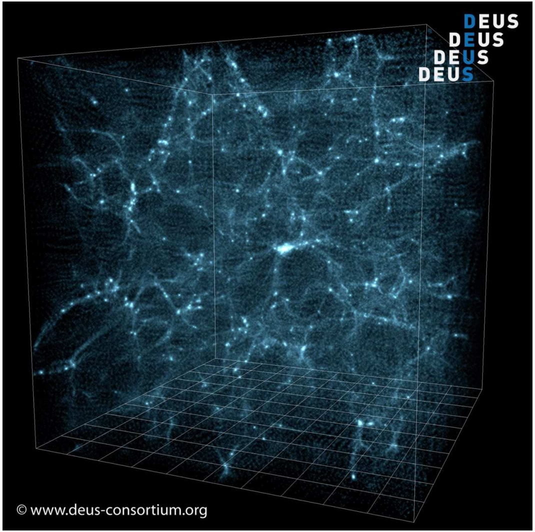
3D

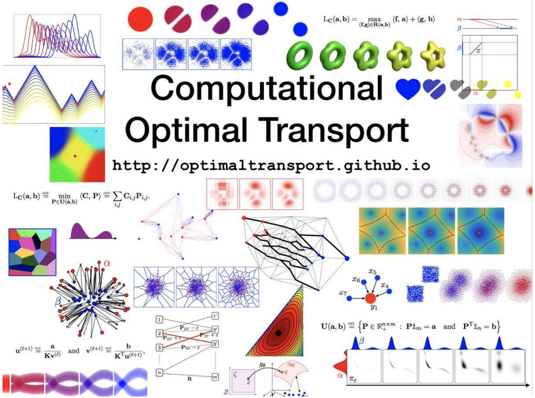
Computer graphics and videogames: small fixed-size vectors and matrices



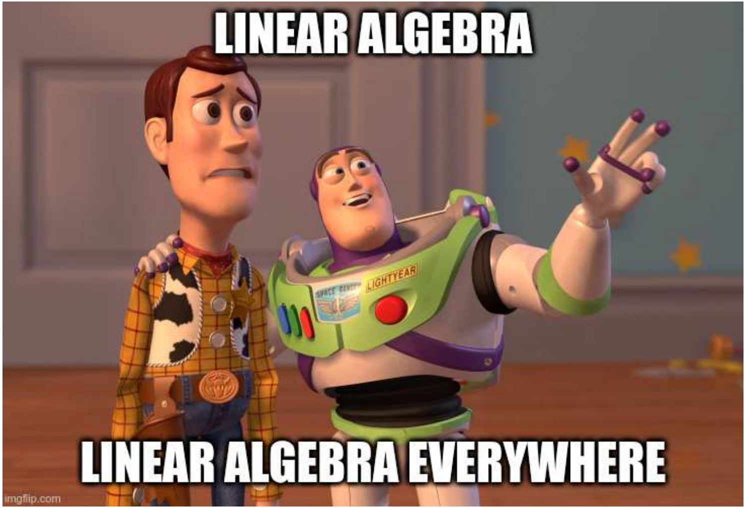


Personal use case with optimal transport: up to 6D dynamic arrays with billions of elements





Summary: multidimensional stuff are useful



Standardization

A standardization effort

On `mdspan`

- P0332: Relaxed Incomplete Multidimensional Array Type Declaration
- P0009: `mdspan`: A Non-Owning Multidimensional Array Reference
- P1684: `mdarray`: An Owning Multidimensional Array Analog of `mdspan`

On linear algebra in general

- P1417: Historical lessons for C++ linear algebra library standardization
- P1166: What do we need from a linear algebra library?
- P1385: A proposal to add linear algebra support to the C++ standard library
- P1635: A Design for an Inter-Operable and Customizable Linear Algebra Library

On BLAS (Basic Linear Algebra Subprograms)

- P1673: A free function linear algebra interface based on the BLAS
- P1674: Evolving a Standard C++ Linear Algebra Library from the BLAS

Miscellaneous

- P1416: Linear Algebra for Data Science and Machine Learning
- P1770: On vectors, tensors, matrices, and hypermatrices

Naming

Originally proposed in P0009 (with array instead of span)

- sci_array
- numeric_array
- vla_array
- multidimensional_array
- multidim_array
- mdarray/md_array
- multiarray/multi_array (multi means something different in multi-containers)

Alternatives

- ndarray/nd_array: because the rank is finite and known at compile-time (used in mathematics and in python)
- hyperarray/hyper_array: because the mathematical generalization of a matrix is called an hypermatrix (same with hypercubes, hyperspheres, . . .)

Layered approach

Standardization by layers

- non-owning references to arrays
- basic algorithms/operations
- owning arrays
- overloaded operators
- geometric aspects
- ...

