Solutions of Homework #4: *Proof Techniques*

Q1. Show that $\sqrt[3]{3}$ is irrational.

Answer

Proof by contradiction: Assume that $\sqrt[3]{3} = \frac{p}{q}$ in its simplest form, i.e., both p and q do not have a common divisor and therefore **the fraction $\frac{p}{q}$ cannot be simplified further**. Thus,

$$
\begin{aligned}
& 3 = \frac{p^3}{q^3} && (1)\\
\rightarrow\quad & p^3 = 3q^3 && (2)\\
\rightarrow\quad & 3|p^3 && (3)\\
\rightarrow\quad & 3|p. && (4)
\end{aligned}
$$

Where $3|p$ means that p is divisible by 3. . . . . (I)

From (I), $p = 3k$ for some integer k. Substituting in (2):

$$
\begin{aligned}
& (3k)^3 = 3q^3\\
\rightarrow\quad & 27k^3 = 3q^3\\
\rightarrow\quad & q^3 = 9k^3\\
\rightarrow\quad & 3|q^3\\
\rightarrow\quad & 3|q.
\end{aligned}
$$

Thus, q is also divisible by 3. . . . . (II)

From (I) and (II), the fraction $\frac{p}{q}$ is *not* in its simplest form for it can be simplified further by dividing both the numerator and the denominator by 3 which contradicts the original assumption.

Q.2 Let A be a set of cardinality n. Let $P(A)$ be the power set, that is, the set of *all* subsets of A. Prove by induction that the cardinality of $P(A)$ is 2^n, that is

$$|P(A)| = 2^n.$$

Answer

<u>Proof by induction on n</u>

BASIS CASE ($n = 1$): Since $n = 1$, $|A| = 1$. Let $A = \{a\}$, then $P(A) = \{\emptyset, \{a\}\}$. Therefore, $|P(A)| = 2$.

INDUCTION STEP: Assume $P(n)$ is *true*, i.e., $|A| = n \rightarrow |P(A)| = 2^n$. We need to prove that $P(n+1)$ is also *true*:

$$\begin{aligned} & A = \{a_1, a_2, \ldots, a_n\} \\ \rightarrow \quad & P(A) = \{S_1, S_2, \ldots, S_{2^n}\} \end{aligned}$$

Let

$$\begin{aligned} & B = A \cup \{a_{n+1}\} = \{a_1, a_2, \ldots, a_n, a_{n+1}\} \\ \rightarrow \quad & P(B) = \{S_1, S_2, \ldots, S_{2^n}\} \cup \{S_1 \cup \{a_{n+1}\}, S_2 \cup \{a_{n+1}\}, \ldots, S_{2^n} \cup \{a_{n+1}\}\} \\ \rightarrow \quad & |P(B)| = 2 \times |P(A)| = 2 \times 2^n = 2^{n+1} \end{aligned}$$

That is, the power set of the extended set B contains all subsets of the initial set A as well as their extensions with the added element a_{n+1}. Therefore, $P(n+1)$ is *true*.

Consider, for example, $A = \{a_1, a_2\}$, then $P(A) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}\}$. If

$$\begin{aligned} & B = A \cup \{a_3\} = \{a_1, a_2, a_3\} \\ \rightarrow \quad & P(B) = \{\emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_2, a_3\}, \{a_1, a_3\}, \{a_1, a_2, a_3\}\} \\ & = \{\emptyset, \{a_1\}, \{a_2\}, \{a_1, a_2\}\} \cup \{\emptyset \cup \{a_3\}, \{a_1\} \cup \{a_3\}, \{a_2\} \cup \{a_3\}, \{a_1, a_2\} \cup \{a_3\}\} \\ \rightarrow \quad & |P(B)| = 2 \times |P(A)| = 2 \times 2^2 = 2^3 = 8 \end{aligned}$$

Q.3 Prove by induction on $n \geq 1$

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1.$$

Answer

Proof by induction on $n \geq 1$

BASIS CASE ($n = 1$): L.H.S.$=\sum_{i=1}^{1} i \cdot i! = 1 \cdot 1! = 1$, R.H.S.$=2! - 1 = 1$ therefore, $P(1)$ is *true*.

INDUCTION STEP: Assume $P(k)$ is *true*, i.e., $\sum_{i=1}^{k} i \cdot i! = (k+1)! - 1$. We prove that $P(k+1)$ is also *true*, i.e., $\sum_{i=1}^{k+1} i \cdot i! = (k+2)! - 1$ as follows:

$$\begin{aligned}
\sum_{i=1}^{k+1} i \cdot i! &= \sum_{i=1}^{k} i \cdot i! + (k+1) \cdot (k+1)! \\
&= \big((k+1)! - 1\big) + (k+1) \cdot (k+1)! \\
&= (k+1)!(1 + k + 1) - 1 \\
&= (k+2)! - 1.
\end{aligned}$$

Q.4 The *harmonic number* H_n is defined as for $n \geq 1$

$$H_n = \sum_{k=1}^{n} \frac{1}{k}.$$

Prove by induction that

$$H_{2^n} \geq 1 + \frac{n}{2}$$

whenever n is nonnegative natural number.

Answer

Proof by induction on n

BASIS CASE ($n = 0$): $H_1 = \sum_{k=1}^{1} \frac{1}{k} = 1 \geq 1 + \frac{0}{2}$.

INDUCTION STEP: Assume $P(n)$ is *true*, i.e., $H_{2^n} \geq 1 + \frac{n}{2}$. We need to prove that $P(n+1)$, which is $H_{2^{n+1}} \geq 1 + \frac{n+1}{2}$, is also *true*:

$$\begin{aligned}
H_{2^{n+1}} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} + \frac{1}{2^n+1} \cdots + \frac{1}{2^{n+1}} \\
&= H_{2^n} + \frac{1}{2^n+1} \cdots + \frac{1}{2^{n+1}} \\
&\geq (1 + \frac{n}{2}) + \frac{1}{2^n+1} \cdots + \frac{1}{2^{n+1}} \\
&\geq (1 + \frac{n}{2}) + \frac{1}{2^{n+1}} \cdots + \frac{1}{2^{n+1}} \\
&= (1 + \frac{n}{2}) + 2^n \cdot \frac{1}{2^{n+1}} \\
&= (1 + \frac{n}{2}) + \frac{1}{2} \\
&= 1 + \frac{n+1}{2}.
\end{aligned}$$

Q.5 Derive an explicit formula for the following recurrence for $n \geq 2$

$$a_n = \frac{n-1}{3}\ a_{n-1}$$

with $a_1 = 1$.

Answer

$$\begin{aligned}
a_n &= \frac{n-1}{3}\ a_{n-1} \\
&= \frac{n-1}{3} \times \frac{n-2}{3}\ a_{n-2} \\
&= \frac{n-1}{3} \times \frac{n-2}{3} \times \frac{n-3}{3}\ a_{n-3} \\
&\vdots \\
&= \underbrace{\frac{n-1}{3} \times \frac{n-2}{3} \times \frac{n-3}{3} \ldots \times \frac{3}{3} \times \frac{2}{3} \times \frac{1}{3}}_{(n-1)\ terms}\ a_1 \\
&= \underbrace{\frac{(n-1)!}{3^{n-1}}} \times 1 \\
&= \frac{(n-1)!}{3^{n-1}}.
\end{aligned}$$