Assignment 7

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***Abstract*—This a simple document finding foot of perpendicular on a plane through A.**

Download latex-tikz codes from

https://github.com/EE5609/Assignment−7

1 Problem

Write the equation of the line through $\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ and perpendicular to the plane $3x + 2y - 6z - 2 = 0$. Determine the coordinates of the point in which the plane is met by this line.

2 Solution

Given a point $\mathbf{A} = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}$ and a plane $\begin{pmatrix} 3 & 2 & -6 \end{pmatrix} \mathbf{x} = 2$. We know that the equation of a plane is given by

$$\mathbf{n}^{\mathbf{T}} \mathbf{x} = c \tag{2.0.1}$$

Hence, normal vector $\mathbf{n}$ is given by

$$\mathbf{n} = \begin{pmatrix} 3 \\ 2 \\ -6 \end{pmatrix} \tag{2.0.2}$$

Let $\mathbf{m_1}$ and $\mathbf{m_2}$ be two vectors that are normal to normal vector $\mathbf{n}$. Let $\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, then if

$$\mathbf{n}^{\mathbf{T}} \mathbf{m} = 0 \tag{2.0.3}$$

$$\begin{pmatrix} 3 & 2 & -6 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \tag{2.0.4}$$

Taking $a = 1$, $b = 0$, we get $c = \frac{1}{2}$, and hence

$$\mathbf{m_1} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{2.0.5}$$

Take $a = 0$ and $b = 1$, we get $c = \frac{1}{3}$, and hence

$$\mathbf{m_2} = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{3} \end{pmatrix} \tag{2.0.6}$$

Since foot of perpendicular is the point where the plane is met by a line perpendicular to the same plane. So, to get foot of perpendicular, we solve

$$\mathbf{M} \mathbf{x} = \mathbf{b} \tag{2.0.7}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix}, b = \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \tag{2.0.8}$$

To solve (2.0.7), we perform singular value decomposition on $\mathbf{M}$ given as

$$\mathbf{M} = \mathbf{U} \mathbf{S} \mathbf{V}^{\mathbf{T}} \tag{2.0.9}$$

Substituting the value of $\mathbf{M}$ from (2.0.9) in (2.0.7), we get

$$\mathbf{U} \mathbf{S} \mathbf{V}^{\mathbf{T}} \mathbf{x} = \mathbf{b} \tag{2.0.10}$$

$$\implies \mathbf{x} = \mathbf{V} \mathbf{S}_{+} \mathbf{U}^{\mathbf{T}} \mathbf{b} \tag{2.0.11}$$

where, $\mathbf{S}_{+}$ is Moore-Pen-rose Pseudo-Inverse of $\mathbf{S}$. Columns of $\mathbf{U}$ are eigen-vectors of $\mathbf{M}\mathbf{M}^{\mathbf{T}}$, columns of $\mathbf{V}$ are eigenvectors of $\mathbf{M}^{\mathbf{T}}\mathbf{M}$ and $\mathbf{S}$ is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^{\mathbf{T}}\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^{\mathbf{T}}\mathbf{M}$.

$$\mathbf{M}^{\mathbf{T}}\mathbf{M} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \tag{2.0.12}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left|\mathbf{M^TM}-\lambda\mathbf{I}\right| = 0 \tag{2.0.13}$$

$$\left|\begin{pmatrix}\frac{5}{4}-\lambda & \frac{1}{6}\\ \frac{1}{6} & \frac{10}{9}-\lambda\end{pmatrix}\right| = 0 \tag{2.0.14}$$

$$\left(\frac{5}{4}-\lambda\right)\left(\frac{10}{9}-\lambda\right)-\frac{1}{36} = 0 \tag{2.0.15}$$

$$\left(\lambda-\frac{34}{25}\right)(\lambda-1) = 0 \tag{2.0.16}$$

Hence,

$$\lambda_1 = \frac{34}{25}, \lambda_2 = 1 \tag{2.0.17}$$

Eigen-vector corresponding to $\lambda = \frac{34}{25}$,

$$\mathbf{v_1} = \begin{pmatrix}\frac{3}{2}\\ 1\end{pmatrix} \tag{2.0.18}$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_2} = \begin{pmatrix}-\frac{2}{3}\\ 1\end{pmatrix} \tag{2.0.19}$$