The mdspan proposal: synopsis

Proposed **mdspan** synopsis as of P0009R10

```
namespace std {
  // [mdspan.extents], class template extents
  template<ptrdiff_t... Extents>
    class extents;

  // [mdspan.layout], Layout mapping policies
  class layout_left;
  class layout_right;
  class layout_stride;

  // [mdspan.accessor.basic]
  template<class ElementType>
    class accessor_basic;

  // [mdspan.basic], class template mdspan
  template<class ElementType, class Extents, class LayoutPolicy = layout_right,
           class AccessorPolicy = accessor_basic<ElementType>>
    class basic_mdspan;

  template<class T, ptrdiff_t... Extents>
    using mdspan = basic_mdspan<T, extents<Extents...>>;

  // ...
}
```

The mdspan proposal: extents

Proposed extents synopsis as of P0009R10

```
template<ptrdiff_t... Extents>
class extents {
public:
  using index_type = ptrdiff_t;

  // [mdspan.extents.cons], Constructors and assignment
  constexpr extents() noexcept = default;
  constexpr extents(const extents&) noexcept = default;
  constexpr extents& operator=(const extents&) noexcept = default;

  template<ptrdiff_t... OtherExtents>
    constexpr extents(const extents<OtherExtents...>&) noexcept;
  template<class... IndexType>
    constexpr extents(IndexType...) noexcept;
  template<class IndexType>
    constexpr extents(const array<IndexType, rank_dynamic()>&) noexcept;
  template<ptrdiff_t... OtherExtents>
    constexpr extents& operator=(const extents<OtherExtents...>&) noexcept;

  // [mdspan.extents.obs], Observers of the domain multidimensional index space
  static constexpr size_t rank() noexcept;
  static constexpr size_t rank_dynamic() noexcept;
  static constexpr index_type static_extent(size_t);
  constexpr index_type extent(size_t) const noexcept;

private:
  static constexpr size_t dynamic_index(size_t) noexcept; // exposition only
  array<index_type, rank_dynamic()> dynamic_extents_{}; // exposition only
};
```

The `mdspan` proposal: `layout`

Proposed `layout_left` synopsis as of P0009R10

```cpp
struct layout_left {
  template<class Extents>
  class mapping {
    using index_type = typename Extents::index_type; // exposition only
  public:
    constexpr mapping() noexcept = default;
    constexpr mapping(const mapping&) noexcept = default;
    constexpr mapping(const Extents&) noexcept;
    template<class OtherExtents> constexpr mapping(const mapping<OtherExtents>&) noexcept;

    constexpr mapping& operator=(const mapping&) noexcept = default;
    template<class OtherExtents> constexpr mapping& operator=(const mapping<OtherExtents>&) noexcept;

    constexpr Extents extents() const noexcept { return extents_; }
    constexpr index_type required_span_size() const noexcept;
    template<class... Indices> index_type operator()(Indices...) const noexcept;
    static constexpr bool is_always_unique() noexcept { return true; }
    static constexpr bool is_always_contiguous() noexcept { return true; }
    static constexpr bool is_always_strided() noexcept { return true; }
    constexpr bool is_unique() const noexcept { return true; }
    constexpr bool is_contiguous() const noexcept { return true; }
    constexpr bool is_strided() const noexcept { return true; }
    index_type stride(size_t) const noexcept;
    template<class OtherExtents> constexpr bool operator==(const mapping<OtherExtents>&) const noexcept;
    template<class OtherExtents> constexpr bool operator!=(const mapping<OtherExtents>& rhs) const noexcept

  private:
    Extents extents_{}; // exposition only
  };
};
```

The mdspan proposal: accessor

Proposed accessor_basic synopsis as of P0009R10

```
template<class ElementType>
  struct accessor_basic {
    using offset_policy = accessor_basic;
    using element_type = ElementType;
    using reference = ElementType&;
    using pointer = ElementType*;

    constexpr typename offset_policy::pointer
      offset(pointer p, ptrdiff_t i) const noexcept;

    constexpr reference access(pointer p, ptrdiff_t i) const noexcept;

    constexpr pointer decay(pointer p) const noexcept;
  };
```


The mdspan proposal: mdspan (1/3)

Proposed mdspan synopsis as of P0009R10

```
template<class ElementType, class Extents, class LayoutPolicy, class AccessorPolicy>
class basic_mdspan {
public:

  // Domain and codomain types
  using extents_type = Extents;
  using layout_type = LayoutPolicy;
  using accessor_type = AccessorPolicy;
  using mapping_type = typename layout_type::template mapping_type<extents_type>;
  using element_type = typename accessor_type::element_type;
  using value_type = remove_cv_t<element_type>;
  using index_type = ptrdiff_t ;
  using difference_type = ptrdiff_t;
  using pointer = typename accessor_type::pointer;
  using reference = typename accessor_type::reference;

  // [mdspan.basic.cons], basic_mdspan constructors, assignment, and destructor
  constexpr basic_mdspan() noexcept = default;
  constexpr basic_mdspan(const basic_mdspan&) noexcept = default;
  constexpr basic_mdspan(basic_mdspan&&) noexcept = default;
  template<class... IndexType>
    explicit constexpr basic_mdspan(pointer p, IndexType... dynamic_extents);
  template<class IndexType, size_t N>
    explicit constexpr basic_mdspan(pointer p, const array<IndexType, N>& dynamic_extents);
  constexpr basic_mdspan(pointer p, const mapping_type& m);
  constexpr basic_mdspan(pointer p, const mapping_type& m, const accessor_type& a);
  template<class OtherElementType, class OtherExtents, class OtherLayoutPolicy, class OtherAccessorPolicy>
    constexpr basic_mdspan(
      const basic_mdspan<OtherElementType, OtherExtents, OtherLayoutPolicy, OtherAccessorPolicy>& other
    );

  // ...
};
```

