Weak Coin Flipping

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2018 End
(0v8a)

Abstract

We investigate weak coin flipping, a fundamental cryptographic primitive where two distrustful parties need to remotely establish a shared random bit. A cheating player can try to bias the output bit towards a preferred value. For weak coin flipping the players have known opposite preferred values. A weak coin-flipping protocol has a bias ϵ if neither player can force the outcome towards his/her preferred value with probability more than $\frac{1}{2} + \epsilon$. While it is known that classically $\epsilon = 1/2$, Mochon showed in 2007 [1] that quantumly weak coin flipping can be achieved with arbitrarily small bias (near perfect) but the best known explicit protocol has bias 1/6 (also due to Mochon [2]). We propose a framework to construct new explicit protocols achieving biases beyond 1/6. In particular, we construct explicit unitaries for protocols with bias up to 1/10 (the first improvement of its kind in the last thirteen years). To go beyond, we introduce what we call the Elliptic Monotone Align (EMA) algorithm which, together with the framework, allows us to construct protocols with arbitrarily small biases. This solves the open problem of quantum weak coin flipping (in the absence of noise).

Contents

[*]responsible for all mistakes

1 Introduction

We investigate coin flipping[1], a fundamental cryptographic primitive where two distrustful parties need to remotely generate a shared unbiased random bit. A cheating player can try to bias the output bit towards a preferred value. For weak coin flipping the players have known opposite preferred values. A weak coin-flipping (WCF) protocol has a bias ϵ if neither player can force the outcome towards his/her preferred value with probability more than $\frac{1}{2}+\epsilon$. For strong coin-flipping there are no a priori preferred values and the bias is defined similarly. Restricting to classical resources, neither weak nor strong coin flipping is possible under information-theoretic security, as there always exists a player [3] who can force any outcome with probability 1. However, in a quantum world, strong coin-flipping protocols with bias strictly less than $\frac{1}{2}$ have been shown and the best known explicit protocol has bias $\frac{1}{4}$ [4]. Nevertheless, Kitaev showed a lower bound of $\frac{1}{\sqrt{2}}-\frac{1}{2}$ for the bias of any quantum strong coin flipping, so an unbiased protocol is not possible.

As for weak coin flipping, the current best known explicit protocol—the Dip Dip Boom protocol—is due to Mochon [2] and has bias 1/6. In a breakthrough result, he even proved the existence of a quantum weak coin-flipping protocol with arbitrarily low bias $\epsilon > 0$, hence showing that near-perfect weak coin flipping is theoretically possible [1]. This fundamental result for quantum cryptography, unfortunately, was proved non-constructively, by elaborate successive reductions (80 pages) of the protocol to different versions of so-called point games, a formalism introduced by Kitaev [5] in order to study coin flipping. Consequently, the structure of the protocol whose existence is proved is lost. A systematic verification of this by independent researchers recently led to a simplified proof [6] (*only* 50 pages) but eleven years later, an explicit weak coin-flipping protocol is still unknown, despite various expert approaches ranging from the distillation of a protocol using the proof of existence to numerical search [7]. Further, weak coin flipping provides, via black-box reductions, optimal protocols for strong coin flipping [8] and bit commitment (another fundamental cryptographic primitive) [9], making the absence of an explicit protocol even more frustrating.

We construct a framework that allows us to convert simple point games (i.e. corresponding to known protocols, back) into explicit quantum protocol defined in terms of unitaries and projectors. We use the said framework to convert a bias 1/10 point game into its corresponding explicit protocol making it the first improvement of its kind in the last thirteen years since Mochon's Dip Dip Boom protocol (bias 1/6) [2].

Our second contribution, the Elliptic Monotone Align (EMA) algorithm, can provably find the unitaries required for implementing protocols with arbitrary biases, including the ones with $\epsilon \to 0$. In effect, the framework supplemented by the EMA algorithm allows us to solve quantum weak coin flipping.

2 Prior Art

We start with stating the results up to a certain point from Ahoronov et al's [6] paper (which completely formalises Mochon's results [1] and simplifies one of the results proved in its appendix; we build on this improved proof in our work). We will motivate the statements as we go along. It is unlikely that the next section will make perfect sense unless one has already read Aharonov's et al's and/or Mochon's article. The ideas should get clear in the following sections as we use them to construct explicit protocols. Logically, however, we have tried to keep everything consistent (even though the presentation thereof may not be optimal).

We define $\mathbb{R}_{\geq} := [0,\infty)$, $\mathbb{R}_{>} := (0,\infty)$ and similarly $\mathbb{R}_{\leq} := (-\infty,0], \mathbb{R}_{<} := (-\infty,0)$. A note about the colour scheme. We use purple for intuitive and non-technical discussions and, in this section, blue for statements/corrections that we have added to the results from Aharonov et al's article.

2.1 WCF protocol as an SDP and its Dual

Any weak coin flipping protocol can be expressed in the following general form (we will not prove this claim here; see [4]).

Definition 1 (WCF protocol with bias ϵ)**.** For n even, an n-message WCF protocol between two players, Alice and Bob, is described by

- three Hilbert spaces with $\mathcal{A}$, $\mathcal{B}$ corresponding to Alice and Bob's private workspaces (Bob does not have any access to $\mathcal{A}$ and Alice to $\mathcal{B}$), and a message space $\mathcal{M}$;
- an initial product state $|\psi_0\rangle = |\psi_{A,0}\rangle \otimes |\psi_{M,0}\rangle \otimes |\psi_{B,0}\rangle \in \mathcal{A}\otimes\mathcal{M}\otimes\mathcal{B}$;
- a set of n unitaries $\{U_1,\dots U_n\}$ acting on $\mathcal{A}\otimes\mathcal{M}\otimes\mathcal{B}$ with $U_i = U_{A,i}\otimes\mathbb{I}_{\mathcal{B}}$ for i odd and $U_i = \mathbb{I}_{\mathcal{A}}\otimes U_{B,i}$ for i even;
- a set of honest states $\{|\psi_i\rangle : i\in[n]\}$ defined by $|\psi_i\rangle = U_iU_{i-1}\dots U_1|\psi_0\rangle$;
- a set of n projectors $\{E_1,\dots E_n\}$ acting on $\mathcal{A}\otimes\mathcal{M}\otimes\mathcal{B}$ with $E_i = E_{A,i}\otimes\mathbb{I}_{\mathcal{B}}$ for i odd, and $E_i = \mathbb{I}_{\mathcal{A}}\otimes E_{B,i}$ for i even, such that $E_i|\psi_i\rangle = |\psi_i\rangle$;
- two final positive operator valued measure (POVM) $\{\Pi_A^{(0)},\Pi_A^{(1)}\}$ acting on $\mathcal{A}$ and $\{\Pi_B^{(0)},\Pi_B^{(1)}\}$ acting on $\mathcal{B}$.

[1] For the latest version visit: https://atulsingharora.github.io/QIP_19. We intend to release an update on October 17, 2018.

The WCF protocol proceeds as follows:

- In the beginning, Alice holds $|\psi_{A,0}\rangle |\psi_{M,0}\rangle$ and Bob $|\psi_{B,0}\rangle$.
- For $i = 1$ to n:
  - If i is odd, Alice applies U_i and measures the resulting state with the POVM $\{E_i, \mathbb{I} - E_i\}$. On the first outcome, Alice sends the message quibts to Bob; on the second outcome, she ends the protocol by outputting "0", i.e, Alice declares herself to be the winner.
  - If i is even, Bob applies U_i and measures the resulting state with the POVM $\{E_i, \mathbb{I} - E_i\}$. On the first outcome, Bob sends the message qubits to Alice; on the second outcome, he ends the protocol by outputting "1", i.e., Bob declares himself to be the winner.
  - Alice and Bob measure their part of the state with the final POVM and output the outcome of their measurements. Alice wins on outcome "0" and Bob on outcome "1".

The WCF protocol has the following properties:

- Correctness: When both players are honest, Alice and Bob's outcomes are always the same: $\Pi_A^{(0)} \otimes \mathbb{I}_{\mathcal{M}} \otimes \Pi_B^{(1)} |\psi_n\rangle = \Pi_A^{(1)} \otimes \mathbb{I}_{\mathcal{M}} \otimes \Pi_B^{(0)} |\psi_n\rangle = 0$.
- Balanced: When both players are honest, they win with probability 1/2: $P_A = \left|\Pi_A^{(0)} \otimes \mathbb{I}_{\mathcal{M}} \otimes \Pi_B^{(0)} |\psi_n\rangle\right|^2 = \frac{1}{2}$ and $P_B = \left|\Pi_A^{(1)} \otimes \mathbb{I}_{\mathcal{M}} \otimes \Pi_B^{(1)} |\psi_n\rangle\right|^2 = \frac{1}{2}$.
- ϵ biased: When Alice is honest, the probability that both players agree on Bob winning is $P_B^* \leq \frac{1}{2} + \epsilon$. And conversely, if Bob is honest, the probability that both players agree on Alice winning is $P_A^* \leq \frac{1}{2} + \epsilon$.

To be able to define the bias, we need P_A^* and P_B^* which correspond to the best possible cheating strategy of the opponent. The primal semi-definite program (SDP) formalises this statement.

Theorem 2 (Primal)**.**
$P_B^* = \max Tr((\Pi_A^{(1)} \otimes \mathbb{I}_{\mathcal{M}})\rho_{AM,n})$ *over all* $\rho_{AM,i}$ *satisfying the constraints*

- $Tr_{\mathcal{M}}(\rho_{AM,0}) = Tr_{\mathcal{MB}}(|\psi_0\rangle\langle\psi_0|) = |\psi_{A,0}\rangle\langle\psi_{A,0}|$;
- *for* i *odd,* $Tr_{\mathcal{M}}(\rho_{AM,i}) = Tr_{\mathcal{M}}(E_i U_i \rho_{AM,i-1} U_i^\dagger E_i)$;
- *for* i *even,* $Tr_{\mathcal{M}}(\rho_{AM,i}) = Tr_{\mathcal{M}}(\rho_{AM,i-1})$.

$P_A^* = \max Tr((\mathbb{I}_{\mathcal{M}} \otimes \Pi_B^{(0)})\rho_{MB,n})$ *over all* $\rho_{BM,i}$ *satisfying the constraints*

- $Tr_{\mathcal{M}}(\rho_{MB,0}) = Tr_{\mathcal{AM}}(|\psi_0\rangle\langle\psi_0|) = |\psi_{B,0}\rangle\langle\psi_{B,0}|$;
- *for* i *even,* $Tr_{\mathcal{M}}(\rho_{MB,i}) = Tr_{\mathcal{M}}(E_i U_i \rho_{MB,i-1} U_i^\dagger E_i)$;
- *for* i *odd,* $Tr_{\mathcal{M}}(\rho_{MB,i}) = Tr_{\mathcal{M}}(\rho_{MB,i-1})$.

A feasible solution to an optimisation problem is one which satisfies the constraints but is not necessarily optimal. A feasible solution to the primal problems will give a lower bound on P_A^* and P_B^*. If we consider the duals instead, it is known that, a feasible solution will give an upper bound on P_A^* and P_B^*. This will certify how good the protocol is.

Theorem 3 (Dual)**.**
$P_B^* = \min Tr(Z_{A,0} |\psi_{A,0}\rangle\langle\psi_{A,0}|)$ *over all* $Z_{A,i}$ *under the constraints*

1. $\forall i$, $Z_{A,i} \geq 0$;
2. *for* i *odd,* $Z_{A,i-1} \otimes \mathbb{I}_{\mathcal{M}} \geq U_{A,i}^\dagger E_{A,i}(Z_{A,i} \otimes \mathbb{I}_{\mathcal{M}})E_{A,i}U_{A,i}$;
3. *for* i *even,* $Z_{A,i-1} = Z_{A,i}$;
4. $Z_{A,n} = \Pi_A^{(1)}$.

$P_A^* = \min Tr(Z_{B,0} |\psi_{B,0}\rangle\langle\psi_{B,0}|)$ *over all* $Z_{B,i}$ *under the constraints*

1. $\forall i$, $Z_{B,i} \geq 0$;
2. *for* i *even,* $\mathbb{I}_{\mathcal{M}} \otimes Z_{B,i-1} \geq U_{B,i}^\dagger E_{B,i}(\mathbb{I}_{\mathcal{M}} \otimes Z_{B,i})E_{B,i}U_{B,i}$;

3. *for i odd, $Z_{B,i-1} = Z_{B,i}$;*

4. $Z_{B,n} = \prod_B^{(0)}$.

We add one more constraint to the above dual SDPs.

5. *$|\psi_{A,0}\rangle$ is an eigenvector of $Z_{A,0}$ with eigenvalue $\alpha > 0$ and $|\psi_{B,0}\rangle$ is an eigenvector of $Z_{B,0}$ with eigenvalue $\beta > 0$.*

Definition 4 (dual feasible points)**.** We call *dual feasible points* any two sets of matrices $\{Z_{A,0}, \ldots, Z_{A,n}\}$ and $\{Z_{B,0}, \ldots, Z_{B,n}\}$ that satisfy the corresponding conditions 1 to 5 as listed in Theorem 3.

It turns out that strong duality holds for the primal problems which means that there will be a cheating strategy for Alice and Bob matching the upper bound on P_A^* and P_B^* respectively.

Proposition 5. *$P_A^* = \inf \alpha$ and $P_B^* = \inf \beta$ where the infimum is over all dual feasible points and β, α are defined in constraint 5 of the definition of the dual feasible points.*

2.2 (Time Dependent) Point Games with EBM transitions

The basic idea here is to remove all inessential information, that is the basis information, from the two aforesaid dual problems. Kitaev's genius was to achieve this by considering, at a given step, the dual variable Z_A, Z_B as observables with $|\psi\rangle$ governing the probability. This combines the evolution of the certificates on cheating probabilities with the evolution of the honest state — the state obtained when both players follow the protocol (nobody cheats). Originally, using a similar manoeuvre, Kitaev settled solvability of the quantum strong coin flipping problem by giving a bound on ϵ. To make this insight precise, first "prob" is defined.

Definition 6 (prob)**.** Consider $Z \geq 0$ and let $\Pi^{[z]}$ represent the projector on the eigenspace of eigenvalue $z \in \mathrm{sp}(Z)$. We have $Z = \sum_z z\Pi^{[z]}$. Let $|\psi\rangle$ be a (not necessarily normalised) vector. We define the function with finite support $\mathrm{prob}[Z, \psi] : [0, \infty) \to [0, \infty)$ as

$$\mathrm{prob}[Z, \psi](z) = \begin{cases} \langle\psi|\, \Pi^{[z]}\, |\psi\rangle & \text{if } z \in \mathrm{sp}(Z) \\ 0 & \text{else.} \end{cases}$$

If $Z = Z_A \otimes \mathbb{I}_{\mathcal{M}} \otimes Z_B$, using the same notation, we define the 2−variate function with finite support $\mathrm{prob}[Z_A, Z_B, \psi] : [0, \infty) \times [0, \infty) \to [0, \infty)$ as

$$\mathrm{prob}[Z_A, Z_B, \psi](z_A, z_B) = \begin{cases} \langle\psi|\, \Pi^{[z_A]} \otimes \mathbb{I}_{\mathcal{M}} \otimes \Pi^{[z_B]}\, |\psi\rangle & \text{if } (z_A, z_B) \in \mathrm{sp}(Z_A) \times \mathrm{sp}(Z_B), \\ 0 & \text{else.} \end{cases}$$

Think of the aforesaid 2–variate function as assigning a weight on each point of the plane. Going from one such configuration to another is what we would intuitively refer to as a "move" for the moment. Notice that at an odd step i, the dual variable Z_B doesn't change while Z_A does (see Theorem 3). The constraint equation in this step is $Z_{A,i-1} \otimes \mathbb{I}_M \geq U_i^\dagger \left(Z_{A,i} \otimes \mathbb{I}_M\right) U_i$. The honest state can be expressed as $|\psi_i\rangle = U_i\, |\psi_{i-1}\rangle$ but this acts on the complete $\mathcal{A} \otimes \mathcal{M} \otimes \mathcal{B}$ space. Applying the aforesaid method of removing the basis information using the prob method, and appending the fixed $Z_{B,i-1} = Z_{B,i}$, we conclude that $\mathrm{prob}(Z_{A,i-1} \otimes \mathbb{I}_M \otimes Z_{B,i}, |\psi_{i-1}\rangle) \to \mathrm{prob}(Z_{A,i} \otimes \mathbb{I}_M \otimes Z_{B,i}, |\psi_i\rangle)$ should constitute an "allowed move" as it is simply re-expressing the dual SDP in a basis independent form. For the dual, we are assuming the protocol is given to us, i.e. U_i (unitary operations), $\Pi_{A/B}$ (measurements) and $|\psi_0\rangle$ (initial state) are specified, and we have to find the appropriate Zs. However, when we discuss the notion of an "allowed move" we are moving towards a framework which will free us from discussing a specific protocol. This motivates the following definitions.

Definition 7 (EBM line transition)**.** Let $g, h : [0, \infty) \to [0, \infty)$ be two functions with finite supports. The line transition $g \to h$ is EBM if there exist two matrices $0 \leq G \leq H$ and a (not necessarily normalised) vector $|\psi\rangle$ such that $g = \mathrm{prob}[G, \psi]$ and $h = \mathrm{prob}[H, \psi]$.

Definition 8 (EBM transition)**.** Let $p, q : [0, \infty) \times [0, \infty) \to$ be two functions with finite supports. The transition $p \to q$ is an

- EBM horizontal transition if for all $y \in [0, \infty)$, $p(., y) \to q(., y)$ is an EBM line transition, and
- EBM vertical transition if for all $x \in [0, \infty)$, $p(x, .) \to q(x, .)$ is an EBM line transition.

It turns out that when one writes the dual, the order of the constraints gets inverted, i.e. the condition associated with the final measurements and states appears first and the condition associated with the initial state appears in the end. We expect the final state to be like an EPR state and, intuitively, expect two points (in terms of the 2–variate function as described earlier) to be associated with it (TODO: check if it is indeed intuitive). This makes it plausible that we will start with two points in the dual when it is expressed in the aforementioned basis independent way. The initial state of the protocol is unentangled. This we expect should correspond to a single point. This helps us accept that we end with a single point in basis independent expression of the dual. The rules for moving these points must be related to the dual constraints. We already formalised these conditions into EBM transitions. The notation

$$[x_g, y_g](x, y) = \begin{cases} 1 & x_g = x \text{ and } y_g = y \\ 0 & \text{else} \end{cases}$$

will be useful for formalising the complete description into what Mochon dubbed an "Expressible by Matrices" (Time Dependent) point game.

Definition 9 (EBM point game)**.** An EBM point game is a sequence of functions $\{p_0, p_1, \ldots, p_n\}$ with finite support such that

- $p_0 = 1/2[0,1] + 1/2[1,0]$
- for all even i, $p_i \to p_{i+1}$ is an EBM vertical transition;
- for all odd i, $p_i \to p_{i+1}$ is an EBM horizontal transition;
- $p_n = 1[\beta, \alpha]$ for some $\alpha, \beta \in [0,1]$. We call $[\beta, \alpha]$ the final point of the EBM point game.

Since we started with a WCF protocol, considered its dual and re-expressed it as a TDPG (which is just a basis independent representation), the following shouldn't come as a surprise.

Proposition 10 (WCF $\implies$ EBM point game)**.** *Given a WCF protocol with cheating probabilities P_A^* and P_B^*, along with a positive real number $\delta > 0$, there exists an EBM point game with final point $[P_B^* + \delta, P_A^* + \delta]$.*

What is slightly more non-trivial is that given this TDPG one can construct a WCF protocol. This means that by using only "allowed moves" one can be sure that there exists a corresponding sequence of unitaries U_i, the measurements $\Pi_{A/B}$ and the initial state $|\psi_0\rangle$ complemented by the dual variables $Z_{A,i}$ and $Z_{B,i}$ which certify the bias corresponding to the coordinates of the final point in the point game. This establishes the equivalence between TDPGs and WCF protocols. The precise statement is as follows.

Theorem 11 (EBM to protocol)**.** *Given an EBM point game with final point $[\beta, \alpha]$, there exists a WCF protocol and two dual feasible points proving that the optimal cheating probabilities are $P_A^* \le \alpha$ and $P_B^* \le \beta$.*

Our first contribution is related to this part. We construct protocols from EBM point games in a slightly different way which results in two important improvements. The first improvement makes the protocol more practical as the message register gets decoupled from Alice and Bob's registers after each round. (In Mochon/Aharonov et al's version the message register is highly entangled and stays that way until the very end.) The second improvement is due to the addition of a cheat detection measurement at every round (similar to Mochon's improved Dip Dip Boom protocol) which allows us to consider certain matrices with infinite eigenvalues in a well defined way. These pave the path for converting the bias 1/10 point game (due to Mochon; will be introduced later) into a protocol.

2.3 (Time Dependent) Point Games with valid transitions

While the problem has been simplified by the removal of the basis information, it is still hard to know which transitions are allowed, i.e. are EBM transitions. This is because finding the matrices certifying that a transition is EBM is not easy. The goal of this section is to find another criterion for establishing that a transition is EBM. This criterion is at the heart of coin flipping. It would turn out that the set of EBM functions (closely related to EBM transitions) form a closed convex cone. The dual of this cone happens to be the set operator monotone functions (these are a generalisation of monotone functions, $x \ge y \implies f(x) \ge f(y)$, to $X \ge Y \implies f(X) \ge f(Y)$ where X and Y are now matrices; to appreciate the non-triviality note that $f(x) = x^2$ is not an operator monotone[2];). These functions have a very nice and simple characterisation. This is what leads to the key simplification for WCF. To be able to harness this, one can use the known fact that for a closed convex cone, the dual of the dual is the original cone itself (also called bi-dual). So this dual of operator monotone functions, i.e. the bi-dual of the cone of EBM functions which equals the cone of EBM functions. The dual of operator monotone functions has an easy description because operator monotone functions have an easy description. Combining these, one obtains an easy characterisation of EBM functions which allows one to construct interesting WCF protocols. The catch is that we establish that the two cones, the cone of EBM functions and the dual of the cone of operator monotone functions, are the same but given a point in the second cone we don't have a recipe for finding the matrices certifying it is an EBM. Without the matrices we can't implement the protocol even though we know the matrices must exist as the cones are the same. These notions will now be formalised.

[2] try $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \ge \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

2.3.1 Formalising the equivalence between transitions and functions

Working with functions instead of transitions will be rather useful as will be evident from the next subsection.

Definition 12 (K, EBM functions)**.** A function $a : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with finite support is an *EBM function* if the line transition $a^- \to a^+$ is EBM, where $a^+ : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $a^- : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ denote, respectively, the positive and the negative part of a ($a = a^+ - a^-$).

We denote by K the set of EBM functions.

Definition 13 (K_Λ, EBM functions on $[0, \Lambda]$)**.** For any finite Λ, a function $a : [0, \Lambda) \to \mathbb{R}$ with finite support is an *EBM function with support on* $[0, \Lambda]$ if the line transition $a^- \to a^+$ is EBM with its spectrum in $[0, \Lambda]$, where $a^- : [0, \Lambda) \to \mathbb{R}_{\geq 0}$ and $a^+ : [0, \Lambda) \to \mathbb{R}_{\geq 0}$ denote, respectively, the positive and the negative part of a.

We denote the set of EBM functions with support on $[0, \Lambda]$ by K_Λ.

It is evident that if the functions g, h denoting the transition $g \to h$ have no common support then the function description uniquely captures the said transition. In this section we will restrict to such transitions and will therefore use them interchangeably. In later sections we will revisit this notion.

To be able to talk about different characterisations of EBM functions it is useful to abstract it (the characterisation) into a property $\mathcal{P}$ which the function must satisfy. Using this we can define games which use these $\mathcal{P}$ functions. This is done to be able to handle subtleties which arise in proving that the set of EBM functions is the same as the set of $\mathcal{P}$ functions for specific $\mathcal{P}$s.

Definition 14 (Horizontal and vertical $\mathcal{P}$-functions)**.** A $\mathcal{P}$-function $a : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a function with finite support that has the property $\mathcal{P}$.

A function $t : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a

- *horizontal $\mathcal{P}$-function* if for all $y \geq 0$, $t(., y)$ is a $\mathcal{P}$-function;
- *vertical $\mathcal{P}$-function* if for all $x \geq 0$, $t(x, .)$ is a $\mathcal{P}$-function.

Definition 15 (point games with $\mathcal{P}$-functions)**.** A point game with $\mathcal{P}$-functions is a set $\{t_1, \dots, t_n\}$ of n $\mathcal{P}$-functions alternatively horizontal and vertical such that

- $\frac{1}{2}[0, 1] + \frac{1}{2}[1, 0] + \sum_{i=1}^{n} t_i = [\beta, \alpha]$;
- $\forall j \in \{1, \dots, n\}$, $\frac{1}{2}[0, 1] + \frac{1}{2}[1, 0] + \sum_{i=1}^{j} t_i \geq 0$.

We call $[\beta, \alpha]$ the final point of the game.

We note the following before looking at $\mathcal{P}$ functions in more detail.

Lemma 16 (point game with EBM functions $\implies$ point game with EBM transitions)**.** *Given a point game with n EBM functions and final point $[\beta, \alpha]$ we can construct a point game with n EBM transitions and final point $[\beta, \alpha]$.*

2.3.2 Operator monotone functions and valid functions

This discussion is essential to understand our second contribution. The set of EBM functions forms a convex cone. To see this we recall the definition of a convex cone.

Definition 17 (convex cone)**.** A set C in a vector space V is a cone if for all $x \in C$ and for all $\lambda > 0$, $\lambda x \in C$. It is convex if for all $x, y \in C$, $x + y \in C$.

Noting that the state $|\psi\rangle$ in the definition of an EBM function (which in turn invokes an EBM transition) is unnormalised the set of EBM functions is easily seen to form a cone. By taking a direct sum one can establish convexity as well. The vector space of interest here will be given by the span of the basis $\{[x_g]\}_{x_g \in [0,\infty)}$ where $[x_g](x) = \delta_{x_g, x}$. The notation is similar to the one introduced earlier. We will use it shortly.

Lemma 18. *K is a convex cone. Also, for any $\Lambda \in (0, \infty)$, K_Λ is a convex cone.*

To establish an alternative characterisation of the cone of EBM functions we need to define what is called a dual cone.

Definition 19 (dual cone)**.** Let C be a cone in a normed vector space V. We denote by V' the space of continuous linear functionals from V to $\mathbb{R}$. The dual cone of a set $C \subseteq V$ is

$$C^* = \{\Phi \in V' | \forall a \in C, \Phi(a) \geq 0\}.$$

For our purpose linear functionals can be thought of simply as functions which map objects in the cone to a non-negative real number with the added property of being linear in its argument.

We now formally define operator monotone functions.

Definition 20 (operator monotone functions)**.** A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is operator monotone if for all $0 \leq X \leq Y$ we have $f(X) \leq f(Y)$.

Definition 21 (operator monotone functions on $[0, \Lambda]$)**.** A function $f : [0, \Lambda] \to \mathbb{R}$ is operator monotone on $[0, \Lambda]$ if for all $0 \leq X \leq Y$ with spectrum in $[0, \Lambda]$ we have $f(X) \leq f(Y)$.

The pivotal result of this (sub)section is the equivalence between the cone of operator monotone functions and the dual cone of EBM functions.

Lemma 22. *$\Phi \in K^*$ if and only if f_Φ is operator monotone in $[0, \infty]$. Also, for any $\Lambda \in (0, \infty)$, $\Phi \in K^*_\Lambda$ if and only if f_Φ is operator monotone on $[0, \Lambda]$.*

(NB: We need to use the bijection between Φ (a linear functional from $V \to \mathbb{R}$) and a function on reals (from $\mathbb{R} \to \mathbb{R}$) given by the identification $f_\Phi(x) = \Phi([x])$ to make such a statement)

The proof of this crucial result is not too hard (almost trivial in one direction) and follows from the respective definitions with some work for unpacking. What makes this connection interesting is the following beautiful characterisation of operator monotone functions introduced by Löwner (in 1934, see [10]).

Lemma 23 (characterisation of operator monotone functions)**.** *Any operator monotone function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ can be written as*

$$f(x) = c_0 + c_1 x + \int_0^\infty \frac{\lambda x}{\lambda + x} d\omega(\lambda)$$

for a measure ω satisfying $\int_0^\infty \frac{\lambda}{1+\lambda} d\omega(\lambda) < \infty$.

Lemma 24 (characterisation of operator monotone functions on $[0, \Lambda]$)**.** *Any operator monotone function $f : [0, \Lambda] \to \mathbb{R}$ can be written as*

$$f(x) = c_0 + c_1 x + \int \frac{\lambda x}{\lambda + x} d\omega(\lambda)$$

with the integral ranging over $\lambda \in (-\infty, -\Lambda) \cup (0, \infty)$ satisfying $\int \frac{\lambda}{1+\lambda} d\omega(\lambda) < \infty$ where ω is a measure.

(NB: I added the integrability part. Should be there for the limiting cases to match but it wasn't there in Ahoronov et al's paper)

As will become clear when we discuss the dual of the cone of operator monotones, it suffices to consider operator monotones of the form $\lambda x / (\lambda + x)$ (which basically is because ω is a measure).

So far the statements from Aharonov et al's paper were made in the same order as they had originally appeared. We will now re-order these a little with an eye on our end-goal (as opposed to the one of Mochon/Aharonov). It is known that the bi-dual of a cone is the closure of the cone we started with.

Fact 25. *Let $C \subseteq V$ be a convex cone, then $C^{**} = cl(C)$ where C^* is the dual cone of C.*[3]

The astute reader would've guessed where we are going with this discussion. We define, from hindsight, the bi-dual of EBM functions to be the cone of valid functions. Since the dual of EBM functions has an easy characterisation, the bi-dual also has an easy characterisation which is why we are interested in it.

Definition 26 (Λ valid functions)**.** A function $a : [0, \Lambda] \to \mathbb{R}$ with finite support on $[0, \Lambda]$ is Λ valid if $a \in K^{**}_\Lambda$.

To be able to use the aforementioned fact we note that the cone of interest, the cone of EBM functions, is closed. As one can imagine, proving this is easier if the matrices involved have a bounded spectrum. We consider only these for now. This means that the cone of valid functions is the same as the cone of EBM functions. We state these precisely in the following statements.

Lemma 27. *For $\Lambda \in (0, \infty)$, K_Λ is closed (which implies $K^{**}_\Lambda = K_\Lambda$).*

Corollary 28. *For $\Lambda \in (0, \infty)$, $K_\Lambda = \{a \in V | \forall \Phi \in K^*_\Lambda, \Phi(a) \geq 0\}$.*

Corollary 29 (EBM on $[0, \Lambda]$ is equivalent to Λ valid)**.** *A function $a : [0, \Lambda] \to \mathbb{R}$ with finite support on $[0, \Lambda]$ is EBM on $[0, \Lambda]$ if and only if $\sum_x a(x) = 0$, $\sum_x x a(x) \geq 0$ and $\forall \lambda \in (-\infty, -\Lambda] \cup (0, \infty)$, $\sum_x \frac{\lambda x}{\lambda + x} a(x) \geq 0$.*

In the last statement, the characterisation of operator monotone functions was used which we introduced earlier. Note that all the statements made here assume that the matrices used in EBM functions have a finite spectrum. Our EMA algorithm heavily relies on this part of the analysis which is due to Aharonov et al.

It is worth pointing out that Mochon outlines this scheme used by Aharonov et al but himself proceeds by using matrix perturbation theory for proving a similar result.

[3] See [Boyd and Vandenberghe 2004] for proofs of these facts.

2.3.3 Strictly valid functions are EBM functions

To be able to simplify the conditions one needs to check, it is useful to relax the condition on the spectrum of the matrices involved. This is evident from range of λ one needs to use in the characterisation of operator monotone functions (compare Lemma 24 and Lemma 23).

It is easy to describe the interior of the dual of a cone. It is also possible to relate the interior with the closure of the cone, but in finite dimensions. This reasoning fails for infinite dimensions. They still serve as motivation for the definition of valid and strictly valid functions.

Fact 30. *Let C be a convex set, then $int(C) = int(cl(C))$.*

Fact 31. *Let C be a cone in the finite-dimensional vector space V, then $int(C^*) = \{\Phi \in V' | \forall a \in C - \{0\}, \Phi(a) > 0\}$.*

Definition 32 (valid function)**.** A function $a : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with finite support is valid if for every operator monotone function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ we have $\sum_{x \in \text{supp}(h)} f(x)a(x) \geq 0$.

Definition 33 (strictly valid function)**.** A function $a : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with finite support is strictly valid if for every non-constant operator monotone function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ we have $\sum_{x \in \text{supp}(a)} f(x)a(x) > 0$.

One can use the characterisation of operator monotone functions to explicitly characterise the set of valid and strictly valid functions.

Lemma 34. *Let $a : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a function with finite support such that $\sum_x a(x) = 0$. The function a is a strictly valid function if and only if for all $\lambda > 0$, $\sum_x \frac{-a(x)}{\lambda + x} > 0$ and $\sum_x x.a(x) > 0$.*
(We added the last condition else the merge (discussed later) becomes a strictly valid function but it can be shown that no bounded matrices exist for which it is EBM.)

The function a is valid if and only if for all $\lambda > 0$, $\sum_x \frac{-a(x)}{\lambda + x} \geq 0$ and $\sum_x x.a(x) \geq 0$.

The set of strictly valid functions can be shown to also be Λ valid for some finite Λ. This means that it would also be EBM on $[0, \Lambda]$ which in turn means it would be an EBM function. We hence have the following.

Lemma 35. *Any strictly valid function is an EBM function.*

2.3.4 From valid functions to EBM functions

If we construct a point game with valid functions we can convert it into a game with EBM functions with an arbitrarily small overhead on the bias. The trick is to raise the coordinates all the final points (ones with positive weight) a little at each step, to convert a valid function into a strictly valid function.

Theorem 36 (valid to EBM)**.** *Given a point game with $2m$ valid functions and final point $[\beta, \alpha]$ and any $\epsilon > 0$, we can construct a point game with $2m$ EBM functions and final point $[\beta + \epsilon, \alpha + \epsilon]$.*

Lemma 37. *Fix $\epsilon > 0$. Given a point game with $2m$ valid functions and final point $[\beta, \alpha]$ we can construct a point game with $2m$ strictly valid functions and final point $[\beta + \epsilon, \alpha + \epsilon]$.*

2.3.5 Examples of valid line transitions

We go back to transitions to discuss some simple valid and strictly valid line transitions which are defined similar to the corresponding functions.

Definition 38 (Valid and strictly valid line transitions)**.** Let $g, h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be two functions with finite support. The transition $g \to h$ is valid (resp., strictly valid) if the function $h - g$ is valid (resp., strictly valid).

We focus our attention to the simplest cases. The first is to increase the coordinate a point. The second considers the case of merging two points into one. The third is about splitting a single point into two.

Example 39 (Point raise)**.** $p[x_g] \to p[x_h]$ with $x_h \geq x_g$.

Example 40 (Point merge)**.** $p_{g_1}[x_{g_1}] + p_{g_2}[x_{g_2}] \to (p_{g_1} + p_{g_2})[x_h]$ with $x_h \geq \frac{p_{g_1} x_{g_1} + p_{g_2} x_{g_2}}{p_{g_1} + p_{g_2}}$.

Example 41 (Point split)**.** $p_g[x_g] \to p_{h_1}[x_{h_1}] + p_{h_2}[x_{h_2}]$ with $p_g = p_{h_1} + p_{h_2}$ and $\frac{p_g}{x_g} \geq \frac{p_{h_1}}{x_{h_1}} + \frac{p_{h_2}}{x_{h_2}}$.

The merge and split can be generalised to many points and be shown to have the same form.

