EE5609: Matrix Theory
Assignment 12

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Download codes from

https://github.com/Atul 191/EE5609/Assignment12

1 Question

Let A,B be $n \times n$ matrices such that $BA + B^2 = I - BA^2$ where I is the $n \times n$ identity matrix. Which of the following is always correct

1) A is non singular
2) B is non singular
3) A+B is non singular
4) AB is non singular

2 Solution

Statement	**Solution**
Given Condition	$$BA + B^2 = I - BA^2 \tag{2.0.1}$$
Solution by Theory	We will first provide theoretical proof

Theory	As per definition of invertible matrix,A matrix 'B' in our case is defined as invertible if there exists left and right inverse of B such that BC=CB=I In that case C is called the two sided inverse of B and B is said to be invertible. Now refer(2.0.1) we get $$BA + B^2 = I - BA^2 \quad (2.0.2)$$ $$\implies BA + B^2 + BA^2 = I \quad (2.0.3)$$ $$\implies I = B\left(A + B + A^2\right) \quad (2.0.4)$$ $$(2.0.5)$$ Let C= $\left(A + B + A^2\right)$ rewrite (2.0.4) as $$I = BC \quad (2.0.6)$$ Also $$I = \left(A + B + A^2\right)B \quad (2.0.7)$$ Let D= $\left(A + B + A^2\right)$ rewrite (2.0.7) as $$I = DB \quad (2.0.8)$$ Now we can write $$D = DI \quad (2.0.9)$$ Ref (2.0.6) $$= D(BC) \quad (2.0.10)$$ $$= (DB)C \quad (2.0.11)$$ $$(2.0.12)$$ Ref (2.0.8) $$= IC \quad (2.0.13)$$ $$= C \quad (2.0.14)$$ $$\implies D = C \quad (2.0.15)$$ Hence by definition stated above we imply that Left inverse=Right inverse. So by looking at (2.0.4),we imply that B has a left and right inverse $$\implies I = BB^{-1} \quad (2.0.16)$$ $$\implies \text{B is invertible} \quad (2.0.17)$$ $\therefore$ B is non singular.Hence Option 2 is correct
Solution by examples	We will check each respective options through examples

Option 3	Let us take $$A = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.0.18}$$ $$B = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} \tag{2.0.19}$$ Take L.H.S of (2.0.1) $$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} \tag{2.0.20}$$ $$= \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.0.21}$$ Take R.H.S of (2.0.1) $$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.0.22}$$ $$= \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.0.23}$$ Our assumption satisfies (2.0.1). Now $$A + B = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} + \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} \tag{2.0.24}$$ $$= \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.0.25}$$ $\therefore \left\lvert A + B \right\rvert = 0$ the respective option is Singular. Hence Option 3 is incorrect
Option 1	Now let us take $$A = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} B = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} \tag{2.0.26}$$ Substituting(2.0.26) in (2.0.1) Take L.H.S of (2.0.1) $$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} + \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix} \tag{2.0.27}$$ $$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.0.28}$$ Take R.H.S of (2.0.1) $$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} - \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \tag{2.0.29}$$ $$= \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \tag{2.0.30}$$ Our assumption satisfies (2.0.1) But $\left\lvert A \right\rvert = 0$ $\therefore$ the respective option is Singular. Hence Option 1 is incorrect

Option 4	Similarly $$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad (2.0.31)$$ $$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (2.0.32)$$ Here also $\left\|AB\right\| = 0$ $\therefore$ the AB option is also Singular. Hence Option 4 is incorrect also
Correct Answer	So we conclude that Option 2 is correct by eliminating other options

Table1:Solution