The mdspan proposal: mdspan (2/3)

Proposed **mdspan** synopsis as of P0009R10

```
template<class ElementType, class Extents, class LayoutPolicy, class AccessorPolicy>
class basic_mdspan {
public:
  // ...
  constexpr basic_mdspan& operator=(const basic_mdspan&) noexcept = default;
  constexpr basic_mdspan& operator=(basic_mdspan&&) noexcept = default;
  template<class OtherElementType, class OtherExtents, class OtherLayoutPolicy, class OtherAccessorPolicy>
    constexpr basic_mdspan& operator=(
      const basic_mdspan<OtherElementType, OtherExtents, OtherLayoutPolicy, OtherAccessorPolicy>& other
    ) noexcept;

  // [mdspan.basic.mapping], basic_mdspan mapping domain multidimensional index to access codomain element
  constexpr reference operator[](index_type) const noexcept;
  template<class... IndexType>
    constexpr reference operator()(IndexType... indices) const noexcept;
  template<class IndexType, size_t N>
    constexpr reference operator()(const array<IndexType, N>& indices) const noexcept;
  // ...
};
```

The mdspan proposal: mdspan (3/3)

Proposed **mdspan** synopsis as of P0009R10

```cpp
template<class ElementType, class Extents, class LayoutPolicy, class AccessorPolicy>
class basic_mdspan {
public:
  // ...
  accessor_type accessor() const;

  static constexpr int rank() noexcept;
  static constexpr int rank_dynamic() noexcept;
  static constexpr index_type static_extent(size_t r) noexcept;

  constexpr Extents extents() const noexcept;
  constexpr index_type extent(size_t r) const noexcept;
  constexpr index_type size() const noexcept;
  constexpr index_type unique_size() const noexcept;

  // [mdspan.basic.codomain], basic_mdspan observers of the codomain
  constexpr span<element_type> span() const noexcept;
  constexpr pointer data() const noexcept;

  static constexpr bool is_always_unique() noexcept;
  static constexpr bool is_always_contiguous() noexcept;
  static constexpr bool is_always_strided() noexcept;

  constexpr mapping_type mapping() const noexcept;
  constexpr bool is_unique() const noexcept;
  constexpr bool is_contiguous() const noexcept;
  constexpr bool is_strided() const noexcept;
  constexpr index_type stride(size_t r) const;
};
```

The `mdspan` proposal: summary

`mdspan` is the product of

- an element type `ElementType`
- an `Extents` to describe its shape
- a `LayoutPolicy` to specify the mapping between the indices and storage
- an `AccessorPolicy` to specify how individual and contiguous elements can be accessed in memory

Simplifications...

- it is non-owning
- it has been designed for contiguous memory storage

The `mdspan` proposal: specifying the shape

Example `mdspan` declaration

```cpp
// As of P0009R10
using myspan = std::mdspan<double, extents<3, std::dynamic_extent, 7>>;

// With P0332: Relaxed Incomplete Multidimensional Array Type Declaration
using myspan = std::mdspan<double[][3][7]>; // already allowed
using myspan = std::mdspan<double[3][][7]>; // currently not allowed
```

Standardization

- The Library Evolution Working Group (LEWG) was largely in favor of `std::mdspan<double[3][][7]>`
- The Evolution Working Group (EWG) was against a language change because of unintended side effects in other contexts

Zero-sized arrays

- *"Its value N specifies the array bound, i.e., the number of elements in the array; N shall be greater than zero."*
- $\Rightarrow$ `double[3][7][0]` not allowed
- However `std::array<double, 0>` is a compiling special case

The shapes of multi-dimensional arrays

What are we looking for?

A way to specify the shapes of multi-dimensional arrays

Goals

- Genericity: cover as much as possible of the parameter space
- Performance: both in terms of computing time and memory
- Expressivity: easy to and understand, read, and write in a concise way

Design

The shapes of multi-dimensional arrays

What are we looking for?

A way to specify the shapes of multi-dimensional arrays

Design goals: the GPE principle

- **G**enericity: cover as much as possible of the parameter space
- **P**erformance: both in terms of computing time and memory
- **E**xpressivity: easy to and understand, read, and write in a concise way

However...