2.4 TIPGs.

Mochon's Dip Dip Boom protocol, the one with bias 1/6, can be expressed already as a (time dependent) point game. However, it is possible to simplify the point game formalism even further and it is in this simplified formalism Mochon constructs his family of point games that achieve an arbitrarily small bias. Instead of worrying about the entire sequence of horizontal and vertical transitions, one can focus on just two functions as described below.

Definition 42 (TIPG)**.** A TIPG is a valid horizontal function a and a valid vertical function b such that

$$a + b = 1[\beta, \alpha] - \frac{1}{2}[0, 1] - \frac{1}{2}[1, 0]$$

for some $\alpha, \beta > 1/2$. We call the point $[\beta, \alpha]$ the final point of the game.

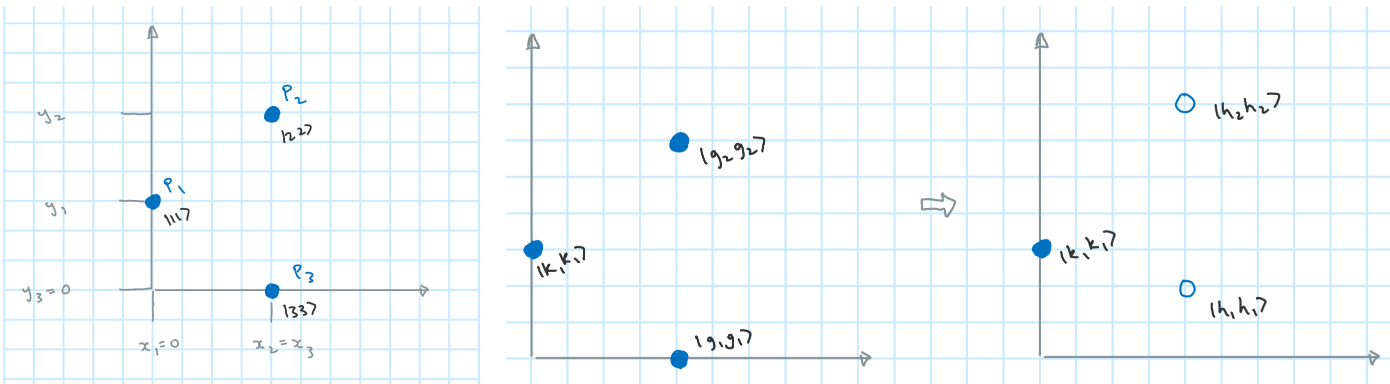
The main difference here is that we don't worry about the sequence in which one must apply the transitions to obtain the final configuration. This justifies the name TIPG which stands for a Time Independent Point Game. It is not too hard to see that if we have a valid point game we can combine the horizontal functions and the vertical functions to obtain a and b. It is a little counter-intuitive in fact to learn that one can convert a TIPG into a valid (time dependent) point game with an arbitrarily small cost on the bias. It is counter-intuitive because it is not clear that one can flesh out a time ordered sequence as one can, and in fact does for Mochon's point games, run into causal loops that is you expect a point to be present to create another point which in turn is required to produce the first point. The trick that is used to fix this problem is known as the "catalyst state". One deposits a little bit of weight wherever there's negative weight for a, for instance, and then one can implement a scaled down round of a and b. The scaling is proportional to the weight that is placed to start with. Repeating this procedure multiple times yields the required final state along with the "catalyst state" which stays unchanged. Absorbing the catalyst state leads to a small increase in the bias. The number of rounds increases with how small one wants this increase in bias to be.

Theorem 43 (TIPG to valid point games)**.** *Given a TIPG with a valid horizontal function a and a valid vertical function b such that $a + b = 1[\beta, \alpha] - \frac{1}{2}[0, 1] - \frac{1}{2}[1, 0]$, we can construct, for all $\epsilon > 0$, a valid point game with final point $[\beta + \epsilon, \alpha + \epsilon]$ where the number of transitions depends on ϵ.*

It is important to state that the conversion from a TIPG to a valid (time dependent) point game, TDPG, is easy and explicit.

A word about resource usage. The size (dimension) of the physical system we use will depend on the number of points involved in the point games linearly. The number of rounds on the other hand needs to be calculated with more care as it depends on the choice of the catalyst state. These calculations with respect to Mochon's game and in general have been performed by Aharonov et al in their article and we will not be able to discuss it here.

We have stated enough results to be able to commence the discussion of our work.



(a) Frame of a TDPG

(b) The points which are unchanged from one frame to another are labelled by $\{k_i\}$. Among the points that change, the initial ones are labelled by $\{g_i\}$ and the final ones by $\{h_i\}$.

Figure 1: Illustrations for the Canonical Form

Part I

Bias $1/10$

3 TDPG → Explicit Protocol, Framework (TEF)

We strongly recommend that the reader looks at the third section titled "The illustrated guide to point games" from Mochon's [1] article, if (s)he hasn't already, before proceeding.

3.1 Motivation and Conventions

We wish to construct a protocol such that its dual matches a given TDPG. The main difference in our construction, compared to the one used by Aharonov et al and Mochon, is the introduction of a message register that decouples after each round and of suitably adapted projectors. Consequently, the non-trivial constraint that the dual matrices must satisfy would be similar to, but not quite the same as, the EBM condition.

Keep Definition 6 in the mind. Intuitively, the most natural way of constructing Zs and a $|\psi\rangle$ given an arbitrary frame (think of a TDPG as a sequence of frames) is to construct an entangled state that encodes the weight and define Zs to contain the coordinates corresponding to the weight. Let us make this idea more precise.

Definition (Canonical Form)**.** The tuple $(|\psi\rangle, Z^A, Z^B)$ is said to be in the Canonical Form with respect to a set of points in a frame of a TDPG if (see Figure 1a) $|\psi\rangle = \sum_i \sqrt{P_i}\,|ii\rangle_{AB} \otimes |.\rangle_M$, $Z^A = (\sum x_i |i\rangle\langle i|_A) \otimes |.\rangle\langle .|_M$ and $Z^B = (\sum y_i |i\rangle\langle i|_B) \otimes |.\rangle\langle .|_M$ where $|.\rangle_M$ represent extra uncoupled registers which might be present.

It is easy to see that the 'label' $|ii\rangle$ will correspond to a point with coordinates x_i, y_i and weight P_i in the frame (see Definition 6). It is tempting to imagine that we systematically construct, from each frame of a TDPG, a canonical form of $|\psi\rangle$ s and Zs. The unitaries can be deduced from the evolution of $|\psi\rangle$. This approach has two problems, (1) it doesn't manifestly mean that the unitaries would be decomposable into moves by Alice and Bob who communicate only through the messaging register and (2) the constraints imposed consecutive Zs, of the form $Z_{n-1} \otimes \mathbb{I} \geq U_n^\dagger (Z_n \otimes \mathbb{I}) U_n$, are not satisfied in general. This construction will ensure these issues are dissolved.

The framework will output variables in the reverse time convention indexed as, for example, $|\psi_{(i)}\rangle$, $Z_{(i)}$, $U_{(i)}$. The variables at the i^{th} step of the protocol (which follows the forward time convention) would be given by $|\psi_i\rangle = |\psi_{(N-i)}\rangle$, $Z_i = Z_{(N-i)}$ and $U_i = U_{(N-i)}^\dagger$. Note that the results so obtain extend naturally to the case where U_i may not be unitary and contains projections.

Basic Moves Work Out of the Box

Recall the three basic moves of a TDPG were given by

1. Raise: $p_1[x, y] \to p_1[x', y]$ s.t. $x' \geq x$.

2. Merge: $p_1[x_1, y] + p_2[x_2, y] \to p_1 + p_2 \left[\frac{p_1 x_1 + p_2 x_2}{p_1 + p_2}, y\right]$

3. Split: $(p_1 + p_2)\left[\left(\frac{p_1 w_1 + p_2 w_2}{p_1 + p_2}\right)^{-1}, y\right] \to p_1[x_1, y] + p_2[x_2, y]$ where $w_1 = 1/p_1$ and $w_2 = 1/p_2$.

We will construct the explicit Unitaries that implement these moves which in turn (when generalised to n points) are enough to construct the former best known protocol from its TDPG. Note, however, that these moves do not exhaust the set of moves and more advanced moves will be constructed to go beyond the 1/6 limit.

3.2 The Framework

Intuition

Imagine a canonical description is given. Let the labels on the points one wants to transform be indexed by $\{g_i\}$ and let us also assume that one wishes to apply an x-transition (meaning Alice will perform the non-trivial step). Let the labels of the points that one wishes to leave untouched be given by $\{k_i\}$ (see Figure 1b). We can write the state as

$$\left|\psi_{(1)}\right\rangle = \left(\sum_i \sqrt{p_{g_i}} \left|g_i g_i\right\rangle_{AB} + \sum_i \sqrt{p_{k_i}} \left|k_i k_i\right\rangle_{AB}\right) \otimes \left|m\right\rangle_M .$$

We want Bob to send his part of $|g_i\rangle$ states to Alice through the message register. One way is that he conditionally swaps to obtain the following

$$\left|\psi_{(2)}\right\rangle = \sum_i \sqrt{p_{g_i}} \left|g_i g_i\right\rangle_{AM} \otimes \left|m\right\rangle_B + \sum_i \sqrt{p_{k_i}} \left|k_i k_i\right\rangle_{AB} \otimes \left|m\right\rangle_M .$$

This should at most force all the points to align along the y-axis but no non-trivial constraint should arise (speaking with hindsight). Let $\{h_i\}$ be the labels of the new points after the transformation. We will assume that h and g index orthonormal vectors. Alice can update the probabilities and labels by locally performing a unitary to obtain

$$\left|\psi_{(3)}\right\rangle = \sum_i \sqrt{p_{h_i}} \left|h_i h_i\right\rangle_{AM} \otimes \left|m\right\rangle_B + \sum_i \sqrt{p_{k_i}} \left|k_i k_i\right\rangle_{AB} \otimes \left|m\right\rangle_M .$$

It is precisely this step which yields the non-trivial constraint. Bob must now accept this by 'unswapping' to get

$$\left|\psi_{(4)}\right\rangle = \left(\sum_i \sqrt{p_{h_i}} \left|h_i h_i\right\rangle_{AB} + \sum_i \sqrt{p_{k_i}} \left|k_i k_i\right\rangle_{AB}\right) \otimes \left|m\right\rangle_M$$

which leaves Bob's Z in essentially the standard form (we will see). Remember that in the actual protocol the sequence will get reversed as described above.

Note that we add a few extra frames to the final TDPG to go from a given frame to the next of the initial TDPG. This is irrelevant as the bias stays the same but we mention it to avoid confusion.

Formal Description and Proofs

1. **First frame.**

$$\begin{aligned}
\left|\psi_{(1)}\right\rangle &= \left(\sum_i \sqrt{p_{g_i}} \left|g_i g_i\right\rangle_{AB} + \sum_i \sqrt{p_{k_i}} \left|k_i k_i\right\rangle_{AB}\right) \otimes \left|m\right\rangle_M \\
Z^A_{(1)} &= \sum_i x_{g_i} \left|g_i\right\rangle \left\langle g_i\right|_A + \sum_i x_{k_i} \left|k_i\right\rangle \left\langle k_i\right|_A \\
Z^B_{(1)} &= \sum_i y_{g_i} \left|g_i\right\rangle \left\langle g_i\right|_B + \sum_i y_{k_i} \left|k_i\right\rangle \left\langle k_i\right|_B .
\end{aligned}$$

Proof. Follows from the assumption of starting with a Clean Canonical Form. □

2. **Bob sends to Alice.** With $y \geq \max\{y_{g_i}\}$ the following is a valid choice

$$\begin{aligned}
\left|\psi_{(2)}\right\rangle &= \sum_i \sqrt{p_{g_i}} \left|g_i g_i\right\rangle_{AM} \otimes \left|m\right\rangle_B + \sum_i \sqrt{p_{k_i}} \left|k_i k_i\right\rangle_{AB} \otimes \left|m\right\rangle_M \\
U^{(1)} &= U^{\mathrm{SWP}\{\vec{g},m\}}_{BM} \\
Z^A_{(2)} &= Z^A_{(1)} \\
Z^B_{(2)} &= y\mathbb{I}^{\{\vec{g},m\}}_B + \sum_i y_{k_i} \left|k_i\right\rangle \left\langle k_i\right|_B .
\end{aligned}$$

Proof. We have to prove: (1) $\left|\psi_{(2)}\right\rangle = U^{(2)}\left|\psi_{(1)}\right\rangle$ and (2) $U^{(1)\dagger}\left(Z^B_{(2)}\otimes\mathbb{I}_M\right)U^{(1)} \geq \left(Z^B_{(1)}\otimes\mathbb{I}_M\right)$.
(1) It follows trivially from the defining action of $U^{(1)}$.
(2) For convenience, let momentarily $U = U^{(1)}$ and note that $U^\dagger = U$ so that we can write

$$\begin{aligned} U\left(Z^B_{(2)}\otimes\mathbb{I}_M\right)U &= y\left(U\left(\mathbb{I}_B^{\{\vec{g},m\}}\otimes\mathbb{I}_M^{\{\vec{g},m\}}\right)U + U\underbrace{\left(\mathbb{I}_B^{\{\vec{g},m\}}\otimes\mathbb{I}_M^{\{\vec{k},\vec{h}\}}\right)}_{\text{outside } U\text{'s action space}}U\right) + U\underbrace{\left(\sum y_{k_i}\left|k_i\right\rangle\left\langle k_i\right|\otimes\mathbb{I}\right)}_{\text{outside } U\text{'s action space}}U \\ &= Z_{(2)}\otimes\mathbb{I}_M \geq Z_{(1)}\otimes\mathbb{I}_M \end{aligned}$$

so long as $y \geq y_{g_i}$ which is guaranteed by the choice of y. ☐

3. **Alice's non-trivial step.** We claim that the following is a valid choice,

$$\begin{aligned} \left|\psi_{(3)}\right\rangle &= \sum_i \sqrt{p_{h_i}}\left|h_ih_i\right\rangle_{AM}\otimes\left|m\right\rangle_B + \sum_i \sqrt{p_{k_i}}\left|k_ik_i\right\rangle_{AB}\otimes\left|m\right\rangle_M \\ E^{(2)}U^{(2)} &= E^{(2)}\left(\left|w\right\rangle\left\langle v\right| + \text{other terms acting on span}\{\left|h_ih_i\right\rangle,\left|g_ig_i\right\rangle\}\right)_{AM} \\ Z^A_{(3)} &= \sum_i x_{h_i}\left|h_i\right\rangle\left\langle h_i\right| + \sum_i x_{k_i}\left|k_i\right\rangle\left\langle k_i\right| \\ Z^B_{(3)} &= Z^B_{(2)} \end{aligned}$$

where

$$\left|v\right\rangle = \frac{\sum_i\sqrt{p_{g_i}}\left|g_ig_i\right\rangle}{\sqrt{\sum_i p_{g_i}}},\ \left|w\right\rangle = \frac{\sum_i\sqrt{p_{h_i}}\left|h_ih_i\right\rangle}{\sqrt{\sum_i p_{h_i}}},\ E^{(2)} = \left(\sum\left|h_i\right\rangle\left\langle h_i\right|_A + \sum\left|k_i\right\rangle\left\langle k_i\right|_A\right)\otimes\mathbb{I}_M$$

subject to the condition

$$\sum x_{h_i}\left|h_ih_i\right\rangle\left\langle h_ih_i\right| \geq \sum x_{g_i}E^{(2)}U^{(2)}\left|g_ig_i\right\rangle\left\langle g_ig_i\right|U^{(2)\dagger}E^{(2)} \tag{1}$$

and of course the conservation of probability, viz. $\sum p_{g_i} = \sum p_{h_i}$.

Proof. We must show that (1) $E^{(2)}\left|\psi_{(3)}\right\rangle = U^{(2)}\left|\psi_{(2)}\right\rangle$ and (2) $Z^A_{(3)}\otimes\mathbb{I}_M \geq E^{(2)}U^{(2)}\left(Z^A_{(2)}\otimes\mathbb{I}_M\right)U^{(2)\dagger}E^{(2)}$
(1) Observing $E^{(2)}\left|\psi_{(3)}\right\rangle = \left|\psi_{(3)}\right\rangle$ the statement holds almost trivially by construction of $U^{(2)}$.
(2) Consider the space $\mathcal{H} = \text{span}\left\{\left|g_1g_1\right\rangle,\left|g_2g_2\right\rangle\ldots,\left|h_1h_1\right\rangle,\left|h_2,h_2\right\rangle\ldots\right\}$. We will separate all expressions (they are nearly diagonal) into the $\mathcal{H}$ space (which gets non-diagonal) and the rest. We start with the RHS,

$$Z^A_{(2)}\otimes\mathbb{I}_M = \underbrace{\sum x_{g_i}\left|g_ig_i\right\rangle\left\langle g_ig_i\right|}_{\text{I}} + \sum x_{g_i}\left|g_i\right\rangle\left\langle g_i\right|\otimes\left(\mathbb{I}-\left|g_i\right\rangle\left\langle g_i\right|\right) + \sum x_{k_i}\left|k_i\right\rangle\left\langle k_i\right|\otimes\mathbb{I},$$

where only term I is in the operator space spanned by $\mathcal{H}$. Note that all the terms are still diagonal. Next consider the LHS, without the Us,

$$Z^A_{(3)}\otimes\mathbb{I}_M = \underbrace{\sum x_{h_i}\left|h_ih_i\right\rangle\left\langle h_ih_i\right|}_{\text{I}} + \sum x_{h_i}\left|h_i\right\rangle\left\langle h_i\right|\otimes\left(\mathbb{I}-\left|h_i\right\rangle\left\langle h_i\right|\right) + \sum x_{k_i}\left|k_i\right\rangle\left\langle k_i\right|\otimes\mathbb{I},$$

which also has only term I in the $\mathcal{H}$ operator space. Consequently, only on these will U have a non-trivial action. Let us first evaluate the non-$\mathcal{H}$ part where we only need to apply the projector. The result after separating equations where possible is

$$\begin{aligned} \sum x_{h_i}\left|h_i\right\rangle\left\langle h_i\right|\otimes\left(\mathbb{I}-\left|h_i\right\rangle\left\langle h_i\right|\right) &\geq 0 \\ \sum(x_{k_i}-x_{k_i})\left|k_i\right\rangle\left\langle k_i\right|\otimes\mathbb{I} &\geq 0 \end{aligned}$$

which essentially only implies

$$x_{h_i} \geq 0.$$

Finally the non-trivial part yields

$$\sum x_{h_i}\left|h_ih_i\right\rangle\left\langle h_ih_i\right| \geq \sum x_{g_i}EU\left|g_ig_i\right\rangle\left\langle g_ig_i\right|U^\dagger E$$

which completes the proof. ☐

4. **Bob accepts Alice's change.** The following is valid.

$$
\begin{aligned}
\left|\psi_{(4)}\right\rangle &= \left(\sum_i \sqrt{p_{h_i}}\left|h_i h_i\right\rangle_{AB} + \sum_i \sqrt{p_{k_i}}\left|k_i k_i\right\rangle_{AB}\right)\otimes|m\rangle_M \\
E^{(3)}U^{(3)} &= E^{(3)}U_{BM}^{\mathrm{SWP}\{\vec{h},m\}} \\
Z_{(4)}^A &= Z_{(3)}^A \\
Z_{(4)}^B &= y\sum_i |h_i\rangle\langle h_i| + \sum_i y_{k_i}|k_i\rangle\langle k_i|_B
\end{aligned}
$$

where $E^{(3)} = \left(\sum|h_i\rangle\langle h_i| + \sum|k_i\rangle\langle k_i|\right)_B \otimes \mathbb{I}_M$.

Proof. We have to prove: (1) $E^{(3)}\left|\psi_{(4)}\right\rangle = U^{(3)}\left|\psi_{(3)}\right\rangle$ and (2) $Z_{(4)}^B\otimes\mathbb{I}_M \geq E^{(3)}U^{(3)}\left(Z_{(3)}^B\otimes\mathbb{I}_M\right)U^{(3)\dagger}E^{(3)}$.
(1) This can be proven again, by a direct application of EU (defined to be $E^{(3)}U^{(3)}$ for the proof).
(2) Note that

$$
\begin{aligned}
EU\left(\mathbb{I}_B^{\{\vec{g},m\}}\otimes\mathbb{I}_M^{\{\vec{h},\vec{g},\vec{k},m\}}\right)U^\dagger E &= EU\left(\mathbb{I}_B^{\{m\}}\otimes\mathbb{I}_M^{\{\vec{h},\vec{g},\vec{k},m\}}\right)U^\dagger E + E\left(\mathbb{I}_B^{\{\vec{g}\}}\otimes\mathbb{I}_M^{\{\vec{h},\vec{g},\vec{k},m\}}\right)E \\
&= EU\left(\mathbb{I}_B^{\{m\}}\otimes\mathbb{I}_M^{\{\vec{h},m\}}\right)U^\dagger E \\
&= \sum|h_i\rangle\langle h_i|\otimes\mathbb{I}_M^{\{m\}}.
\end{aligned}
$$

Since the other term in $Z_3^B\otimes\mathbb{I}$ is anyway in the non-action space of U it follows that

$$
EU(Z_3^B\otimes\mathbb{I})U^\dagger E = y\sum|h_i\rangle\langle h_i|\otimes\mathbb{I}_M^{\{m\}} + \sum y_{k_i}|k_i\rangle\langle k_i|\otimes\mathbb{I}_M.
$$

It only remains to show that $Z_{(4)}^B\otimes\mathbb{I}_M \geq E^{(3)}U^{(3)}\left(Z_{(3)}^B\otimes\mathbb{I}_M\right)U^{(3)\dagger}E^{(3)}$ which it obviously is because $y\sum|h_i\rangle\langle h_i|\otimes\mathbb{I}_M \geq y\sum|h_i\rangle\langle h_i|\otimes\mathbb{I}_M^{\{m\}}$ and the y_{k_i} term is common. $\square$

3.3 Important Special Case: The Blinkered Unitary

So far we have not specified the non-trivial $U^{(2)}$ (which we will call U from now) beyond requiring it to have a certain action on the honest state. We now define an important class of U, we call the Blinkered Unitary, as

$$
U = |w\rangle\langle v| + |v\rangle\langle w| + \sum|v_i\rangle\langle v_i| + \sum|w_i\rangle\langle w_i| + \mathbb{I}^{\text{outside }\mathcal{H}}
$$

and can even drop the last term as we are restricting our analysis to the $\mathcal{H}$ operator space, where $|v\rangle, \{|v_i\rangle\}$ form a complete orthonormal basis and so do $|w\rangle, \{|w_i\rangle\}$ wrt span$\{|g_i g_i\rangle\}$ and span$\{|v_i v_i\rangle\}$ respectively. The blinkered unitary can be used to implement the two non-trivial operations of the set of basic moves.

- Merge: $g_1, g_2 \to h_1$
  We can construct from the very definitions

$$
|v\rangle = \frac{\sqrt{p_{g_1}}|g_1g_1\rangle + \sqrt{p_{g_2}}|g_2g_2\rangle}{N},\ |v_1\rangle = \frac{\sqrt{p_{g_2}}|g_1g_1\rangle - \sqrt{p_{g_1}}|g_2g_2\rangle}{N},\ |w\rangle = |h_1h_1\rangle
$$

  with $N = \sqrt{p_{g_1}+p_{g_2}}$ and even

$$
U = |w\rangle\langle v| + |v\rangle\langle w| + |v_1\rangle\langle v_1|\ (= U^\dagger).
$$

  I would need

$$
EU|g_1g_1\rangle = \frac{\sqrt{p_{g_1}}|w\rangle}{N},\ EU|g_2g_2\rangle = \frac{\sqrt{p_{g_2}}|w\rangle}{N}
$$

  because the constraint was (plugging in the m and n)

$$
x_h|h_1h_1\rangle\langle h_1h_1| \geq \sum x_{g_i} EU|g_ig_i\rangle\langle g_ig_i|U^\dagger E
$$

  which becomes

$$
x_h \geq \frac{p_{g_1}x_{g_1} + p_{g_2}x_{g_2}}{N^{22}}.
$$

  This is precisely the merge condition Mochon derives. This can be readily generalised to an $m \to 1$ point merge condition by simply constructing appropriate vectors (which we leave for the appendix).

- Split: $g_1 \to h_1, h_2$

$$|v\rangle = |g_1 g_1\rangle \,,\; |w\rangle = \frac{\sqrt{p_{h_1}}\,|h_1 h_1\rangle + \sqrt{p_{h_2}}\,|h_2 h_2\rangle}{N},\; |w_1\rangle = \frac{\sqrt{p_{h_2}}\,|h_1 h_1\rangle - \sqrt{p_{h_1}}\,|h_2 h_2\rangle}{N}$$

with $N = \sqrt{p_{h_1} + p_{h_2}}$ and

$$U = |v\rangle\langle w| + |w\rangle\langle v| + |w_1\rangle\langle w_1| = U^\dagger.$$

We evaluate $EU\,|g_1 g_1\rangle = |w\rangle$ which upon being plugged into the constraint yields

$$x_{h_1}\,|h_1 h_1\rangle\langle h_1 h_1| + x_{h_2}\,|h_2 h_2\rangle\langle h_2 h_2| - x_{g_1}\,|w\rangle\langle w| \geq 0.$$

This yields the matrix equation

$$\begin{bmatrix} x_{h_1} & \\ & x_{h_2} \end{bmatrix} - \frac{x_{g_1}}{N^2}\begin{bmatrix} p_{h_1} & \sqrt{p_{h_1}p_{h_2}} \\ \sqrt{p_{h_1}p_{h_2}} & p_{h_2} \end{bmatrix} \geq 0$$

$$\mathbb{I} \geq \frac{x_{g_1}}{N^2}\begin{bmatrix} \frac{p_{h_1}}{x_{h_1}} & \sqrt{\frac{p_{h_1}}{x_{h_1}}\frac{p_{h_2}}{x_{h_2}}} \\ \sqrt{\frac{p_{h_1}}{x_{h_1}}\frac{p_{h_2}}{x_{h_2}}} & \frac{p_{h_2}}{x_{h_2}} \end{bmatrix} \qquad \left(\begin{array}{l}\text{using the } F - M \geq 0 \\ \implies \mathbb{I} - \sqrt{F}^{-1} M \sqrt{F}^{-1} \geq 0\end{array}\right)$$

$$\frac{x_{g_1}}{N^2}\left(\frac{p_{h_1}}{x_{h_1}} + \frac{p_{h_2}}{x_{h_2}}\right) \leq 1 \qquad \left(\begin{array}{l}\text{using the } |\psi\rangle\langle\psi| \text{ trick} \\ \text{and demanding } 1 \geq \langle\psi|\psi\rangle\end{array}\right).$$

The last statement is the same constraint for a split as the one derived by Mochon. This also readily generalises to the case of $1 \to N$ splits which again we defer to the appendix.

- General $m \to n$: $g_1, g_2 \ldots g_m \to h_1, h_2 \ldots h_n$
It is not too hard to show that in general one obtains the constraint

$$\frac{1}{\langle x_g \rangle} \geq \left\langle \frac{1}{x_h} \right\rangle$$

using the appropriate blinkered unitary (which also we show in the appendix).

This class of unitary is enough to convert the 1/6 game into an explicit protocol. However, for games given by Mochon that go beyond 1/6 this class falls short. One way of seeing this is that the general $m \to n$ blinkered transition effectively behaves like an $m \to 1$ merge followed by a $1 \to n$ split, which are a set of moves that are insufficient to break the 1/6 limit (at least using Mochon's games).

4 Games and Protocols

We will now describe two games, the bias 1/6 game and the bias 1/10 game, from the family of games constructed by Mochon to show that arbitrarily small bias is achievable. Mochon parametrises his games by k which determines the number of points involved in the non-trivial step. The bias he obtains is $\epsilon = 1/(4k+2)$. We consider games with $k = 1$ and $k = 2$, yielding the aforementioned bias.

4.1 Mochon's Approach

4.1.1 Assignments

Recall that a function

$$\sum_{z \in \{x_1, x_2, \ldots x_n\}} p(z)[z]$$

is valid if

$$\sum_{z \in \{x_1, x_2, \ldots x_n\}} \left(\frac{-1}{\lambda + z}\right) p(z) \geq 0, \qquad \sum_{z \in \{x_1, x_2, \ldots x_n\}} z p(z) \geq 0, \qquad \sum_{z \in \{x_1 \ldots x_n\}} p(z) = 0$$

for all $\lambda > 0$ where $x_i \geq 0$. Checking if a generic assignment for p satisfies these infinite constraints in not always easy. Mochon had used a constructive approach here and we will build on to it. Let us state these results with some precision (proven in the appendix, well most) where as above n numbers are assumed to be represented by x_i and each $x_i \geq 0$.

Lemma (Mochon's Denominator). $\sum_{i=1}^{n} \frac{1}{\prod_{j \neq i}(x_j - x_i)} = 0$ *for* $n \geq 2$.

Lemma (Mochon's f-assignment Lemma). $\sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j \neq i}(x_j - x_i)} = 0$ *where* $f(x_i)$ *is a polynomial of order* $k \leq n - 2$.

Definition (Mochon's TIPG assignment)**.** Given a set of n points $0 < x_1 < x_2 \cdots < x_n$, a polynomial $f(x)$ with order k at most $n-2$ and $f(-\lambda) \geq 0$ for all $\lambda \geq 0$, the probability weights for a TIPG assignment is $p(x_i) = -\frac{f(x_i)}{\prod_{j\neq i}(x_j - x_i)}$.

Mochon was able to show that 'Mochon's TIPG assignment' makes for a valid function (in the TIPG formalism), given by

$$\sum_{i=1}^{n} p(x_i)[x_i, y]$$

where the notion of validity has been extended to a pair of points. As we will see soon, the power of this construction lies in the fact that we can easily construct polynomials that have roots at arbitrary locations. This allows us to create interesting repeating structures called ladders (due to Mochon) which we can terminate using these polynomials to obtain a game with a finite set of points. These ladders play a pivotal role in achieving smaller biases and the ability to obtain finite ladders is essential for being able to obtain a physical process that would yield the said bias.

We now build a little on Mochon's notation and results.

Definition (Mochon's TDPG assignment)**.** Given Mochon's TIPG assignment, let $\{i\}$ be the set of indices for which $p(x_i) < 0$ and $\{k\}$ be the remaining indices with respect to $\{1, 2, \dots n\}$. The TDPG assignment (in accordance with the notation used in TEF) is given as

$$\begin{aligned} \{x_{g_1}, x_{g_2} \dots\} &= \{x_i\} \\ \{p_{g_1}, p_{g_2} \dots\} &= \{-p(x_i)\} \\ \{x_{h_1}, x_{h_2} \dots\} &= \{x_k\} \\ \{p_{h_1}, p_{h_2} \dots\} &= \{p(x_k)\}. \end{aligned}$$

With these in place we make some observations about how initial and final averages behave under such an assignment.

Proposition. *$N_h^2 = N_g^2$ where $N_g^2 = \sum p_{g_i}$ and $N_h^2 = \sum p_{h_i}$ for Mochon's TDPG assignment.*

Proof. We have to show that $N_h^2 - N_g^2 = \sum p_{h_i} - \sum p_{g_i} = 0$ which is the same as showing $\sum_{i=1}^{n} p(x_i) = 0$ which holds because we just showed that $\sum_{i=1}^{n} f(x_i)/\prod_{j\neq i}(x_j - x_i) = 0$ (Mochon's f-assignment Lemma). □

Proposition. *$\langle x_h \rangle - \langle x_g \rangle = 0$ for a Mochon's TDPG assignment with $k \leq n-3$ where $\langle x_h \rangle = \frac{1}{N_h^2}\sum p_{h_i} x_{h_i}$ and $\langle x_g \rangle = \frac{1}{N_g^2}\sum p_{g_i} x_{g_i}$.*

Proof. This is a direct consequence of Mochon's f-assignment lemma. Let h be the $n-3$ order polynomial defined by Mochon's TDPG assignment so that $\langle x_h \rangle - \langle x_g \rangle \propto \sum p_{h_i} x_i - \sum p_{g_i} x_{g_i} = \sum_{i=1}^{n} p(x_i) x_i = \sum_{i=1}^{n} \frac{h(x_i)x_i}{\prod_{j\neq i}(x_j - x_i)} = \sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j\neq i}(x_j - x_i)} = 0$ because f is an $n-2$ order polynomial. □

Lemma. *$\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{j\neq i}(x_j - x_i)} = (-1)^{n-1}$ for $n \geq 2$.*

Proposition. *$\langle x_h \rangle - \langle x_g \rangle = \frac{1}{N_h^2} = \frac{1}{N_g^2}$ for a Mochon's TDPG assignment with $k = n-2$ and coefficient of x^{n-2} ± 1 in $f(x)$. As above here $\langle x_h \rangle = \frac{1}{N_h^2}\sum p_{h_i} x_{h_i}$ and $\langle x_g \rangle = \frac{1}{N_g^2}\sum p_{g_i} x_{g_i}$.*

We will see that typically $N_h = N_g$ are quite large and the average only slightly increase, if at all. We are now in a position to discuss Mochon's games.

4.1.2 Typical Game Structure

We assume an equally spaced n-point lattice given by $x_j = x_0 + j\delta x$ where $\delta x = \delta y$ is small and x_0 would essentially give a bound on P_B^* which will be determined by following the constraints; similarly $y_j = y_0 + j\delta y$ and we also define $\Gamma_{k+1} = y_{n-k} = x_{n-k}$. Let $P(x_j)$ be the probability weight associated with the point $[x_j, 0]$ s.t.

$$\sum_{i=1}^{n} P(x_j) = \frac{1}{2}, \sum_{j=1}^{n} \frac{P(x_j)}{x_j} = \frac{1}{2}.$$

Similarly with the point $[0, y_j]$ we associate $P(y_j)$ where $y_j = x_j$ as we will also assume that $x_0 = y_0$. These choices explicitly impose symmetry between Alice and Bob which in turn entails that we have to do only half the analysis.

