Solutions of HW#2: *Language of Mathematics*

Q1. Prove that for any sets A and B,

$$\overline{A \cap B} = \bar{A} \cup \bar{B}.$$

Answer

To prove that two sets are equal, we need to prove that each set is a subset of the other:

i) To prove that $\bar{A} \cup \bar{B} \subseteq \overline{A \cap B}$:

$$\begin{aligned} \forall x \in \bar{A} \cup \bar{B} \quad &\Rightarrow \quad x \in \bar{A} \vee x \in \bar{B} \\ &\Rightarrow \quad x \notin A \vee x \notin B \\ &\Rightarrow \quad x \notin A \cap B \\ &\Rightarrow \quad x \in \overline{A \cap B} \end{aligned}$$

Thus, all elements of $\bar{A} \cup \bar{B}$ are also elements of $\overline{A \cap B}$. That is,

$$\bar{A} \cup \bar{B} \subseteq \overline{A \cap B} \cdots\cdots\cdots (1)$$

ii) Likewise, to prove that $\overline{A \cap B} \subseteq \bar{A} \cup \bar{B}$, we follow the exact same steps but in the reverse order thus proving:

$$\overline{A \cap B} \subseteq \bar{A} \cup \bar{B} \cdots\cdots\cdots (2)$$

From $(1), (2)$: $\overline{A \cap B} = \bar{A} \cup \bar{B}$.

Q2. Let x and y be *integers*. Determine whether the following relations are reflexive, symmetric, antisymmetric, or transitive:

i)$x \equiv y \mod 5$;

ii) $xy \leq 1$;

iii) $x = y^3$.

Justify your statements. Finally, determine which of the above relations are equivalence and partial order relations. For equivalence relations, construct the equivalence classes.

Answer

i)$R = \{(x, y)|x \equiv y \mod 5\}$:

- Reflexive: **(YES)** since $x \equiv x \mod 5$.
- Symmetric: **(YES)** since $x \equiv y \mod 5 \Rightarrow x - y = 5k \Rightarrow y - x = -5k$ $\Rightarrow y \equiv x \mod 5$.
- Antisymmetric: **(NO)** since, for example, $(1, 6)$ and $(6, 1)$ are both in R.
- Transitive: **(YES)** since if $(x \equiv y \mod 5)$ and $(y \equiv z \mod 5)$, then $x - y = 5k$ and $y - z = 5l \Rightarrow x - z = 5(k + l) \Rightarrow x \equiv z \mod 5$.

Thus, R is an equivalence relation. The equivalence classes are:

$$[0] = \{\cdots, -10, -5, 0, 5, 10, \cdots\}$$
$$[1] = \{\cdots, -9, -4, 1, 6, 11, \cdots\}$$
$$[2] = \{\cdots, -8, -3, 2, 7, 12, \cdots\}$$
$$[3] = \{\cdots, -7, -2, 3, 8, 13, \cdots\}$$
$$[4] = \{\cdots, -6, -1, 4, 9, 14, \cdots\}.$$

ii) $R = \{(x, y)|xy \leq 1\}$:

- Reflexive: **(NO)** since, for example, $(2, 2) \notin R$ (this is because $2^2 = 4 > 1$).
- Symmetric: **(YES)** since $xy = yx$.
- Antisymmetric: **(NO)** since, for example, $(1, 0)$ and $(0, 1)$ are both in R.
- Transitive: **(NO)** since, for example, $(-1, 1) \in R \wedge (1, -2) \in R$ but $(-1, -2) \notin R$.

Thus, R is neither an equivalence relation nor a partial order relation.

iii) $R = \{(x, y)|x = y^3\}$:

- Reflexive: **(NO)** since, for example, $(2, 2) \notin R$.
- Symmetric: **(NO)** since, for example, $(8, 2) \in R$ but $(2, 8) \notin R$.
- Antisymmetric: **(YES)** since if $x = y^3$ and $y = x^3$, then $x = y$.
- Transitive: **(NO)** since if $x = y^3$ and $y = z^3$, then $x = z^9 \neq z^3$.