One cannot have everything at the same time

Software design

Software architecture

The art of balancing and compromising between genericity, performance, and expressivity.

Design goals

- **G**enericity: cover as much as possible of the parameter space
- **P**erformance: both in terms of computing time and memory
- **E**xpressivity: easy to and understand, read, and write in a concise way

Design complexity

- Focusing on only one axis drastically simplifies the problem (for example focusing on expressivity in a high level language)
- Focusing on two axes can still lead to reasonable level of complexity (for example performance and expressivity)
- Addressing the three axes at the same time is what can make the design a real challenge

C++

The standard C++ library has to handle the three axes.

$\Rightarrow$ Still no `std::matrix` in 2020. . . .

What do we want (ideally) for a n-dimensional array?

Expressivity (pseudo-code)

```
std::ndarray<double> a;
std::ndarray<double[0]> b;
std::ndarray<double[3][4][5]> c;
std::ndarray<double[3][][5]> d;
std::ndarray<double[3][][5], properties...> d;
```

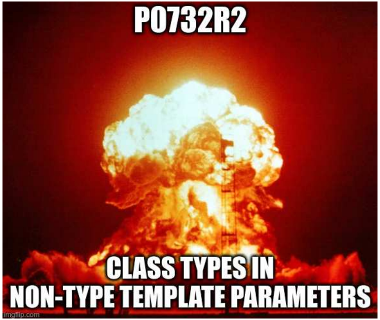
Performance

No computing-time/memory overhead of generic solutions compared to handcrafted solutions designed and optimized for specific scenarios

Genericity

- Arbitrary memory layouts
- Arbitrary indexing schemes
- Arbitrary finite ranks
- Combinations of static and dynamic dimensions
- Zero-sized case

What do we need? (hint: C++20)

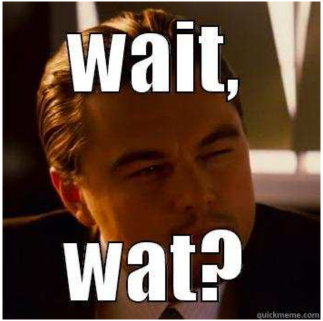


EDSL

What if. . .

. . . the following was perfectly possible in C++20. . .

- `std::ndarray<double, shape[4]()[3][5]> myarray;`
- or even: `std::ndarray<contents<double>[4]()[3][5]> myarray;`



Reverse-engineering from the expressivity goal

With shape:

`std::ndarray<double, shape[4]()[3][5]> myarray;`

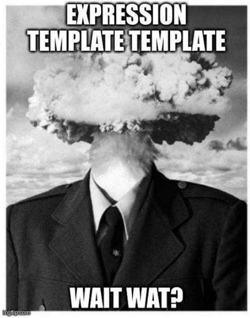
- shape cannot be a type, it has to be a variable
- `std::ndarray` should be a class template of the form `template <class, auto>`
- shape should be a variable of a type with `operator()` and `operator[]`
- `operator()`/`operator[]` should return a value of the same type so that `operator()`/`operator[]` can be called again recursively until all dimensions are specified

With contents:

`std::ndarray<contents<double>[4]()[3][5]> myarray;`

- contents cannot be a type, it has to be a variable template
- `std::ndarray` should be a class template of the form `template <auto>`
- `contents<T>` should be a variable of a type with `operator()` and `operator[]`
- `operator()`/`operator[]` should return a value of the same type so that `operator()`/`operator[]` can be called again recursively until all dimensions are specified

Introducing expression template template



Principle

- `shape` builds an expression template
- The result of the expression template is then injected as a template parameter

Warning

- Takes EDSL (Embedded Domain Specific Languages) to a whole new level

Building the EDSL: tags and shaper constructors

Tagging operators to know what operator has been called

```
// A tag marking the function call operator
template <class... Args> struct function_call_operator {};

// A tag marking the subscripting operator
template <class... Args> struct subscripting_operator {};
```

Shaper class template

```
// The type of a shape remembering the sequence of operations it was made from
template <class... Ops> class shaper {
    // Types
    public:
    using type = shaper<Ops...>;
    using index_sequence = std::index_sequence_for<Ops...>;

    // Lifecycle
    public:
    constexpr shaper() noexcept = default;
    template <class... Args> constexpr shaper(shaper<Args...> other, std::size_t i) noexcept
    : _data(other.to_array(i)) {}

    // Implementation details
    private:
    std::array<std::size_t, sizeof...(Ops)> _data;

    // ...
};
```