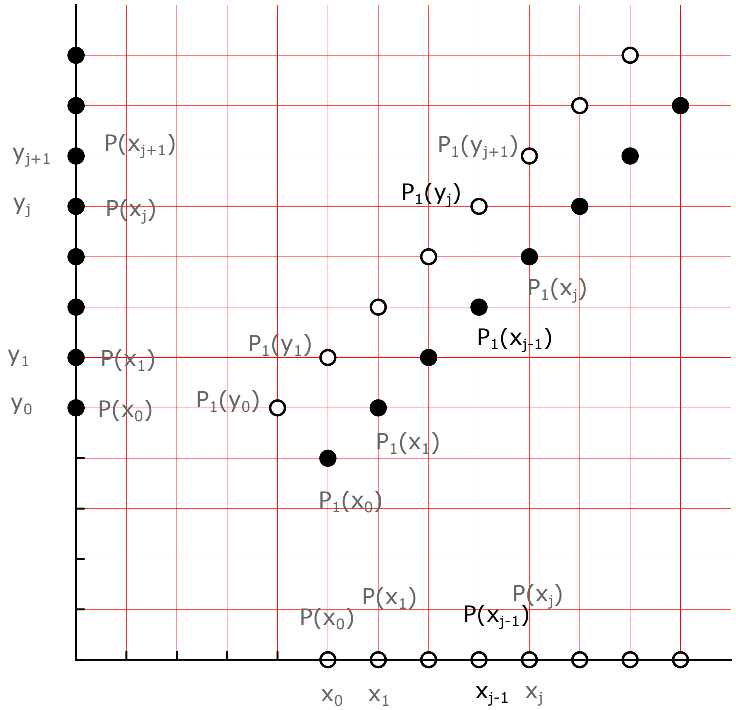


Figure 2: Building a TDPG/TIPG using merge moves

4.2 Bias 1/6

4.2.1 Game

With reference to Figure 2 we need to satisfy $P(x_{j-1}) + P_1(y_j) = P_1(x_{j-1})$ which is probability conservation and $P_1(y_j)y_j \leq P_1(x_{j-1})y_{j-2}$ which is the merge condition. Both of these are automatically satisfied if we make a Mochon's denominator based assignment as follows

$$\begin{aligned} 0 &\leftrightarrow x_{g_1} \\ y_j &\leftrightarrow x_{g_2} \\ y_{j-2} &\leftrightarrow x_{h_1} \end{aligned}$$

$$\begin{aligned} P(x_{j-1}) \leftrightarrow p_{g_1} &= \frac{c(x_{j-1})}{y_j y_{j-2}} \\ P_1(y_j) \leftrightarrow p_{g_2} &= \frac{c(x_{j-1})}{(y_j - y_{j-2})(y_j)} = \frac{c(x_{j-1})}{2y_j\delta y} \\ P_1(x_{j-1}) \leftrightarrow p_{h_1} &= \frac{c(x_{j-1})}{(y_j - y_{j-2})(y_{j-2})} = \frac{c(x_{j-1})}{2y_{j-2}\delta y} \end{aligned}$$

where the function $c(x_{j-1})$ must be chosen so that $P_1(y_j) = P_1(x_j)$ which entails

$$\frac{c(x_{j-1})}{\cancel{2}y_j\cancel{\delta y}} = \frac{c(x_j)}{\cancel{2}y_{j-1}\cancel{\delta y}}$$

and that in turn is solved by $c(x_j) = \frac{c_0\delta x}{x_j}$ where we used $x_j = y_j$, $\delta x = \delta y$ and added a δx to make $\sum P(x_j)$ into an integral. Plugging this back we have

$$P_1(x_j) = \frac{c_0}{2x_j x_{j-1}},\ P(x_j) = \frac{c_0\delta x}{x_{j-1}x_j x_{j+1}}$$

Since they involve a sum we will do this in the limit $\delta x \to 0$ and $\Gamma \to \infty$ to avoid dealing with summing a series.

$$\sum_{j=0}^{n} P(x_j) = \frac{1}{2} \to c_0 \int_{x_0}^{\Gamma} \frac{dx}{x^3} = \frac{c_0}{(-2)}\left[\frac{1}{\Gamma^2} - \frac{1}{x_0^2}\right] = \frac{1}{2}$$

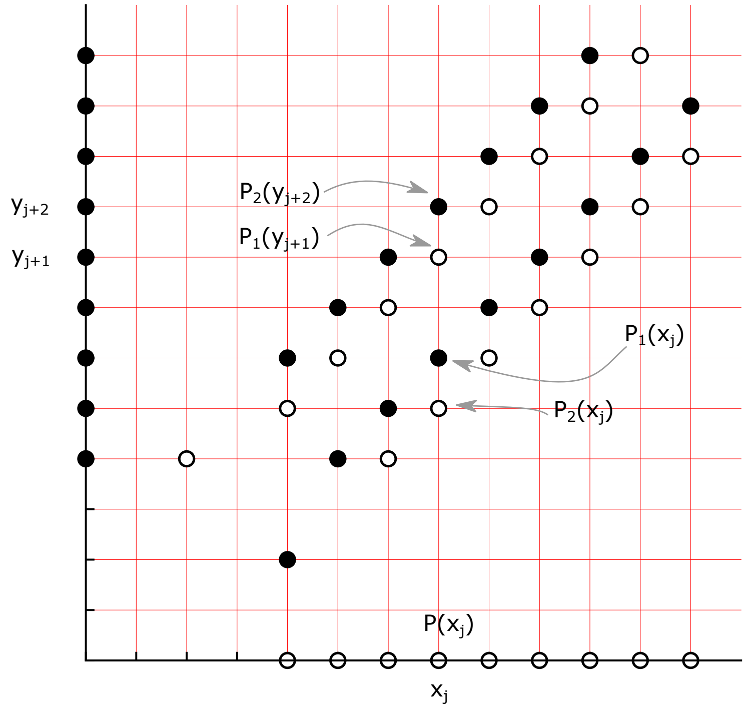


Figure 3: 1/10 game: The $3 \to 2$ move based TIPG for bias 1/10

which entails $c_0 = x_0^2$. The next condition will yield x_0

$$\sum_{j=0}^{n} \frac{P(x_j)}{x_j} = \frac{1}{2} \to x_0^2 \int_{x_0}^{\Gamma} \frac{dx}{x^4} = \frac{x_0^2}{(-3)} \left[\frac{1}{\Gamma^3} - \frac{1}{x_0^3} \right] = \frac{1}{3x_0} = \frac{1}{2}$$

which means

$$x_0 = \frac{2}{3} \implies \epsilon = \frac{1}{6}.$$

Of course a more careful analysis must be done to show these things exactly. Aside from the integration step one must also set $c_0(x) = (\Gamma_{n+1} - x)$ in order to terminate the ladder which turns the terminating step on the ladder into a raise. At the moment, however, we satisfy ourselves with this and move on to the more interesting 1/10 game.

4.2.2 Protocol

Although we could only claim that one can construct the protocol once the unitaries are known, the basic idea is that one starts with a split, then a raise by Alice and Bob, followed by a merge by Bob, then a merge by Alice and so on until only two points remain. Bob can also start as the description is symmetric. These two can then be raised to the same location and merged. The coordinates of these points will tend to $[\frac{2}{3}, \frac{2}{3}]$ as calculated above. The only creative part left would be the choice of labels that make the description neater from the point of view of the explicit protocol.

4.3 Bias 1/10 Game

With respect to Figure 3 we use Mochon's assignment with $f(y_i) = (y_{-2} - y_i)\,(\Gamma_1 - y_i)\,(\Gamma_2 - y_i)$ as

$$\frac{f(y_j)c'(x_j)}{\prod_{k \neq j}(y_k - y_j)}.$$

Following the scheme as described above the probabilities become

$$\begin{aligned}
P_2(y_{j+2}) &= \frac{-f(y_{j+2})c(x_j)}{4.3(\delta y)^2 y_{j+2}} \\
P_1(y_{j+1}) &= \frac{-f(y_{j+1})c(x_j)}{3.2(\delta y)^2 y_{j+1}} \\
P_1(x_j) &= \frac{-f(y_{j-1})c(x_j)}{3.2(\delta y)^2 y_{j-1}} \\
P_2(x_j) &= \frac{-f(y_{j-2})c(x_j)}{4.3(\delta y)^2 y_{j-2}} \\
P(x_j) &= \frac{f(0)c(x_j)\delta y}{y_{j+2}y_{j+1}y_{j-1}y_{j-2}}
\end{aligned}$$

where we added the minus sign to account for the fact that f will be negative for coordinates between y_{-2} and Γ_1. Imposing the symmetry constraints $P_1(y_j) = P_1(x_j)$ we get

$$\frac{f(y_j)c(x_{j-1})}{\cancel{3.2(\delta y)^2} y_j} = \frac{f(y_{j-1})c(x_j)}{\cancel{3.2(\delta y)^2} y_{j-1}}$$

which means

$$c(x_j) = \frac{c_0 f(x_j)}{x_j}$$

where c_0 is a constant. This also entails that $P_2(y_j) = P_2(x_j)$, viz. it satisfies the second symmetry constraint. Finally we can evaluate

$$P(x_j) = \frac{f(0)f(x_j)\delta x}{x_{j+2}x_{j+1}x_j x_{j-1}x_{j-2}} = \frac{c_0 x_0(x_0 - x_j)}{x_j^5}\delta x + \mathcal{O}(\delta x^2)$$

which means that

$$\sum P(x_j) = \frac{1}{2} = \sum \frac{P(x_j)}{x_j} \to \int_{x_0}^{\Gamma} \frac{(x_0 - x)dx}{x^5} = \int_{x_0}^{\Gamma} \frac{(x_0 - x)dx}{x^6}.$$

We can evaluate this as

$$\begin{aligned}
x_0 \int_{x_0}^{\Gamma} \left(\frac{1}{x^5} - \frac{1}{x^6}\right) dx &= \int_{x_0}^{\Gamma} \left(\frac{1}{x^4} - \frac{1}{x^5}\right) dx \\
\left[\frac{1}{4x_0^3} - \frac{1}{5x_0^4}\right] &= \left[\frac{1}{3x_0^3} - \frac{1}{4x_0^4}\right] \\
\left[\frac{1}{4} - \frac{1}{3}\right] &= \left[\frac{1}{5} - \frac{1}{4}\right]\frac{1}{x_0} \\
x_0 \frac{3-4}{\cancel{4}.3} &= \frac{4-5}{5.\cancel{4}} \\
x_0 = \frac{3}{5} &\implies \epsilon = \frac{3}{5} - \frac{1}{2} = \frac{1}{10}.
\end{aligned}$$

4.4 Bias $1/10$ Protocol

4.4.1 The $3 \to 2$ Move

In this section we will introduce as many parameters as possible within the TEF to implement the largest class of $3 \to 2$ moves. However, we will use our insight to choose an appropriate basis so that the parameters are small which in turn simplifies the analysis.

Recall that

$$|v\rangle = \frac{\sqrt{p_{g_1}}\,|g_1\rangle + \sqrt{p_{g_2}}\,|g_2\rangle + \sqrt{p_{g_3}}\,|g_3\rangle}{N_g}$$

and let

$$\begin{aligned}
|v_1\rangle &= \frac{\sqrt{p_{g_3}}\,|g_2\rangle - \sqrt{p_{g_2}}\,|g_3\rangle}{N_{v_1}} \\
|v_2\rangle &= \frac{-\frac{(p_{g_2}+p_{g_3})}{\sqrt{p_{g_1}}}\,|g_1\rangle + \sqrt{p_{g_2}}\,|g_2\rangle + \sqrt{p_{g_3}}\,|g_3\rangle}{N_{v_2}}
\end{aligned}$$

where $N_{v_1}^2 = p_{g_3} + p_{g_2}$ and $N_{v_2}^2 = \frac{(p_{g_2}+p_{g_3})^2}{p_{g_1}} + p_{g_2} + p_{g_3}$. Recall that

$$
\begin{aligned}
|w\rangle &= \frac{\sqrt{p_{h_1}}\,|h_1\rangle + \sqrt{p_{h_2}}\,|h_2\rangle}{N_h} \\
|w_1\rangle &= \frac{\sqrt{p_{h_2}}\,|h_1\rangle - \sqrt{p_{h_1}}\,|h_2\rangle}{N_h}.
\end{aligned}
$$

Now we define

$$
\begin{aligned}
|v_1'\rangle &= \cos\theta\,|v_1\rangle + \sin\theta\,|v_2\rangle \\
|v_2'\rangle &= \sin\theta\,|v_1\rangle - \cos\theta\,|v_2\rangle
\end{aligned}
$$

where we know (from hindsight) that $\cos\theta \approx 1$. The full unitary (which is manifestly unitary) we define to be

$$
U = |w\rangle\langle v| + (\alpha\,|v_1'\rangle + \beta\,|w_1\rangle)\,\langle v_1'| + |v_2'\rangle\langle v_2'| + (\beta\,|v_1'\rangle - \alpha\,|w_1\rangle)\,\langle w_1| + |v\rangle\langle w|
$$

where $|\alpha|^2 + |\beta|^2 = 1$ for $\alpha, \beta \in \mathbb{C}$. There is some freedom in choosing U in the sense that $\alpha\,|v\rangle + \beta\,|w_1\rangle$ would also work (then $|v\rangle\langle w| \to |v_1\rangle\langle w|$) because these don't influence the constraint equation. That is what we evaluate now. We will need terms of the form $EU\,|g_i\rangle$ with $E = \mathbb{I}^{\{h_i\}}$. This entails that on the $\{|g_i\rangle\}$ space

$$
\begin{aligned}
E_h U E_g &= |w\rangle\langle v| + \beta\,|w_1\rangle\langle v_1'| \\
&= |w\rangle\langle v| + \beta\,|w_1\rangle\left(\cos\theta\,\langle v_1| + \sin\theta\,\langle v_2|\right).
\end{aligned}
$$

Consequently I have

$$
\begin{aligned}
E_h U\,|g_{11}\rangle &= \frac{\sqrt{p_{g_1}}}{N_g}\,|w\rangle + \left[\cos\theta.0 - \sin\theta\frac{p_{g_2}+p_{g_3}}{\sqrt{p_{g_1}}N_{v_2}}\right]\beta\,|w_1\rangle \\
E_h U\,|g_{22}\rangle &= \frac{\sqrt{p_{g_2}}}{N_g}\,|w\rangle + \left[\cos\theta\frac{\sqrt{p_{g_3}}}{N_{v_1}} + \sin\theta\frac{\sqrt{p_{g_2}}}{N_{v_2}}\right]\beta\,|w_1\rangle \\
E_h U\,|g_{33}\rangle &= \frac{\sqrt{p_{g_3}}}{N_g}\,|w\rangle + \left[-\cos\theta\frac{\sqrt{p_{g_2}}}{N_{v_1}} + \sin\theta\frac{\sqrt{p_{g_3}}}{N_{v_2}}\right]\beta\,|w_1\rangle.
\end{aligned}
$$

Recall that the constraint equation was

$$
\sum x_{h_i}\,|h_i\rangle\langle h_i| - \sum x_{g_i} E_h U\,|g_i\rangle\langle g_i|\,U^\dagger E_h \geq 0
$$

where the first sum becomes

$$
\begin{bmatrix} \langle x_h\rangle & \frac{\sqrt{p_{h_1}p_{h_2}}}{N_h^2}(x_{h_1}-x_{h_2}) \\ \text{h.c.} & \frac{p_{h_2}x_{h_1}+p_{h_1}x_{h_2}}{N_h^2} \end{bmatrix}
$$

in the $|w\rangle, |w_1\rangle$ basis. Since we plan to use the $3 \to 2$ move with one point on the axis, we take $x_{g_1} = 0$. Consequently we need only evaluate

$$
\begin{aligned}
x_{g_2} E_h U\,|g_2\rangle\langle g_2|\,U^\dagger E_h &\doteq x_{g_2} \begin{bmatrix} \frac{p_{g_2}}{N_g^2} & \beta\left(\cos\theta\frac{\sqrt{p_{g_3}p_{g_2}}}{N_g N_{v_1}} + \sin\theta\frac{p_{g_2}}{N_g N_{v_2}}\right) \\ \text{h.c.} & \left(\cos\frac{\sqrt{p_{g_3}}}{N_{v_1}} + \sin\theta\frac{\sqrt{p_{g_2}}}{N_{v_2}}\right)^2 |\beta|^2 \end{bmatrix} \\
x_{g_3} E_h U\,|g_3\rangle\langle g_3|\,U^\dagger E_h &\doteq x_{g_3} \begin{bmatrix} \frac{p_{g_3}}{N_g^2} & \beta\left(-\cos\theta\frac{\sqrt{p_{g_2}p_{g_3}}}{N_g N_{v_1}} + \sin\theta\frac{p_{g_3}}{N_g N_{v_2}}\right) \\ \text{h.c.} & \left(-\cos\frac{\sqrt{p_{g_2}}}{N_{v_1}} + \sin\frac{\sqrt{p_{g_3}}}{N_{v_2}}\right)^2 |\beta|^2 \end{bmatrix}
\end{aligned}
$$

which means that the constraint equation becomes

$$
\begin{bmatrix} \langle x_h\rangle - \langle x_g\rangle & \frac{\sqrt{p_{h_1}p_{h_2}}}{N_h^2}(x_{h_1}-x_{h_2}) - \beta\cos\theta\frac{\sqrt{p_{g_2}p_{g_3}}}{N_g N_{v_1}}(x_{g_2}-x_{g_3}) - \beta\sin\theta\,\langle x_g\rangle\,\frac{N_g}{N_{v_2}} \\ \text{h.c.} & \frac{p_{h_2}x_{h_1}+p_{h_1}x_{h_2}}{N_h^2} - |\beta|^2\left[\frac{\cos^2\theta}{N_{v_1}^2}(p_{g_3}x_{g_2}+p_{g_2}x_{g_3}) + \frac{\sin^2\theta}{(N_{v_2}^2/N_g^2)}\,\langle x_g\rangle + \frac{2\cos\theta\sin\theta\sqrt{p_{g_3}p_{g_2}}}{N_{v_1}N_{v_2}}\,(x_{g_2}-x_{g_3})\right] \end{bmatrix} \geq 0.
$$

We already showed that Mochon's transition is average non-decreasing viz. $\langle x_h\rangle - \langle x_g\rangle \geq 0$. We will set the off-diagonal elements of the matrix above to zero and show that the second diagonal element, the second eigenvalue therefore, is positive.

Setting the off-diagonal to zero one can obtain θ by solving the quadratic in terms of β although the expression will not be particularly pretty. To establish existence and positivity we need to simplify our expressions.

So far everything was exact even though the basis and techniques were chosen based on experience. Now we claim that $\theta\frac{N_g}{N_{v_2}} \approx \mathcal{O}(\delta y)$ at most (where $\delta y = \delta x$ is the lattice spacing) and since δy will be taken to be small we can take the small $\theta\frac{N_g}{N_{v_2}}$ limit and to first order in it the constraints become

$$\frac{\frac{\sqrt{p_{h_1}p_{h_2}}}{N_h^2}(x_{h_1}-x_{h_2}) - \beta\frac{\sqrt{p_{g_2}p_{g_3}}}{N_g N_{v_1}}(x_{g_2}-x_{g_3})}{\beta\langle x_g\rangle} = \theta\frac{N_g}{N_{v_2}} + \mathcal{O}(\delta y^2)$$

and

$$\frac{p_{h_2}x_{h_1}+p_{h_1}x_{h_2}}{N_h^2} - |\beta|^2\left[\frac{p_{g_3}x_{g_2}+p_{g_2}x_{g_3}}{N_{v_1}^2} + 2\theta\frac{N_g}{N_{v_2}}\frac{\sqrt{p_{g_3}p_{g_2}}}{N_g N_{v_1}}(x_{g_2}-x_{g_3})\right] + \mathcal{O}(\delta y^2) \geq 0.$$

If our claim is wrong when we evaluate $\theta\frac{N_g}{N_{v_2}}$ we will get zero order terms but as we show in the following section $\theta\frac{N_g}{N_{v_2}} = 0.\delta y + \mathcal{O}(\delta y^2)$ in fact.

4.4.2 Validity of the $3 \to 2$ Move

With respect to Figure 3 we have

$$\begin{aligned}
P_2(y_{j+2}) &= p_{h_2} = \frac{-f(y_{j+2})}{4.3\delta y^2 y_{j+2}}\\
P_1(y_{j+1}) &= p_{g_3} = \frac{-f(y_{j+1})}{3.2\delta y^2 y_{j+1}}\\
P_1(x_j) &= p_{h_1} = \frac{-f(y_{j-1})}{3.2\delta y^2 y_{j-1}}\\
P_2(x_j) &= p_{g_2} = \frac{-f(y_{j-2})}{4.3\delta y^2 y_{j-2}}\\
P(x_j) &= p_{g_1} = \frac{f(0)\delta y}{y_{j+2}y_{j+1}y_{j-1}y_{j-2}}
\end{aligned}$$

where we assumed $f(0) > 0$ and $f(y) < 0$ for $y > y_0'$, $y_0' = y_0 + \delta y$. We also scaled by δy to make $P(x_j)$ into a nice density. So far everything is exact. We will now convert all expressions to first order in δy. To this end we note

$$\begin{aligned}
f(y_{j+m}) &= f(y_j) + \frac{\partial f}{\partial y}m\delta y + \mathcal{O}(\delta y^2)\\
\frac{1}{y_{j+m}} &= (y_j + m\delta y)^{-1} = \frac{1}{y_j}\left(1 + m\frac{\delta y}{y_j}\right)^{-1} = \frac{1}{y_j} - m\frac{\delta y}{y_j^2} + \mathcal{O}(\delta y^2)
\end{aligned}$$

where $\frac{\partial f}{\partial y}$ refers to $\frac{\partial f(y)}{\partial y}|_{y_j}$. We define and evaluate

$$\begin{aligned}
P_k^m &= \frac{-f(y_{j+m})}{k\delta y^2 y_{j+m}}\\
&= \frac{1}{k\delta y^2}\left[-f(y_j) - \frac{\partial f}{\partial y}m\delta y + \mathcal{O}(\delta y^2)\right]\left[\frac{1}{y_j} - m\frac{\delta y}{y_j^2} + \mathcal{O}(\delta y^2)\right]\\
&= \frac{1}{k\delta y^2}\left[-\frac{f}{y_j} - m\frac{\delta y}{y_j}\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right) + \mathcal{O}(\delta y^2)\right]\\
&= \frac{1}{k y_j\delta y^2}\left[-f - m\delta y\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right) + \mathcal{O}(\delta y^2)\right]
\end{aligned}$$

where f means $f(y_j)$. In this notation

$$\begin{aligned}
p_{h_2} &= P_{12}^2,\, p_{h_1} = P_6^{-1}\\
p_{g_2} &= P_{12}^{-2},\, p_{g_3} = P_6^1.
\end{aligned}$$

With an eye at the off-diagonal condition we evaluate

$$P_{k_1}^{m_1}P_{k_2}^{m_2} = \frac{1}{k_1k_2}\left(\frac{1}{y_j\delta y^2}\right)^2\left[f^2 + f\delta y\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right)(m_1+m_2) + \mathcal{O}(\delta y^2)\right]$$

and

$$P_{k_1}^{m_1} + P_{k_2}^{m_2} = \frac{1}{y_j\delta y^2}\left[-\left(\frac{1}{k_1}+\frac{1}{k_2}\right)f - \left(\frac{m_1}{k_1}+\frac{m_2}{k_2}\right)\delta y\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right) + \mathcal{O}(\delta y^2)\right].$$

We now evaluate

$$
\begin{aligned}
\sqrt{p_{h_1}p_{h_2}} &= \sqrt{P_{12}^2 P_6^{-1}} = \frac{1}{y_j\delta y^2}\sqrt{\frac{1}{12.6}\left[f^2 + f\delta y\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right) + \mathcal{O}(\delta y^2)\right]} \\
N_h^2 &= P_{12}^2 + P_6^{-1} = \frac{1}{y_j\delta y^2}\left[-\left(\frac{1}{12}+\frac{1}{6}\right)f - \cancelto{0}{\left(\frac{2}{12}-\frac{1}{6}\right)}\delta y\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right) + \mathcal{O}(\delta y^2)\right] \\
&= \frac{1}{4y_j\delta y^2}\left[-f + \mathcal{O}(\delta y^2)\right]
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\sqrt{p_{g_2}p_{g_3}} &= \sqrt{P_{12}^{-2} P_6^{1}} = \frac{1}{y_j\delta y^2}\sqrt{\frac{1}{12.6}\left[f^2 - f\delta y\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right) + \mathcal{O}(\delta y^2)\right]} \\
N_g^2 &= P_{12}^{-2} + P_6^{1} + p_{g_1} = \frac{1}{4y_j\delta y^2}\left[-f + \mathcal{O}(\delta y^2)\right] + \left[\frac{f(0)\delta y}{y_j^4} + \mathcal{O}(\delta y^2)\right] \\
&= \frac{1}{4y_j\delta y^2}\left[-f + \mathcal{O}(\delta y^2)\right] \\
N_{v_1}^2 &= \frac{1}{4y_j\delta y^2}\left[-f + \mathcal{O}(\delta y^2)\right]
\end{aligned}
$$

where even though it seems like we have neglected p_{g_1} when we take the ratios the meaning of keeping first order in δy would become precise. We can actually take $\beta = 1$ and obtain

$$
\begin{aligned}
\theta\frac{N_g}{N_{v_2}} &= \frac{4\sqrt{\frac{1}{12.6}}(-3\delta y)\left[f.(\cancel{1} + \frac{\delta y}{2f}\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right)) - f.(\cancel{1} - \frac{\delta y}{2f}\left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right)) + \mathcal{O}(\delta y^2)\right]}{\langle x_g\rangle} \\
&= 0 + \mathcal{O}(\delta y^2).
\end{aligned}
$$

This shows that to first order the off-diagonal term is zero for $\theta = 0$.

Now we will show that the second diagonal element is positive to first order in δy. Using the fact that $\theta\frac{N_g}{N_{v_2}} = \mathcal{O}(\delta y^2)$ we have

$$
\frac{p_{h_2}x_{h_1} + p_{h_1}x_{h_2}}{N_h^2} - \frac{p_{g_3}x_{g_2} + p_{g_2}x_{g_3}}{N_{v_1}^2} + \mathcal{O}(\delta y^2) \geq 0
$$

as the positivity condition. This becomes

$$
\begin{aligned}
&= \frac{P_{12}^2 y_{j-1} + P_6^{-1} y_{j+2}}{N_h^2} - \frac{P_6^1 y_{j-2} + P_{12}^{-2} y_{j+1}}{N_{v_1}^2} + \mathcal{O}(\delta y^2) \\
&= \left(\frac{4y_j\delta y^2}{-f}\right)\frac{1}{y_j\delta y^2} \\
&\quad \left\{\frac{1}{12}\left[-f - 2\delta y\gamma\right](y_j - \delta y) + \frac{1}{6}\left[-f + \gamma\delta y\right](y_j + 2\delta y) - \left(\frac{1}{6}\left[-f - \delta y\gamma\right](y_j - 2\delta y) + \frac{1}{12}\left[-f + 2\delta y\gamma\right](y_j + \delta y\right)\right\} + \mathcal{O}(\delta y^2) \\
&= \frac{-2}{3f}\left\{\frac{1}{2}(\cancel{-fy} + f\delta y - 2y\delta y\gamma) + (\cancel{-fy} - 2f\delta y + y\delta y\gamma) - \left((\cancel{-fy} + 2f\delta y - y\delta y\gamma) + \frac{1}{2}(\cancel{-fy} - f\delta y + 2y\delta y\gamma)\right)\right\} + \mathcal{O}(\delta y^2) \\
&= \frac{-2}{3f}\left\{(f\delta y - 2y\delta y\gamma) + 2(-2f\delta y + y\delta y\gamma)\right\} + \mathcal{O}(\delta y^2) \\
&= \frac{-2}{3f}\left\{-3f\delta y\right\} + \mathcal{O}(\delta y^2) = 2\delta y + \mathcal{O}(\delta y^2) \geq 0
\end{aligned}
$$

where $\gamma = \left(\frac{\partial f}{\partial y} - \frac{f}{y_j}\right)$ and we suppressed the index j in y_j for simplicity. This establishes the validity of the $3 \to 2$ transition for a closely spaced lattice.

Note that only the proof of validity was done perturbatively to first order in δy. The unitary itself is known exactly (θ can be obtained by solving the quadratic).

Using $f(y) = (y_0' - y)(\Gamma_1 - y)(\Gamma_2 - y)$ we can implement the last two moves in Figure 3 as they form a $3 \to 1$ merge and a $2 \to 1$ merge (possibly followed by a raise). The only remaining task is implementing the $2 \to 2$ move in the last step because we assumed here that $\sqrt{p_{g_2}} \neq 0$ (else the vectors which we assumed are orthonormal, cease to be so).

4.4.3 The $2 \to 2$ Move and its validity

We claim that the $2 \to 2$ move can be implemented using

$$U = |w\rangle\langle v| + (\alpha|v\rangle + \beta|w_1\rangle)\langle v_1| + |v\rangle\langle w| + (\beta|v\rangle - \alpha|w_1\rangle)\langle w_1|$$

where as before $|\alpha|^2 + |\beta|^2 = 1$,

$$|v\rangle = \frac{1}{N_g}\left(\sqrt{p_{g_1}}|g_1\rangle + \sqrt{p_{g_2}}|g_2\rangle\right),$$

$$|w\rangle = \frac{1}{N_h}\left(\sqrt{p_{h_1}}|h_1\rangle + \sqrt{p_{h_2}}|h_2\rangle\right),$$

$$|v_1\rangle = \frac{1}{N_g}\left(\sqrt{p_{g_2}}|g_1\rangle - \sqrt{p_{g_1}}|g_2\rangle\right),$$

and

$$|w_1\rangle = \frac{1}{N_h}\left(\sqrt{p_{h_2}}|h_1\rangle - \sqrt{p_{h_1}}|h_2\rangle\right).$$

We evaluate the constraint equation using

$$E_hU|g_{11}\rangle = \frac{\sqrt{p_{g_1}}|w\rangle + \beta e^{-i\phi_g}e^{i\phi_h}\sqrt{p_{g_2}}|w_1\rangle}{N_g}$$

$$E_hU|g_{22}\rangle = \frac{\sqrt{p_{g_2}}|w\rangle - \beta e^{-i\phi_g}e^{i\phi_h}\sqrt{p_{g_1}}|w_1\rangle}{N_g}$$

and

$$E_hU|g_{11}\rangle\langle g_{11}|U^\dagger E_h = \frac{1}{N_g^2}\begin{array}{c|cc} & \langle w| & \langle w_1| \\ \hline |w\rangle & p_{g_1} & \beta e^{i(\phi_h-\phi_g)}\sqrt{p_{g_2}p_{g_1}} \\ |w_1\rangle & \text{h.c.} & |\beta|^2 p_{g_2} \end{array}$$

(similarly for $L|g_{22}\rangle\langle g_{22}|L^\dagger$) as

$$\begin{bmatrix} \langle x_h\rangle - \langle x_g\rangle & \frac{1}{N_g^2}\left[\sqrt{p_{h_1}p_{h_2}}(x_{h_1} - x_{h_2}) - \beta\sqrt{p_{g_1}p_{g_2}}(x_{g_1} - x_{g_2})\right] \\ \text{h.c.} & \frac{1}{N_g^2}\left[p_{h_2}x_{h_1} + p_{h_1}x_{h_2} - |\beta|^2(p_{g_2}x_{g_1} + p_{g_1}x_{g_2})\right] \end{bmatrix} \geq 0$$

where we absorbed the phase freedom in β, a free parameter, which will be fixed shortly. We will use the same strategy as above and take the first diagonal element to be zero. Our burden would be to first show that

$$\sqrt{\frac{p_{h_1}p_{h_2}}{p_{g_1}p_{g_2}}}\frac{(x_{h_1} - x_{h_2})}{(x_{g_1} - x_{g_2})} = \beta \leq 1$$

and subsequently

$$\frac{1}{N_g^2}\left[p_{h_2}x_{h_1} + p_{h_1}x_{h_2} - |\beta|^2(p_{g_2}x_{g_1} + p_{g_1}x_{g_2})\right] \geq 0.$$

What makes this situation special (compared to the $3 \to 2$ merge) is that $f(y_{j-2}) = 0$ which we use to write

$$f(y_{j+k}) = \left.\frac{\partial f}{\partial y}\right|_{y_{j-2}}(k+2)\delta y = -(k+2)\alpha\delta y$$

where

$$\alpha = -\left.\frac{\partial f}{\partial y}\right|_{y_{j-2}} = (\Gamma_1 - y_{j-2})(\Gamma_2 - y_{j-2}).$$

Using the axis situation as depicted in Figure 4 we note that

$$p_{h_1} = P_1(x_j) = \frac{-f(y_{j-1})}{3.2\delta y^2 y_{j-1}} = \frac{\alpha + \mathcal{O}(\delta y)}{6\delta y y_j}$$

$$p_{h_2} = P_2(y_{j+2}) = \frac{-f(y_{j+2})}{4.3\delta y^2 y_{j+2}} = \frac{\alpha + \mathcal{O}(\delta y)}{3\delta y y_j}$$

$$x_{h_1} = y_{j-1},\ x_{h_2} = y_{j+2}$$

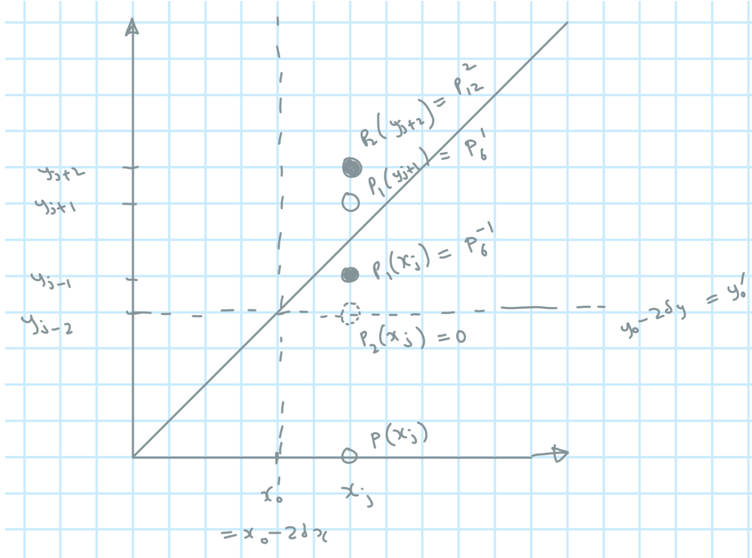


Figure 4: Final Move: The special $2 \to 2$ Transition

while

$$
\begin{aligned}
p_{g_1} &= P(x_j) = \frac{f(0)\delta y}{y_{j+2}y_{j+1}y_{j-1}y_{j-2}} = \frac{f(0)\delta y + \mathcal{O}(\delta y^2)}{y_j^4} \\
p_{g_2} &= P_1(y_{j+1}) = \frac{-f(y_{j+1})}{3.2\delta y^2 y_{j+1}} = \frac{\alpha + \mathcal{O}(\delta y)}{2\delta y y_j} \\
x_{g_1} &= 0,\, x_{g_2} = y_{j+1}.
\end{aligned}
$$

This entails

$$
\begin{aligned}
\beta = \sqrt{\frac{p_{h_1}p_{h_2}}{p_{g_1}p_{g_2}}\frac{(x_{h_1}-x_{h_2})}{(x_{g_1}-x_{g_2})}} &= \sqrt{\frac{\alpha^{\cancel{2}} + \mathcal{O}(\delta y)}{\cancel{6}.3\delta y^2 \cancel{y_j^2}}\frac{\cancel{2}\cancel{\delta y}\cancel{y_j^4}y_j}{\cancel{\delta y}(f(0)\cancel{\alpha} + \mathcal{O}(\delta y))}\frac{\cancel{(3\delta y)^2}}{\cancel{y_j^2 + \mathcal{O}(\delta y)}}} \\
= \sqrt{\frac{y_0'\alpha + \mathcal{O}(\delta y)}{f(0)}} &= \sqrt{\frac{(\Gamma_1 - y_{j-2})(\Gamma_2 - y_{j-2}) + \mathcal{O}(\delta y)}{\Gamma_1\Gamma_2}} \leq 1
\end{aligned}
$$

where we used $f(0) = y_0'\Gamma_1\Gamma_2$ and assumed δy is small compared Γs (which is the case) for the inequality in the last step to hold.

The second condition can be proven similarly

$$
\begin{aligned}
\frac{1}{N_g^2}\left[p_{h_2}x_{h_1} + p_{h_1}x_{h_2} - |\beta|^2\left(p_{g_2}x_{g_1} + p_{g_1}x_{g_2}\right)\right] &\geq \frac{1}{N_g^2}\left[p_{h_2}x_{h_1} + p_{h_1}x_{h_2} - p_{g_2}x_{g_1}\right] \\
&= \frac{1}{N_g^2}\left[\frac{\alpha + \mathcal{O}(\delta y)}{3\delta y y_j}y_{j-1} + \frac{\alpha + \mathcal{O}(\delta y)}{6\delta y y_j}y_{j+2} - \frac{f(0)\delta y + \mathcal{O}(\delta y^2)}{y_j^4}y_{j+1}\right] \\
&= \frac{1}{3\delta y N_g^2}\left[(\alpha + \mathcal{O}(\delta y))\left(\frac{3}{2}\right) - \underbrace{\frac{f(0)\delta y^2 + \mathcal{O}(\delta y^3)}{y_j^3}}_{\in \mathcal{O}(\delta y^2)}\right] \\
&= \frac{1}{2\delta y N_g^2}\left[(\Gamma_1 - y_{j-2})(\Gamma_2 - y_{j-2}) + \mathcal{O}(\delta y)\right] \geq 0
\end{aligned}
$$

where the last step holds for δy small enough.