Thus, R is neither an equivalence relation nor a partial order relation.

Q3. Determine whether the following function is bijection from $\mathbf{R}$ to $f(\mathbf{R})$:

i) $f(x) = x^4$
ii) $f(x) = \cos^2(x)$
iii) $f(x) = \frac{x+3}{x+1}$.

Answer

Note that since the ranges of the functions are $f(\mathbf{R})$, all the functions cover the range and, consequently, they are onto.

i) $f(x) = x^4$:
- One-to-one: **(NO)** since, for example, $2^4 = (-2)^4 = 16$.
- Onto: **(YES)** see above.
- Bijection: **(NO)** since it is not one-to-one and onto.

ii) $f(x) = \cos^2(x)$:
- One-to-one: **(NO)** since, for example, $\cos^2(\pi/2) = \cos^2(3\pi/2) = 0$.
- Onto: **(YES)** see above.
- Bijection: **(NO)** since it is not one-to-one and onto.

iii) $f(x) = \frac{x+3}{x+1} = \frac{x+1+2}{x+1} = 1 + \frac{2}{x+1}$:
- One-to-one: **(YES)** since if $f(x_1) = f(x_2) \Rightarrow 1 + \frac{2}{x_1+1} = 1 + \frac{2}{x_2+1} \Rightarrow x_1 = x_2$.
- Onto: **(YES)** see above.
- Bijection: **(YES)** since it is both one-to-one and onto.

Q4. Let $g(x) = \lfloor x \rfloor$. Find

- $g^{-1}(\{0\})$;
- $g^{-1}(\{x : \ 0 < x < 1\})$

Answer

- $g^{-1}(\{0\}) = \{x | 0 \leq x < 1\}$ (i.e. the values whose floor is 0).
- $g^{-1}(\{x : \ 0 < x < 1\}) = \{\} = \phi$ (since the floor has to be a whole number).

Q5. What are the values of the following:

$$\sum_{i=1}^{500} 6^{i+1},$$

$$\sum_{i=1}^{2} \sum_{j=1}^{3} (i + j)$$

$$\sum_{j=0}^{8} j \cdot 3^j.$$

Answer

i)

$$\sum_{i=1}^{500} 6^{i+1} = 6\sum_{i=1}^{500} 6^i = 6\Big(\sum_{i=0}^{500} 6^i - 1\Big) = 6\Big(\frac{6^{501}-1}{6-1} - 1\Big)$$

ii)

$$\sum_{i=1}^{2}\sum_{j=1}^{3}(i+j) = \sum_{i=1}^{2}\sum_{j=1}^{3} i + \sum_{i=1}^{2}\sum_{j=1}^{3} j = \sum_{j=1}^{3}(1+2) + \sum_{i=1}^{2}(1+2+3) = 3\times(1+2)+2\times(1+2+3) = 21$$

iii)

$$\sum_{j=0}^{8} j\cdot 3^j = 0\times 3^0+1\times 3+2\times 3^2+3\times 3^3+4\times 3^4+5\times 3^5+6\times 3^6+7\times 3^7+8\times 3^8 = 73812.$$

OR

$$\sum_{j=0}^{n} r^j = \frac{r^{n+1}-1}{r-1}.$$

Differentiating both sides with respect to r:

$$\sum_{j=0}^{n} j\cdot r^{j-1} = \frac{(n+1)(r-1)r^n - (r^{n+1}-1)}{(r-1)^2}$$

$$\Rightarrow \sum_{j=0}^{n} j\cdot r^{j} = \frac{(n+1)(r-1)r^{n+1} - (r^{n+2}-r)}{(r-1)^2}$$

$$\Rightarrow \sum_{j=0}^{8} j\cdot 3^{j} = \frac{9\times 2\times 3^9 - (3^{10}-3)}{2^2} = 73812.$$