Building the EDSL: shaper operations

Shaper class template

```cpp
// The type of a shape remembering the sequence of operations it was made from
template <class... Ops> class shaper {
    // ...
    // Operators
    public:
    constexpr auto operator()() const noexcept {
        return shaper<Ops..., function_call_operator<>>(*this, -1);
    }
    constexpr auto operator()(std::size_t i) const noexcept {
        return shaper<Ops..., function_call_operator<std::size_t>>(*this, i);
    }
    constexpr auto operator[](std::size_t i) const noexcept {
        return shaper<Ops..., subscripting_operator<std::size_t>>(*this, i);
    }
    // Management
    public:
    static constexpr std::size_t rank() noexcept {
        return sizeof...(Ops);
    }
    constexpr std::size_t at(std::size_t i) const noexcept {
        return _data[i];
    }
    template <class... X> constexpr auto to_array(X... x) const noexcept {
        return to_array(index_sequence(), x...);
    }
    template <std::size_t... I, class... X> constexpr auto to_array(std::index_sequence<I...>, X... x) const noexcept {
        return std::array<std::size_t, sizeof...(Ops) + sizeof...(X)>{{std::get<I>(_data)..., x...}};
    }
};

// Default shaper instance
inline constexpr shaper shape;
```

Building the EDSL: indexed dynamic extent

Indexing helper

```
// Index constant alias template
template <std::size_t I> using index_constant = std::integral_constant<std::size_t, I>;

// Index variable template
template <std::size_t I> inline constexpr index_constant<I> index;
```

A dynamic extent

```
// A dynamic extent
template <auto...> struct indexed_dynamic_extent;

// A dynamic extent with no default value
template <std::size_t Index> struct indexed_dynamic_extent<Index> {
    using is_constant = std::false_type;
    constexpr auto operator[](index_constant<Index>) const noexcept {
        return *this;
    }
};

// A dynamic extent with a default value for initialization
template <std::size_t Index, std::size_t Value> struct indexed_dynamic_extent<Index, Value>
: std::integral_constant<std::size_t, Value> {
    using is_constant = std::false_type;
    constexpr auto operator[](index_constant<Index>) const noexcept {
        return *this;
    }
};
```

Building the EDSL: indexed static extent

A static extent

```
// A static extent with its value
template <std::size_t Index, std::size_t Value>
struct indexed_static_extent
: std::integral_constant<std::size_t, Value> {
    using is_constant = std::true_type;
    constexpr auto operator[](index_constant<Index>) const noexcept {
        return *this;
    }
};
```

Building the EDSL: indexed extent maker

Indexed extent maker

```
// A helper to build an indexed extent from a tagged operator
template <class, std::size_t, auto...>
struct indexed_extent_maker;

// Converts the function call operator into a dynamic extent
template <std::size_t I, auto... Args>
struct indexed_extent_maker<function_call_operator<>, I, Args...> {
    using type = indexed_dynamic_extent<I>;
};

// Converts the function call operator into a dynamic extent with default value
template <std::size_t I, std::size_t V, auto... Args>
struct indexed_extent_maker<function_call_operator<std::size_t>, I, V, Args...> {
    using type = indexed_dynamic_extent<I, V>;
};

// Converts the subscripting operator into a static extent
template <std::size_t I, std::size_t V, auto... Args>
struct indexed_extent_maker<subscripting_operator<std::size_t>, I, V, Args...> {
    using type = indexed_static_extent<I, V>;
};

// Alias template to build an indexed extent from a tagged operator
template <class Op, std::size_t I, auto... Args>
using make_indexed_extent = typename indexed_extent_maker<Op, I, Args...>::type;
```

Building the EDSL: extent policy

Extent policy class template

```cpp
// The extents, static and dynamic, of a N-dimensional array
template <class... Extents>
struct extent_policy
: Extents... {
    using index_sequence = std::index_sequence_for<Extents...>;
    using Extents::operator[]...;
    static constexpr std::size_t rank() noexcept {
        return sizeof...(Extents);
    }
};

// Extent policy maker declaration
template <auto, class, class>
struct extent_policy_maker;

// Specialization of extent policy maker for a shape, a shaper, and an index sequence
template <auto S, class... Ops, std::size_t... I>
struct extent_policy_maker<S, shaper<Ops...>, std::index_sequence<I...>> {
    using type = extent_policy<make_indexed_extent<Ops, I, S.at(I)>...>;
};

// Make an extent policy type from a shape variable
template <auto S, class T = decltype(S), class I = typename T::index_sequence>
using make_extent_policy = typename extent_policy_maker<S, typename T::type, I>::type;
```

Building the EDSL: extend

Extend class template

```cpp
// A class to specify the dynamic size along an axis
template <std::size_t Index>
class indexed_extender
{
    // Constants
    public:
    using is_extender = std::true_type;

    // Lifecycle
    public:
    constexpr indexed_extender() noexcept = default;
    explicit constexpr indexed_extender(std::size_t size) noexcept
    : _value(size) {
    }

    // Assignment and access
    public:
    constexpr indexed_extender operator=(std::size_t size) const noexcept {
        return indexed_extender(size);
    }
    constexpr operator std::size_t() const {
        return _value;
    }
    constexpr auto operator[](index_constant<Index>) const noexcept {
        return *this;
    }

    // Implementation details
    private:
    std::size_t _value;
};
```