The $2 \to 2$ move corresponding to the leftmost and bottommost set of points can be shown to be implementable similarly.

Part II
Elliptic Monotone Algorithm (EMA)

5 Canonical Forms Revisited

We note that to construct the unitaries involved in the bias 1/10 protocol we did not follow any systematic recipe. We now switch gears and construct an algorithm that can generate the required unitary for any given Λ-valid function (see Definition 26). Note that corresponding to any WCF protocol with valid functions, one can find a WCF protocol with strictly valid functions (see Lemma 37). All strictly valid functions are Λ-valid for some finite Λ (see Lemma 35, Corollary 29). Thus we do not lose generality by restricting to Λ-valid functions.

In this section we will formalise the non-trivial constraint Equation (1) into two forms which we call the Canonical Projective Form (CPF) and the Canonical Orthogonal Form (COF). The CPF is always well defined but the corresponding COF may contain diverging eigenvalues. However since we restrict to Λ-valid functions, as we will see, the COF will also always be well defined. We need the COF as it is this that we use in the Elliptic Monotone Algorithm (EMA) algorithm.

5.1 The Canonical Projective Form (CPF) and the Canonical Orthogonal Form (COF)

We will always use the convention $p_{g_i}, p_{h_i} > 0$. This is important else in some of the statements one can find trivial counter-examples. We begin with formally stating the main result of Section 3.

Proposition. *For an x-transition (where Alice performs the non-trivial step)*

$$\sum_{i=1}^{n_g} p_{g_i}[x_{g_i}] \to \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}]$$

to be implementable under the TDPG to Explicit protocol Framework (TEF) one must find a $U^{(2)}$ that satisfies the inequality

$$\sum_{i=1}^{n_h} x_{h_i} |h_i h_i\rangle \langle h_i h_i|_{AM} \geq \sum_{i=1}^{n_g} x_{g_i} E_h^{(2)} U^{(2)} |g_i g_i\rangle \langle g_i g_i|_{AM} U^{(2)\dagger} E_h^{(2)} \tag{2}$$

and the honest action constraint

$$U^{(2)} |v\rangle = |w\rangle$$

where $|h_i\rangle$ and $|g_i\rangle$ are orthonormal basis vectors, $|v\rangle = \mathcal{N}\left(\sum \sqrt{p_{g_i}} |g_i g_i\rangle_{AM}\right)$ and $|w\rangle = \mathcal{N}\left(\sum \sqrt{p_{h_i}} |h_i h_i\rangle_{AM}\right)$ for $\mathcal{N}(|\psi\rangle) = |\psi\rangle / \sqrt{\langle\psi|\psi\rangle}$, $E_h = (\sum_{i=1}^{n_h} |h_i\rangle\langle h_i|_A + \sum |k_i\rangle\langle k_i|_A) \otimes \mathbb{I}_M$ with $U^{(2)}$'s non-trivial action restricted to span $\{\{|g_i g_i\rangle_{AM}\}, \{|h_i h_i\rangle_{AM}\}\}$ (recall $|k_i\rangle$ were the points we left unchanged in a transition).

Note that the number of points initially and finally may differ.

To facilitate further discussion we formalise the aforesaid condition into an object and its property. First, however, we define the following notation.

Definition 44 (Transition)**.** Consider two finitely supported functions $g, h : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$. A transition is defined as $g = \sum_{i=1}^{n_g} p_{g_i}[x_{g_i}] \to h = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}]$ where $[y](x) := \delta_{xy}$ and $p_{g_i} > 0$, $p_{h_i} > 0$.

Definition 45 (Canonical Projective Form (CPF) for a give transition)**.** For a given transition the *Canonical Projective Form (CPF)* is given by the set of $m \times m$ matrices X_h, X_g, U, D and m dimensional vectors $|v\rangle$, $|w\rangle$ where

$$X_h := \sum x_{h_i} |h_i\rangle\langle h_i|, \; X_g := \sum x_{g_i} |g_i\rangle\langle g_i|,$$

$$|w\rangle := \sum \sqrt{p_{h_i}} |h_i\rangle, \; |v\rangle := \sum \sqrt{p_{g_i}} |g_i\rangle,$$

$$D := X_h - E_h U X_g U^\dagger E_h$$

and U is unitary which satisfies

$$U|v\rangle = |w\rangle$$

for $E_h = \sum |h_i\rangle\langle h_i|$, orthonormal basis vectors $\{|g_1\rangle, |g_2\rangle \ldots |g_{n_g}\rangle, |h_1\rangle, |h_2\rangle \ldots |h_{n_h}\rangle\}$, $m = n_g + n_h$.

Definition 46 (legal CPF)**.** A CPF is *legal* if $D \geq 0$.

In this language then our objective is to find a legal CPF for a given transition.

Surprising as it may seem it suffices to *restrict ourselves to real unitaries viz. orthogonal matrices*. This will be justified in the next section but we will already make this restriction in everything that follows (unless stated otherwise). In this section we will try to reach an equivalence between a legal CPF and what we call the legal Canonical Orthogonal Form (COF).

The latter will be, roughly speaking, an inequality of the form $X_h - OX_gO^T \geq 0$ where $X_h = \text{diag}\{x_{h_1}, x_{h_2} \dots, x_{h_{n_h}}, \xi, \xi \dots\}$ and $X_g = \text{diag}\{x_{g_1}, x_{g_2} \dots, x_{g_{n_g}}, 0, 0 \dots\}$. It is easy to see that if we can find an O that satisfies the COF for a given transition then the same O would satisfy the TEF inequality. It is almost trivial to note that a Λ valid function will admit matrices of the COF form but we will show this later. Proving the other way, i.e. every legal CPF entails the corresponding COF must also be legal, is more non-trivial. Doing this requires handling the infinities and the matrix sizes more carefully. We will only sketch an argument for this as we don't use it in the algorithm.

Definition 47 ((n,ξ) Canonical Orthogonal Form (COF) for a transition, ξ COF for a transition)**.** For a given transition and two numbers $n \geq \max(n_h, n_g)$, $\xi \geq \max(x_{h_1}, x_{h_2} \dots x_{h_{n_h}})$ an (n,ξ) *Canonical Orthogonal Form (COF)* is given by the set of $n \times n$ matrices X_h, X_g, O, D and vectors $|v\rangle$, $|w\rangle$ where

$$X_h := \text{diag}\{x_{h_1}, x_{h_2} \dots, x_{h_{n_h}}, \xi, \xi \dots\},$$

$$X_g := \text{diag}\{x_{g_1}, x_{g_2} \dots, x_{g_{n_g}}, 0, 0 \dots\},$$

$$|v\rangle := \sum_{i=1}^{n_g} \sqrt{p_{g_i}} \, |i\rangle \, ,$$

$$|w\rangle := \sum_{i=1}^{n_h} \sqrt{p_{h_i}} \, |i\rangle \, ,$$

$$D := X_h - OX_gO^T$$

and the matrix O is orthogonal which satisfies

$$|v\rangle = O \, |w\rangle \, .$$

A ξ *Canonical Orthogonal Form (COF)* is an (n,ξ) COF with $n = n_h + n_g - 1$.

Definition 48 (n legal COF, legal COF)**.** An (n,ξ) COF is an n *legal COF* if $D \geq 0$ in the limit $\xi \to \infty$. A *legal COF* is a ξ COF such that $D \geq 0$ in the limit $\xi \to \infty$.

Imagine you found a legal COF corresponding to some transition. One can then sandwich D between a positive matrix as EDE to get

$$\left[\begin{array}{c|ccc} X_h & & & \\ \hline & 1 & & \\ & & \ddots & \\ & & & 1 \end{array}\right] - \underbrace{\left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & \xi^{-1/2} & & \\ & & & & \ddots & \\ & & & & & \xi^{-1/2} \end{array}\right]}_{:=E} UX_gU^\dagger \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & \xi^{-1/2} & & \\ & & & & \ddots & \\ & & & & & \xi^{-1/2} \end{array}\right].$$

Note that $D \geq 0 \iff EDE \geq 0$ because $E > 0$ (which means one can write $EDE = (E\sqrt{D})(\sqrt{D}E)$ which in turn is of the A^TA form). From the legality of the COF, $D \geq 0$ in the limit $\xi \to \infty$ and in this limit E becomes a projector. After some relabelling (and appropriately expanding the space to $m = n_g + n_h$ dimensions) the inequality reduces to a CPF. This observation readily extends to the n legal case where $n \leq n_g + n_h$. It turns out that one can, and we will show this later, always express an n' legal COF as an n legal COF with $n \leq n_g + n_h$ (in fact we can prove that $n \leq n_g + n_h - 1$). We have established the following statement.

Proposition 49. *Consider a transition. If there exists an n legal COF corresponding to it then there exists a legal CPF for the said transition.*

How about the reverse? Given a legal CPF can we find the corresponding n legal COF? We are given

$$D = \left[\begin{array}{c|ccc} X_h & & & \\ \hline & 0 & & \\ & & \ddots & \\ & & & 0 \end{array}\right] - \underbrace{\left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{array}\right]}_{=E_h} U \left[\begin{array}{c|c} 0 & \\ \hline & X_g \end{array}\right] U^\dagger \left[\begin{array}{ccc|ccc} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ \hline & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{array}\right] \geq 0.$$

Replacing the appended diagonal zeros in the first matrix (one containing X_h) with 1s yields an equivalent inequality. Next note that we can conjugate by a permutation matrix to get

$$\left[\begin{array}{c|c} 0 & \\ \hline & X_g \end{array}\right] = \tilde{U} \left[\begin{array}{c|c} X_g & \\ \hline & 0 \end{array}\right] \tilde{U}.$$

Finally we write the diagonal zeros in E_h as $1/\xi^{1/2}$ and use the reverse of the trick above to recover an m legal COF where recall $m := n_g + n_h$. This sketches the proof of the following statement.

Proposition 50. *Consider a transition. If there exists legal CPF corresponding to it then there exists an m legal COF for the said transition (where recall $m := n_g + n_h$).*

5.2 From EBM to EBRM to COF

We briefly summarise, at the cost being redundant, how Ahoronov et al prove that valid functions are equivalent to the EBM functions (assuming the operator monotones are on/the spectrum of the matrices is in $[0, \Lambda]$). They do this is by showing that the set of EBM functions forms a convex cone K. Then they take the dual of this cone to get K^*. *This dual happens to be the set of operator monotone functions.* Then they find the bi-dual K^{**} and define the objects in this to be valid functions. They then show that $K = K^{**}$ which is to say that valid functions are equivalent to EBM functions. Note that all of this is assuming the aforesaid $[0, \Lambda]$ condition.

This is an extremely useful result because checking if a function is EBM is hard. Checking if a function is valid is a piece of cake because mathematical wizards have neatly characterised the set of operator monotone functions.

It turns out that one can do even better. Instead of EBM functions, consider EBRM functions where the matrices are additionally restricted to be real. Let this set be given by K'. It turns out that its dual K'^* is also the set of operator monotone functions viz. $K'^* = K^*$ (we will show this shortly; one of the authors had asked this on stackoverflow and Tobias Fritz at the Max Planck Institute answered it showing it to be almost trivial). Ahoronov et al's proof for $K = K^{**}$ can be applied to the real case as is to get $K' = K^{**}$ (granted we assume the same $[0, \Lambda]$ condition).

Since Mochon's point games (and even the ones built later) use valid functions, the aforesaid simplification justifies why it suffices to restrict to real matrices.

We will use the definition of Prob (Definition 6), EBM line transition (Definition 7), EBM function (Definition 12, Definition 13), Operator Monotone functions (Definition 20, Definition 21) and their characterisation (Lemma 23, Lemma 24), Λ valid functions (Definition 26) and finally its equivalence with EBM functions (Corollary 29).

Equivalence of EBM and EBRM

First we define EBRM transitions and EBRM functions similar to their EBM analogues except with the further restriction that the matrices and vectors involved are real.

Definition 51 (EBRM transitions)**.** Let $g, h : [0, \infty) \to [0, \infty)$ be two functions with finite supports. The transition $g \to h$ is EBRM if there exist two real matrices $0 \le G \le H$ and a (not necessarily normalised) vector $|\psi\rangle$ such that $g = \text{prob}[G, \psi]$ and $h = \text{prob}[H, \psi]$.

Definition 52 (K', EBRM functions; K'_Λ, EBRM functions on $[0, \Lambda]$)**.** A function $a : [0, \infty) \to \mathbb{R}$ with finite support is an EBRM function if the transition $a^- \to a^+$ is EBRM, where $a^+ : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ and $a^- : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ denote, respectively, the positive and the negative part of a ($a = a^+ - a^-$). We denote by K' the set of EBRM functions.

For any finite $\Lambda \in (0, \infty)$, a function $a : [0, \Lambda) \to \mathbb{R}$ with finite support is an EBRM function with support on $[0, \Lambda]$ if the transition $a^- \to a^+$ is EBRM with its spectrum in $[0, \Lambda]$, where a^+ and a^- denote, respectively, the positive and the negative part of a (again, $a = a^+ - a^-$). We denote by K'_Λ the set of EBRM functions with support on $[0, \Lambda]$.

Definition 53 (Real operator monotone functions)**.** A function $f : (0, \infty) \to \mathbb{R}$ is real operator monotone if for all real matrices $0 \le A \le B$ we have $f(A) \le f(B)$.

A function $f : (0, \Lambda) \to \mathbb{R}$ is real operator monotone on $[0, \Lambda]$ if for all real matrices $0 \le A \le B$ with spectrum in $[0, \Lambda]$ we have $f(A) \le f(B)$.

Lemma 54. *$K^*_\Lambda = K'^*_\Lambda$ and $K^* = K'^*$, i.e. the set of operator monotones on $[0, \Lambda]$ = the set of real operator monotones on $[0, \Lambda]$ and the set of operator monotones = the set of real operator monotones.*

Proof sketch. Ahoronov showed that K^*_Λ is the set of operator monotone functions on $[0, \Lambda]$. His proof can be adapted here by restricting to real matrices which entails that K'^*_Λ is the set of real operator monotone functions on $[0, \Lambda]$. ☐

Corollary 55. *$K'_\Lambda = K'^{**}_\Lambda = K^{**}_\Lambda = K_\Lambda$, i.e. the set of EBRM functions on $[0, \Lambda]$ = the set of Λ valid functions (dual of EBRM functions) = the set of Λ valid functions (dual of EBM functions) = the set of EBM functions on $[0, \Lambda]$.*

Corollary 56. *Any strictly valid function is EBRM.*

We now sketch the proof the lemma. It is clear that the set of real operator monotones must contain the set of operator monotones because operator monotones are by definition required to work in the restricted real case as well. The set of real operator monotones might be bigger but that doesn't happen to be the case. This is because we can encode an $n \times n$ hermitian matrix into a $2n \times 2n$ real symmetric matrix. This is achieved by replacing for each complex number $\alpha + i\beta$ with the matrix

$$\alpha \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \beta \begin{bmatrix} & -1 \\ 1 & \end{bmatrix}.$$

Note that the matrices have the exact same properties as 1 and i respectively. This corresponds to (after some permutation) writing a complex matrix $W = W_{\Re} + iW_{\Im}$ as a real symmetric

$$W' = \begin{bmatrix} W_{\Re} & -W_{\Im} \\ W_{\Im} & W_{\Re} \end{bmatrix}$$

where $W_{\Re}$ and $W_{\Im}$ are real. For Hermitian $W^\dagger = W$ we must have $W_{\Im} = -W_{\Im}$ which makes $W' = W'^T$ a symmetric matrix. The spectrum of W and W' are the same. This is established most easily by looking at the diagonal decomposition. $W = USU^\dagger$ which would get transformed to $W' = U'S'U'^\dagger$. Since S is real S' is also real with doubly degenerate eigenvalues (except for the degenerecy already present in S). Thus if we have $0 \le A \le B$ then we would also have $0 \le A' \le B'$ as $A - B$ and $A' - B'$ would have the same spectrum where we used A' and B' to represent real symmetric analogues of the hermitian matrices A and B. The other way is trivial. Hence we have an equivalence which means that requiring a function to be operator monotone on complex matrices is the same as requiring it to be operator monotone on real symmetric matrices of twice the size (at most). This means that the set of real operator monotones is the same as the set of operator monotones.

EBRM to COF | Mochon's Variant

Lemma 57. *For every EBRM transition $g \to h$ with spectrum in $[a,b]$ there exists an orthogonal matrix O, diagonal matrices X_h, X_g (with no multiplicities except possibly those of a and b) of size at most $n_g + n_h - 1$ such that*

$$O \underbrace{\begin{bmatrix} x_{g_1} & & & & \\ & \ddots & & & \\ & & x_{g_{n_g}} & & \\ & & & a & \\ & & & & \ddots \end{bmatrix}}_{:=X_g} O^T \le \begin{bmatrix} x_{h_1} & & & & \\ & \ddots & & & \\ & & x_{h_{n_h}} & & \\ & & & b & \\ & & & & \ddots \end{bmatrix} = X_h,$$

and the vector $|\psi\rangle := \sum_{i=1}^{n_h} \sqrt{p_{h_i}} |i\rangle = \sum_{i=1}^{n_g} \sqrt{p_{g_i}} O |i\rangle$.

Proof. An EBRM entails that we are given $G \le H$ with their spectrum in $[a,b]$ and a $|\psi\rangle$ such that

$$g = \text{Prob}[G, |\psi\rangle] = \sum_{i=1}^{n_g} p_{g_i}[x_{g_i}]$$

and

$$h = \text{Prob}[H, |\psi\rangle] = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}]$$

with $p_{g_i}, p_{h_i} > 0$ and $x_{g_i} \ne x_{g_j}$, $x_{h_i} \ne x_{h_j}$ for $i \ne j$ but the dimension and multiplicities can be arbitrary. First we show that one can always choose the eigenvectors $|g_i\rangle$ of G with eigenvalue x_{g_i} such that

$$|\psi\rangle = \sum_{i=1}^{n_g} \sqrt{p_{g_i}} |g_i\rangle.$$

Consider P_{g_i} to be the projector on the eigenspace with eigenvalue x_{g_i}. Note that

$$|g_i\rangle := \frac{P_{g_i} |\psi\rangle}{\sqrt{\langle\psi| P_{g_i} |\psi\rangle}}$$

fits the bill. Similarly we choose/define $|h_i\rangle$ so that

$$|\psi\rangle = \sum_{i=1}^{n_h} \sqrt{p_{h_i}} |h_i\rangle.$$

Consider now the projector onto the $\{|g_i\rangle\}$ space

$$\Pi_g = \sum_{i=1}^{n_g} |g_i\rangle \langle g_i| .$$

Note that this will not have all eigenvectors with eigenvalues $\in \{x_{g_i}\}$. Similarly we define

$$\Pi_h = \sum_{i=1}^{n_h} |h_i\rangle \langle h_i| .$$

We further define $G' := \Pi_g G \Pi_g + a(\mathbb{I} - \Pi_g)$ and $H' := \Pi_h H \Pi_h + b(\mathbb{I} - \Pi_h)$. These definitions are useful as we can show

$$G' \leq H'.$$

From $G = \Pi_g G \Pi_g + (\mathbb{I} - \Pi_g) G (\mathbb{I} - \Pi_g)$ we can conclude that $\Pi_g G \Pi_g + a(\mathbb{I} - \Pi_g) \leq G$. This entails $G' \leq G$. Using a similar argument one can also establish $H \leq H'$. Combining these we get $G' \leq H'$.
Consider the projector

$$\Pi := \text{projector on span}\{\{g_i\}_{i=1}^{n_g}, \{h_i\}_{i=1}^{n_h}\}$$

and note that this has at most $n_g + n_h - 1$ dimension because $|\psi\rangle$ lives in the span of $\{g_i\}$ and in the span of $\{h_i\}$ so one of the basis vectors at least is not independent. Now note that

$$G'' := \Pi G' \Pi \leq \Pi H' \Pi =: H''$$

because we can always conjugate an inequality by a positive semi-definite matrix on both sides. Note also that $\Pi |\psi\rangle = |\psi\rangle$ which means the matrices and the vectors have the claimed dimension. We will now establish that $\text{Prob}[H'', |\psi\rangle] = h$ and $\text{Prob}[G'', |\psi\rangle] = g$. For this we first write the projector tailored to the g basis as $\Pi = \Pi_g + \Pi_{g_\perp}$ where $\Pi_{g_\perp}$ is meant to enlarge the space to the $\text{span}\{h_i\}_{i=1}^{n_h}$. With this we evaluate

$$\begin{aligned} G'' &= (\Pi_g + \Pi_{g_\perp}) \left[\Pi_g G \Pi_g + a(\mathbb{I} - \Pi_g)\right] (\Pi_g + \Pi_{g_\perp}) \\ &= \Pi_g G \Pi_g + a \Pi_{g_\perp} . \end{aligned}$$

Manifestly then $\text{Prob}[G'', |\psi\rangle] = g$. By a similar argument one can establish the h claim. Note that that G'' and H'' have no multiplicities except possibly in a and b respectively. Thus we conclude we can always restrict to the claimed dimension and form. □

Corollary 58. *For every EBRM transition the corresponding COF is legal.*

6 Ellipsoids

6.1 The inequality as containment of ellipsoids

Consider an unnormalised vector $|u\rangle = \sum_j u_j |h_j\rangle$ with $u_j \in \mathbb{R}$. Note that the set

$$\{|u\rangle \mid \langle u| X_h |u\rangle = 1\}$$

describes the surface of an ellipsoid. This is easy to see as the constraint corresponds to

$$x_{h_1} u_1^2 + x_{h_2} u_2^2 + \cdots = 1$$

which is of the form

$$\frac{u_1^2}{a_1^2} + \frac{u_2^2}{a_2^2} + \cdots = 1$$

which is the equation of an ellipsoid in the variables $\{u_i\}$ with axes $a_1 = 1/\sqrt{x_{h_1}}, a_2 = 1/\sqrt{x_{h_2}} \ldots$. An inequality would correspond to points inside or outside the ellipsoid. It is also useful to note that if we start with some arbitrary (even unnormalised) vector $|u\rangle$ then the point on the ellipse along this direction will be given by

$$\mathcal{E}_h(|u\rangle) = \frac{|u\rangle}{\sqrt{\langle u| X_h |u\rangle}}.$$

Finally, note also that the set $\left\{|u\rangle \mid \langle u| U X_g U^\dagger |u\rangle = 1\right\}$ also corresponds to the equation of an ellipse with axes $\left\{1/\sqrt{x_{g_i}}\right\}$ except that it is rotated. This follows from the fact that if we use $|u'\rangle = U |u\rangle$ then the equation reduces to the standard form in the u_i' variables which can then be used to obtain u_is by the aforesaid relations which is a rotation. We can define a similar map from a vector $|u\rangle$ to a point on the rotated ellipse as

$$\mathcal{E}_g(|u\rangle) = \frac{|u\rangle}{\sqrt{\langle u| U X_g U^\dagger |u\rangle}}.$$

With this understanding in place we are ready to get a visual interpretation of our equation. The statement that

$$\begin{aligned} & X_h - U X_g U^\dagger \geq 0 \\ \iff & \langle u| X_h |u\rangle - \langle u| U X_g U^\dagger |u\rangle \geq 0 && \forall |u\rangle \\ \iff & \langle u| U X_g U^\dagger |u\rangle \leq 1 && \forall \{|u\rangle \,|\, \langle u| X_h |u\rangle = 1\} \end{aligned}$$

which in turn corresponds to the statement that every point denoted by $|u\rangle$ that is on the h ellipse must be on or inside the g ellipse. Note that if $\langle x_h\rangle - \langle x_g\rangle = 0$ then for $|u\rangle = |w\rangle$ the inequality saturates. This in turn means that even for $\mathcal{E}_h(|w\rangle)$ the inequality is saturated as it is the same vector up to scaling. The difference is that $\mathcal{E}_h(|w\rangle)$ represents a point on the h ellipsoid. Since the inequality is saturated it means that the ellipsoids must touch at this point. Thus $\mathcal{E}_g(|w\rangle) = \mathcal{E}_h(|w\rangle)$ which one can check explicitly as well.

6.2 Convex Geometry Tools | Weingarten Map and the Support Function

For a normalised direction vector $|u\rangle$ the support function corresponding to an ellipsoid X is given by

$$h(u) = \sqrt{\langle u| X^{-1} |u\rangle} = \sqrt{\sum x_i^{-1} u_i^2}. \tag{3}$$

The derivative of the support function yields the point on the ellipsoid where the tangent plane corresponding to the direction $|u\rangle$ touches the said ellipsoid. It is

$$\partial_i h(u) = \frac{x_i^{-1} u_i}{h(u)}.$$

The most remarkable of all these is the fact is that

$$h\partial_j \partial_i h(u) = \left(-\frac{x_j^{-1} x_i^{-1} u_i u_j}{h^2} + x_i^{-1} \delta_{ij} \right) \tag{4}$$

contains as eigenvalues the radii of curvature at the aforesaid point and as eigenvectors the directions of principle curvature. If instead of the normal you know the point at which you'd like to evaluate this object then one can use the gradient to first find this normal and then apply the aforesaid. The normal at a point of contact $|c\rangle = \sum c_i |i\rangle$ is $|u(c)\rangle = \mathcal{N}\left(\sum x_i c_i |i\rangle\right)$. The results discussed here were deduced as special cases of those discussion in §2.5 of the book on convex bodies by R. Schneider [11].

7 Elliptic Monotone Align (EMA) Algorithm

Solving the weak coin flipping (WCF) problem can be reduced to finding explicit matrices for a given EBM transition $g = \sum_{i=1}^{n_g} p_{g_i}[x_{g_i}] \to h = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}]$ where g and h have disjoint support or, equivalently, for a given EBM function $a = h - g = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}] - \sum_{i=1}^{n_g} p_{g_i}[x_{g_i}]$. Here we describe an algorithm, the elliptic monotone align (EMA) algorithm, which runs by converting the given problem into the same problem of one less dimension iteratively until it is solved.

7.1 Notation

At step k of the iteration, the transition $g \to h$ and the associated function $a = h - g$ used below will be given by $g^{(k)} \to h^{(k)}$ and $a^{(k)}$ respectively. It will remain fixed for the said step. We will therefore not write an explicit dependence on it in the following definitions. We consider the extended real line $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ with $1/\infty = -1/\infty := 0$. We will also need the extended half line $\bar{\mathbb{R}}_\geq := \mathbb{R}_\geq \cup \{\infty\}$ and $\bar{\mathbb{R}}_> := \mathbb{R}_> \cup \{\infty\}$. We will use $\mathcal{N}(|\psi\rangle) := |\psi\rangle = \sqrt{\langle\psi|\psi\rangle}$. We will usually denote by $[x_{\min}, x_{\max}]$ the smallest interval that contains $\mathrm{supp}(a)$. We will call this interval the *support domain* for a. Similarly, we would refer to the smallest interval containing $\mathrm{supp}(g) \cup \mathrm{supp}(h)$ as the *transition support domain* for (the transition) $g \to h$. We will use the variables $\chi, \xi \in \mathbb{R}$ to be such that they denote an interval $[\chi, \xi] \supseteq [x_{\min}, x_{\max}]$. We would refer to this interval as the *spectral domain*.

Definition 59 (f_λ on (α, β))**.** $f_\lambda : (\alpha, \beta) \to \mathbb{R}$ is defined for $\lambda \in \mathbb{R}\backslash[-\beta, -\alpha]$ as

$$f_\lambda(x) := \frac{-1}{\lambda + x}.$$

Definition 60 (f_λ on $[\alpha, \beta]$)**.** $f_\lambda : [\alpha, \beta] \to \bar{\mathbb{R}}$ is defined for $\lambda \in \mathbb{R}\backslash(-\beta, -\alpha)$. For $\lambda \in \mathbb{R}\backslash[-\beta, -\alpha]$ we define

$$f_\lambda(x) := \frac{-1}{\lambda + x}.$$

For $\lambda = -\beta$ and $-\alpha$ we retain the same definition as above except when $x = \beta$ and α respectively in which case we define

$$\begin{aligned} f_{-\beta}(\beta) &:= \infty \\ f_{-\alpha}(\alpha) &:= -\infty. \end{aligned}$$

Remark 61. Values for $f_{-\beta}(\beta)$ and $f_{-\alpha}(\alpha)$ are obtained by taking for x, respectively, the left limit (approaching from the left to β) and right limit (approaching from the right to α). Also note that the operator monotone $f(x) = x$ is not included in the aforesaid family of functions.

Definition 62 ($l_\gamma, l^1_\gamma, a_\gamma$)**.** Consider the transition $g \to h$ and let $a = h - g$. For $\gamma \in (0, 1]$ we define the finitely supported function $h_\gamma : \mathbb{R} \to \mathbb{R}_\geq$, $a_\gamma(x) : \mathbb{R} \to \mathbb{R}$ as

$$h_\gamma(x) := h(x/\gamma)$$
$$a_\gamma(x) := h_\gamma(x) - g(x).$$

Let $S_\gamma = [x_{\min}(\gamma), x_{\max}(\gamma)]$ be the support domain of a_γ. We define $l_\gamma : \mathbb{R}\backslash[-x_{\max}(\gamma), -x_{\min}(\gamma)] \to \mathbb{R}$

$$l_\gamma(\lambda) := \sum_{x \in \text{supp}(a_\gamma)} a_\gamma(x) f_\lambda(x)$$

where f_λ is defined on S_γ.

We define

$$l^1_\gamma := \sum_{x \in \text{supp}(a_\gamma)} a_\gamma(x) x.$$

Remark 63. h_γ and g might have overlapping support for certain values of γ which we handle by generalising some of the earlier results in the next subsection.

Definition 64 ($m(\gamma, \chi, \xi)$)**.** We define $m : ((0, 1], \mathbb{R}, \mathbb{R}) \to \{0, 1\}$ to be

$$m(\gamma, \chi, \xi) := \begin{cases} 0 & \text{if any of the following root conditions hold} \\ 1 & \text{else.} \end{cases}$$

where the first root condition is satisfied if there exists a $\lambda \in \mathbb{R}\backslash(-\xi, -\chi)$ such that $l_\gamma(\lambda) = 0$, and the second root condition is satisfied if $l^1_\gamma = 0$.

Definition 65 (Matrix Instance, $\underline{\text{X}} \to$ Function Instance, $\underline{\text{x}}$)**.** For a *Matrix Instance* defined to be the tuple $\underline{\text{X}} := (X_h, X_g, |w\rangle, |v\rangle)$ where X_h, X_g are diagonal matrices and $|w\rangle, |v\rangle$ are vectors on $\mathbb{R}^n$ for some n with equal norm, i.e. $\langle w|w\rangle = \langle v|v\rangle$, we define the *Function Instance* to be the tuple $\underline{\text{x}} : \{g, h, a\}$ where $h = \text{Prob}[X_h, |w\rangle]$, $g = \text{Prob}[X_g, |v\rangle]$ and $a = h - g$.

Definition 66 (Attributes of the Function Instance, $\underline{\text{x}}$)**.** For a given set $\underline{\text{x}} := (g, h, a)$ as Definition 65 we define the attributes $n_h, n_g, \{p_{g_i}\}, \{p_{h_i}\}, \{x_{g_i}\}, \{x_{h_i}\}$ as they appear by declaring $g \to h$ to be a transition, i.e.,

- n_h as the number of times h is non-zero,
- n_g as the number of times g is non-zero,
- $\{p_{h_i}\}, \{x_{h_i}\}, \{p_{g_i}\}, \{x_{g_i}\}$ implicitly as

  $$h = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}],\ g = \sum_{i=1}^{n_g} p_{g_i}[x_{g_i}]$$

  (for $p_{h_i}, p_{g_i} > 0$).

The *support domain* for a is denoted by $[x_{\min}, x_{\max}]$, i.e., the attributes $x_{\min}, x_{\max}$ are defined to be such that $[x_{\min}, x_{\max}]$ is the smallest interval containing $\text{supp}(a)$.

Remark 67. Note that $x_{\min}$ and $x_{\max}$ may not be $x_{\min} = \min\{\{x_{h_i}\}, \{x_{g_i}\}\}$ and $x_{\max} = \max\{\{x_{h_i}\}, \{x_{g_i}\}\}$ respectively because there can be cancellations in the evaluation of $h - g = a$.

Definition 68 (Attributes of the Matrix Instance, $\underline{\text{X}}$)**.** We associate the following with a matrix instance.

- *Spectral domain*: For a set $\underline{\text{X}}$ as defined in definition 65 we denote the *spectral domain* by $[\chi, \xi]$ where the attributes χ, ξ are such that $[\chi, \xi]$ is the smallest interval containing $\text{spec}\{X_g \oplus X_h\}$.
- *Solution*: We say that O is a *solution* to the matrix instance $\underline{\text{X}} = \{X_h, X_g, |w\rangle, |v\rangle\}$ if $X_h \geq O X_g O^T$ and $O|v\rangle = |w\rangle$.
- *Notation*: With respect to a standard orthonormal basis $\{|i\rangle\}$, we will use the *notation* $X_h := \sum_{i=1}^k y_{h_i} |i\rangle\langle i|$, $X_g := \sum_{i=1}^k y_{g_i} |i\rangle\langle i|$, $|w\rangle := \sum_{i=1}^k \sqrt{q_{h_i}} |i\rangle$, $|v\rangle := \sum_{i=1}^k \sqrt{q_{g_i}} |i\rangle$.

Remark 69. We will index the Matrix Instance and the corresponding function instance as $\underline{\text{X}}^{(k)} = \{X_h^{(k)}, X_g^{(k)}, |w^{(k)}\rangle, |v^{(k)}\rangle\}$ and $\underline{\text{X}}^{(k)} \to \underline{\text{x}}^{(k)} = \{h^{(k)}, g^{(k)}, a^{(k)}\}$ respectively. The associated attributes will be implicitly assumed to be correspondingly indexed, e.g., as $\chi^{(k)}, \xi^{(k)}$ and $n_h^{(k)}, n_g^{(k)}, x_{\min}^{(k)}, x_{\max}^{(k)}$.

Remark 70. We introduce two different symbol sets p, x and q, y as it allows us to describe the proof more neatly by allowing two ways of indexing the same object. We use p, x for $\underline{\text{x}}$ and q, y for $\underline{\text{X}}$ which are essentially the same.

7.2 Lemmas for EMA

7.2.1 Generalisations

It might be useful to keep the bigger picture, Figure 5, in mind.

Definition 71 (Canonical Orthogonal Form (COF) with spectrum in $[\chi,\xi]$)**.** For a given transition $g \to h$ let $[\chi,\xi]$ be such that it contains $\text{supp}(g)$ and $\text{supp}(h)$. We define the Canonical Orthogonal Form (COF) with its spectrum in $[\chi,\xi]$ by the set of $n \times n$ matrices X_h, X_g, O, D and vectors $|v\rangle, |w\rangle$ where

$$X_h := \text{diag}\{x_{h_1}, x_{h_2} \dots, x_{h_{n_h}}, \xi, \xi \dots\},$$

$$X_g := \text{diag}\{x_{g_1}, x_{g_2} \dots, x_{g_{n_g}}, \chi, \chi \dots\},$$

$$|v\rangle := \sum_{i=1}^{n_g} \sqrt{p_{g_i}}\, |i\rangle\,,$$

$$|w\rangle := \sum_{i=1}^{n_h} \sqrt{p_{h_i}}\, |i\rangle\,,$$

$$D := X_h - O X_g O^\dagger,$$

the matrix O is orthogonal which satisfies

$$|v\rangle = O\,|w\rangle$$

and $n = n_g + n_h - 1$.