Normalizing, the eigen vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, we get

$$\mathbf{v_1} = \frac{1}{\sqrt{13}}\begin{pmatrix}3\\ 2\end{pmatrix} \tag{2.0.20}$$

$$\mathbf{v_2} = \frac{1}{\sqrt{13}}\begin{pmatrix}-2\\ 3\end{pmatrix} \tag{2.0.21}$$

Hence,

$$\mathbf{V} = \frac{1}{\sqrt{13}}\begin{pmatrix}3 & -2\\ 2 & 3\end{pmatrix} \tag{2.0.22}$$

Now calculating the eigenvectors corresponding to $\mathbf{MM^T}$

$$\mathbf{MM^T} = \begin{pmatrix}1 & 0\\ 0 & 1\\ \frac{1}{2} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{3}\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}\end{pmatrix} \tag{2.0.23}$$

Eigen values of $\mathbf{MM^T}$ can be found out as

$$\left|\mathbf{MM^T}-\lambda\mathbf{I}\right| = 0 \tag{2.0.24}$$

$$\left|\begin{pmatrix}1-\lambda & 0 & \frac{1}{2}\\ 0 & 1-\lambda & \frac{1}{3}\\ \frac{1}{2} & \frac{1}{3} & \frac{13}{36}-\lambda\end{pmatrix}\right| = 0 \tag{2.0.25}$$

$$\tag{2.0.26}$$

Hence,

$$\lambda_3 = 0, \lambda_4 = 1, \lambda_5 = \frac{49}{36} \tag{2.0.27}$$

Eigen-vector corresponding to $\lambda = 0$,

$$\mathbf{v_3} = \begin{pmatrix}-3\\ -2\\ 6\end{pmatrix} \tag{2.0.28}$$

Eigen-vector corresponding to $\lambda = 1$,

$$\mathbf{v_4} = \begin{pmatrix}-2\\ 3\\ 0\end{pmatrix} \tag{2.0.29}$$

Eigen-vector corresponding to $\lambda = \frac{49}{36}$,

$$\mathbf{v_5} = \begin{pmatrix}18\\ 12\\ 13\end{pmatrix} \tag{2.0.30}$$

Normalizing, the eigen vectors $\mathbf{v_3}$, $\mathbf{v_4}$ and $\mathbf{v_5}$, we get

$$\mathbf{v_3} = \frac{1}{7}\begin{pmatrix}-3\\ -2\\ 6\end{pmatrix} = \begin{pmatrix}-\frac{3}{7}\\ -\frac{2}{7}\\ \frac{6}{7}\end{pmatrix} \tag{2.0.31}$$

$$\mathbf{v_4} = \frac{1}{\sqrt{13}}\begin{pmatrix}-2\\ 3\\ 0\end{pmatrix} = \begin{pmatrix}-\frac{2}{\sqrt{13}}\\ \frac{3}{\sqrt{13}}\\ 0\end{pmatrix} \tag{2.0.32}$$

$$\mathbf{v_5} = \frac{1}{7\sqrt{13}}\begin{pmatrix}18\\ 12\\ 13\end{pmatrix} = \begin{pmatrix}\frac{18}{7\sqrt{13}}\\ \frac{12}{7\sqrt{13}}\\ \frac{13}{7\sqrt{13}}\end{pmatrix} \tag{2.0.33}$$

Hence,

$$\mathbf{U} = \begin{pmatrix}-\frac{3}{7} & -\frac{2}{\sqrt{13}} & \frac{18}{7\sqrt{13}}\\ -\frac{2}{7} & \frac{3}{\sqrt{13}} & \frac{12}{7\sqrt{13}}\\ \frac{6}{7} & 0 & \frac{13}{7\sqrt{13}}\end{pmatrix} \tag{2.0.34}$$

Now $\mathbf{S}$ corresponding to eigenvalues is as follows,

$$\mathbf{S} = \begin{pmatrix}\frac{\sqrt{34}}{5} & 0\\ 0 & 1\\ 0 & 0\end{pmatrix} \tag{2.0.35}$$