Building the EDSL: extenders

Extenders class template

```
// A class grouping extenders
template <class...>
struct extenders;

// A class grouping extenders specialized for extenders
template <std::size_t... I>
struct extenders<indexed_extender<I>...>
: indexed_extender<I>... {
    using indexed_extender<I>::operator[]...;
    explicit constexpr extenders(indexed_extender<I>... i) noexcept
    : indexed_extender<I>(i)... {
    }
};

// Extenders deduction guide
template <std::size_t... I>
extenders(indexed_extender<I>... i) -> extenders<indexed_extender<I>...>;

// Extend variable template
template <std::size_t I>
inline constexpr indexed_extender<I> extend(-1);
```

Building the EDSL: ndarray

Ndarray class template

```
// Generic ndarray declaration
template <class...>
class basic_ndarray;

// Simplified ndarray definition
template <class Type, class ExtentPolicy>
class basic_ndarray<Type, ExtentPolicy>
{
    // Types and constants
    public:
    using value_type = Type;
    using extent_policy = ExtentPolicy;

    // Lifecycle
    public:
    template <class... Extenders>
    explicit constexpr basic_ndarray(Extenders... e) noexcept;

    // Access
    public:
    constexpr extents<extent_policy::rank()> shape();

    // ...
};

// Ndarray alias template
template <class Type, auto Shape, class... Args>
using ndarray = basic_ndarray<Type, make_extent_policy<Shape>, Args...>;
```


Illustration

Usage in practice

```
int main(int argc, char* argv[])
{
    // A ndarray of:
    // Dimension-0: a static size of 3
    // Dimension-1: a dynamic size whose initial value is given in constructor
    // Dimension-2: a static size of 4
    // Dimension-3: a dynamic size with a default initial value of 2
    // Dimension-4: a static size of 3
    // And then the constructor initializes the dimension 1 to size 5
    ndarray<int, shape[3]()[4](2)[3]> myarray(extend<1> = 5);
    return 0;
}
```

Extents

The problem with extents

Static only

When all dimensions are static, extents can just be an integer sequence ⇒ `sizeof(extents)` should be the size of an empty struct.

Dynamic only

When all dimensions are dynamic, extents can just be a runtime array of size `rank` ⇒ `sizeof(extents)` = `sizeof(std::size_t)` × `rank`.

Mixed static/dynamic

Well, it's a little bit more complicated...

Optimization goal

- Memory footprint of only dynamic dimensions
- Optimal runtime access of a particular dimension
- Bonus: optimized compilation time

Challenges

Access

- `extents[3]`: dynamic access to the extent along axis 3
- `extents[n]`: dynamic access to the extent along axis n (runtime variable)
- `extents[index<3>]`: static access to the extent along axis 3
- `extents[index<N>]`: static access to the extent along axis N (constant expression)

Challenges

- `extents[n]`: avoiding $\mathcal{O}(n)$ scaling for static dimensions
- `extents[index<N>]`: avoiding $\mathcal{O}(N)$ scaling in compilation-time

Log-tuple trick: introduction

Straightforward approach

`get<N>(tuple)` has to iterate over the first `N` types.

Advanced approach

There is a way to exploit overload resolution to have $\mathcal{O}(\log(N))$ compile-time access.

Indexing

```
// Index constant type
template <std::size_t I>
struct index_constant: std::integral_constant<std::size_t, I> {};

// Index constant variable template
template <std::size_t I>
inline constexpr index_constant<I> index = {};
```

Log-tuple trick: elements

Element wrappers

```
// A basic element wrapper
template <class T>
struct tuple_element_wrapper {
    constexpr tuple_element_wrapper(const T& x): value(x) {}
    // Other constructors to be defined
    T value;
};

// An indexed tuple element
template <std::size_t I, class T>
struct tuple_element: tuple_element_wrapper<T> {
    constexpr tuple_element(const T& x): tuple_element_wrapper<T>(x) {}
    constexpr T& operator[](index_constant<I>) {
        return static_cast<wrapper<T>&>(*this).value;
    }
    constexpr const T& operator[](index_constant<I>) const {
        return static_cast<const wrapper<T>&>(*this).value;
    }
};
```

Log-tuple trick: tuple

Tuple

```cpp
// Base class declaration
template <class Sequence, class... T>
struct tuple_base;

// Base class specialization for index sequence
template <std::size_t... I, class... T>
struct tuple_base<std::index_sequence<I...>, T...>
: tuple_element<I, T>... {
    using index_sequence = std::index_sequence<I...>;
    using tuple_element<I, T>::operator[]...;
    constexpr tuple_base(const T&... x): tuple_element<I, T>(x)... {}
    // Other constructors to be defined
};

// Actual tuple implementation
template <class... T>
struct tuple: tuple_base<std::index_sequence_for<T...>, T...> {
    using base = tuple_base<std::index_sequence_for<T...>, T...>;
    using base::base;
    using base::operator[];
};
template <class... T>
tuple(const T&...) -> tuple<T...>;
```

Result

`mytuple[index<3>]` leverages overload resolution to access the element at compile-time.