Definition 72 (Legal COF with spectrum in $[\chi,\xi]$)**.** A COF with spectrum in $[\chi,\xi]$ is legal if $D \geq 0$.

Definition 73 (Operator monotone functions on $[\chi,\xi]$)**.** A function $f : [\chi,\xi] \to \mathbb{R}$ is operator monotone on $[\chi,\xi]$ if for all real symmetric matrices H, G with $\text{spec}(H \oplus G) \in [\chi,\xi]$ and $H \geq G$ we have $f(H) \geq f(G)$.

Claim 74. $f(x)$ is an operator monotone function on $[\chi,\xi]$ if and only if $f'(x') = f(x' - x_0)$ is an operator monotone function on $[\chi + x_0, \xi + x_0]$.

Proof. Consider real symmetric matrices $H \geq G$ with $\text{spec}(H \oplus G) \in [\chi,\xi]$ and let $f(x)$ be operator monotone on $[\chi,\xi]$. We must consider $f'(x') = f(x = x' - x_0)$ which is the same as $f'(x + x_0) = f(x)$. We will show that f' is an operator monotone on $[\chi + x_0, \xi + x_0]$. Note that $H' := H + x_0\mathbb{I}$ and $G' := G + x_0\mathbb{I}$ are such that $H' \geq G'$ and $\text{spec}(H' \oplus G') \in [\chi + x_0, \xi + x_0]$. Note that $f'(H') = f(H)$ and $f'(G') = f(G)$ because

$$\begin{aligned} f'(H') &= f'(H + x_0\mathbb{I}) \\ &= O_h f'(H_d + x_0\mathbb{I}) O_h^T \\ &= O_h f(H_d) O_h^T \\ &= f(H) \end{aligned}$$

and similarly for G where $H = O_h H_d O_h^T$ for O_h orthogonal and H_d diagonal. Since f is operator monotone on $[\chi,\xi]$ we have $f(H) \geq f(G)$ which entails $f'(H') \geq f'(G')$. Since this holds for all H', G' with their $\text{spec}(H' \oplus G') \in [\chi + x_0, \xi + x_0]$ we can conclude that f' is an operator monotone on $[\chi + x_0, \xi + x_0]$. The other way follows by setting $\chi + x_0$ to χ, $\xi + x_0$ to ξ, x_0 to $-x_0$ but since all these were arbitrary to start with, the reasoning goes through unchanged. □

Corollary 75 (Characterisation of operator monotone functions on $[0,\Lambda]$)**.** *Any operator monotone function $f : [\chi,\xi] \to \mathbb{R}$ can be written as*

$$f(x) = c_0 + c_1 x - \int \frac{1}{\lambda + x} d\tilde{\omega}(\lambda)$$

with the integral ranging over $\lambda \in (-\infty, -\Lambda) \cup (0, \infty)$ satisfying $\int \frac{1}{\lambda(1+\lambda)} d\tilde{\omega}(\lambda) < \infty$.

Proof. Consider the characterisation given in Lemma 24 according to which we had $f(x) = c_0' + c_1 x + \int \frac{\lambda}{\lambda + x} d\omega(\lambda)$ with $\int \frac{\lambda}{1+\lambda} d\omega(\lambda) < \infty$. We can write

$$\begin{aligned} f(x) &= c_0' + c_1 x + \int \left(\lambda - \frac{\lambda^2}{\lambda + x}\right) d\omega(\lambda) \\ &= c_0 + c_1 x - \int \frac{\lambda^2 d\omega(\lambda)}{\lambda + x} \end{aligned}$$

where with $d\tilde{\omega} = \lambda^2 d\omega(\lambda)$ we obtain the claimed form. Note that the finiteness of $\int \frac{\lambda}{1+\lambda} d\omega$ is necessary to conclude that $c_0 = c_0' + \int \frac{\lambda}{1+\lambda} d\omega$ is also finite. □

Corollary 76 (Characterisation of operator monotone functions on $[\chi, \xi]$)**.** *Any operator monotone function $f' : [\chi, \xi] \to \mathbb{R}$ can be written as*

$$f'(x') = c'_0 + c'_1 x' - \int \frac{1}{\lambda' + x'} d\tilde{\omega}'(\lambda')$$

with the integral ranging over $\lambda' \in (-\infty, -\xi) \cup (-\chi, \infty)$ satisfying $\int \frac{1}{(\lambda'+\chi)(1+\lambda'+\chi)} d\tilde{\omega}'(\lambda') < \infty$.

Proof. We will follow the convention that $x' \in [\chi, \xi]$ while the unprimed $x \in [0, \xi - \chi]$. From Claim 74 we know that $f(x)$ is operator monotone on $[0, \xi - \chi]$ if and only if $f'(x') = f(x' - \chi)$ is operator monotone on $[\chi, \xi]$ where $x' = x + \chi$. Since we already have a characterisation for $f(x)$ we can characterise $f'(x')$ as $f(x' - \chi)$. From Corollary 75 we have

$$\begin{aligned} f'(x') &= c_0 + c_1(x' - \chi) - \int \frac{d\tilde{\omega}(\lambda)}{\lambda + x' - \chi} \\ &= c'_0 + c_1 x' - \int \frac{d\tilde{\omega}'(\lambda')}{\lambda' + x'} \end{aligned}$$

where $\lambda' = \lambda - \chi$. Since we had $\lambda \in (-\infty, -\xi - \chi) \cup (0, \infty)$ it entails $\lambda' \in (-\infty, -\xi) \cup (-\chi, \infty)$. The condition on the integral $\int \frac{d\tilde{\omega}(\lambda)}{\lambda(\lambda+x)} < \infty$ can be expressed in terms of λ' as $\int \frac{d\tilde{\omega}'(\lambda')}{(\lambda'+\chi)(1+\lambda'+\chi)} < \infty$ with $d\tilde{\omega}'(\lambda') = d\tilde{\omega}(\lambda' + \chi)$. With $c_1 = c'_1$ we obtain the claimed form. □

Remark 77. Since in Corollary 76 $d\tilde{\omega}'$ is a measure, to establish (χ, ξ) validity of functions, it would suffice to restrict our attention to operator monotones $f'(x') = x'$, $f'(x') = -\frac{1}{\lambda'+x'}$ with $x' \in [\chi, \xi]$, $\lambda' \in (-\infty, -\xi) \cup (-\chi, \infty)$.

Definition 78 ((χ, ξ) valid function)**.** A finitely supported function $a : \mathbb{R} \to \mathbb{R}$ with $\text{supp}(a) \in [\chi, \xi]$ is (χ, ξ) valid if for every operator monotone function f on $[\chi, \xi]$ we have $\sum_{x \in \text{supp}(a)} a(x) f(x) \geq 0$.

Corollary 79 ($a(x)$ is (χ, ξ) valid $\iff$ $a(x' - x_0)$ is $(\chi + x_0, \xi + x_0)$ valid)**.** *A finitely supported function $a : \mathbb{R} \to \mathbb{R}$ with $supp(a) \in [\chi, \xi]$ is (χ, ξ) valid if and only if the function $a'(x') := a(x' - x_0) : \mathbb{R}_{\geq} \to \mathbb{R}$ is $(\chi - x_0, \xi - x_0)$ valid.*

Proof. a is (χ, ξ) valid entails $\sum_{x \in supp(a)} a(x) f(x) \geq 0$ for all f operator monotone on $[\chi, \xi]$. We can write the sum as $\sum a(x' - x_0) f(x' - x_0) \geq 0$. Using Claim 74 we note that $f'(x') = f(x' - x_0)$ is operator monotone on $[\chi + x_0, \xi + x_0]$. For $a'(x') = a(x' - x_0)$ we thus have $\sum a'(x') f'(x') \geq 0$ which means $a'(x')$ is a $(\chi + x_0, \xi + x_0)$ valid function. The other way follows similarly. □

Definition 80 (EBRM on $[\chi, \xi]$)**.** A finitely supported function $a : \mathbb{R} \to \mathbb{R}$ is EBRM on $[\chi, \xi]$ if there exist real symmetric matrices $H \geq G$ with their spectrum in $[\chi, \xi]$ and a vector $|w\rangle$ such that $a = \text{Prob}[H, |w\rangle] - \text{Prob}[G, |w\rangle]$.

Corollary 81 ($a(x)$ is EBRM on $[\chi, \xi]$ $\iff$ $a(x + \chi)$ is EBRM on $[0, \xi - \chi]$)**.** *A finitely supported function $a : \mathbb{R} \to \mathbb{R}$ with $supp(a) \in [\chi, \xi]$ is EBRM on $[\chi, \xi]$ if and only if the function $a'(x) := (a + \chi) : \mathbb{R}_{\geq} \to \mathbb{R}$ is EBRM on $[0, \xi - \chi]$.*

Proof. If a is EBRM on $[\chi, \xi]$ it follows that there exist real symmetric matrices with $H \geq G$ and a vector $|w\rangle$ such that $\text{spec}[H \oplus G] \in [\chi, \xi]$ and $a = \text{Prob}[H, |w\rangle] - \text{Prob}[G, |w\rangle]$. Clearly, $H' := H - \chi \mathbb{I} \geq G - \chi \mathbb{I} =: G'$ and $a'(x) = \text{Prob}[H', |w\rangle] - \text{Prob}[G', |w\rangle] = a(x + \chi)$ with $\text{spec}[H' \oplus G'] \in [0, \xi - \chi]$. This means a' is EBRM on $[0, \xi - \chi]$. The other way follows similarly. □

Lemma 82 ($a(x)$ is (χ, ξ) valid function $\iff$ $a(x)$ is EBRM on $[\chi, \xi]$)**.** *A finitely supported function $a : \mathbb{R} \to \mathbb{R}$ with $supp(a) \in [\chi, \xi]$ being (χ, ξ) valid is equivalent to it being EBRM on $[\chi, \xi]$.*

Proof. From Corollary 79 we know that $a(x)$ being (χ, ξ) valid is equivalent to $a(x - \chi)$ being $\Lambda = \xi - \chi$ valid. From Corollary 55 we know that $a(x - \chi)$ is equivalently EBRM on $[0, \xi - \chi]$. Finally using Corollary 81 we know that $a(x - \chi)$ being EBRM on $[0, \xi - \chi]$ is equivalent to $a(x)$ being EBRM on $[\chi, \xi]$. □

Lemma 83 (EBRM function $\iff$ EBRM transition even with common support)**.** *If we write an EBRM function a with spectrum in $[\chi', \xi']$ as $a = h - g$ with $h, g : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ which may have common support then $g \to h$ is an EBRM transition with spectrum in $[\chi, \xi]$ and with (the smallest) matrix size (at most) $n_g + n_h - 1$ where $[\chi, \xi]$ is the smallest interval containing $[\chi', \xi']$ and $supp(h) \cup supp(g)$.*

Conversely, if $g \to h$ is an EBRM transition with spectrum in $[\chi, \xi]$ with $h, g : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ which may have common support then $a = h - g$ is an EBRM function with its spectrum in $[\chi, \xi]$ (the smallest) matrix size at most $n_g + n_h - 1$.

Proof. To prove the first statement we write $a = a^+ - a^-$ where $a^+ = \sum_{i=1}^{n'_h} p'_{h_i}[x_{h_i}]$, $a^- = \sum_{i=1}^{n'_g} p'_{g_i}[x_{g_i}]$, for $a^+, a^- : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$, represent the positive and the negative part of a. Note that a^+ and a^- by virtue of this definition can't have any common support. Consider $\Delta = \sum_{i=1}^{n_\Delta} c_i[x_i] : \mathbb{R}_{\geq} \to \mathbb{R}_{\geq}$ to be such that $h = a^+ + \Delta$ and $g = a^- + \Delta$. This is always the case because $h - g = a$. Consider the case where $\text{supp}(\Delta) \cap \text{supp}(a) = \emptyset$. In this case $n_g = n'_g + n_\Delta$ and $n_h = n'_h + n_\Delta$. Since a is an EBRM function we have a legal COF, viz $O' X'_g O'^T \leq X'_h$ and $|w'\rangle = O' |v'\rangle$, of dimension ($n' = n'_g + n'_h - 1$) from Lemma 57. To obtain the matrices corresponding to $g \to h$ we expand the space to $n = n_g + n_h - 1$ dimensions and define $X_g = X'_g \oplus X$, $X_h = X'_h \oplus X$, $O = O' \oplus \mathbb{I}$, $|v\rangle = |v'\rangle + \sum_{i=n'}^{n} \sqrt{c_{i+1-n'}} |i\rangle$ where $X = \text{diag}\{x_1, x_2 \dots x_{n_\Delta}\}$. This is just an elaborate way of adding the points in Δ to the matrices and the vectors in such a way that the part corresponding to Δ remains unchanged. The other cases can be similarly demonstrated with the only difference being in the relation between n_g, n'_g and n_h, n'_h. Suppose

$$
\begin{array}{ccc}
a(x) \text{ is } \Lambda - \text{valid} & \overset{\text{Corollary 55}}{\Longleftrightarrow} & a(x) \text{ is } \Lambda - \text{EBRM} \\
\Updownarrow \text{Cor. } 79 & & \Updownarrow \text{Cor. } 81 \\
a(x'-\chi) \text{ is } [\chi,\xi] - \text{valid} & \overset{\text{Lemma 82}}{\Longleftrightarrow} & a(x'-\chi) \text{ is } [\chi,\xi] - \text{EBRM} \\
 & & \Updownarrow \text{Lemma } 83
\end{array}
$$

$$
\begin{array}{ccc}
h \to g \text{ is } [\chi',\xi'] \text{ EBRM} & \overset{\text{Lemma 57}}{\Longrightarrow} & [\chi',\xi'] - \text{COF} \\
\text{if } a = h - g \text{ and } [\chi,\xi] \cup \text{spec}(g+h) \subset [\chi',\xi'] & \overset{\text{trivial}}{\Longleftarrow} &
\end{array}
$$

Figure 5: Generalisation schematized.

Δ is non-zero only at one point. If Δ adds a point where a^- had a point then it will not contribute to increasing the number of points in g that is $n_g = n_g'$ but it will increase the number in h that is $n_h = n_h' + 1$. This means that we have one extra dimension to find the matrices certifying $g \to h$ is EBRM which is precisely what is needed to append that extra idle point as described above. Similarly one can reason for adding a point where a^+ had a point and finally extend it to the most general case of $\text{supp}(\Delta) \cap \text{supp}(a) \neq \emptyset$ which may involve multiple points.

We now prove the converse. Since $g \to h$ is an EBRM transition from Lemma 57 we know that it admits a legal COF, that is $OX_gO^T \leq X_h$ and $O\,|v\rangle = |w\rangle$ with dimension $n_g + n_h - 1$. To be able to show that $a = h - g = a^+ - a^-$ (where a^+ and a^- are again the positive and negative part of a) is an EBRM function it suffices to show that a is a valid function. This follows directly from the COF and operator monotones as $Of(X_g)O^T \leq f(X_h)$ implies $\langle v|\, f(X_g)\,|v\rangle \leq \langle w|\, f(X_h)\,|w\rangle$ which in turn is $\sum h(x)f(x) - \sum g(x)f(x) \geq 0$ and that is the same as $\sum a(x)f(x) \geq 0$ for all f operator monotone on the spectrum of $X_h \oplus X_g$, viz a is valid. From Lemma 82 we conclude that a is also EBRM with size at most $n_g + n_h - 1$ (actually we can make a stronger statement by saying the size should be at most $n_g' + n_h' - 1$ where $|\text{supp}(a^+)| = n_h'$ and $|\text{supp}(a^-)| = n_g'$). $\square$

Our achievement so far has been schematized in Figure 5

7.2.2 For the finite part

Fact 84 (Weyl's Monotonicity Theorem). *If H is positive semi-definite and A is Hermitian then $\lambda_j^{\downarrow}(A+H) \geq \lambda_j^{\downarrow}(A)$ for all j where $\lambda_j^{\downarrow}(M)$ represents the j^{th} largest eigenvalue of the Hermitian matrix M.*

Corollary 85. *If $H \geq G$ then $\lambda_j^{\downarrow}(H) \geq \lambda_j^{\downarrow}(G)$ for all j.*

Claim 86 (Continuity of l). Let $[x_{\min}, x_{\max}]$ be the smallest interval containing $\text{supp}(a)$. $l(\lambda)$ is continuous in the intervals $\lambda \in (-x_{\min}, \infty]$ and $\lambda \in [-\infty, -x_{\max})$ (see Definition 62).

Proof. Since $l(\lambda)$ is just a rational function of λ it suffices to show that the denominator doesn't become zero in the said range. The roots of the denominator are of the form $\lambda + x = 0$ for $x \in \{\{x_{g_i}\}, \{x_{h_i}\}\}$. Hence the largest root will be $\lambda = -x_{\min}$ and the smallest $\lambda = -x_{\max}$. Neither of the intervals defined in the statement contain any roots and therefore we can conclude that $l(\lambda)$ will be continuous therein. Note that the function f_λ on $[x_{\min}, x_{\max}]$ is not even defined for λ in $(-x_{\max}, -x_{\min})$. $\square$

Lemma 87 (Tightening with the matrix spectrum unknown). *Consider a finitely supported valid function a. Let $[x_{\min}(\gamma), x_{\max}(\gamma)]$ be the smallest interval containing supp(a_γ). Consider $m(\gamma, x_{\min}(\gamma), x_{\max}(\gamma))$ as a function of γ (see Definition 64). m has at least one root in the interval* $(0,1]$.

Proof. To prove the claim it suffices to show that $l_\gamma(\lambda)$ has a root in the range $(0,\infty)$ for some $\gamma \in (0,1]$. Note that we are given a valid function a which means $\text{supp}(a) \in \mathbb{R}_{\geq}$.

We assume that $l_{\gamma=1}(\lambda) > 0$ for all $\lambda \in (0,\infty)$ because if this was not the case then we trivially have $\gamma = 1$ as a root, i.e. $m(1, x_{\min}(1), x_{\max}(1)) = 0$.

Notice that since $\sum h(x) = \sum g(x)$ we have

$$
\begin{aligned}
\lambda l(\lambda) &= \sum h(x)(\lambda f_\lambda(x) + 1) - \sum g(x)(\lambda f_\lambda(x) + 1) \\
&= \sum h(x)\frac{x}{\lambda + x} - \sum g(x)\frac{x}{\lambda + x}.
\end{aligned}
$$

Therefore for the remainder of this proof we redefine $f_\lambda = \frac{1}{\lambda}\frac{x}{\lambda+x}$ without changing the value of l or by extension l_γ (the $1/\lambda$ factor is partly why we restricted λ to $(0,\infty)$ instead of the more general $(-x_{\min},\infty)$). Note that $\lim_{\gamma\to 0^+} l_\gamma(\lambda) < 0$ for all $\lambda \in (0,\infty)$ because $h_\gamma(x) = h(x/\gamma)$ which means $\lim_{\gamma\to 0}\sum h_\gamma(x)f_\lambda(x) = \lim_{\gamma\to 0}\sum h(x)f_\lambda(\gamma x) = 0$ since $\lim_{x\to 0} f_\lambda(x) = 0$. This in turn means $\lim_{\gamma\to 0^+} l_\gamma(\lambda) = -\sum g(x)f_\lambda(x) < 0$.

Further, each term constituting $l_\gamma(\lambda)$ is finite for $\lambda \in (0,\infty)$ since for $\lambda > 0$ the denominators are of the form $\lambda + x$ which are always positive. Hence $l_\gamma(\lambda)$ as a function of $\lambda \in [0,\infty)$ and $\gamma \in (0,1]$ is continuous. By continuity then between $\gamma = 0^+$ and $\gamma = 1$ there should be a root.

It remains to justify why we extended the range of λ from $(0,\infty)$ to $(-\infty,-x_{\max}) \cup (-x_{\min},\infty)$ in the definition of m (see Definition 64) as it appears in the statement of the lemma. This is due to the fact that $l_\gamma(\lambda)$ is continuous for λ in the stated range, see Lemma 86, and so there might be a root which appears in the extended range. If this is the case we would like to use this possibly higher value of γ (because for a small enough value a non-negative root must appear due to the aforesaid reasoning). This can help us avoid infinities (we will explain this later). $\square$

Lemma 88 (Matrix spectrum from a valid function)**.** *Consider a valid function a, i.e. an a such that $l(\lambda) \geq 0$ and $l^1 \geq 0$ for $\lambda \in [0,\infty)$ (see Definition 62) and let $[\chi,\xi]$ be such that for $\lambda \in [-\infty,-\xi) \cup (-\chi,\infty]$ we have $l(\lambda) \geq 0$.*

There exists a legal COF, corresponding to the function a, with its spectrum contained in $[\chi,\xi]$.

Proof. Since $l(\lambda) \geq 0$ for $\lambda \in (-\infty,-\xi) \cup (-\chi,\infty)$ and $l^1 \geq 0$ we know from Corollary 76 that a is (χ,ξ) valid. From Lemma 82 we know that a is EBRM on $[\chi,\xi]$. Finally from Lemma 57 we know that there exists a legal COF with spectrum in $[\chi,\xi]$. $\square$

Lemma 89 ($H \geq G \iff f_\lambda(H) \geq f_\lambda(G)$)**.** *Let H, G be real symmetric matrices and $[\chi,\xi]$ be the smallest interval containing spec$[H \oplus G]$ and f_λ be on (χ,ξ). $H \geq G$ if and only if $f_\lambda(H) \geq f_\lambda(G)$.*

Proof. $H \geq G \implies f_\lambda(H) \geq f_\lambda(G)$ because f_λ is an operator monotone function for matrices with spectrum in $[\chi,\xi]$. We prove the converse. We find the inverse function of f_λ and show that it is also an operator monotone. Start with recalling that for $x \in [\chi,\xi]$ we have

$$y = f_\lambda(x) = \frac{-1}{\lambda + x} \implies x = -\frac{1}{y} - \lambda$$

where $\lambda \in \mathbb{R}\backslash[\chi,\xi]$. Thus $f_\lambda^{-1}(y) = -\frac{1}{y} - \lambda$. For a given λ either $f_\lambda(\chi)$ and $f_\lambda(\xi)$ are both greater than zero or both less than zero. Hence the operator monotones $f'_{\lambda'}(y)$ on $[f_\lambda(\chi), f_\lambda(\xi)]$ permit $\lambda' = 0$. Consequently $f'_{\lambda'=0}(y) = \frac{-1}{y}$ is an operator monotone on $[f_\lambda(\chi), f_\lambda(\xi)]$. A constant is also an operator monotone. Thus we conclude $f_\lambda^{-1}(y)$ is an operator monotone on the required interval establishing the converse. $\square$

7.2.3 For Wiggle-v; the infinite part

Lemma 90 (Strict inequality under f_λ)**.** *$H > G$ if and only if $f_\lambda(H) > f_\lambda(G)$ where f_λ is on $(\chi,\xi) \supset$ spec$[H \oplus G]$.*

Proof. Note that $H > G \iff H' := H + \lambda\mathbb{I} > G + \lambda\mathbb{I} =: G'$ where $\lambda \in \bar{\mathbb{R}}\backslash[-\xi,-\chi]$ (by definition of f_λ on (χ,ξ)). There can be two cases, either both the matrices are strictly positive or both are strictly negative. Let us assume the former (the other follows similarly). We have

$$\begin{aligned} & H' > G' > 0 \\ \iff & \mathbb{I} > H'^{-1/2} G' H'^{-1/2} \\ \iff & \mathbb{I} < H'^{1/2} G'^{-1} H'^{1/2} \\ \iff & H'^{-1} < G'^{-1} \end{aligned}$$

where the first inequality follows from the fact that multiplication by a positive matrix doesn't affect the inequality (hint: it only changes the vectors $|w\rangle$ we use to show $\langle w|(H' - G')|w\rangle > 0$ but maps the set of rays to themselves as the norm of the vector might change), the second follows from the fact that one can diagonalise the matrices (identity stays the same) and then it is just a set of inequalities involving real numbers, and the third follows from again multiplication by a positive matrix. The last one is the same as $f_\lambda(H) > f_\lambda(G)$. $\square$

Corollary 91 (Tightness preservation under f_λ)**.** *Let $H \geq G$ and f_λ be on $(\chi,\xi) \supset$ spec$[H \oplus G]$. There exists a $|w\rangle$ such that $\langle w|(H - G)|w\rangle = 0$ if and only if there exists a $|w_\lambda\rangle$ such that $\langle w_\lambda|(f_\lambda(H) - f_\lambda(G))|w_\lambda\rangle = 0$.*

Proof. The contrapositive of the aforesaid condition is that $f_\lambda(H) > f_\lambda(G)$ if and only if $H > G$ which holds due to Lemma 90. $\square$

Lemma 92 (Extending tightness preservation under f_λ to apparently divergent situations)**.** *Let X_h, X_g be diagonal matrices with spec$[X_h] \in (\chi,\xi]$, spec$[X_g] \in [\chi,\xi)$ and let f_λ be on $[\chi,\xi]$. Let, further, O be an orthogonal matrix such that $X_h \geq O X_g O^T$.*

If there exists a $|w\rangle$ such that $\langle w|\left(f_{-\xi}(X_h) - O f_{-\xi}(X_g) O^T\right)|w\rangle = 0$ then there exists a $|w_\lambda\rangle$ such that $\langle w_\lambda|\left(f_\lambda(X_h) - O f_\lambda(X_g) O^T\right)|w_\lambda\rangle = 0$ for $\lambda \in \mathbb{R}\backslash[\chi,\xi]$. The other way also holds.

Similarly, if there exists a $|w\rangle$ such that $\langle w|\left(f_{-\chi}(X_h) - O f_{-\chi}(X_g) O^T\right)|w\rangle = 0$ then there exists a $|w_\lambda\rangle$ such that $\langle w_\lambda|\left(f_\lambda(X_h) - O f_\lambda(X_g) O^T\right)|w_\lambda\rangle = 0$ for $\lambda \in \mathbb{R}\backslash[\chi,\xi]$. The other way also holds.

Proof. The trouble with this version of the tightness statement is that X_h has an eigenvalue ξ (if it doesn't then it reduces to the previous statement) which means that $f_{-\xi}(X_h)$ is not well defined. We assume that X_h can be expressed as

$$X_h = \begin{bmatrix} X'_h & \\ & \xi\mathbb{I}'' \end{bmatrix}$$

where X'_h has no eigenvalue equal to ξ and $\mathbb{I}''$ is the identity matrix in the subspace. We can write

$$\begin{aligned} X_h > OX_gO^T \iff & \begin{bmatrix} f_\lambda(X'_h) & \\ & f_\lambda(\xi\mathbb{I}'') \end{bmatrix} > Of_\lambda(X_g)O^T \text{ for } \lambda \in \mathbb{R}\backslash[-\xi,-\chi] \\ \iff & \begin{bmatrix} f_\lambda(X'_h) & \\ & \mathbb{I}'' \end{bmatrix} > \begin{bmatrix} \mathbb{I}' & \\ & f_\lambda(\xi\mathbb{I}'')^{-1/2} \end{bmatrix} Of_\lambda(X_g)O^T \begin{bmatrix} \mathbb{I}' & \\ & f_\lambda(\xi\mathbb{I}'')^{-1/2} \end{bmatrix} \text{ for } \lambda \in \mathbb{R}\backslash[-\xi,-\chi] \end{aligned}$$

where in the last line the expression has a well defined limit for $\lambda = -\xi$. This establishes the contrapositive variant of the statement we wanted to prove (similar to the strategy used for proving Corollary 91) once we note the following. If $\langle w| \left(f_{-\xi}(X_h) - Of_{-\xi}(X_g)O^T\right)|w\rangle = 0$ it is easy to see that $\begin{bmatrix} 0 & \\ & \mathbb{I}'' \end{bmatrix}|w\rangle = 0$ otherwise due to the constraint on the spectrum of X_g the aforesaid expression would be ∞. This entails that

$$\langle w| \left(\begin{bmatrix} f_{-\xi}(X'_h) & \\ & \mathbb{I}'' \end{bmatrix} - \begin{bmatrix} \mathbb{I}' & \\ & f_{-\xi}(\xi\mathbb{I}'')^{-1/2} \end{bmatrix} Of_{-\xi}(X_g)O^T \begin{bmatrix} \mathbb{I}' & \\ & f_{-\xi}(\xi\mathbb{I}'')^{-1/2} \end{bmatrix} \right)|w\rangle = 0.$$

One can similarly prove the case for $f_{-\chi}(X_g)$. □

7.3 The Algorithm

We start with motivating the exact step of the algorithm and then provide a proof or justification for the claims made in that step.

7.3.1 Phase 1: Initialisation

We are given a Λ-valid transition $g \to h$ and the EBRM function $a = h - g$. (Remark: We will use below the notation used in the definition of a transition.)

Since the function is EBRM we know there are matrices $H \geq G$ and a vector $|\psi\rangle$ such that $a = \text{Prob}[H, |\psi\rangle] - \text{Prob}[G, |\psi\rangle]$. We also know that the maximum matrix size we need to consider is $n_g + n_h - 1$. We want to know the spectrum of the matrices involved to proceed.

The picture we have in mind is the following. We know that $H \geq G$ in terms of ellipsoids means that the H ellipsoid is inside the G ellipsoid (the order gets reversed). We will try to expand the H ellipsoid (which means scaling down the matrix H) until it touches the G ellipsoid. When they touch we know that the corresponding spectrum of the matrices is optimal in some sense. This would be trivial if we already knew H and G but it serves as a good picture nonetheless.

What we do know is the function $a = h - g$. We use the equivalence between EBRM and valid functions to perform the aforesaid tightening procedure even without knowing the matrices. We will use $a_\gamma = h_\gamma - g$ where $h_\gamma(x) = h(x/\gamma)$ and check if a_γ stays valid as we shrink γ from one to zero. We stop the moment we see any signature of tightness. Using this a_γ we determine the spectrum of the matrices certifying the EBRM claim.

We start with tightening till we find some operator monotone labelled by λ for which $l_{\gamma'}(\lambda)$ disappears. This captures the notion of the ellipsoids touching as after applying this operator monotone, along the $|w\rangle$ direction, the ellipsoids must touch.

Tightening procedure: Let $[x_{\min}(\gamma'), x_{\max}(\gamma')]$ be the support domain for $a_{\gamma'}$. Let $\gamma \in (0,1]$ be the largest root of $m(\gamma', x_{\min}(\gamma'), x_{\max}(\gamma'))$. Let $x_{\max} := x_{\max}(\gamma)$ and $x_{\min} := x_{\min}(\gamma)$.

First we must show that there would indeed be a root of m as a function of γ' in the range $(0,1]$. This is a direct consequence of Lemma 87. Second we must show that if we can find the matrices certifying a_γ is EBRM we can find the matrices certifying a is EBRM. This follows from the observation that $\gamma X_h \geq OX_gO^T$ implies that $X_h \geq \gamma X_h \geq OX_gO^T$.

We found a signature of tightness. Now we find the spectrum of the matrices involved.

Spectral domain for the representation: Find the smallest interval $[\chi, \xi]$ such that $l_\gamma(\lambda) \geq 0$ for $\lambda \in \bar{\mathbb{R}}\backslash[\chi, \xi]$. If $\text{supp}(g), \text{supp}(h)$ is not contained in $[\chi, \xi]$ then from all expansions of $[\chi, \xi]$ that contain the aforesaid sets, pick the smallest. Relabel this interval to $[\chi, \xi]$.

The interval so obtained will contain the spectrum of the matrices that certify a_γ is EBRM. This is justified by Lemma 88 using the fact that $l^1_\gamma \geq 0$ due to the previous step.

We need our matrices to be positive to be able to use the elliptic picture. We therefore shift the spectrum of the matrices so that the smallest eigenvalue required is one (where we could have used any positive number).

Shift: Transform

$$a(x) \to a'(x') := a(x' + \chi - 1)$$

where instead of 1 any positive constant would do (justified by Corollary 81). Similarly transform

$$g(x) \to g'(x') := g(x' + \chi - 1)$$
$$h(x) \to h'(x') := h(x' + \chi - 1).$$

Relabel a' to be a, g' to be g and h' to be h. (Remark: We do not deduce h and g from a as its positive and negative part because they might now have common support due to the tightening procedure.)

We use Corollary 81 to deduce that if $a(x)$ is EBRM with spectrum in $[\chi, \xi]$ then $a'(x') = a(x'+\chi-1)$ is EBRM with spectrum in $[1, \xi - \chi + 1]$. We must also show that if we can find the matrices for certifying a' is EBRM then we can find the matrices certifying a is EBRM. This is a direct consequence of the fact that $X'_h \geq OX'_gO^T \iff X_h - (\chi-1)\mathbb{I} \geq O(X_g - (\chi-1)\mathbb{I})O^T$. The orthogonal matrix, O, which is of primary interest remains unchanged.

With the spectrum determined and adjusted to the elliptic picture, which we will put to use soon, we fix everything except the orthogonal matrix by using the Canonical Orthogonal Form (up to a permutation).

The matrices: For $n := n_g + n_h - 1$ we define $n \times n$ matrices with spectrum in $[\chi, \xi]$ and n dimensional vectors as

$$\begin{aligned} X_g^{(n)} &= \mathrm{diag}[\chi, \chi, \dots x_{g_1}, x_{g_2} \dots, x_{g_{n_g}}], \\ X_{h_\gamma}^{(n)} &= \mathrm{diag}[\gamma x_{h_1}, \gamma x_{h_2}, \dots, \gamma x_{h_{n_h}}, \xi, \xi, \dots], \\ \left|v^{(n)}\right\rangle &\doteq \left[0, 0 \dots, \sqrt{p_{g_1}}, \sqrt{p_{g_2}}, \dots, \sqrt{p_{g_{n_g}}}\right], \\ \left|w^{(n)}\right\rangle &\doteq \left[\sqrt{p_{h_1}}, \sqrt{p_{h_2}}, \dots, \sqrt{p_{h_{n_h}}}, 0, 0 \dots\right] \end{aligned}$$

where $g = \sum_{i=1}^{n_g} p_{g_i}[x_{g_i}]$ and $h = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}]$. Note that n_g and n_h may be different.

We use Lemma 83 to deduce that $g \to h$ is a valid transition from the validity of a. Then we use Lemma 57 to write the diagonal matrices as described above given the valid transition $g \to h$, upto a permutation. Our objective is to find a matrix $O^{(n)}$ such that $O^{(n)}\left|v^{(n)}\right\rangle = \left|w^{(n)}\right\rangle$ while satisfying the inequality $X_h^{(n)} \geq O^{(n)}X_g^{(n)}O^{(n)T}$.

We now remove all the redundant information and pack it into a form which we can iteratively reduce to a simpler form.

Bootstrapping the iteration:

- Basis: $\left\{\left|t_{h_i}^{(n+1)}\right\rangle\right\}$ where $\left|t_{h_i}^{(n+1)}\right\rangle := |i\rangle$ for $i = 1, 2 \dots n$ where $|i\rangle$ refers to the standard basis in which the matrices and the vectors were originally written.
- Matrix Instance: $\underline{\mathrm{X}}^{(n)} = \{X_h^{(n)}, X_g^{(n)}, \left|w^{(n)}\right\rangle, \left|v^{(n)}\right\rangle\}$.

7.3.2 Phase 2: Iteration

We will take as input the matrices X_g, X_h together with the vectors $|w\rangle, |v\rangle$ and churn out the same objects with one less dimension. We will also output objects that, once we have iteratively reduced the problem to triviality, can be put together to find the elusive orthogonal matrix O. See Figure 6 for a schematic reference.

