UNIT-1

Set Theory Questions and Solutions

Question 1: In a class of 40 students, 22 play hockey, 26 play basketball, and 14 play both. How many students do not play either of the games?

Solution:
Let H be the set of students playing hockey, and B be the set playing basketball.
$n(H) = 22$, $n(B) = 26$, $n(H \cap B) = 14$.
$n(H \cup B) = n(H) + n(B) - n(H \cap B) = 22 + 26 - 14 = 34$.
Students not playing either = Total students - $n(H \cup B) = 40 - 34 = 6$.

Question 2: If set A = {1, 3, 5, 7, 9} and set B = {1, 2, 3, 4, 5}, find A ∪ B and A ∩ B.
Solution:
$A \cup B$ (union) is the set of elements that are in A, or B, or both.
$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$.
$A \cap B$ (intersection) is the set of elements that are in both A and B.
$A \cap B = \{1, 3, 5\}$.

Question 3: In a survey of 60 people, 25 liked tea, 30 liked coffee, and 10 liked both. How many people liked only tea?
Solution:
Number of people who liked only tea = Number who liked tea - Number who liked both.
= 25 - 10 = 15 people liked only tea.

Question 4: For sets A = {x | x is an integer, 1 ≤ x ≤ 6} and B = {x | x is an even integer, 2 ≤ x ≤ 8}, find the set A - B.
Solution:
$A = \{1, 2, 3, 4, 5, 6\}$, $B = \{2, 4, 6, 8\}$.
A - B (difference) is the set of elements in A that are not in B.
$A - B = \{1, 3, 5\}$.

Question 5: Given sets X = {a, b, c, d} and Y = {b, d, e, f}, find the symmetric difference of X and Y (denoted as X Δ Y).
Solution:
$X \Delta Y$ is the set of elements in either X or Y, but not in their intersection.
Intersection $X \cap Y = \{b, d\}$.
$X \Delta Y = (X \cup Y) - (X \cap Y) = \{a, b, c, d, e, f\} - \{b, d\} = \{a, c, e, f\}$.

Question 6: If set C = {2, 4, 6, 8} and set D = {6, 8, 10, 12}, what are the sets C ∩ D and C ∪ D?
Solution:
C ∩ D (intersection) is the set of elements common to both C and D.
C ∩ D = {6, 8}.
C ∪ D (union) is the set of all elements in C, or D, or both.
C ∪ D = {2, 4, 6, 8, 10, 12}.

Question 7: A survey of 100 students found that 70 students like pizza, 75 like burgers, and 60 like both. How many students like neither pizza nor burgers?
Solution:
Let P be the set of students who like pizza, and B be the set who like burgers.
n(P ∪ B) = n(P) + n(B) - n(P ∩ B) = 70 + 75 - 60 = 85.
Students who like neither = Total students - n(P ∪ B) = 100 - 85 = 15.

Question 8: For sets E = {1, 3, 5, 7, 9} and F = {0, 1, 2, 3, 4}, find the set E ∪ F and the set E - F.
Solution:
E ∪ F (union) is the set of elements in E, or F, or both.
E ∪ F = {0, 1, 2, 3, 4, 5, 7, 9}.
E - F (difference) is the set of elements in E that are not in F.
E - F = {5, 7, 9}.

Question 9: Given set G = {a, e, i, o, u} and set H = {a, e, y}, find the symmetric difference G Δ H.
Solution:
G Δ H is the set of elements in either G or H, but not in their intersection.
Intersection G ∩ H = {a, e}.
G Δ H = (G ∪ H) - (G ∩ H) = {a, e, i, o, u, y} - {a, e} = {i, o, u, y}.

Question 10: In a group of 50 people, 28 like tea, 26 like coffee, and 12 like both. Find the number of people who like only coffee.
Solution:
Number of people who like only coffee = Number who like coffee - Number who like both.
= 26 - 12 = 14 people like only coffee.

