Assignment 5

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***Abstract*—This document is about tracing a Curve**

Download all Latex code from

https://github.com/Atul191/Assignment 5

1 Problem

Trace the following

$$x^2 - 3xy + y^2 + 10x - 10y + 21 = 0 \tag{1.0.1}$$

2 Solution

The given quadratic equation can be written in the matrix form as

$$\mathbf{x}^T \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix} \mathbf{x} + 2\begin{pmatrix} 5 & -5 \end{pmatrix} \mathbf{x} + 21 = 0 \tag{2.0.1}$$

Calculating the parameters,we get

$$\left|\mathbf{V}\right| = \begin{vmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{vmatrix} = -\frac{5}{4} \tag{2.0.2}$$

Since, $\left|\mathbf{V}\right| < 0$, therefore the given equation represents a hyperbola.

The characteristic equation of $\mathbf{V}$ will be

$$\left|\mathbf{V} - \lambda\mathbf{I}\right| = \begin{vmatrix} 1-\lambda & -\frac{3}{2} \\ -\frac{3}{2} & 1-\lambda \end{vmatrix} = 0 \tag{2.0.3}$$

$$\implies 4\lambda^2 - 8\lambda - 5 = 0 \tag{2.0.4}$$

$$\implies \lambda_1 = \frac{5}{2}, \lambda_2 = -\frac{1}{2} \tag{2.0.5}$$

The eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.6}$$

$$\implies \mathbf{V} - \lambda\mathbf{I}\mathbf{p} = 0 \tag{2.0.7}$$

For $\lambda_1 = \frac{5}{2}$

$$\mathbf{V} - \lambda\mathbf{I} = \begin{pmatrix} 1-\frac{5}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1-\frac{5}{2} \end{pmatrix} \tag{2.0.8}$$

$$= \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \tag{2.0.9}$$

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_2=R_2-R_1} \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.10}$$

$$\xleftrightarrow{R_1=R_1/-\frac{3}{2}} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.0.11}$$

Substituting (2.0.11) in (2.0.7) we get

$$\mathbf{p_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \tag{2.0.12}$$

Therefore the normalized eigen vector will be

$$\mathbf{p_1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.0.13}$$

For $\lambda_2 = -\frac{1}{2}$

$$\mathbf{V} - \lambda\mathbf{I} = \begin{pmatrix} 1+\frac{1}{2} & -\frac{3}{2} \\ -\frac{3}{2} & 1+\frac{1}{2} \end{pmatrix} \tag{2.0.14}$$

$$= \begin{pmatrix} \frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \tag{2.0.15}$$

$$\begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{pmatrix} \xleftrightarrow{R_2=R_2+R_1} \begin{pmatrix} -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow{R_1=R_1/\frac{3}{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \tag{2.0.17}$$

Substituting (2.0.17) in (2.0.7) we get

$$\mathbf{p_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{2.0.18}$$

Therefore the normalized eigen vector will be

$$\mathbf{p_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.0.19}$$

Eigen decomposition

Since $\mathbf{V} = \mathbf{V}^T$there exists an orthogonal matrix P such that

$$\mathbf{P}\mathbf{P}^T = \mathbf{I} \tag{2.0.20}$$

$$\mathbf{P}\mathbf{V}\mathbf{P}^T = \mathbf{D} = diag\left(\lambda_1\lambda_2\right) \tag{2.0.21}$$

or equivalently

$$\mathbf{V} = \mathbf{P}\mathbf{D}\mathbf{P}^T \tag{2.0.22}$$

As

$$\mathbf{P} = \begin{pmatrix} p_1 & p_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \tag{2.0.23}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.24}$$

$$\implies \mathbf{D} = \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \tag{2.0.25}$$

$$\mathbf{C} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.26}$$

$$\implies \mathbf{C} = \begin{pmatrix} -\frac{4}{5} & -\frac{6}{5} \\ -\frac{6}{5} & -\frac{4}{5} \end{pmatrix} \begin{pmatrix} -5 \\ 5 \end{pmatrix} \tag{2.0.27}$$

$$= \begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{2.0.28}$$

$\therefore$ Centre C is given by:

$$\begin{pmatrix} -2 \\ 2 \end{pmatrix} \tag{2.0.29}$$

Now Equation (1.0.1) can be written as

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - \mathbf{f} \tag{2.0.30}$$

$$\tag{2.0.31}$$

where y is given by:

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \tag{2.0.32}$$

So

$$\mathbf{y}^T \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{y} = -1 \tag{2.0.33}$$

$$\implies \mathbf{y}^T \begin{pmatrix} \frac{5}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \mathbf{y} + 1 = 0 \tag{2.0.34}$$

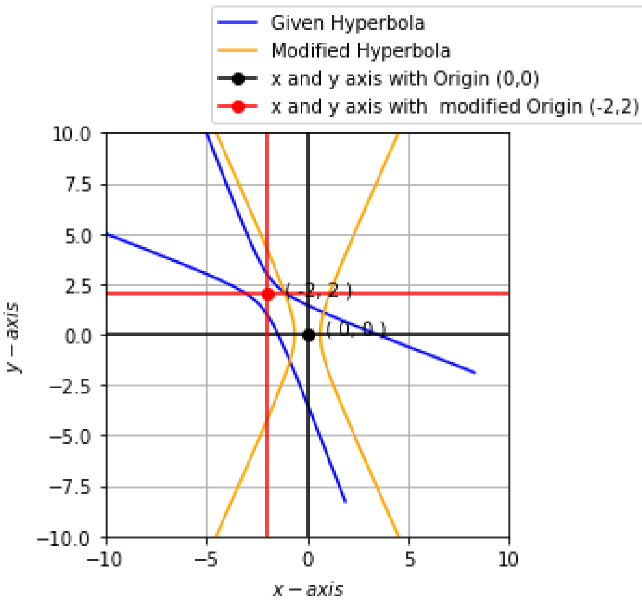

Fig. 1: Hyperbola plot when origin is shifted