- Objective: Find the objects $\left|u_h^{(k)}\right\rangle, \bar{O}_g^{(k)}, \bar{O}_h^{(k)}$ and $s^{(k)}$ (which together relate $O^{(k)}$ to $O^{(k-1)}$ where $O^{(k)}$ solves $\underline{\mathrm{X}}^{(k)}$ and $O^{(k-1)}$ solves $\underline{\mathrm{X}}^{(k-1)}$ that is yet to be defined)
- Input: We will assume we are given
  - Basis: $\left\{\left|t_{h_i}^{(k+1)}\right\rangle\right\}$
  - Matrix Instance: $\underline{\mathrm{X}}^{(k)} = \{X_h^{(k)}, X_g^{(k)}, \left|w^{(k)}\right\rangle, \left|v^{(k)}\right\rangle\}$ with attribute $\chi^{(k)} > 0$
  - Function Instance: $\underline{\mathrm{X}}^{(k)} \to \underline{\mathrm{x}}^{(k)} = \{h^{(k)}, g^{(k)}, a^{(k)}\}$
- Output:
  - Basis: $\left\{\left|u_h^{(k)}\right\rangle, \left|t_{h_i}^{(k)}\right\rangle\right\}$
  - Matrix Instance: $\underline{\mathrm{X}}^{(k-1)} = \{X_h^{(k-1)}, X_g^{(k-1)}, \left|w^{(k-1)}\right\rangle, \left|v^{(k-1)}\right\rangle\}$ with attribute $\chi^{(k-1)} > 0$

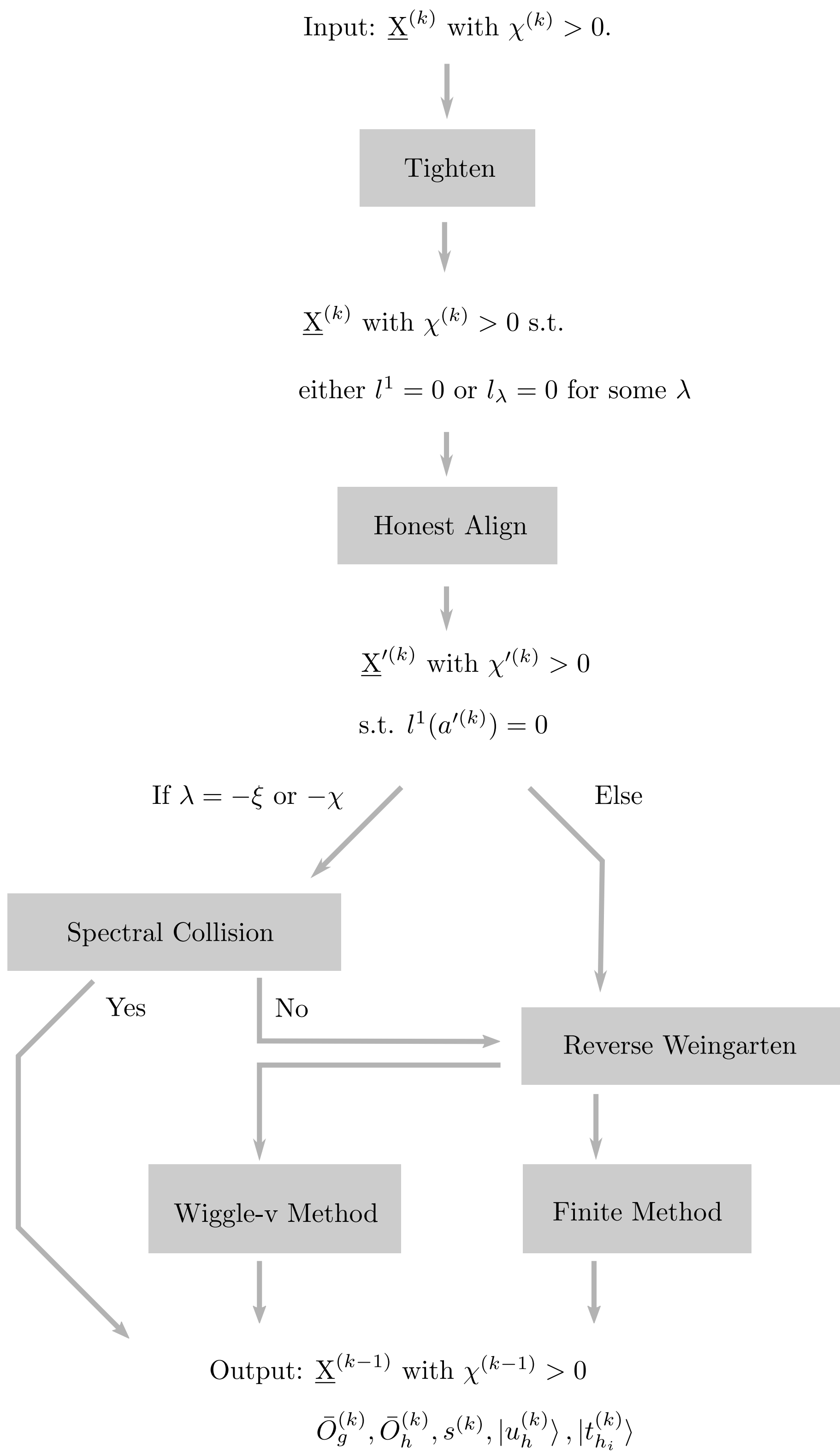


Figure 6: Overview of the main step, the iteration, of the algorithm (excluding the boundary condition).

- Function Instance: $\underline{\mathrm{X}}^{(k-1)} \to \underline{\mathrm{x}}^{(k-1)} = \{h^{(k-1)}, g^{(k-1)}, a^{(k-1)}\}$
- Unitary Constructors: Either $\bar{O}_g^{(k)}$ and $\bar{O}_h^{(k)}$ are returned or $\bar{O}^{(k)}$ is returned. If $\bar{O}^{(k)}$ is returned, set $\bar{O}_g^{(k)} := \bar{O}^{(k)}$ and $\bar{O}_h^{(k)} = \mathbb{I}$.
- Relation: If $s^{(k)}$ is not specified, define $s^{(k)} := 1$.
  If $s^{(k)} = 1$ then use
  $$O^{(k)} := \bar{O}_h^{(k)} \left(\left| u_h^{(k)} \right\rangle \left\langle u_h^{(k)} \right| + O^{(k-1)} \right) \bar{O}_g^{(k)}$$
  else use
  $$O^{(k)} := \left[\bar{O}_h^{(k)} \left(\left| u_h^{(k)} \right\rangle \left\langle u_h^{(k)} \right| + O^{(k-1)} \right) \bar{O}_g^{(k)} \right]^T.$$

  Our task is to solve the matrix instance $\underline{\mathrm{X}}^{(k)}$, i.e. find a real unitary $O^{(k)}$ such that $X_h^{(k)} \geq O^{(k)} X_g^{(k)} O^{(k)T}$ and $O^{(k)} \left| v^{(k)} \right\rangle = \left| w^{(k)} \right\rangle$. We will assume that the solution exists and show that the solution to the smaller instance, denoted by $\underline{\mathrm{X}}^{(k-1)}$ must also exist. More precisely, we will show that $O^{(k)}$ must have the form $O^{(k)} = \left(\left| u_h^{(k)} \right\rangle \left\langle u_h^{(k)} \right| + O^{(k-1)} \right) \bar{O}^{(k)}$ (for a solution to exist) which satisfies the aforesaid constraints granted we can find $O^{(k-1)}$ which acts on a $k-1$ dimensional Hilbert space orthogonal to $\left| u_h^{(k)} \right\rangle$ and satisfies constraints of the same form in the smaller dimension, viz. $X_h^{(k-1)} \geq O^{(k-1)} X_g^{(k-1)} O^{(k-1)T}$ and $O^{(k-1)} \left| v^{(k-1)} \right\rangle = \left| w^{(k-1)} \right\rangle$. Hence the assumption that $O^{(k)}$ has a solution allows us to deduce that $O^{(k-1)}$ must also have a solution. This will allow us to iteratively solve the problem.
  In certain trivial cases, where a point appears both before and after a transition viz. $X_g^{(k)}$ and $X_h^{(k)}$ have a common eigenvalue, the solution will have the form $O^{(k)} = \bar{O}_h^{(k)} \left(\left| u_h^{(k)} \right\rangle \left\langle u_h^{(k)} \right| + O^{(k-1)} \right) \bar{O}_g^{(k)}$. Finally, in one of the "infinite" cases denoted by the "Wiggle-v method" the solution will have the form $O^{(k)} = \left[\left(\left| u_h^{(k)} \right\rangle \left\langle u_h^{(k)} \right| + O^{(k-1)} \right) \bar{O}^{(k)} \right]^T$.

- Algorithm:

If we reach a stage where the vector constraints have disappeared then we can simply stop.

- **Boundary condition: If** $n_g = 0$ and $n_h = 0$ **then** set $k_0 = k$ and **jump to** phase 3.

  We assumed that an $O^{(k)}$ satisfying the constraints (listed right after the input/output section) exists. In this case it means that there exists an $O^{(k)}$ such that $X_h^{(k)} \geq O^{(k)} X_g^{(k)} O^{(k)T}$ as there's no vector $\left| v^{(k)} \right\rangle, \left| w^{(k)} \right\rangle$ to impose further constraints. Using Corollary 85 with $H = X_h^{(k)}$ and $G = O^{(k)} X_g^{(k)} O^{(k)T}$ we conclude that $O^{(k)}$ need only be a permutation matrix. Note that this step can never be entered right after the $\underline{\mathrm{X}}^{(n)}$ instance as we start with assuming $n_g, n_h > 0$. Further, since the protocol by construction always returns X_h and X_g in the ascending order the permutation matrix will be $\mathbb{I}$.

Finally, we deal with the interesting case. We again use the picture where the H ellipsoid is contained inside the G ellipsoid. We will expand the H ellipsoid (which corresponds to shrinking the H matrix) until it touches the G ellipsoid as before by working with the function a.

- **Tighten**: Define $X_{h_{\gamma'}}^{(k)} := \gamma' X^{(k)}$. Let γ be the largest root of $m(\gamma', \chi_{\gamma'}^{(k)}, \xi_{\gamma'}^{(k)})$ for $a^{(k)}$ where $\chi_{\gamma'}^{(k)}, \xi_{\gamma'}^{(k)}$ are such that $[\chi_{\gamma'}^{(k)}, \xi_{\gamma'}^{(k)}]$ is the smallest interval containing $\mathrm{spec}[X_{h_{\gamma'}}^{(k)} \oplus X_g^{(k)}]$. Relabel $X_{h_\gamma}^{(k)}$ to $X_h^{(k)}$, $\chi_\gamma^{(k)}$ to $\chi^{(k)}$ and $\xi_\gamma^{(k)}$ to $\xi^{(k)}$ for notational ease. Similarly relabel $a_\gamma^{(k)}$ to $a^{(k)}$, $h_\gamma^{(k)}$ to $h^{(k)}$, $l_\gamma^{(k)}$ to $l^{(k)}$. Update $x_{\min}$ and $x_{\max}$ to be such that $\mathrm{supp}(a^{(k)}) \in [x_{\min}^{(k)}, x_{\max}^{(k)}]$ is the smallest such interval. Define $s^{(k)} := 1$.
  Our burden again is to show that m as a function of γ' will have a root. Unlike the first tightening procedure this time we know the spectrum of the matrices involved. Since we are given (by assumption) that the matrix instance has a solution we know that $l_{\gamma'=1}(\lambda) \geq 0$ and $l_{\gamma'=1}^1 \geq 0$ for $\lambda \in \bar{\mathbb{R}} \backslash [\chi_{\gamma'=1}^{(k)}, \xi_{\gamma'=1}^{(k)}]$ using Lemma 82. We also know that $\chi_{\gamma'}^{(k)} > 0$ which means that $a^{(k)}$ (as deduced from the function instance of $\underline{\mathrm{X}}^{(k)}$) is a valid function. This observation lets us conclude that m as a function of γ' will have a root in the required range because the reasoning behind a similar claim proved in Lemma 87 goes through unchanged.

The tightening procedure guarantees we will be able to find a λ which corresponds to an operator monotone such that after applying this function the ellipsoids, which we don't even know completely yet, must touch along the $|w\rangle$ direction. This piece of information is key to reducing the problem to a smaller instance of itself. Recall the picture with the H ellipsoid contained inside the G ellipsoid. If we know that they, in addition, touch at some known point then it must be so that the inner ellipsoid is more curved than the outer ellipsoid. When expressed algebraically, this condition essentially becomes that requirement that an ellipsoid $H^{(k-1)}$ that encodes the curvature of the ellipsoid $H^{(k)}$ at the point of contact must be contained inside the corresponding $G^{(k-1)}$ ellipsoid which encodes the curvature of the $G^{(k)}$ ellipsoid. The vector condition also reduces similarly. Subtleties arise when λ happens to have boundary values in its allowed range as this yields infinities and this has an interesting consequence.

– **Honest align**: **If** $l^{1(k)} = 0$ **then** define $\eta = -\chi^{(k)} + 1$

$$X_h'^{(k)} := X_h^{(k)} + \eta, \quad X_g'^{(k)} := X_g + \eta.$$

Else: Pick a root λ of the function $l^{(k)}(\lambda')$ in the domain $\mathbb{R}\backslash(-\xi^{(k)}, -\chi^{(k)})$. In the following two cases we will consider the function f_λ on $[\chi^{(k)}, \xi^{(k)}]$.

* If $\lambda \neq -\chi^{(k)}$ then: Let $\eta = -f_\lambda(\chi^{(k)}) + 1$ where any positive constant could be chosen instead of 1. Define

$$X_h'^{(k)} := f_\lambda(X_h^{(k)}) + \eta, \quad X_g'^{(k)} := f_\lambda(X_g^{(k)}) + \eta.$$

* If $\lambda = -\chi^{(k)}$ then: Update $s^{(k)} = -1$. Let $\eta = -f_\lambda(\xi^{(k)}) - 1$ where any positive constant could be chosen instead of 1. Define

$$X_h'^{(k)} := X_g''^{(k)}, \quad X_g'^{(k)} := X_h''^{(k)},$$

where

$$X_h''^{(k)} := -f_\lambda(X_h^{(k)}) - \eta, \quad X_g''^{(k)} := -f_\lambda(X_g^{(k)}) - \eta$$

and make the replacement

$$\left|v^{(k)}\right\rangle \to \left|w^{(k)}\right\rangle$$
$$\left|w^{(k)}\right\rangle \to \left|v^{(k)}\right\rangle.$$

If we have $\lambda = -\chi^{(k)}$ or $-\xi^{(k)}$ it means that at least one of the matrices (among $X_g^{(k)}$ and $X_h^{(k)}$ under f_λ) would diverge. We must remove eigenvalues common to both matrices as isolating the divergence makes it easier to handle.

– **Remove spectral collision: If** $\lambda = -\chi^{(k)}$ or $\lambda = -\xi^{(k)}$ **then**

If it so happens that the coordinate and the probability associated is the same we must leave the associated vector unchanged (up to a relabelling). The following simply formalises this procedure and encodes the remaining non-trivial part into a problem of one less dimension.

1. **Idle point: If** for some j', j, we have $q_{g_{j'}}^{(k)} = q_{h_j}^{(k)}$ and $y_{g_{j'}}^{(k)} = y_{h_j}^{(k)}$ **then** the solution is given by

$$\left\{\left|u_h^{(k)}\right\rangle, \left|t_{h_1}^{(k)}\right\rangle, \left|t_{h_2}^{(k)}\right\rangle, \dots \left|t_{h_{k-1}}^{(k)}\right\rangle\right\} \overset{\text{componentwise}}{:=} \left\{\left|t_{h_j}^{(k+1)}\right\rangle, \left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_{j-1}}^{(k+1)}\right\rangle, \left|t_{h_{j+1}}^{(k+1)}\right\rangle, \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\},$$

$$\bar{O}^{(k)} := \sum_{i=1}^{k} |a_i\rangle \left\langle t_{h_i}^{(k+1)}\right|,$$

where

$$\{|a_1\rangle, |a_2\rangle \dots |a_k\rangle\} \overset{\text{componentwise}}{:=}$$

$$\begin{cases} \left\{\left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_{j-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'}}^{(k+1)}\right\rangle, \left|t_{h_j}^{(k+1)}\right\rangle, \left|t_{h_{j+1}}^{(k+1)}\right\rangle, \dots, \left|t_{h_{j'-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'+1}}^{(k+1)}\right\rangle \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\} & j < j' \\ \left\{\left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_{j'-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'+1}}^{(k+1)}\right\rangle \dots \left|t_{h_{j-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'}}^{(k+1)}\right\rangle, \left|t_{h_j}^{(k+1)}\right\rangle, \left|t_{h_{j+1}}^{(k+1)}\right\rangle \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\} & j > j' \\ \left\{\left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\} & j = j', \end{cases}$$

and

$$X_h^{(k-1)} := \sum_{i \neq j} y_{h_i}^{(k)} \left|t_{h_i}^{(k+1)}\right\rangle \left\langle t_{h_i}^{(k+1)}\right|,$$

$$X_g^{(k-1)} := \bar{O}^{(k)} X_g^{(k)} \bar{O}^{(k)T} - y_{h_j} \left|t_{h_j}^{(k+1)}\right\rangle \left\langle t_{h_j}^{(k+1)}\right|,$$

$$\left|w^{(k-1)}\right\rangle = \mathcal{N}\left[\left|w^{(k)}\right\rangle - \sqrt{p_{h_j}} \left|t_{h_j}^{(k+1)}\right\rangle\right], \left|v^{(k-1)}\right\rangle = \mathcal{N}\left[\bar{O}^{(k)} \left|v^{(k)}\right\rangle - \sqrt{p_{h_j}} \left|t_{h_j}^{(k+1)}\right\rangle\right].$$

(This specifies $\underline{\mathrm{X}}^{(k-1)} := \{X_h^{(k-1)}, X_g^{(k-1)}, \left|w^{(k-1)}\right\rangle, \left|v^{(k-1)}\right\rangle\}$.)
Jump to **End**.

In this proof by x_{h_i} we mean y_{h_i}, similarly by x_{g_i} we mean y_{h_i}; we apologise for the inconvenience. We want to find an $O^{(k)}$ such that $X_h^{(k)} \geq O^{(k)} X_g^{(k)} O^{(k)T}$ and $O^{(k)} \left|v^{(k)}\right\rangle = \left|w^{(k)}\right\rangle$. We will do this in two stages. First, we re-arrange the entries of $X_g^{(k)}$ as $X_g'^{(k)} := O_p^{(k)} X_g^{(k)} O_p^{(k)T}$ and define $\left|v_p^{(k)}\right\rangle := O_p^{(k)} \left|v\right\rangle$ for an $O_p^{(k)}$ to be specified later. The re-arrangement will be such that $x_{g_{j'}}$ sits at the j,j location while the rest of the elements of $X_g'^{(k)}$ are arranged in the increasing order. Second, we will solve our initial problem under the assumption that $j = j'$. The non-trivial part here would be showing that we can take $O^{(k)}$ to have the form $\left(\left|j\right\rangle\left\langle j\right| + O^{(k-1)}\right)\bar{O}^{(k)}$ without loss of generality.

Let us start with the first step. We will denote the orthogonal matrix $O = \sum_i \left|b_i\right\rangle\left\langle a_i\right|$ by $\{\left|a_1\right\rangle, \left|a_2\right\rangle, \dots \left|a_k\right\rangle\} \to \{\left|b_1\right\rangle, \left|b_2\right\rangle, \dots \left|b_k\right\rangle\}$ where $\{\left|b_i\right\rangle\}$ and $\{\left|a_i\right\rangle\}$ each constitute an orthonormal basis. Using this notation then for the case $j < j'$, we define $O_p^{(k)}$ by

$$\left\{\left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\} \to \\ \left\{\left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_{j-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'}}^{(k+1)}\right\rangle, \left|t_{h_j}^{(k+1)}\right\rangle, \left|t_{h_{j+1}}^{(k+1)}\right\rangle, \dots, \left|t_{h_{j'-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'+1}}^{(k+1)}\right\rangle \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\},$$

for $j' < j$ we define it by

$$\left\{\left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\} \to \\ \left\{\left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_{j'-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'+1}}^{(k+1)}\right\rangle \dots \left|t_{h_{j-1}}^{(k+1)}\right\rangle, \left|t_{h_{j'}}^{(k+1)}\right\rangle, \left|t_{h_j}^{(k+1)}\right\rangle, \left|t_{h_{j+1}}^{(k+1)}\right\rangle \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\}$$

and if $j' = j$ we set $O_p^{(k)} = \mathbb{I}^{(k)}$.

For the second step, we solve the main problem under the assumption that $j' = j$. We are given $X_g^{'(k)} = \mathrm{diag}\{x'_{g_1}, x'_{g_2} \dots x'_{g_k}\}$ and $X_h^{(k)} = \mathrm{diag}\{x_{h_1}, x_{h_2} \dots x_{h_k}\}$ which are such that $x_{h_j} = x'_{g_j}$; $\left|v'^{(k)}\right\rangle \doteq (\sqrt{q'_{g_1}}, \sqrt{q'_{g_2}}, \dots \sqrt{q'_{g_k}})^T$ and $\left|w^{(k)}\right\rangle \doteq (\sqrt{q_{h_1}}, \sqrt{q_{h_2}}, \dots \sqrt{q_{h_k}})^T$ are such that $q_{h_j} = q'_{g_j}$. Let us define the matrix instance to be $\underline{\mathrm{X}}'^{(k)} = \{X_h^{(k)}, X_g'^{(k)}, \left|v'^{(k)}\right\rangle, \left|w^{(k)}\right\rangle\}$. We have to find an $O'^{(k)}$ such that $X_h^{(k)} \geq O'^{(k)} X_g'^{(k)} O'^{(k)T}$ and $O'^{(k)} \left|v'^{(k)}\right\rangle = \left|w\right\rangle$. Let $\underline{\mathrm{X}}'^{(k-1)} = \left\{X_h^{(k-1)}, X_g'^{(k-1)}, \left|v'^{(k-1)}\right\rangle, \left|w^{(k-1)}\right\rangle\right\}$ be the matrix instance obtained after removing the j^{th} entry from the vectors, viz. $\left|v'^{(k-1)}\right\rangle := \sum_{i\neq j} \sqrt{q'_{g_i}} \left|t_{h_i}^{(k+1)}\right\rangle$, $\left|w^{(k-1)}\right\rangle := \sum_{i\neq j} \sqrt{q_{h_i}} \left|t_{h_i}^{(k+1)}\right\rangle$ and similarly defining $X_g'^{(k-1)} = \mathrm{diag}\{x'_{g_1}, x'_{g_2} \dots x'_{g_{j-1}}, x'_{g_{j+1}}, \dots x_{g_k}\}$, $X_h^{(k-1)} = \mathrm{diag}\left\{x_{h_1}, x_{h_2} \dots x_{h_{j-1}}, x_{h_{j+1}}, \dots x_{h_k}\right\}$. Note that $a^{(k)} = a^{(k-1)}$ as the j^{th} point gets cancelled. This means that if there is an $O'^{(k)}$ satisfying the aforementioned constraints $a^{(k)}$ is EBRM on the spectral domain of $\underline{\mathrm{X}}^{(k)}$. Since $a^{(k)} = a^{(k-1)}$ we know that $a^{(k-1)}$ is also EBRM on the same domain. From lemma 83 (we will justify that k is large enough separately) we conclude that there must also exist an $O'^{(k-1)}$ which satisfies $X_h^{(k-1)} \geq O'^{(k-1)} X_g'^{(k-1)} O'^{(k-1)T}$ and $O'^{(k-1)} \left|v'^{(k-1)}\right\rangle = \left|w^{(k)}\right\rangle$.

With all this in place we can claim that without loss of generality we can write $O'^{(k)} = \left|t_{h_j}\right\rangle\left\langle t_{h_j}\right| + O'^{(k-1)}$ because if we can find some other $\tilde{O}'^{(k)}$ which satisfies the required constraints then there exists an $O'^{(k-1)}$ which satisfies the corresponding constraints in the smaller dimension and that means we can show $O'^{(k)}$ also satisfies the required constraints,

$$\begin{aligned} X_h^{(k)} = x_{h_j} \left|t_{h_j}^{(k+1)}\right\rangle\left\langle t_{h_j}^{(k+1)}\right| + X_h^{(k-1)} &\geq x_{g_j} \left|t_{h_j}^{(k+1)}\right\rangle\left\langle t_{h_j}^{(k+1)}\right| + O'^{(k-1)} X_g'^{(k-1)} O'^{(k-1)} \\ &= \left(\left|t_{h_j}^{(k+1)}\right\rangle\left\langle t_{h_j}^{(k+1)}\right| + O'^{(k-1)}\right) X_g'^{(k)} \left(\left|t_{h_j}^{(k+1)}\right\rangle\left\langle t_{h_j}^{(k+1)}\right| + O'^{(k-1)}\right)^T \\ &= O'^{(k)} X_g'^{(k)} O'^{(k)T}, \end{aligned}$$

along with

$$O'^{(k)} \left|v'^{(k)}\right\rangle = \sqrt{q'_{g_j}} \left|t_{h_j}^{(k+1)}\right\rangle + O'^{(k-1)} \left|v'^{(k-1)}\right\rangle = \sqrt{q'_{g_j}} \left|t_{h_j}^{(k+1)}\right\rangle + \left|w^{(k-1)}\right\rangle = \left|w^{(k-1)}\right\rangle.$$

It remains to combine the two steps to produce the matrix $\bar{O}^{(k)}$, the vectors $\left\{\left|n_h^{(k)}\right\rangle, \left\{\left|t_{h_i}^{(k)}\right\rangle\right\}\right\}$, along with $\underline{\mathrm{X}}^{(k-1)}$. We use $X_g'^{(k)} = O_p^{(k)} X_g^{(k)} O_p^{(k)T}$ from the first step and substitute it in the inequality which we showed would hold, i.e.

$$X_h^{(k)} \geq O'^{(k)} X_g'^{(k)} O'^{(k)T} = O'^{(k)} O_p^{(k)} X_g O_p^{(k)T} O'^{(k)T}$$

and using $O_p^{(k)}\left|v^{(k)}\right\rangle=\left|v'^{(k)}\right\rangle$ we have

$$O'^{(k)}\left|v'^{(k)}\right\rangle=O'^{(k)}O_p^{(k)}\left|v^{(k)}\right\rangle=\left|w^{(k)}\right\rangle.$$

Comparing the inequality to the form $X_h^{(k)}\geq O^{(k)}X_g^{(k)}O^{(k)T}$, $O^{(k)}\left|v^{(k)}\right\rangle=\left|w^{(k)}\right\rangle$ for

$$O^{(k)}=\left(\left|n_h^{(k)}\right\rangle\left\langle n_h^{(k)}\right|+O^{(k-1)}\right)\bar{O}^{(k)}$$

we get $\bar{O}^{(k)}=O_p^{(k)}$, $\left|n_h^{(k)}\right\rangle=\left|t_{h_j}^{(k+1)}\right\rangle$ and $O^{(k-1)}=O'^{(k-1)}$. Note that this $O^{(k)}$ is consistent with comparing the equality with $O^{(k)}\left|v^{(k)}\right\rangle=\left|w^{(k)}\right\rangle$. The basis for the sub-problem, i.e. the $(k-1)$ dimensional problem, was the same as before except for the fact that we removed $\left|t_{h_j}^{(k+1)}\right\rangle$. Thus we define $\left\{\left|t_{h_1}^{(k)}\right\rangle,\left|t_{h_2}^{(k)}\right\rangle\dots\left|t_{h_{k-1}}^{(k)}\right\rangle\right\}=\left\{t_{h_1}^{(k+1)},t_{h_2}^{(k+1)}\dots t_{h_{j-1}}^{(k+1)},t_{h_{j+1}}^{(k+1)},\dots t_{h_k}^{(k+1)}\right\}$. Identifying $\underline{\mathrm{X}}^{(k-1)}=\left\{X_h^{(k-1)},X_g^{(k-1)},\left|v^{(k-1)}\right\rangle,\left|w^{(k-1)}\right\rangle\right\}$ with $\underline{\mathrm{X}}'^{(k-1)}=\left\{X_h^{(k-1)},X_g'^{(k-1)},\left|v'^{(k-1)}\right\rangle,\left|w^{(k-1)}\right\rangle\right\}$ completes the argument since $O^{(k-1)}$ was already identified with $O'^{(k-1)}$ so we are just labelling here.

2. **Final Extra**: **If** for some j,j' we have $q_{g_{j'}}^{(k)}>q_{h_j}^{(k)}$ and $y_{g_{j'}}^{(k)}=y_{h_j}^{(k)}$ **then** the solution is given by $\underline{\mathrm{X}}^{(k-1)}:=(X_h^{(k-1)},X_g^{(k-1)},\left|w^{(k-1)}\right\rangle,\left|v^{(k-1)}\right\rangle)$ where $X_h^{(k-1)}=\sum_{i=1}^{k-1}y_{h_i}^{(k-1)}\left|t_{h_i}^{(k)}\right\rangle\left\langle t_{h_i}^{(k)}\right|$, $X_g^{(k-1)}=\sum_{i=1}^{k-1}y_{g_i}^{(k-1)}\left|t_{h_i}^{(k)}\right\rangle\left\langle t_{h_i}^{(k)}\right|$, $\left|v^{(k-1)}\right\rangle=\mathcal{N}\left[\sum_{i=1}^{k-1}\sqrt{q_{g_i}^{(k-1)}}\left|t_{h_i}^{(k)}\right\rangle\right]$, $\left|w^{(k-1)}\right\rangle=\mathcal{N}\left[\sum_{i=1}^{k-1}\sqrt{q_{h_i}^{(k-1)}}\left|t_{h_i}^{(k)}\right\rangle\right]$ where the coordinates and weights are given by

$$\begin{aligned}
\left\{q_{h_1}^{(k-1)},\dots q_{h_{k-1}}^{(k-1)}\right\}&\overset{\text{componentwise}}{=}\left\{q_{h_1}^{(k)},q_{h_2}^{(k)}\dots,q_{h_{j-1}}^{(k)},q_{h_{j+1}}^{(k)},\dots q_{h_k}^{(k)}\right\}\\
\left\{q_{g_1}^{(k-1)},\dots q_{g_{k-1}}^{(k-1)}\right\}&\overset{\text{componentwise}}{=}\left\{q_{g_2}^{(k)}\dots,q_{g_{j'-1}}^{(k)},q_{g_{j'}}^{(k)}-q_{h_j}^{(k)},q_{g_{j'+1}}^{(k)},q_{g_{j'+2}}^{(k)}\dots q_{g_k}^{(k)}\right\}\\
\left\{y_{g_1}^{(k-1)},\dots y_{g_{k-1}}^{(k-1)}\right\}&\overset{\text{componentwise}}{=}\left\{y_{g_2}^{(k)},\dots y_{g_k}^{(k)}\right\}\\
\left\{y_{h_1}^{(k-1)},\dots y_{h_{k-1}}^{(k-1)}\right\}&\overset{\text{componentwise}}{=}\left\{y_{h_1}^{(k)},\dots y_{h_{j-1}}^{(k)},y_{h_{j+1}}^{(k)}\dots,y_{h_k}^{(k)}\right\},
\end{aligned}$$

the basis is given by

$$\left\{\left|u_h^{(k)}\right\rangle,\left|t_{h_1}^{(k)}\right\rangle\dots\left|t_{h_{k-1}}^{(k)}\right\rangle\right\}\overset{\text{componentwise}}{=}\left\{\left|t_{h_j}^{(k+1)}\right\rangle,\left|t_{h_1}^{(k+1)}\right\rangle,\left|t_{h_2}^{(k+1)}\right\rangle,\dots\left|t_{h_{j-1}}^{(k+1)}\right\rangle,\left|t_{h_{j+1}}^{(k+1)}\right\rangle,\left|t_{h_{j+2}}^{(k+1)}\right\rangle\dots\left|t_{h_k}^{(k+1)}\right\rangle\right\}.$$

The orthogonal matrices are given by $\bar{O}_h^{(k)}:=\sum\left|t_{h_i}^{(k+1)}\right\rangle\langle a_i|$ where

$$\{|a_1\rangle,\dots|a_k\rangle\}\to\left\{\left|u_h^{(k)}\right\rangle,\left|t_{h_1}^{(k)}\right\rangle\dots\left|t_{h_{k-1}}^{(k)}\right\rangle\right\},$$

$\bar{O}_g^{(k)}:=\tilde{O}^{(k)}\bar{O}_h^{(k)}$ where

$$\begin{aligned}
\tilde{O}^{(k)}:=&\mathcal{N}\left[\sqrt{q_{h_j}^{(k)}}\left|u_h^{(k)}\right\rangle+\sqrt{q_{g_{j'}}^{(k)}-q_{h_j}^{(k)}}\left|t_{h_{j'}}^{(k)}\right\rangle\right]\mathcal{N}\left[\sqrt{q_{g_1}^{(k)}}\left\langle u_h^{(k)}\right|+\sqrt{q_{g_{j'}}^{(k)}}\left\langle t_{h_{j'}}^{(k)}\right|\right]\\
&+\mathcal{N}\left[\sqrt{q_{g_{j'}}^{(k)}-q_{h_j}^{(k)}}\left|u_h^{(k)}\right\rangle-\sqrt{q_{h_j}^{(k)}}\left|t_{h_{j'}}^{(k)}\right\rangle\right]\mathcal{N}\left[\sqrt{q_{g_{j'}}^{(k)}}\left\langle u_h^{(k)}\right|-\sqrt{q_{g_1}^{(k)}}\left\langle t_{h_{j'}}^{(k)}\right|\right]\\
&+\sum_{i\in\{1,\dots k\}\backslash j'}\left|t_{h_i}^{(k)}\right\rangle\left\langle t_{h_i}^{(k)}\right|.
\end{aligned}$$

Jump to **End**.