Now, Moore-Pen-Rose Pseudo inverse of $\mathbf{S}$ is given by,

$$\mathbf{S_+} = \begin{pmatrix}\frac{5}{\sqrt{34}} & 0 & 0\\ 0 & 1 & 0\end{pmatrix} \tag{2.0.36}$$

Hence, we get singular value decomposition of $\mathbf{M}$ as,

$$\mathbf{M} = \begin{pmatrix} -\frac{3}{7} & -\frac{2}{\sqrt{13}} & \frac{18}{7\sqrt{13}} \\ -\frac{2}{7} & \frac{3}{\sqrt{13}} & \frac{12}{7\sqrt{13}} \\ \frac{6}{7} & 0 & \frac{13}{7\sqrt{13}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{34}}{5} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{13}} \begin{pmatrix} 3 & 2 \\ -2 & 3 \end{pmatrix} \tag{2.0.37}$$

Substituting values of (2.0.8), (2.0.22), (2.0.34) and (2.0.35) into (2.0.11), we get

$$\mathbf{U^T b} = \begin{pmatrix} -\frac{3}{7} & -\frac{2}{7} & \frac{6}{7} \\ -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} & 0 \\ \frac{18}{7\sqrt{13}} & \frac{12}{7\sqrt{13}} & \frac{13}{7\sqrt{13}} \end{pmatrix} \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix} \tag{2.0.38}$$

$$\implies \mathbf{U^T b} = \begin{pmatrix} -\frac{1700}{2523} \\ \frac{10}{3} \\ \frac{11}{7} \end{pmatrix} \tag{2.0.39}$$

Now,

$$\mathbf{VS_+} = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} \frac{5}{\sqrt{34}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{2.0.40}$$

$$\implies \mathbf{VS_+} = \frac{1}{\sqrt{13}} \begin{pmatrix} \frac{1500}{583} & -2 & 0 \\ \frac{1000}{583} & 3 & 0 \end{pmatrix} \tag{2.0.41}$$

Now, by (2.0.11), we have

$$\mathbf{x} = \frac{1}{\sqrt{13}} \begin{pmatrix} \frac{1500}{583} & -2 & 0 \\ \frac{1000}{583} & 3 & 0 \end{pmatrix} \begin{pmatrix} -\frac{1700}{2523} \\ \frac{10}{3} \\ \frac{11}{7} \end{pmatrix} \tag{2.0.42}$$

$$\implies \mathbf{x} = \begin{pmatrix} -\frac{114}{49} \\ \frac{120}{49} \end{pmatrix} \tag{2.0.43}$$

Now, we verify our solution using

$$\mathbf{M^T M x} = \mathbf{M^T b} \tag{2.0.44}$$

$$\implies \begin{pmatrix} \frac{5}{4} & \frac{1}{6} \\ \frac{1}{6} & \frac{10}{9} \end{pmatrix} \mathbf{x} = \begin{pmatrix} -\frac{5}{2} \\ \frac{7}{3} \end{pmatrix} \tag{2.0.45}$$

$$\begin{pmatrix} \frac{5}{4} & \frac{1}{6} & -\frac{5}{2} \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \xleftrightarrow{R_1 = \frac{4}{5}R_1} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ \frac{1}{6} & \frac{10}{9} & \frac{7}{3} \end{pmatrix} \tag{2.0.46}$$

$$\xleftrightarrow{R_2 = R_2 - \frac{1}{6}R_1} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & \frac{49}{45} & \frac{8}{3} \end{pmatrix} \tag{2.0.47}$$

$$\xleftrightarrow{R_2 = \frac{45}{49}R_2} \begin{pmatrix} 1 & \frac{2}{15} & -2 \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \tag{2.0.48}$$

$$\xleftrightarrow{R_1 = R_1 - \frac{2}{15}R_2} \begin{pmatrix} 1 & 0 & -\frac{114}{49} \\ 0 & 1 & \frac{120}{49} \end{pmatrix} \tag{2.0.49}$$

Comparing (2.0.49) with (2.0.43), solution is given by,

$$\mathbf{x} = \begin{pmatrix} -\frac{114}{49} \\ \frac{120}{49} \end{pmatrix} \tag{2.0.50}$$