Problems on Tautology

Find if the given propositional logic is a tautology or not.

1) P

Truth table:

P
T
F

The truth table of P contains a False value. *Thus, it can not be a tautology.*

2) P⇒P

We shall draw the truth table for this proposition.

Implication:

$P \Rightarrow Q = \neg P \vee Q$

The simplified expression of the given proposition is: $\neg P \vee P$

Truth Table:

P	¬P	¬P∨P
T	F	T
F	T	T

The truth table of $\neg P \vee P$ consists only of True values. *Therefore, P⇒P is a tautology.*

3) (P ⇒ P) ⇒ P

We shall draw the truth table for this proposition.

Implication:

$P \Rightarrow Q = \neg P \vee Q$

The simplified expression of the given proposition is:

$(\neg P \vee P) \Rightarrow P$

$\neg(\neg P \vee P) \vee P$

$(\neg(\neg P) \wedge \neg P) \vee P$ {By Demorgan's Law}

$(P \wedge \neg P) \vee P$

$(P \wedge \neg P)$= False {Complement laws: – $P \wedge \neg P = F$ }

False $\vee P = P$ {Absorption law}

Thus, $(P \Rightarrow P) \Rightarrow P$ is equivalent to P. We have already solved this in problem-1.
*Therefore, **this is not a Tautology**.*

4) $(p \to q) \to [(p \to q) \to q]$

Solving: $(p \to q) = \neg p \vee q$ {Implication}
Solving: $[(p \to q) \to q]$
$= [(\neg p \vee q) \to q]$
$= [\neg(\neg p \vee q) \vee q]$
$= [(\neg(\neg p) \wedge \neg q) \vee q]$ *{Demorgan's Law}*
$= [(p \wedge \neg q) \vee q]$ *{Involution law}*
$= [(p \vee q) \wedge (\neg q \vee q)]$ *{Distributive law}*
$= [(p \vee q) \wedge T]$ *{Complement law}*
$= (p \vee q)$ *{Absorption law}*
Solving $(p \to q) \to [(p \to q) \to q]$
$\neg(\neg p \vee q) \vee (p \vee q)$
$[\neg(\neg p) \wedge (\neg q)] \vee (p \vee q)$*{Demorgan's Law}*
$(p \wedge \neg q) \vee (p \vee q)$ *{Involution law}*
Thus, final expression is: $(p \wedge \neg q) \vee (p \vee q)$

Truth Table:

p	**q**	**¬q**	**(p∧¬q)**	**(p∨q)**	**(p∧¬q)∨(p∨q)**
T	T	F	F	T	T
T	F	T	T	T	T
F	T	F	F	T	T
F	F	T	F	F	**F**

Since there is a False entry in the truth table, **it implies it is not a Tautology**.

5) $((P \Rightarrow Q) \wedge P) \Rightarrow Q$

Solving $(P \Rightarrow Q)$: $\neg P \vee Q$
Solving $((P \Rightarrow Q) \wedge P)$: $((\neg P \vee Q) \wedge P)$
$= (\neg P \wedge P) \vee (Q \wedge P)$ *{Distributive Law}*
$= (F) \vee (Q \wedge P)$ *{Complement Law}*
$= (Q \wedge P)$ *{Absorption Law}*

Solving $((P \Rightarrow Q) \wedge P) \Rightarrow Q$: $(Q \wedge P) \Rightarrow Q$
$= \neg(Q \wedge P) \vee Q$
$= (\neg Q \vee \neg P) \vee Q$ *{Demorgan Law}*
$= (\neg Q \vee Q) \vee \neg P)$ *{Associative Law}*
$= T \vee (\neg P)$ *{Complement Law}*
$= T$ *{Absorption Law}*

Final CNF is: ***True***
Here, no need of finding the Truth Table. Given logic is a tautology.