We are given $\underline{\mathrm{X}}^{(k)}=(X_h^{(k)},X_g^{(k)},\left|w^{(k)}\right\rangle,\left|v^{(k)}\right\rangle)$ where $X_h^{(k)}=\sum_{i=1}^k y_{h_i}^{(k)}\left|t_{h_i}^{(k+1)}\right\rangle\left\langle t_{h_i}^{(k+1)}\right|$, $X_g^{(k)}=\sum_{i=1}^k y_{g_i}^{(k)}\left|t_{h_i}^{(k+1)}\right\rangle\left\langle t_{h_i}^{(k+1)}\right|$, $\left|v^{(k)}\right\rangle=\sum_{i=1}^k q_{g_i}^{(k)}\left|t_{h_i}^{(k+1)}\right\rangle$, $\left|w^{(k)}\right\rangle=\sum_{i=1}^k q_{h_i}^{(k)}\left|t_{h_i}^{(k+1)}\right\rangle$ which means the corresponding function instance $\underline{\mathrm{x}}^{(k)}=(h^{(k)},g^{(k)},a^{(k)})$ where, in particular we have, $a^{(k)}=\sum_{i\in\{1,\dots k\}\backslash j}q_{h_i}^{(k)}[y_{h_i}]-\sum_{i\in\{1,\dots k\}\backslash j'}q_{g_i}^{(k)}[y_{g_i}]-(q_{g_{j'}}^{(k)}-q_{h_j}^{(k)})[y_{h_j}]$. Since we assume $\underline{\mathrm{X}}^{(k)}$ has a solution it follows that $a^{(k)}$ is $[\chi,\xi]$ valid. Thus the transition $g^{(k-1)}:=a_-^{(k)}\to a_+^{(k)}=:h^{(k-1)}$ is also $[\chi,\xi]$ valid where $g^{(k-1)}$ comprises $n_g^{(k-1)}=n_g^{(k)}$ points and $h^{(k-1)}$ comprises $n_h^{(k-1)}=n_h^{(k)}-1$ points (using the attributes corresponding to the function instance $(h^{(k-1)},g^{(k-1)},h^{(k-1)}-g^{(k-1)})$; The notation would be of the form $g=\sum_{i=1}^{n_g}p_{g_i}[x_{g_i}]$

and $h = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}]$). Since $k = n_g^{(k)} + n_h^{(k)} - 1$ the aforesaid relation yields $k-1 = n_g^{(k-1)} + n_h^{(k-1)} - 1$. We conclude that $\underline{\mathrm{X}}^{(k-1)} := (X_h^{(k-1)}, X_g^{(k-1)}, |w^{(k-1)}\rangle, |v^{(k-1)}\rangle)$ where $X_h^{(k-1)} = \sum_{i=1}^{k-1} y_{h_i}^{(k-1)} \left|t_{h_i}^{(k)}\right\rangle\left\langle t_{h_i}^{(k)}\right|$, $X_g^{(k-1)} = \sum_{i=1}^{k-1} y_{g_i}^{(k-1)} \left|t_{h_i}^{(k)}\right\rangle\left\langle t_{h_i}^{(k)}\right|$, $|v^{(k-1)}\rangle = \mathcal{N}\left[\sum_{i=1}^{k-1}\sqrt{q_{g_i}^{(k-1)}}\left|t_{h_i}^{(k)}\right\rangle\right]$, $|w^{(k-1)}\rangle = \mathcal{N}\left[\sum_{i=1}^{k-1}\sqrt{q_{h_i}^{(k-1)}}\left|t_{h_i}^{(k)}\right\rangle\right]$ will have a solution for

$$\begin{aligned}
\left\{q_{h_1}^{(k-1)}, \dots q_{h_{k-1}}^{(k-1)}\right\} &\overset{\text{componentwise}}{=} \left\{q_{h_1}^{(k)}, q_{h_2}^{(k)} \dots, q_{h_{j-1}}^{(k)}, q_{h_{j+1}}^{(k)}, \dots q_{h_k}^{(k)}\right\} \\
\left\{q_{g_1}^{(k-1)}, \dots q_{g_{k-1}}^{(k-1)}\right\} &\overset{\text{componentwise}}{=} \left\{q_{g_2}^{(k)} \dots, q_{g_{j'-1}}^{(k)}, q_{g_{j'}}^{(k)} - q_{h_j}^{(k)}, q_{g_{j'+1}}^{(k)}, q_{g_{j'+2}}^{(k)} \dots q_{g_k}^{(k)}\right\} \\
\left\{y_{g_1}^{(k-1)}, \dots y_{g_{k-1}}^{(k-1)}\right\} &\overset{\text{componentwise}}{=} \left\{y_{g_2}^{(k)}, \dots y_{g_k}^{(k)}\right\} \\
\left\{y_{h_1}^{(k-1)}, \dots y_{h_{k-1}}^{(k-1)}\right\} &\overset{\text{componentwise}}{=} \left\{y_{h_1}^{(k)}, \dots y_{h_{j-1}}^{(k)}, y_{h_{j+1}}^{(k)} \dots, y_{h_k}^{(k)}\right\}
\end{aligned}$$

as the corresponding function instance $\underline{\mathrm{x}}^{(k-1)}$ is indeed given by $(h^{(k-1)}, g^{(k-1)}, a^{(k-1)} = a^{(k)})$. Here $\{\left|t_{h_i}^{(k)}\right\rangle\}$ constitute an orthonormal basis which we will relate to $\left|t_{h_i}^{(k+1)}\right\rangle$ shortly. We used the fact that $q_{g_1}^{(k)} = 0$ as $y_{g_1}^{(k)} = \chi$ (To see this note that $k-1 > n_g^{(k-1)}$ which means that many q_{g_i} will be zero; by convention we write the smallest eigenvalue, χ first to increase the matrix size so the first $i = 1, 2 \dots \left(k-1-n_g^{(k-1)}\right)$ q_is will be zero.). This means that there must exist an $O^{(k-1)}$ which solves $\underline{\mathrm{X}}^{(k-1)}$.

Let's take a moment to note the following basis change manoeuvre. Note that $X'_h \geq O'X'_gO'^T$ with $O'|v'\rangle = |w'\rangle$ is equivalent to $X_h \geq OX_gO^T$ with $O|v\rangle = |w\rangle$ where $O = \bar{O}_h^T O' \bar{O}_g$, $\bar{O}_g|v\rangle = |v'\rangle$, $\bar{O}_h|w\rangle = |w'\rangle$, $\bar{O}_h X_h \bar{O}_h^T = X'_h$, $\bar{O} X_g \bar{O}_g^T = X'_g$ which is easy to see by a simple substitution.

We first expand the matrix $\underline{\mathrm{X}}^{(k-1)}$ to k dimensions as follows. We already had $X_h^{(k-1)} \geq O^{(k-1)} X_g^{(k-1)} O^{(k-1)T}$ with $O^{(k-1)}\left|v^{(k-1)}\right\rangle = \left|w^{(k-1)}\right\rangle$ which we expand as

$$\underbrace{y_{h_j}^{(k)}\left|u_h^{(k)}\right\rangle\left\langle u_h^{(k)}\right| + X_h^{(k-1)}}_{:=X_h'^{(k)}} \geq \underbrace{\left(\left|u_h^{(k)}\right\rangle\left\langle u_h^{(k)}\right| + O^{(k-1)}\right)}_{:=O'^{(k)}} \underbrace{\left(y_{h_j}^{(k)}\left|u_h^{(k)}\right\rangle\left\langle u_h^{(k)}\right| + X_g^{(k-1)}\right)}_{:=X_g'^{(k)}} \left(\left|u_h^{(k)}\right\rangle\left\langle u_h^{(k)}\right| + O^{(k-1)}\right)^T$$

with $\left|v'^{(k)}\right\rangle = \mathcal{N}\left[\sqrt{q_{h_j}^{(k)}}\left|u_h^{(k)}\right\rangle + \left|v^{(k-1)}\right\rangle\right]$ and $\left|w'^{(k)}\right\rangle = \mathcal{N}\left[\sqrt{q_{h_j}^{(k)}}\left|u_h^{(k)}\right\rangle + \left|w^{(k-1)}\right\rangle\right]$. Note that the matrix instance $\underline{\mathrm{X}}'^{(k)} := (X_h'^{(k)}, X_g'^{(k)}, \left|v'^{(k)}\right\rangle, \left|w'^{(k)}\right\rangle)$ yields $\underline{\mathrm{x}}'^{(k)} = \underline{\mathrm{x}}^{(k)}$. We can now use the equivalence we pointed out above to establish a relation between $X_h^{(k)} \geq O^{(k)} X_g^{(k)} O^{(k)T}$ and $X_h'^{(k)} \geq O'^{(k)} X_g'^{(k)} O'^{(k)T}$ by finding $\bar{O}_g$ and $\bar{O}_h$. We define, somewhat arbitrarily,

$$\left\{\left|u_h^{(k)}\right\rangle, \left|t_{h_1}^{(k)}\right\rangle \dots \left|t_{h_{k-1}}^{(k)}\right\rangle\right\} \overset{\text{componentwise}}{=} \left\{\left|t_{h_j}^{(k+1)}\right\rangle, \left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_{j-1}}^{(k+1)}\right\rangle, \left|t_{h_{j+1}}^{(k+1)}\right\rangle, \left|t_{h_{j+2}}^{(k+1)}\right\rangle \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\}.$$

We require $\bar{O}_h^{(k)}\left|w^{(k)}\right\rangle$ to be $\left|w'^{(k)}\right\rangle$. This is simply a permutation matrix given by $\left\{\left|t_{h_1}^{(k+1)}\right\rangle, \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\} \to \left\{\left|u_h^{(k)}\right\rangle, \left|t_{h_1}^{(k)}\right\rangle \dots \left|t_{h_{k-1}}^{(k)}\right\rangle\right\}$. Note that this yields $\bar{O}_h^{(k)T} X_h'^{(k)} \bar{O}_h^{(k)} = X_h^{(k)}$. It remains to find $\bar{O}_g^{(k)}$ which we demand must satisfy $\bar{O}_g^{(k)}\left|v^{(k)}\right\rangle = \left|v'^{(k)}\right\rangle$. Observe first that $\bar{O}_h^{(k)}\left|v^{(k)}\right\rangle = \sqrt{q_{g_1}^{(k)}}\left|u_h^{(k)}\right\rangle + \sum_{i=2}^{k}\sqrt{q_{g_i}^{(k)}}\left|t_{h_{i-1}}^{(k)}\right\rangle$. We must now apply

$$\begin{aligned}
\tilde{O}^{(k)} :=& \mathcal{N}\left[\sqrt{q_{h_j}^{(k)}}\left|u_h^{(k)}\right\rangle + \sqrt{q_{g_{j'}}^{(k)} - q_{h_j}^{(k)}}\left|t_{h_{j'}}^{(k)}\right\rangle\right]\mathcal{N}\left[\sqrt{q_{g_1}^{(k)}}\left\langle u_h^{(k)}\right| + \sqrt{q_{g_{j'}}^{(k)}}\left\langle t_{h_{j'}}^{(k)}\right|\right] \\
&+ \mathcal{N}\left[\sqrt{q_{g_{j'}}^{(k)} - q_{h_j}^{(k)}}\left|u_h^{(k)}\right\rangle - \sqrt{q_{h_j}^{(k)}}\left|t_{h_{j'}}^{(k)}\right\rangle\right]\mathcal{N}\left[\sqrt{q_{g_{j'}}^{(k)}}\left\langle u_h^{(k)}\right| - \sqrt{q_{g_1}^{(k)}}\left\langle t_{h_{j'}}^{(k)}\right|\right] \\
&+ \sum_{i\in\{1,\dots k\}\backslash j'}\left|t_{h_i}^{(k)}\right\rangle\left\langle t_{h_i}^{(k)}\right|
\end{aligned}$$

to get $\bar{O}_g^{(k)}\left|v^{(k)}\right\rangle = \left|v'^{(k)}\right\rangle$ where we defined $\bar{O}_g^{(k)} := \tilde{O}^{(k)}\bar{O}_h^{(k)}$. (Note the expression could be simplified by using $q_{g_1} = 0$ which in fact is necessary for probability conservation.) Using $y_{h_j}^{(k)} = y_{g_{j'}}^{(k)}$ we can also see that $\bar{O}_g^{(k)T} X_g'^{(k)} \bar{O}_g^{(k)}$ is essentially $X_g^{(k)}$ with $\chi^{(k)}$ at $\left|t_{h_1}^{(k+1)}\right\rangle$ replaced by $y_{g_{j'}} (= y_{h_j})$. One can conclude therefore that

$X'^{(k)}_g \geq \bar{O}^{(k)}_g X^{(k)}_g \bar{O}^{(k)T}_g$. Following the substitution manoeuvre we have

$$\begin{aligned}
& X'^{(k)}_h \geq O'^{(k)} X'^{(k)}_g O'^{(k)T} \geq O'^{(k)} \bar{O}^{(k)}_g X^{(k)}_g \bar{O}^{(k)T}_g O'^{(k)T} \\
\iff & \bar{O}^{(k)T}_h X'^{(k)}_h \bar{O}^{(k)}_h \geq \underbrace{\bar{O}^{(k)T}_h O'^{(k)} \bar{O}^{(k)}_g}_{:=O^{(k)}} X^{(k)}_g \bar{O}^{(k)T}_g O'^{(k)T} \bar{O}^{(k)}_h \\
\iff & X^{(k)}_h \geq O^{(k)} X^{(k)}_g O^{(k)T}
\end{aligned}$$

and similarly

$$\begin{aligned}
& O'^{(k)} \left| v'^{(k)} \right\rangle = \left| w'^{(k)} \right\rangle \\
\iff & O'^{(k)} \bar{O}^{(k)}_g \left| v^{(k)} \right\rangle = \bar{O}^{(k)}_h \left| w^{(k)} \right\rangle \\
\iff & O^{(k)} \left| v^{(k)} \right\rangle = \left| w^{(k)} \right\rangle .
\end{aligned}$$

This completes the proof.

3. **Initial Extra**: **If** for some j, j' we have $q^{(k)}_{g_{j'}} < q^{(k)}_{h_j}$ and $y^{(k)}_{g_{j'}} = y^{(k)}_{h_j}$ **then** the solution is given by $\underline{\mathrm{X}}^{(k-1)} := (X^{(k-1)}_h, X^{(k-1)}_g, \left|w^{(k-1)}\right\rangle, \left|v^{(k-1)}\right\rangle)$ where $X^{(k-1)}_h = \sum_{i=1}^{k-1} y^{(k-1)}_{h_i} \left| t^{(k)}_{h_i} \right\rangle \left\langle t^{(k)}_{h_i} \right|$, $X^{(k-1)}_g = \sum_{i=1}^{k-1} y^{(k-1)}_{g_i} \left| t^{(k)}_{h_i} \right\rangle \left\langle t^{(k)}_{h_i} \right|$, $\left|v^{(k-1)}\right\rangle = \mathcal{N}\left[\sum_{i=1}^{k-1} \sqrt{q^{(k-1)}_{g_i}} \left| t^{(k)}_{h_i} \right\rangle \right]$, $\left|w^{(k-1)}\right\rangle = \mathcal{N}\left[\sum_{i=1}^{k-1} \sqrt{q^{(k-1)}_{h_i}} \left| t^{(k)}_{h_i} \right\rangle \right]$ where the coordinates and weights are given by

$$\begin{aligned}
\left\{ q^{(k-1)}_{h_1}, \dots q^{(k-1)}_{h_{k-1}} \right\} &\overset{\text{componentwise}}{=} \left\{ q^{(k)}_{h_1} \dots, q^{(k)}_{h_{j-1}}, q^{(k)}_{h_j} - q^{(k)}_{g_{j'}}, q^{(k)}_{h_{j+1}}, q^{(k)}_{h_{j+2}} \dots q^{(k)}_{h_{k-1}} \right\} \\
\left\{ q^{(k-1)}_{g_1}, \dots q^{(k-1)}_{g_{k-1}} \right\} &\overset{\text{componentwise}}{=} \left\{ q^{(k)}_{g_1}, q^{(k)}_{g_2} \dots, q^{(k)}_{g_{j'-1}}, q^{(k)}_{g_{j'+1}}, \dots q^{(k)}_{g_k} \right\} \\
\left\{ y^{(k-1)}_{g_1}, \dots y^{(k-1)}_{g_{k-1}} \right\} &\overset{\text{componentwise}}{=} \left\{ y^{(k)}_{g_1}, \dots y^{(k)}_{g_{j'-1}}, y^{(k)}_{g_{j'+1}} \dots, y^{(k)}_{g_k} \right\} \\
\left\{ y^{(k-1)}_{h_1}, \dots y^{(k-1)}_{h_{k-1}} \right\} &\overset{\text{componentwise}}{=} \left\{ y^{(k)}_{h_1}, \dots y^{(k)}_{h_{k-1}} \right\},
\end{aligned}$$

the basis is given by

$$\left\{ \left| u^{(k)}_h \right\rangle, \left| t^{(k)}_{h_1} \right\rangle \dots \left| t^{(k)}_{h_{k-1}} \right\rangle \right\} \overset{\text{componentwise}}{=} \left\{ \left| t^{(k+1)}_{h_j} \right\rangle, \left| t^{(k+1)}_{h_1} \right\rangle, \left| t^{(k+1)}_{h_2} \right\rangle, \dots \left| t^{(k+1)}_{h_{j-1}} \right\rangle, \left| t^{(k+1)}_{h_{j+1}} \right\rangle, \left| t^{(k+1)}_{h_{j+2}} \right\rangle \dots \left| t^{(k+1)}_{h_k} \right\rangle \right\}.$$

The orthogonal matrices are given by $\bar{O}^{(k)}_h := \tilde{O}^{(k)} \sum \left| a_i \right\rangle \left\langle t^{(k+1)}_{h_i} \right|$ where

$$\{ \left| a_1 \right\rangle, \dots \left| a_k \right\rangle \} \overset{\text{componentwise}}{=} \left\{ \left| t^{(k)}_{h_1} \right\rangle, \left| t^{(k)}_{h_2} \right\rangle \dots \left| t^{(k)}_{h_{k-1}} \right\rangle, \left| u^{(k)}_h \right\rangle \right\}.$$

$$\begin{aligned}
\tilde{O}^{(k)} :=& \mathcal{N}\left[\sqrt{q^{(k)}_{g_{j'}}} \left| u^{(k)}_h \right\rangle + \sqrt{q^{(k)}_{h_j} - q^{(k)}_{g_{j'}}} \left| t^{(k)}_{h_j} \right\rangle \right] \mathcal{N}\left[\sqrt{q^{(k)}_{h_k}} \left\langle u^{(k)}_h \right| + \sqrt{q^{(k)}_{g_j}} \left\langle t^{(k)}_{h_j} \right| \right] \\
& + \mathcal{N}\left[\sqrt{q^{(k)}_{h_j} - q^{(k)}_{g_{j'}}} \left| u^{(k)}_h \right\rangle - \sqrt{q^{(k)}_{g_{j'}}} \left| t^{(k)}_{h_j} \right\rangle \right] \mathcal{N}\left[\sqrt{q^{(k)}_{g_j}} \left\langle u^{(k)}_h \right| - \sqrt{q^{(k)}_{h_k}} \left\langle t^{(k)}_{h_j} \right| \right] \\
& + \sum_{i \in \{1, \dots k\} \setminus j} \left| t^{(k)}_{h_i} \right\rangle \left\langle t^{(k)}_{h_i} \right|
\end{aligned}$$

and $\bar{O}^{(k)}_h$ is given by the basis change $\left\{ \left| t^{(k+1)}_{h_1} \right\rangle, \dots \left| t^{(k+1)}_{h_k} \right\rangle \right\} \to \left\{ \left| u^{(k)}_h \right\rangle, \left| t^{(k)}_{h_1} \right\rangle \dots \left| t^{(k)}_{h_{k-1}} \right\rangle \right\}$.
Jump to **End**.

This proof will be very similar to the previous one. We are given $\underline{\mathrm{X}}^{(k)} = (X^{(k)}_h, X^{(k)}_g, \left|w^{(k)}\right\rangle, \left|v^{(k)}\right\rangle)$ where $X^{(k)}_h = \sum_{i=1}^k y^{(k)}_{h_i} \left| t^{(k+1)}_{h_i} \right\rangle \left\langle t^{(k+1)}_{h_i} \right|$, $X^{(k)}_g = \sum_{i=1}^k y^{(k)}_{g_i} \left| t^{(k+1)}_{h_i} \right\rangle \left\langle t^{(k+1)}_{h_i} \right|$, $\left|v^{(k)}\right\rangle = \sum_{i=1}^k q^{(k)}_{g_i} \left| t^{(k+1)}_{h_i} \right\rangle$, $\left|w^{(k)}\right\rangle = \sum_{i=1}^k q^{(k)}_{h_i} \left| t^{(k+1)}_{h_i} \right\rangle$ which means the corresponding function instance $\underline{\mathrm{x}}^{(k)} = (h^{(k)}, g^{(k)}, a^{(k)})$ where, in particular we have,

$$a^{(k)} = \sum_{i \in \{1, \dots k\} \setminus j} q^{(k)}_{h_i} [y_{h_i}] + (q^{(k)}_{h_j} - q^{(k)}_{g_{j'}})[y_{h_j}] - \sum_{i \in \{1, \dots k\} \setminus j'} q^{(k)}_{g_i} [y_{g_i}].$$

Since we assume $\underline{\mathrm{X}}^{(k)}$ has a solution it follows that $a^{(k)}$ is $[\chi,\xi]$ valid. Thus the transition $g^{(k-1)} := a_-^{(k)} \to a_+^{(k)} =: h^{(k-1)}$ is also $[\chi,\xi]$ valid where $g^{(k-1)}$ comprises $n_g^{(k-1)} = n_g^{(k)} - 1$ points and $h^{(k-1)}$ comprises $n_h^{(k-1)} = n_h^{(k)}$ points (using the attributes corresponding to the function instance $(h^{(k-1)}, g^{(k-1)}, h^{(k-1)} - g^{(k-1)})$; The notation would be of the form $g = \sum_{i=1}^{n_g} p_{g_i}[x_{g_i}]$ and $h = \sum_{i=1}^{n_h} p_{h_i}[x_{h_i}]$). Since $k = n_g^{(k)} + n_h^{(k)} - 1$ the aforesaid relation yields $n_g^{(k-1)} + n_h^{(k-1)} - 1 = k-1$. We conclude that $\underline{\mathrm{X}}^{(k-1)} := (X_h^{(k-1)}, X_g^{(k-1)}, \left|w^{(k-1)}\right\rangle, \left|v^{(k-1)}\right\rangle)$ where $X_h^{(k-1)} = \sum_{i=1}^{k-1} y_{h_i}^{(k-1)} \left|t_{h_i}^{(k)}\right\rangle \left\langle t_{h_i}^{(k)}\right|$, $X_g^{(k-1)} = \sum_{i=1}^{k-1} y_{g_i}^{(k-1)} \left|t_{h_i}^{(k)}\right\rangle \left\langle t_{h_i}^{(k)}\right|$, $\left|v^{(k-1)}\right\rangle = \mathcal{N}\left[\sum_{i=1}^{k-1} \sqrt{q_{g_i}^{(k-1)}} \left|t_{h_i}^{(k)}\right\rangle\right]$, $\left|w^{(k-1)}\right\rangle = \mathcal{N}\left[\sum_{i=1}^{k-1} \sqrt{q_{h_i}^{(k-1)}} \left|t_{h_i}^{(k)}\right\rangle\right]$ will have a solution for

$$
\begin{aligned}
\left\{q_{h_1}^{(k-1)}, \dots q_{h_{k-1}}^{(k-1)}\right\} &\stackrel{\text{componentwise}}{=} \left\{q_{h_1}^{(k)} \dots, q_{h_{j-1}}^{(k)}, q_{h_j}^{(k)} - q_{g_{j'}}^{(k)}, q_{h_{j+1}}^{(k)}, q_{h_{j+2}}^{(k)} \dots q_{h_{k-1}}^{(k)}\right\} \\
\left\{q_{g_1}^{(k-1)}, \dots q_{g_{k-1}}^{(k-1)}\right\} &\stackrel{\text{componentwise}}{=} \left\{q_{g_1}^{(k)}, q_{g_2}^{(k)} \dots, q_{g_{j'-1}}^{(k)}, q_{g_{j'+1}}^{(k)}, \dots q_{g_k}^{(k)}\right\} \\
\left\{y_{g_1}^{(k-1)}, \dots y_{g_{k-1}}^{(k-1)}\right\} &\stackrel{\text{componentwise}}{=} \left\{y_{g_1}^{(k)}, \dots y_{g_{j'-1}}^{(k)}, y_{g_{j'+1}}^{(k)} \dots, y_{g_k}^{(k)}\right\} \\
\left\{y_{h_1}^{(k-1)}, \dots y_{h_{k-1}}^{(k-1)}\right\} &\stackrel{\text{componentwise}}{=} \left\{y_{h_1}^{(k)}, \dots y_{h_{k-1}}^{(k)}\right\},
\end{aligned}
$$

as the corresponding function instance $\underline{\mathrm{x}}^{(k-1)}$ is indeed given by $(h^{(k-1)}, g^{(k-1)}, a^{(k-1)} = a^{(k)})$. Here $\{\left|t_{h_i}^{(k)}\right\rangle\}$ constitute an orthonormal basis which we will relate to $\left|t_{h_i}^{(k+1)}\right\rangle$ shortly. We used the fact that $q_{h_k}^{(k)} = 0$ as $y_{h_k}^{(k)} = \xi$. (To see this note that $k-1 > n_h^{(k-1)}$ which means that many q_{h_i} will be zero; by convention we write the smallest eigenvalue, x_{h_1} first all the way till $x_{h_{n_h}}$ and then to increase the matrix size we append zeros so the $i = n_h, n_h+1 \dots k$ will yield $q_{h_i} = 0$.) This means that there must exist an $O^{(k-1)}$ which solves $\underline{\mathrm{X}}^{(k-1)}$.

Let's take a moment to note the following basis change manoeuvre. $X_h' \geq O' X_g' O'^T$ with $O' \left|v'\right\rangle = \left|w'\right\rangle$ is equivalent to $X_h \geq O X_g O^T$ with $O\left|v\right\rangle = \left|w\right\rangle$ where $O = \bar{O}_h^T O' \bar{O}_g$, $\bar{O}_g \left|v\right\rangle = \left|v'\right\rangle$, $\bar{O}_h \left|w\right\rangle = \left|w'\right\rangle$, $\bar{O}_h X_h \bar{O}_h^T = X_h'$, $\bar{O} X_g \bar{O}_g^T = X_g'$ which is easy to see by a simple substitution.

We first expand the matrix $\underline{\mathrm{X}}^{(k-1)}$ to k dimensions as follows. We already had $X_h^{(k-1)} \geq O^{(k-1)} X_g^{(k-1)} O^{(k-1)T}$ with $O^{(k-1)}\left|v^{(k-1)}\right\rangle = \left|w^{(k-1)}\right\rangle$ which we expand as

$$
\underbrace{y_{h_j}^{(k)} \left|u_h^{(k)}\right\rangle \left\langle u_h^{(k)}\right| + X_h^{(k-1)}}_{:=X_h'^{(k)}} \geq \underbrace{\left(\left|u_h^{(k)}\right\rangle \left\langle u_h^{(k)}\right| + O^{(k-1)}\right)}_{:=O'^{(k)}} \underbrace{\left(y_{h_j}^{(k)} \left|u_h^{(k)}\right\rangle \left\langle u_h^{(k)}\right| + X_g^{(k-1)}\right)}_{:=X_g'^{(k)}} \left(\left|u_h^{(k)}\right\rangle \left\langle u_h^{(k)}\right| + O^{(k-1)}\right)^T
$$

with $\left|v'^{(k)}\right\rangle = \mathcal{N}\left[\sqrt{q_{g_{j'}}^{(k)}} \left|u_h^{(k)}\right\rangle + \left|v^{(k-1)}\right\rangle\right]$ and $\left|w'^{(k)}\right\rangle = \mathcal{N}\left[\sqrt{q_{g_{j'}}^{(k)}} \left|u_h^{(k)}\right\rangle + \left|w^{(k-1)}\right\rangle\right]$. Note that the matrix instance $\underline{\mathrm{X}}'^{(k)} := (X_h'^{(k)}, X_g'^{(k)}, \left|v'^{(k)}\right\rangle, \left|w'^{(k)}\right\rangle)$ yields $\underline{\mathrm{x}}'^{(k)} = \underline{\mathrm{x}}^{(k)}$. We can now use the equivalence we pointed out above to establish a relation between $X_h^{(k)} \geq O^{(k)} X_g^{(k)} O^{(k)T}$ and $X_h'^{(k)} \geq O'^{(k)} X_g'^{(k)} O'^{(k)T}$ by finding $\bar{O}_g$ and $\bar{O}_h$. We define, somewhat arbitrarily,

$$
\left\{\left|u_h^{(k)}\right\rangle, \left|t_{h_1}^{(k)}\right\rangle \dots \left|t_{h_{k-1}}^{(k)}\right\rangle\right\} \stackrel{\text{componentwise}}{=} \left\{\left|t_{h_j}^{(k+1)}\right\rangle, \left|t_{h_1}^{(k+1)}\right\rangle, \left|t_{h_2}^{(k+1)}\right\rangle, \dots \left|t_{h_{j-1}}^{(k+1)}\right\rangle, \left|t_{h_{j+1}}^{(k+1)}\right\rangle, \left|t_{h_{j+2}}^{(k+1)}\right\rangle \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\}.
$$

We require $\bar{O}_g^{(k)} \left|v^{(k)}\right\rangle$ to be $\left|v'^{(k)}\right\rangle$. This is simply a permutation matrix given by $\left\{\left|t_{h_1}^{(k+1)}\right\rangle, \dots \left|t_{h_k}^{(k+1)}\right\rangle\right\} \to \left\{\left|u_h^{(k)}\right\rangle, \left|t_{h_1}^{(k)}\right\rangle \dots \left|t_{h_{k-1}}^{(k)}\right\rangle\right\}$. Note that this yields $\bar{O}_g^{(k)T} X_g'^{(k)} \bar{O}_g^{(k)} = X_g^{(k)}$ as $y_{h_j}^{(k)} = y_{g_{j'}}^{(k)}$. It remains to find $\bar{O}_h^{(k)}$ which we demand must satisfy $\bar{O}_h^{(k)} \left|w^{(k)}\right\rangle = \left|w'^{(k)}\right\rangle$. Let us define $\bar{O}_h^{(k)} = \tilde{O}^{(k)} \left(\sum_{i=1}^k \left|a_i\right\rangle \left\langle t_{h_i}^{(k+1)}\right|\right)$. Observe that for $\tilde{O}^{(k)} = \mathbb{I}$ we have $\bar{O}_h^{(k)} \left|w^{(k)}\right\rangle = q_{h_k}^{(k)} \left|u_h^{(k)}\right\rangle + \sum_{i=1}^{k-1} q_{h_i}^{(k)} \left|t_{h_i}^{(k)}\right\rangle$ where

$$
\{\left|a_1\right\rangle, \dots \left|a_k\right\rangle\} \stackrel{\text{componentwise}}{=} \left\{\left|t_{h_1}^{(k)}\right\rangle, \left|t_{h_2}^{(k)}\right\rangle \dots \left|t_{h_{k-1}}^{(k)}\right\rangle, \left|u_h^{(k)}\right\rangle\right\}.
$$

If we define

$$
\begin{aligned}
\tilde{O}^{(k)} :=& \mathcal{N}\left[\sqrt{q_{g_{j'}}^{(k)}} \left|u_h^{(k)}\right\rangle + \sqrt{q_{h_j}^{(k)} - q_{g_{j'}}^{(k)}} \left|t_{h_j}^{(k)}\right\rangle\right] \mathcal{N}\left[\sqrt{q_{h_k}^{(k)}} \left\langle u_h^{(k)}\right| + \sqrt{q_{g_j}^{(k)}} \left\langle t_{h_j}^{(k)}\right|\right] \\
&+ \mathcal{N}\left[\sqrt{q_{h_j}^{(k)} - q_{g_{j'}}^{(k)}} \left|u_h^{(k)}\right\rangle - \sqrt{q_{g_{j'}}^{(k)}} \left|t_{h_j}^{(k)}\right\rangle\right] \mathcal{N}\left[\sqrt{q_{g_j}^{(k)}} \left\langle u_h^{(k)}\right| - \sqrt{q_{h_k}^{(k)}} \left\langle t_{h_j}^{(k)}\right|\right] \\
&+ \sum_{i \in \{1,\dots k\} \setminus j} \left|t_{h_i}^{(k)}\right\rangle \left\langle t_{h_i}^{(k)}\right|
\end{aligned}
$$

we get $\bar{O}_h^{(k)}\left|w^{(k)}\right\rangle = \left|w'^{(k)}\right\rangle$ as desired. We can also see that $\bar{O}_h^{(k)T}X_g'^{(k)}\bar{O}_h^{(k)}$ is essentially X_g with $\xi^{(k)}$ at $\left|t_{h_k}^{(k+1)}\right\rangle$ replaced by y_{h_j}. We therefore conclude that $X_g'^{(k)} \geq \bar{O}_g^{(k)}X_g^{(k)}\bar{O}_g'^{(k)}$. Following the substitution manoeuvre we have

$$\begin{aligned}
&X_h'^{(k)} \geq O'^{(k)}X_g'^{(k)}O'^{(k)T} \geq O'^{(k)}\bar{O}_g^{(k)}X_g^{(k)}\bar{O}_g^{(k)T}O'^{(k)T} \\
\iff &\bar{O}_h^{(k)T}X_h'^{(k)}\bar{O}_h^{(k)} \geq \underbrace{\bar{O}_h^{(k)T}O'^{(k)}\bar{O}_g^{(k)}}_{:=O^{(k)}}X_g^{(k)}\bar{O}_g^{(k)T}O'^{(k)T}\bar{O}_h^{(k)} \\
\iff &X_h^{(k)} \geq O^{(k)}X_g^{(k)}O^{(k)T}
\end{aligned}$$

and similarly

$$\begin{aligned}
&O'^{(k)}\left|v'^{(k)}\right\rangle = \left|w'^{(k)}\right\rangle \\
\iff &O'^{(k)}\bar{O}_g^{(k)}\left|v^{(k)}\right\rangle = \bar{O}_h^{(k)}\left|w^{(k)}\right\rangle \\
\iff &O^{(k)}\left|v^{(k)}\right\rangle = \left|w^{(k)}\right\rangle.
\end{aligned}$$

This completes the proof.

- **Evaluate the Reverse Weingarten Map**:
  1. Consider the point $\left|w^{(k)}\right\rangle/\sqrt{\left\langle w^{(k)}\right|X_h'^{(k)}\left|w^{(k)}\right\rangle}$ on the ellipsoid $X_h'^{(k)}$. Evaluate the normal at this point as $\left|u_h^{(k)}\right\rangle = \mathcal{N}\left(\sum_{i=1}^{n_h^{(k)}}\sqrt{p_{h_i}^{(k)}}x_{h_i}'^{(k)}\left|t_{h_i}^{(k+1)}\right\rangle\right)$. Similarly evaluate $\left|u_g^{(k)}\right\rangle$, the normal at the point $\left|v^{(k)}\right\rangle/\sqrt{\left\langle w^{(k)}\right|X_g'^{(k)}\left|w^{(k)}\right\rangle}$ on the ellipsoid $X_g'^{(k)}$.
  2. Recall that for a given diagonal matrix $X = \sum_i y_i\left|i\right\rangle\left\langle i\right| > 0$ and normal vector $\left|u\right\rangle = \sum_i u_i\left|i\right\rangle$ the Reverse Weingarten map is given by $W_{ij} = \left(-\frac{y_j^{-1}y_i^{-1}u_iu_j}{r^2} + y_i^{-1}\delta_{ij}\right)$ where $r = \sqrt{\sum y_i^{-1}u_i^2}$. Evaluate the Reverse Weingarten maps $W_h'^{(k)}$ and $W_g'^{(k)}$ along $\left|u_h^{(k)}\right\rangle$ and $\left|u_g^{(k)}\right\rangle$ respectively.
  3. Find the eigenvectors and eigenvalues of the Reverse Weingarten maps. The eigenvectors of W_h' form the h tangent (and normal) vectors $\left\{\left\{\left|t_{h_i}^{(k)}\right\rangle\right\}, \left|u_h^{(k)}\right\rangle\right\}$. The corresponding radii of curvature are obtained from the eigenvalues $\left\{\{r_{h_i}^{(k)}\}, 0\right\} = \left\{\{c_{h_i}^{(k)-1}\}, 0\right\}$ which are inverses of the curvature values. The tangents are labelled in the decreasing order of radii of curvature (increasing order of curvature). Similarly for the g tangent (and normal) vectors. Fix the sign freedom in the eigenvectors by requiring $\left\langle t_{h_i}^{(k)}|w^{(k)}\right\rangle \geq 0$ and $\left\langle t_{g_i}^{(k)}|v^{(k)}\right\rangle \geq 0$.
- **Finite Method: If** $\lambda \neq -\xi^{(k)}$ and $\lambda \neq -\chi^{(k)}$, i.e. if it is the finite case **then**
  1. $\bar{O}^{(k)} := \left|u_h^{(k)}\right\rangle\left\langle u_g^{(k)}\right| + \sum_{i=1}^{k-1}\left|t_{h_i}^{(k)}\right\rangle\left\langle t_{g_i}^{(k)}\right|$
  2. $\left|v^{(k-1)}\right\rangle := \bar{O}^{(k)}\left|v^{(k)}\right\rangle - \left\langle u_h^{(k)}\right|\bar{O}^{(k)}\left|v^{(k)}\right\rangle\left|u_h^{(k)}\right\rangle$ and $\left|w^{(k-1)}\right\rangle := \left|w^{(k)}\right\rangle - \left\langle u_h^{(k)}|w^{(k)}\right\rangle\left|u_h^{(k)}\right\rangle$.
  3. Define $X_h^{(k-1)} := \operatorname{diag}\{c_{h_1}^{(k)}, c_{h_2}^{(k)}\ldots, c_{h_{k-1}}^{(k)}\}$, $X_g^{(k-1)} := \operatorname{diag}\{c_{g_1}^{(k)}, c_{g_2}^{(k)}\ldots c_{g_{k-1}}^{(k)}\}$.
  4. **Jump** to **End**.