Runtime element access of a static sequence: introduction

The problem

Considers a tuple $\mathcal{T}$ of types `Types...` (where, for instance each type can be either a static or dynamic extent). How to design a runtime subscript operator for this tuple, so that when provided with an index i and a generic lambda Λ, it would return the result of the lambda applied to the i-th element of the tuple: $\Lambda(\mathcal{T}_i)$?

The trivial approach

```
i == 0 ? lambda(std::get<0>(tuple)) :
  i == 1 ? lambda(std::get<1>(tuple)) :
    i == 2 ? lambda(std::get<2>(tuple)) :
      i == 3 ? lambda(std::get<3>(tuple)) :
        // ...
          lambda(std::get<N - 1>(tuple));
```

The trivial non-branching approach

```
(i == 0) * lambda(std::get<0>(tuple)) +
  (i == 1) * lambda(std::get<1>(tuple)) +
    (i == 2) * lambda(std::get<2>(tuple)) +
      (i == 3) * lambda(std::get<3>(tuple)) +
        // ...
          (i == N - 1) * lambda(std::get<N - 1>(tuple));
```

Runtime element access of a static sequence: complexity

Access complexity

The preceding strategies are $\mathcal{O}(N)$-access strategies. Can we do better?

Dichotomic approach (for $\mathrm{N} = 6$)

```
i < 3
  ? i < 2
    ? i == 0
      ? lambda(std::get<0>(tuple))
      : lambda(std::get<1>(tuple))
    : lambda(std::get<2>(tuple))
  : i < 5
    ? i == 3
      ? lambda(std::get<3>(tuple))
      : lambda(std::get<4>(tuple))
    : lambda(std::get<5>(tuple));
```

Access complexity

The preceding strategy scales as $\mathcal{O}(\log(N))$.

Runtime element access of a static sequence: variadic implementation

Expressivity goal

`overload_sequence(F...)[i](Args...)` should return the result of the i-th function `F` on the arguments `Args...`.

Starting with the log-tuple trick

```
// Index constant helper
template <std::size_t I>
struct index_constant: std::integral_constant<std::size_t, I> {};

// Index constant variable template
template <std::size_t I>
inline constexpr index_constant<I> index = {};

// Element wrapper
template <std::size_t I, class F>
struct overload_sequence_element: F {
    static constexpr index_type<I> index = {};
    constexpr overload_sequence_element(const F& f): F(f) {}
    constexpr F& operator[](index_constant<I>) {return static_cast<F&>(*this);}
    constexpr const F& operator[](index_constant<I>) const {return static_cast<const F&>(*this);}
};
```

Runtime element access of a static sequence: overload sequence

Overload sequence implementation

```cpp
// Overload sequence base declaration
template <class...>
struct overload_sequence_base;

// Overload sequence base definition
template <std::size_t... I, class... F>
struct overload_sequence_base<std::index_sequence<I...>, F...>
: overload_sequence_element<I, F>... {
    using index_sequence = std::index_sequence<I...>;
    using overload_sequence_element<I, F>::operator[]...;
    constexpr overload_sequence_base(const F&... args)
    : overload_sequence_element<I, F>(args)... {}
    static constexpr std::size_t size() {return sizeof...(F);}
};

// Overload sequence
template <class... F>
struct overload_sequence:
overload_sequence_base<std::index_sequence_for<F...>, F...> {
    using base = overload_sequence_base<std::index_sequence_for<F...>, F...>;
    using base::base;
    using base::operator[];
    constexpr overload_sequence_selector<overload_sequence&> operator[](std::size_t i) {
        return overload_sequence_selector<overload_sequence&>(*this, i);
    }
    constexpr overload_sequence_selector<overload_sequence&> operator[](std::size_t i) const {
        return overload_sequence_selector<const overload_sequence&>(*this, i);
    }
};
template <class... F>
overload_sequence(const F&...) -> overload_sequence<F...>;
```

Runtime element access of a static sequence: dichotomic access

Overload sequence selector implementation

```cpp
// Dichotomic access implementation
template <class Sequence>
struct overload_sequence_selector {
    constexpr overload_sequence_selector(Sequence seq, std::size_t i): sequence(seq), index(i) {}
    template <class... Args>
    constexpr void operator()(Args&&... args) {
        this->operator()<0, std::decay_t<Sequence>::size() - 1>(std::forward<Args>(args)...);
    }
    template <int Min, int Max, int Mid = (Min + Max) / 2, class... Args>
    constexpr void operator()(Args&&... args) {
        if constexpr (Min < Max) {
            if (index < Mid) {this->operator()<Min, Mid - 1>(std::forward<Args>(args)...);}
            else if (index > Mid) {this->operator()<Mid + 1, Max>(std::forward<Args>(args)...);}
            else {this->operator()<Mid, Mid>(std::forward<Args>(args)...);}
        } else {
            sequence[index_type<Mid>{}](std::forward<Args>(args)...);
        }
    }
    Sequence sequence;
    std::size_t index;
};
```

Runtime element access of a static sequence: result

Result

For a family of lambdas indexed by i and each associated with an element $\mathcal{T}_i$ of a tuple $\mathcal{T}$, `overload_sequence` provides a runtime generic access in $\mathcal{O}(\log(N))$.