UNIT-2

Problem 2: Given the set A={1,2,3,4,5} and the relation R={(1,2),(2,3),(3,4),(4,5)}, represent this relation using a graph and an adjacency matrix. Check if the relation is reflexive, symmetric, or transitive.

Problem 5: For the set C={a,b,c}, a relation T is given as T={(a,b),(b,a),(b,c)}. Create the corresponding graph and adjacency matrix. Verify if the relation is symmetric and transitive.

Problem 9: For the set H={a,b,c,d}, a relation Y is defined as Y={(a,a),(b,b),(c,c),(d,d)}. Create the corresponding graph and matrix. Determine the properties of this relation.

Solved Examples on Composition of Function

Example 1: For the given functions $f(x) = e^x$ and $g(x) = x^2 + 1$. Find out the values of f(g(x)) and g(f(x)).
Solution:
The domain of both the functions are real numbers, so there is no need to modify the domain for the first function in any case.
For f∘g(x),
$f \circ g(x) = f(g(x))$
$\Rightarrow f \circ g(x) = f(x^2 + 1)$
$\Rightarrow f \circ g(x) =$ *ex2+1* e^{x^2+1}
For g∘ f(x)
$g \circ f(x) = g(f(x))$
$\Rightarrow g \circ f(x) = g(e^x)$
$\Rightarrow g \circ f(x) = (e^x)^2 + 1$
$\Rightarrow g \circ f(x) = e^{2x} + 1$

Example 2: For the given functions $f(x) = 2x$ and $g(x) = x^2 + 1$. Find out the values of f(g(x)) and g(f(x)) at x = 2.
Solution:

The domain of both the functions are real numbers, so there is no need to modify the domain for the first function in any case.
f∘g(x) = f(g(x))
⇒ f∘g(x) = f(x^2 + 1)
⇒ f∘g(x) = 2(x^2 + 1)
At x = 2,
⇒ f(g(2)) = 2(4 + 1)
⇒ f(g(2)) = 10
***g∘ f(x)** = g(f(x))*
⇒ g∘f(x) = g(2x)
⇒ g∘f(x) = $(2x)^2$ + 1
⇒ g∘f(x) = $4x^4$ + 1
At x = 2
g(f(2)) = 4(2^4) + 1
⇒ g(f(2)) = 4(16) + 1
⇒ g(f(2)) = 65

Example 3: For the given functions f(x) = sin(x) and g(x) = x^2. Find out the domain and range for f∘g(x) and g∘f(x).
Solution:
f(x) = sin x has domain as all real numbers and range [-1,1].
While g(x) = x^2 has domain all the real numbers and range R^+.
f∘ g(x) = f(g(x)) = f(x^2) = sin(x^2)
Domain is all real numbers, the range is [-1,1]
g∘ f(x) = g(f(x) = g(sinx) = $\sin^2 x$
Domain is all real numbers, the range is between 0 and 1.

Example 4: For the given functions f(x) = log(x) and g(x) = x + 1. Find out the values of f(g(x)).
Solution:
Domain of f(x) is all positive numbers i.e R^+ and range is all real numbers.
Domain and range for g(x) is all real numbers.
f(g(x)) = f(x + 1) = log(x + 1)
While doing this, the domain of f(x) must be kept in mind as it is logrethmic function it only takes positive values.
x + 1 > 0
x > -1
So,
Domain is (-1, ∞)
Range is all real numbers (R)

UNIT-3

Solved Examples

Problem 1: Prove that in a group G, if $(ab)^2 = a^2b^2$ for all $a, b \in G$, then G is abelian.

Solution:

Given: $(ab)^2 = a^2b^2$ for all $a, b \in G$
Expand $(ab)^2$: $abab = a^2b^2$
Multiply both sides by a^{-1} on the left: $a^{-1}(abab) = a^{-1}(a^2b^2)$
Simplify: $bab = ab^2$
Multiply both sides by b^{-1} on the right: $(bab)b^{-1} = (ab^2)b^{-1}$
Simplify: $ba = ab$
Therefore, G is abelian.

Problem 2: Let G be a group of order 15. Prove that G is cyclic.

Solution:

By Lagrange's theorem, the possible orders of elements in G are 1, 3, 5, and 15.

If there exists an element of order 15, then G is cyclic.

If not, then G has elements of order 3 and 5 (since it can't be all identity elements).

Let a be an element of order 3 and b be an element of order 5.

Consider the subgroup $H = \langle a, b \rangle$. Its order must divide 15.

$|H| \neq 3$ or 5 because it contains elements of both orders.

$|H| \neq 1$ because it's not just the identity.

Therefore, $|H| = 15$, which means $H = G$.

By the fundamental theorem of finite abelian groups, $G \cong Z_3 \oplus Z_5 \cong Z_{15}$. Thus, G is cyclic.

Problem 3: Let G be a group and H be a subgroup of G. Prove that if $[G:H] = 2$, then H is normal in G.

Solution:

$[G:H] = 2$ means there are only two cosets of H in G.

These cosets are H itself and some aH where $a \notin H$.
For any $g \in G$, gH must equal either H or aH.
If $gH = H$, then $g \in H$, so $gHg^{-1} = H$.
If $gH = aH$, then $g = ah$ for some $h \in H$.

In this case, $gHg^{-1} = ahH(ah)^{-1} = aHa^{-1}$

But aHa^{-1} must be either H or aH.

If $aHa^{-1} = aH$, then $Ha^{-1} = H$, which means $a \in H$, contradicting our choice of a.
Therefore, $aHa^{-1} = H$.
Thus, for all $g \in G$, $gHg^{-1} = H$, so H is normal in G.

Problem 4: Let G be a group and let $a, b \in G$. Prove that if $ab = ba$, then $(ab)^n = a^n b^n$ for all $n \in Z$.

Solution:
We'll use induction on n.
Base case: For $n = 0$, $(ab)^0 = e = a^0 b^0$
For $n = 1$, $(ab)^1 = ab = a^1 b^1$

Inductive hypothesis: Assume $(ab)^k = a^k b^k$ for some $k \geq 1$

Inductive step: Consider $(ab)^{k+1}$
$(ab)^{k+1} = (ab)^k(ab) = (a^k b^k)(ab) = a^k(b^k a)b = a^k(ab^k)b = a^{k+1}b^{k+1}$
By induction, $(ab)^n = a^n b^n$ for all $n \geq 0$
For negative integers, note that $(ab)^{-n} = ((ab)^n)^{-1} = (a^n b^n)^{-1} = b^{-n}a^{-n} = a^{-n}b^{-n}$
Therefore, $(ab)^n = a^n b^n$ for all $n \in Z$.

Problem 5: Let G be a group of order pq, where p and q are distinct primes. Prove that G is cyclic if and only if gcd(p-1, q-1) = 1.

Solution:
First, prove that if G is cyclic, then $\gcd(p-1, q-1) = 1$:

If G is cyclic, it has an element of order pq.

By the structure theorem of cyclic groups, $G \cong Z_{pq}$

The number of elements of order pq in Z_{pq} is $\varphi(pq) = (p-1)(q-1)$

This number must equal the number of generators of G, which is $\varphi(pq)$

For this to be true, we must have gcd(p-1, q-1) = 1

Now, prove that if gcd(p-1, q-1) = 1, then G is cyclic:
By Sylow's theorems, G has a unique subgroup P of order p and a unique subgroup Q of order q

Let a generate P and b generate Q

Consider the order of ab:
$(ab)^pq = a^pqb^pq = (a^p)q(bq)^p = e$
So the order of ab divides pq
If ord(ab) = p or q, then ab $\in$ P or ab $\in$ Q, which is impossible
Therefore, ord(ab) = pq, and G is cyclic
Thus, G is cyclic if and only if gcd(p-1, q-1) = 1.