Application

For an heterogeneous sequence of `dynamic_extent` and `static_extent` it provides a $\mathcal{O}(\log(N))$ runtime access with branching.

Beyond logarithmic complexity

Overload sequence selector implementation

```
// Dichotomic access implementation
template <class... Extents>
struct extents {
    static constexpr std::array<bool, sizeof...(Extents)> is_static = {Extents::is_static...};
    static constexpr std::array<std::size_t, sizeof...(Extents)> table = // see after;
    std::array<std::size_t, dynamic_rank> storage = // see after;
    constexpr std::size_t operator[](std::size_t i) {
        return is_static ? table[i] : storage[table[i]];
    }
};
```

Where:

- `storage` contains the dynamic extents known at runtime
- `table` contains:
  - `static_extent::value` if the i-th extent is static
  - the index `j` where the element is stored in `storage` if the i-th extent is dynamic

Access complexity

For an heterogeneous sequence of `dynamic_extent` and `static_extent` it provides a $\mathcal{O}(1)$ runtime access with a branching and a level of indirection.

Optimizing extents

Summary

- Optimizing dynamic-only extents is easy
- Optimizing static-only extents is easy
- Optimizing hybrid extents is tricky

Axes of optimization

- In space: storing only dynamic extents at runtime
- In time: using `overload_sequence` or indirection to optimize runtime access
- In compilation time: using and reusing the log-tuple technique

`mdspan`

The current implementation of `extents.extent(i)` seems to scale as $\mathcal{O}(N)$ (to be confirmed).

Going beyond

Enriching the EDSL: sky is the limit

Layout information

```
std::ndarray<double, +shape[4]()[3][5]>
std::ndarray<double, -shape[4]()[3][5]>
std::ndarray<double, !shape[4]()[3][5]>
```

Symmetries

```
std::ndarray<double, shape[diagonal(4)]()[3][5]>
std::ndarray<double, shape[4]()[diagonal(3)][5]>
std::ndarray<double, shape<triangular>[4]()[3][5]>
```

Axis-based parallelization information

```
std::ndarray<double, shape[distributed(4)]()[3][5]>
std::ndarray<double, shape[distributed(4)]()[3][vectorized(5)]>
std::ndarray<double, shape[4]()[3][vectorized<gpu>(5)]>
```

Operations on shapes

```
std::ndarray<double, shape[4] * shape() * shape[5]>
```

With multidimensional subscript operator

P2128

- P2128: Multidimensional subscript operator
- `operator[](Args&&...)`

Evolution

- `shape[4]()[3][5]` ⇒ `shape[4][][3][5]` (requires zero-parameter operator)
- Other opportunities to enrich the mini-language by combining `operator()` and `operator[]` for different meanings

Towards a generic extent concept

Possible components

- `min`: the minimum allowed size under which it cannot shrink
- `initial`: the initial size at construction
- `threshold`: the maximum number of elements stored without external allocation
- `max`: the maximum allowed size over which is cannot be extended (can be infinite)

Concrete examples

- `static_extent` ⇒ `min == initial == threshold == max`
- `dynamic_extent` ⇒ `min == 0, max == infinity, initial == 0, threshold == 0`
- `fixed_extent` ⇒ `min == 0, max == n, initial == 0, threshold == n`

Generic NTTPs opened a Pandora's box

Metalanguages

- Being able to inject objects as template parameters in C++20 is a game changer for Embedded Domain Specific Languages
- Class templates can now be seen as true mini-compilers: `template <auto> class edsl_compiler ;`

Kind genericity

- The next revolution to come is probably the genericity regarding kinds
- Currently no way to specify universal template parameters (values, types, templates. . .)
- P1985: Universal template parameters
- `template <template auto X>` where `X` can be a value, a type, a template. . .

Conclusion

Summary

Summary

- Linear algebra is everywhere
- Very complex when trying to balance between Genericity, Performance, and Expressivity
- Existing effort with `mdspan`
- C++20 with arbitrary NTTPs allows expressive ways to specify multidimensional shapes
- Expression template template to create mini-languages
- Far more powerful than relaxed incomplete multidimensional array type declaration
- Hybrid dynamic and static extent particularly tricky to implement
- Log-tuple trick useful in many situations
- Extending the EDSL: layout information, symmetries, axis-based parallelization, shape operations
- Conceptifying a generalized extent
- Class NTTPs opened a Pandora's box for EDSL
- Kind genericity will take EDSLs even further

Thank you for your attention

Any question?

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