     Our first burden is to prove that $O^{(k)}$ must have the form $\left(\left|u_h^{(k)}\right\rangle\left\langle u_h^{(k)}\right| + O^{(k-1)}\right)\bar{O}^{(k)}$ for $\bar{O}^{(k)} := \left|u_h^{(k)}\right\rangle\left\langle u_g^{(k)}\right| + \sum_{i=1}^{k-1}\left|t_{h_i}^{(k)}\right\rangle\left\langle t_{g_i}^{(k)}\right|$ if $O^{(k)}$ is to be a solution of the matrix instance $\underline{X}^{(k)}$. This is best explained by imagining that Arthur is trying to find the orthogonal matrix and Merlin already knows the orthogonal matrix but has still been following the steps performed so far. Recall that we are now at a point where

$$\begin{aligned}
\sum a'(x)x &= \left\langle w\right|X_h'\left|w\right\rangle - \left\langle v\right|X_g'\left|v\right\rangle \\
&= \left\langle w\right|X_h'\left|w\right\rangle - \left\langle w\right|OX_g'O^T\left|w\right\rangle \\
&= 0.
\end{aligned}$$

     From Merlin's point of view along the $\left|w\right\rangle$ direction the ellipsoids X_h' and $OX_g'O^T$ touch. Suppose he started with the ellipsoids X_h', X_g' and only subsequently rotated the second one. He can mark the point along the direction $\left|v\right\rangle$ on the X_g' ellipsoid as the point that would after rotation touch the X_h' ellipsoid because as $X_g' \to OX_g'O^T$ the point along the $\left|v\right\rangle$ direction would get mapped to the point along the direction $O\left|v\right\rangle = \left|w\right\rangle$. Now, since the ellipsoids touch it must be so, Merlin deduces, that the normal of the ellipsoid X_g' at the point $\left|v\right\rangle/\sqrt{\left\langle v\right|X_g'\left|v\right\rangle}$

is mapped to the normal of the ellipsoid X'_h at the point $|w\rangle / \sqrt{\langle w| X'_h |w\rangle}$ when X'_g is rotated to OX'_gO^T, i.e. $O|u_g\rangle = |u_h\rangle$.

From Arthur's point of view, who has been following Merlin's reasoning, in addition to knowing that O must satisfy $O|v\rangle = |w\rangle$ he now knows that it must also satisfy $O|u_g\rangle = |u_h\rangle$.

Merlin further concludes that the curvature of the X'_g ellipsoid at the point $|v\rangle / \sqrt{\langle v| X'_g |v\rangle}$ must be more than the curvature of the X'_h ellipsoid at the point $|w\rangle / \sqrt{\langle w| X'_h |w\rangle}$. To be precise, he needs to find a method for evaluating this curvature.

He knows that the brute-force way of doing this is to find a coordinate system with its origin on the said point and then imagining the manifold, locally, as a function from $n-1$ coordinates to one coordinate, call it $x_n(x_1, x_2 \dots x_{n-1})$ (think of a sphere centred at the origin; it can be thought of, locally, as a function from x and y to z given by $z = \sqrt{x^2+y^2}$). The curvature of this object will be a generalisation of the second derivative which forms a matrix with its elements given by $\partial_{x_i}\partial_{x_j} x_n$. Since this matrix is symmetric he knows it can be diagonalised. The directions of the eigenvectors of this matrix he calls the principle directions of curvature where the curvature values are the corresponding eigenvalues.

He recalls that there is a simpler way of evaluating these principle directions and curvatures which uses the Weingarten map. The eigenvectors of the Reverse Weingarten map W'_h, evaluated for X'_h at $|w\rangle$, yield the normal and tangent vectors with the corresponding eigenvalues zero and radii of curvature respectively. Curvature is the inverse of the radius of curvature. Similarly for the Reverse Weingarten map W'_g evaluated for X'_g at $|v\rangle$.

With this knowledge Merlin deduces that he can write, for some $\tilde{O}_{ij} \in \mathbb{R}$ such that $\sum_j \tilde{O}_{ij}\tilde{O}_{jk} = \delta_{ik}$,

$$\begin{aligned} O^{(k)} &= |u_h\rangle\langle u_g| + \sum_{i,j} \tilde{O}_{ij} |t_{h_i}\rangle\langle t_{g_j}| \\ &= \left(|u_h\rangle\langle u_h| + \underbrace{\sum_{i,j} \tilde{O}_{ij} |t_{h_i}\rangle\langle t_{h_j}|}_{=O^{(k-1)}} \right) \left(\underbrace{|u_h\rangle\langle u_g| + \sum_{i} |t_{h_i}\rangle\langle t_{g_i}|}_{=\bar{O}^{(k)}} \right) \end{aligned}$$

where he re-introduced the superscript in the orthogonal operators. He then turns to his intuition about the curvature of the smaller ellipsoid being more than that of the larger ellipsoid. He observes that equivalently, the radius of curvature of the smaller ellipsoid must be smaller than that of the larger ellipsoid. To make this precise he first notes that the Weingarten map W'_g gets transformed to OW'_gO^T when X'_g is rotated as OX'_gO^T. He considers the point $|w\rangle / \sqrt{\langle w| X'_h |w\rangle}$, which is shared by both the X'_h and the OX'_gO^T ellipsoid. It must be so, he reasons, that along all directions in the tangent plane, the X'_h ellipsoid (the smaller one, remember larger X'_h means smaller ellipsoid) must have a smaller radius of curvature than the OX'_gO^T ellipsoid, i.e. for all $|t\rangle \in \text{span}\{|t_{h_i}\rangle\}$, $\langle t| W'_h |t\rangle \le \langle t| OW'_gO^T |t\rangle$. Restricting his attention to the tangent space he deduces the statement is equivalent to $W'_h \le OW'_gO^T$. He writes this out explicitly as $\sum c_{h_i}^{-1} |t_{h_i}\rangle\langle t_{h_i}| \le \sum c_{g_i}^{-1} O |t_{g_i}\rangle\langle t_{g_i}| O^T$. Now he uses the form of O he had deduced to obtain $\sum c_{h_i}^{-1} |t_{h_i}\rangle\langle t_{h_i}| \le \sum c_{g_i}^{-1} O^{(k-1)} |t_{h_i}\rangle\langle t_{h_i}| O^{(k-1)T}$. From this he is able to deduce that the inequality $X_h^{(k-1)} \ge O^{(k-1)} X_g^{(k-1)} O^{(k-1)T}$ must hold.

Merlin's reasoning entails, Arthur summarises, that $O^{(k)}$ must always have the form

$$O^{(k)} = \left(\left|u_h^{(k)}\right\rangle \left\langle u_h^{(k)}\right| + O^{(k-1)} \right) \bar{O}^{(k)}$$

and that $O^{(k-1)}$ must satisfy the constraint

$$X_h^{(k-1)} \ge O^{(k-1)} X_g^{(k-1)} O^{(k-1)T}.$$

Merlin, surprised by the similarity of the constraint he obtained with the one he started with, extends his reasoning to the vector itself. He knows that $O^{(k)} \left|v^{(k)}\right\rangle = \left|w^{(k)}\right\rangle$ but now he substitutes for $O^{(k)}$ to obtain $\left(\left|u_h^{(k)}\right\rangle \left\langle u_h^{(k)}\right| + O^{(k-1)} \right) \bar{O}^{(k)} \left|v^{(k)}\right\rangle = \left|w^{(k)}\right\rangle$. He observes that $O^{(k-1)}$ can not influence the $\left|u_h^{(k)}\right\rangle$ component of the vector $\bar{O}^{(k)} \left|v^{(k)}\right\rangle$. He thus projects out the $\left|u_h^{(k)}\right\rangle$ component to obtain

$$O^{(k-1)} \underbrace{\left(\bar{O}^{(k)} \left|v^{(k)}\right\rangle - \langle u_h| \bar{O}^{(k)} \left|v^{(k)}\right\rangle |u_h\rangle \right)}_{=\left|v^{(k-1)}\right\rangle} = \underbrace{\left|w^{(k)}\right\rangle - \left\langle u_h^{(k)} | w^{(k)} \right\rangle \left|u_h^{(k)}\right\rangle}_{=\left|w^{(k-1)}\right\rangle}.$$

With this, Arthur realises, he can reduce his problem involving a k-dimensional orthogonal matrix into a smaller problem in $k-1$ dimensions with exactly the same form. Since Merlin's orthogonal matrix was any arbitrary solution, and since the constraints involved do not depend explicitly on the solution (only on the initial problem), Arthur concludes that this reduction must hold for all possible solutions.

– **Wiggle-v Method: If** $\lambda = -\xi^{(k)}$ or $\lambda = -\chi^{(k)}$ **then**
The aforesaid method relies on matching the normals. It works well so long as the correct operator monotone (the monotone that yields X'_h and X'_g for which $|w\rangle/\sqrt{\langle w| X'_h |w\rangle}$ is a point on both X'_h and OX'_gO^T) doesn't yield infinities. If the operator monotone yields infinities it means that one of the directions involved has infinite curvature which in turn means that the component of the normal along this direction can be arbitrary. To see this, imagine having a line contained inside an ellipsoid (both centred at the origin) touching its boundaries. The line can be thought of as an ellipse with infinite curvature along one of the directions. The normal of the line at its tip is arbitrary and therefore we can't require the usual condition that normals of the two curves must coincide. The solution is to consider the sequence leading to the aforesaid situation.

1. $\left|u_h^{(k)}\right\rangle$ is renamed to $\left|\bar{u}_h^{(k)}\right\rangle$, $\left|u_g^{(k)}\right\rangle$ remains the same.
2. Let $\tau = \cos\theta := \left\langle u_g^{(k)}|v^{(k)}\right\rangle / \left\langle \bar{u}_h^{(k)}|w^{(k)}\right\rangle$. Let $\left|\bar{t}_h^{(k)}\right\rangle$ be an eigenvector of $X_h'^{(k)-1}$ with zero eigenvalue (comment: this will also be perpendicular to $\left|w^{(k)}\right\rangle$). Redefine
$$\left|u_h^{(k)}\right\rangle := \cos\theta \left|\bar{u}_h^{(k)}\right\rangle + \sin\theta \left|\bar{t}_h^{(k)}\right\rangle,$$
$$\left|t_{h_k}^{(k)}\right\rangle = s\left(-\sin\theta \left|\bar{u}_h^{(k)}\right\rangle + \cos\theta \left|\bar{t}_h^{(k)}\right\rangle\right)$$
where the sign $s \in \{1, -1\}$ is fixed by demanding $\left\langle t_{h_k}^{(k)}|w^{(k)}\right\rangle \geq 0$.
3. $\bar{O}^{(k)}$ and $\left|v^{(k-1)}\right\rangle, \left|w^{(k-1)}\right\rangle$ are evaluated as step i and ii of the finite case (a).
4. Define
$$X_h'^{(k-1)} := \mathrm{diag}\{c_{h_1}^{(k)}, c_{h_2}^{(k)}, \dots, c_{h_{k-1}}^{(k)}\}, \qquad X_g'^{(k-1)} := \mathrm{diag}\{c_{g_1}^{(k)}, c_{g_2}^{(k)}, \dots, c_{g_{k-1}}^{(k)}\}.$$
Let $[\chi'^{(k-1)}, \xi'^{(k-1)}]$ denote the smallest interval containing $\mathrm{spec}[X_h'^{(k-1)} \oplus X_g'^{(k-1)}]$. Let $\lambda' = -\chi'^{(k-1)} + 1$ where instead of 1 any positive number would also work. Consider $f_{\lambda''}$ on $[\chi'^{(k-1)}, \xi'^{(k-1)}]$. Let $\eta = -f_{\lambda'}(\chi'^{(k-1)}) + 1$. Define
$$X_h^{(k-1)} := f_{\lambda'}(X_h'^{(k-1)}) + \eta, \qquad X_g^{(k-1)} := f_{\lambda'}(X_g'^{(k-1)}) + \eta.$$
5. **Jump** to **End**.
We start with the case $\lambda = -\xi^{(k)}$. The other case with $\lambda = -\chi^{(k)}$ follows analogously. For the moment just imagine $\eta = 0$ for simplicity; for $\eta \neq 0$ the argument goes through essentially unchanged. Note that(because $\langle w| f_{-\xi}(X_h) |w\rangle - \langle v| f_{-\xi}(X_g) |v\rangle$ is zero we can conclude that $y_{h_i}^{(k)} = \xi$ implies $q_{h_i} = 0$. After the application of the map $f_{-\xi}$ these $y_{h_i}^{(k)}$s and $y_{g_i}^{(k)}$s would become infinities but $\left\langle t_{h_i}^{(k+1)}|w\right\rangle$ and $\left\langle t_{g_i}^{(k+1)}|v\right\rangle$ would be zero where we suppressed the superscripts for $\left|v^{(k)}\right\rangle$ and $\left|w^{(k)}\right\rangle$. Since the eigenvalues are arranged in the ascending order in $X_h^{(k)}$ (in the $\left\{\left|t_{h_i}^{(k+1)}\right\rangle\right\}$ basis) we have $y_{h_k}^{(k)} = \xi$ and the corresponding vector is $\left|t_{h_k}^{(k+1)}\right\rangle =: \left|\bar{t}_h\right\rangle$. It would be useful to define $\left|\tilde{t}_{h_i}\right\rangle = \left|t_{h_i}\right\rangle$ for $i = 1, 2, \dots j-1$ and $\left|\bar{t}_{h_l}\right\rangle = \left|t_{h_i}\right\rangle$ for $i = j, j+1, \dots k$, $l = (i-j)+1$ where j is the smallest i for which $x_{h_i} = \xi$ (their existence is a straight forward consequence of dimension counting, $k \geq n_g + n_h - 1$). This allows us to speak of the subspace with eigenvalue ξ of $X_h^{(k)}$ easily. We will focus on the two dimensional plane spanned by $|w\rangle$ and $\left|\bar{t}_h\right\rangle$.
Consider the M-view (Merlin's point of view). Since M has a solution $O^{(k)}$ to the matrix instance
$$\underline{\mathrm{X}}^{(k)} = \{X_h^{(k)}, X_g^{(k)}, \left|w^{(k)}\right\rangle, \left|v^{(k)}\right\rangle\}$$
his solution is also a solution to the matrix instance
$$\underline{\mathrm{X}}^{(k)}(\lambda) := \left\{f_\lambda(X_h^{(k)}), f_\lambda(X_g^{(k)}), \left|w^{(k)}\right\rangle, \left|v^{(k)}\right\rangle\right\}$$
for $\lambda \leq -\xi$ but close enough to $-\xi$ such that $f_\lambda(X_h), f_\lambda(X_g) > 0$. This is a consequence of f_λ being operator monotone. Using Corollary 91 and Lemma 92 we know that since the ellipsoids corresponding to the matrix instance $\underline{\mathrm{X}}(-\xi)$ touch along $|w\rangle$ (as we are given that $\langle w| f_{-\xi}(X_h) |w\rangle - \langle w| Of_{-\xi}(X_g)O^T |w\rangle = \langle w| f_{-\xi}(X_h) |w\rangle - \langle v| f_{-\xi}(X_g) |v\rangle = 0$) there must also exist some vector $|c(\lambda)\rangle$ such that $\langle c(\lambda)| f_\lambda(X_h) |c(\lambda)\rangle - \langle c(\lambda)| Of_\lambda(X_g)O^T |c(\lambda)\rangle = 0$ that is the ellipsoids corresponding to the matrix instance $\underline{\mathrm{X}}(\lambda)$ touch along the said direction. (Caution: Do not confuse $|c(\lambda)\rangle$ with c_{h_i}/c_{g_i}. The latter are used for curvature values and the former refers to the contact vector just defined.) Note that to match the other conditions of the lemma it suffices to assume that X_h and X_g don't have a common eigenvalue which in turn is guaranteed by the "remove spectral collision" part.
It is easy to convince oneself that $\lim_{\lambda \to -\xi} |c(\lambda)\rangle = |w\rangle$ (hint: argue along the lines $f_\lambda(X_h)$ is very close to $f_{-\xi}(X_h)$ and so the vectors should also be very close which satisfy the condition). Note that we can write
$$|w\rangle = \sum_{i=1}^{j-1} q_{h_i} \left|\tilde{t}_{h_i}\right\rangle$$

because $\langle \bar{t}_{h_i}|w\rangle = 0$. There is no such restriction on $|c(\lambda)\rangle$ which can have the more general form $|c(\lambda)\rangle = \sum_{i=1}^{j-1} c(\lambda)_i \left|\tilde{t}_{h_i}\right\rangle + \sum_{i=j}^{k} c(\lambda)_i \left|\bar{t}_{h_l}\right\rangle$ where $l = (i-j)+1$. Restating one of the limit conditions, for $i = j, j+1 \dots k$, we must have the $\lim_{\lambda\to-\xi} c(\lambda)_i = 0$. At this point we use the fact that if O is a solution it entails that

$$\acute{O}(\lambda) := \left(\sum_{i=1}^{j-1} \left|\tilde{t}_{h_i}\right\rangle \left\langle \tilde{t}_{h_i}\right| + \sum_{i,m=1}^{k-j+1} Q(\lambda)_{im} \left|\bar{t}_{h_i}\right\rangle \left\langle \bar{t}_{h_m}\right| \right) O$$

is also a solution, where $Q(\lambda)$ is an orthogonal matrix in the space spanned by $\left\{\left|\bar{t}_{h_i}\right\rangle\right\}$. This is a consequence of the fact that $\{\left|\bar{t}_{h_i}\right\rangle\}$ spans an eigenspace (with the same eigenvalue, $f_\lambda(\xi)$,) of $f_\lambda(X_h)$. We can use this freedom to ensure that the point of contact always has the form

$$|c(\lambda)\rangle = \sum_{i=1}^{j-1} c(\lambda)_i \left|\tilde{t}_{h_i}\right\rangle + \bar{c}(\lambda) \left|\bar{t}_h\right\rangle$$

where $\bar{c}(\lambda) = \sqrt{\sum_{i=j}^{k} c(\lambda)_i^2}$ which must vanish in the limit $\lambda \to -\xi$ as its constituents disappear in the said limit. Similarly $\lim_{\lambda\to-\xi} c(\lambda)_i = q_{h_i}$.

Next we evaluate the normals $|u_h(\lambda)\rangle$ at $|c(\lambda)\rangle$ for the ellipsoid represented by $f_\lambda(X_h)$ and similarly the normal $|\bar{u}_h\rangle$ at $|w\rangle$ for the ellipsoid represented by $f_{-\xi}(X_h)$ to show that $\lim_{\lambda\to-\xi} |u_h(\lambda)\rangle \neq |\bar{u}_h\rangle$ (see Figure 7). Notice that the right-most term in $|u_h(\lambda)\rangle = \mathcal{N}\left[\sum_{i=1}^{j-1} f_\lambda(y_{h_i}) c(\lambda)_i \left|\tilde{t}_{h_i}\right\rangle + f_\lambda(\xi)\bar{c}(\lambda)\left|\bar{t}_h\right\rangle\right]$ has $f_\lambda(\xi)$ approaching infinity and $\bar{c}(\lambda)$ approaching zero as λ tends to $-\xi$. This is why it can have a finite component along $\left|\bar{t}_h\right\rangle$. On the other hand, $|\bar{u}_h\rangle = \mathcal{N}\left[\sum_{i=1}^{j-1} f_{-\xi}(y_{h_i}) q_{h_i} \left|\tilde{t}_{h_i}\right\rangle\right]$ which has no component along $\left|\bar{t}_h\right\rangle$. Since $\lim_{\lambda\to-\xi} f_\lambda(y_{h_i}) = f_{-\xi}(y_{h_i})$ and $\lim_{\lambda\to-\xi} c(\lambda)_i = q_{h_i}$ for $i \in \{1, 2 \dots j-1\}$, we can write

$$\lim_{\lambda\to-\xi} |u_h(\lambda)\rangle = \cos\theta\, |\bar{u}_h\rangle + \sin\theta \left|\bar{t}_h\right\rangle := |u_h\rangle .$$

Evidently, we must use $|u_h\rangle$ instead of $|\bar{u}_h\rangle$ to be able to use the reasoning of the finite method. However, we do not know $\cos\theta$ yet.

Our strategy is to proceed as in the finite method with the assumption that $|c(\lambda)\rangle$ is known (which it isn't as we only know it exists and how it behaves in the limit of $\lambda \to -\xi$) and then use a consistency condition to find $\cos\theta$ in terms of known quantities. At this point we re-introduce the superscripts as we will reduce the dimension of the problem as we proceed. Let the normal and tangents at $O^T|c(\lambda)\rangle$ for $f_\lambda(X_g)$ be given by $\left\{\left|u_g^{(k)}(\lambda)\right\rangle, \left\{t_{g_i}^{(k)}(\lambda)\right\}\right\}$. Similarly at $|c(\lambda)\rangle$ for $f_\lambda(X_h)$ the normal and tangents are $\left\{\left|u_h^{(k)}(\lambda)\right\rangle, \left\{t_{h_i}^{(k)}(\lambda)\right\}\right\}$. From the finite method we know that $O^{(k)}(\lambda) := \left(|u_h(\lambda)\rangle\langle u_h(\lambda)| + O^{(k-1)}\right)\bar{O}^{(k)}$ where $\bar{O}^{(k)} = \left|u_h^{(k)}(\lambda)\right\rangle\left\langle u_g^{(k)}(\lambda)\right| + \sum_i \left|t_{h_i}^{(k)}\right\rangle\left\langle t_{g_i}^{(k)}\right|$ can be used to reduce the problem into a smaller instance of itself. In particular, we must have $\left\langle u_h^{(k)}(\lambda)|w\right\rangle = \left\langle u_h^{(k)}(\lambda)\right| O^{(k)}(\lambda)\, |v\rangle = \left\langle u_g^{(k)}(\lambda)|v\right\rangle$ because $O^{(k-1)}$ can influence only the subspace spanned by $\left\{\left|t_{h_i}^{(k)}\right\rangle\right\}$ and the component of the vectors $|w\rangle$ and $O^{(k)}|v\rangle$ along $\left|u_h^{(k)}(\lambda)\right\rangle$ must match for consistency.

We can determine $\cos\theta$ by taking the limit of the aforesaid condition as $\langle u_h|w\rangle = \langle u_g|v\rangle$ where we again suppressed the superscripts. Substituting $|u_h\rangle = \cos\theta\,|\bar{u}_h\rangle + \sin\theta\left|\bar{t}_h\right\rangle$ we obtain

$$\cos\theta = \frac{\langle u_g|v\rangle}{\langle \bar{u}_h|w\rangle}.$$

It now remains to find the limit of the reverse Weingarten maps. The reverse Weingarten map for $f_\lambda(X_g)$ along the normal $|u_g(\lambda)\rangle$ is not of concern because it has a well defined limit as $\lambda \to -\xi$. We consider the case for $f_\lambda(X_h)$ along the normal $|u_h(\lambda)\rangle$. Note that the support function as defined in Equation (3) is finite in the limit $\lambda \to -\xi$ (use the definition of the normal to get $\sum x_i^{-1}u_i^2 = \sum x_i^{-1}x_i^2c_i^2 = \sum x_i c_i^2 = \langle c|X|c\rangle$, plug in $|c\rangle = |w\rangle$, $X = f_{-\xi}(X_h)$ and then use the fact that $\langle w|f_{-\xi}(X_h)|w\rangle - \langle v|f_{-\xi}(X_g)|v\rangle = 0$ which means both must be finite by noting that we already dealt with the troublesome case of $\infty - \infty$ in the "remove spectral collision" part). Let us denote it by $h(\lambda)$. Now the reverse Weingarten map as defined in Equation (4) is given by

$$(W_h(\lambda))_{im} = -\frac{1}{h(\lambda)^2}\frac{u_{h_i}(\lambda)u_{h_m}(\lambda)}{f_\lambda(y_{h_i})f_\lambda(y_{h_m})} + \frac{\delta_{im}}{f_\lambda(x_{h_i})}.$$

Since $\lim_{\lambda\to-\xi}|u(\lambda)\rangle$ is well defined, $\lim_{\lambda\to-\xi} h(\lambda)$ is finite, we only need to show that $\lim_{\lambda\to-\xi} 1/f_\lambda(y_{h_i})$ is well defined. (We assumed η is zero so $f_{-\xi}(y_{h_i}) \neq 0$. If η is not zero we must consider $f_{-\xi}(y_{h_i}) + \eta$ everywhere but

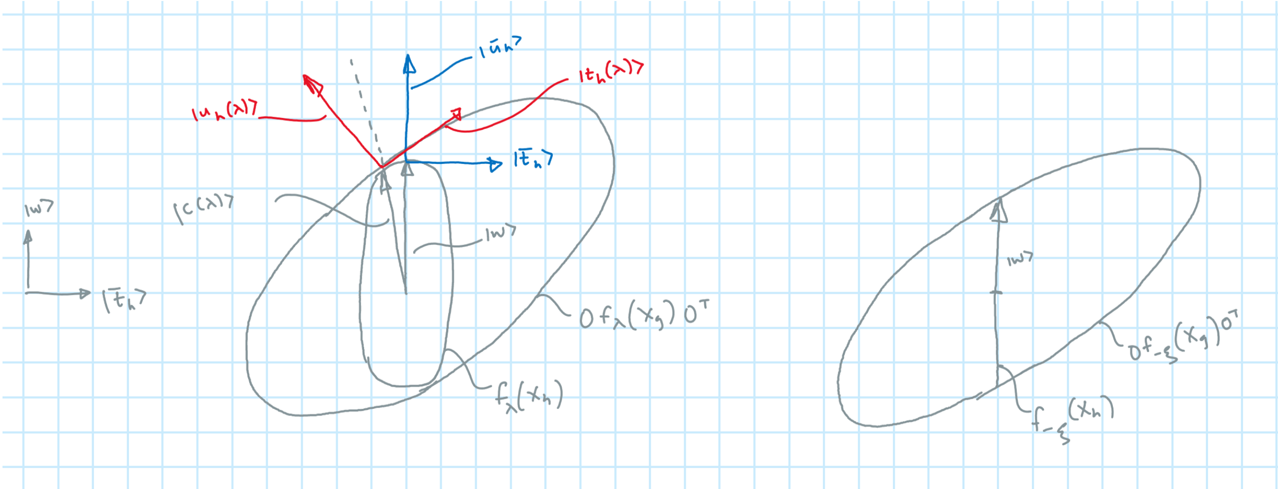


Figure 7: A sequence leading to infinite curvature.

that changes no argument.) For $i = 1, 2 \dots j-1$, $f_{-\xi}(y_{h_i})$ is finite but for $i = j, j+1 \dots k$, $f_{-\xi}(y_{h_i})$ is not well defined however $1/f_{-\xi}(y_{h_i}) = 0$. We therefore conclude that

$$\lim_{\lambda\to-\xi} (W_h(\lambda))_{im} = \begin{cases} -\frac{1}{h^2}\frac{u_{h_i}u_{h_m}}{f_{-\xi}(y_{h_i})f_{-\xi}(y_{h_m})} + \frac{\delta_{im}}{f_{-\xi}(y_{h_i})} & i,m \in \{1,2\dots j-1\} \\ 0 & i,m \in \{j, j+1 \dots k\} \end{cases} := (W_h)_{im}$$

which is simply the reverse Weingarten map evaluated for $f_{-\xi}(X_h)$ along $|u_h\rangle = \cos\theta\,|\bar{u}_h\rangle + \sin\theta\,|\bar{t}_h\rangle$ and $\cos\theta = \langle u_g|v\rangle / \langle \bar{u}_h|w\rangle$. It remains to relate W_h with the reverse Weingarten map, $\bar{W}_h$, evaluated for $f_{-\xi}(X_h)$ along $|\bar{u}_h\rangle$. Surprisingly, it is easy to see that $W_h = \bar{W}_h$ because only the $\cos\theta\,|\bar{u}_h\rangle$ part contributes to the non-zero portion of W_h and the $\cos\theta$ factor gets cancelled due to the h^2 term. Further, recall that the normal vector is always an eigenvector of the reverse Weingarten map evaluated along it, with eigenvalue zero. This tells us that if there is(are) tangent(s) with zero radius of curvature then the normal is not uniquely defined. This confirms what we already knew. Now since both $|\bar{u}_h\rangle$, $|\bar{t}_h\rangle$ have zero eigenvalues for $\bar{W}_h(=W_h)$ and $|u\rangle = \cos\theta\,|\bar{u}_h\rangle + \sin\theta\,|\bar{t}_h\rangle$ we define $|t_h\rangle := s\left(\sin\theta\,|\bar{u}_h\rangle - \cos\theta\,|\bar{t}_h\rangle\right)$ to span the same space so that $|u\rangle$ is the correct normal vector (as we deduced earlier in our discussion) and $|t_h\rangle$ is the correct tangent vector corresponding to the point $|w\rangle$ of $f_{-\xi}(X_h)$.

The final step is to convert the condition on the reverse Weingarten map into a condition on the Weingarten map (inverse of the reverse Weingarten map). After extracting the tangent vectors appropriately, one simply needs to add a constant before inverting to obtain the Weingarten map condition. This is done in the last step. This completes the proof of the wiggle-v method for $\lambda = -\xi$.
To see how the same reasoning applies to the $\lambda = -\chi$ case first note that for $\lambda \geq -\chi$ we have $f_\lambda(X_h), f_\lambda(X_g) < 0$ (assuming $\eta = 0$ as before). The condition $f_\lambda(X_h) \geq O f_\lambda(X_g) O^T$ can then be expressed as $-f_\lambda(X_g) \geq -O^T f_\lambda(X_h) O$ with $O^T|w\rangle = |v\rangle$ which can now be reasoned analogous to the aforementioned analysis.

– **End**: Restart the current phase (phase 2) with the newly obtained $(k-1)$ sized objects.
We end with giving the dimension argument. The dimension after every iteration is $k-1 \geq n_g^{(k-1)} + n_h^{(k-1)} - 1$ if we start with the assumption that $k \geq n_g^{(k)} + n_h^{(k)} - 1$. The reason is that either $n_g^{(k-1)} = n_g^{(k)} - 1$ or $= n_g^{(k)}$. Similarly, either $n_h^{(k-1)} = n_h^{(k)} - 1$ or $= n_h^{(k)}$. Justification of this is simply that we remove at least one component from the two vectors (from the $n_g^{(k)}$ for the usual wiggle-v). To see this, note that in the finite case we remove one from both as we write express the vector in a new basis. This new basis is the space where the vector has finite support. We then remove one of the components in the sub-problem. In the infinite case, it is possible that we remove one and add one for $n_h^{(k-1)}$, assuming it is the usual wiggle-v, but we necessarily reduce $n_g^{(k-1)}$ as this is similar to the finite case. For the other wiggle-v, g and h get swapped but the counting stays the same.

7.3.3 Phase 3: Reconstruction

Let k_0 be the iteration at which the algorithm stops. Using the relation $O^{(k)} = \bar{O}_g^{(k)}\left(\left|u_h^{(k)}\right\rangle\left\langle u_h^{(k)}\right| + O^{(k-1)}\right)\bar{O}_h^{(k)}$ (or its transpose if $s^{(k)} = -1$), evaluate $O^{(k_1)}$ from $O^{(k_0)} := \mathbb{I}_{k_0}$, then $O^{(k_2)}$ from $O^{(k_1)}$, then $O^{(k_3)}$ from $O^{(k_2)}$ and so on until $O^{(n)}$ is obtained which solves the matrix instance $\underline{\mathrm{X}}^{(n)}$ we started with. In terms of EBRM matrices, the solution is given by $H = X_h^{(n)}$, $G = O^{(n)} X_g O^{(n)T}$, and $|w\rangle = \left|w^{(n)}\right\rangle$.

8 Conclusion

In the first part, we described a framework which we used to construct the unitaries required to implement the bias 1/10 protocol. In the second part, we described the EMA algorithm which allows us to find the unitaries corresponding to arbitrary Λ valid moves, which combined with the framework, allows us to solve quantum WCF.

An important open problem that remains is to account for noise in both the quantum formalism and in the EMA algorithm.

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A Blinkered $m \to n$ Transition

Recall that the unitary I had described was of the form $U = |w\rangle\langle v| + |v\rangle\langle w| + \sum |v_i\rangle\langle v_i| + \sum |w_i\rangle\langle w_i|$. It is evident that having a scheme for generating these $|v_i\rangle$ and $|w_i\rangle$ will be useful as we explore more complicated transitions. More precisely, I will need to complete a set containing one vector into a complete orthonormal basis. Let me do this first and then return to the analysis of a $3 \to 2$ merge.

Completing an Orthonormal Basis

Consider an orthonormal complete set of basis vectors $\{|g_i\rangle\}$, and a vector $|v\rangle = \frac{\sum_i \sqrt{p_i}|g_i\rangle}{\sqrt{\sum_i p_i}}$. I will describe a scheme for constructing vectors $|v_i\rangle$ s.t. $\{|v\rangle, \{|v_i\rangle\}\}$ is a complete orthonormal set of basis vectors. Formally, I can do this inductively. Instead, I will do this by examples for that makes it intuitive and demonstrates the generalisable argument right away. The first I define to be

$$|v_1\rangle = \frac{\sqrt{p_1}\,|g_1\rangle - \frac{p_1}{\sqrt{p_2}}\,|g_2\rangle}{\sqrt{p_1 + \frac{p_1^2}{p_2}}} \left(= \frac{\sqrt{p_1}\,|g_1\rangle - \sqrt{p_2}\,|g_2\rangle}{\sqrt{p_1+p_2}}, \text{ the familiar one} \right)$$

which is manifestly normalised and orthogonal to $|v\rangle$, i.e. $\langle v|v_1\rangle = p_1 - p_1 = 0$. The next vector is

$$|v_2\rangle = \frac{\sqrt{p_1}\,|g_1\rangle + \sqrt{p_2}\,|g_2\rangle - \frac{(p_1+p_2)}{\sqrt{p_3}}\,|g_3\rangle}{\sqrt{p_1 + p_2 + \frac{(p_1+p_2)^2}{p_3}}}$$

which is again manifestly normalised and orthogonal to $|v_1\rangle$ because $\langle v_2|v_1\rangle = \langle v|v_1\rangle$. $\langle v|v_2\rangle = p_1 + p_2 - (p_1+p_2) = 0$. Similarly one can construct the $(k+1)^{\text{th}}$ basis vector as

$$|v_k\rangle = \frac{\sum_{i=1}^k \sqrt{p_k}\,|g_k\rangle - \frac{\sum_{i=1}^k p_k}{\sqrt{p_{k+1}}}\,|g_{k+1}\rangle}{N_k}$$

where the $N_k = \sqrt{\sum_{i=1}^k p_k + \frac{(\sum_{i=1}^k p_k)^2}{p_{k+1}}}$ and obtain the full set.

The Analysis

Back to the analysis. Recall that the constraint equation was

$$\underbrace{\sum x_{h_i}\,|h_{ii}\rangle\langle h_{ii}|}_{\text{I}} + \underbrace{x\mathbb{I}^{\{g_{ii}\}}}_{\text{II}} \geq \underbrace{\sum x_{g_i} U\,|g_{ii}\rangle\langle g_{ii}|\,U^\dagger}_{\text{III}}$$

where I have introduced the notation $|h_{ii}\rangle = |h_i h_i\rangle$ in the interest of efficiency. The $g_1, g_2, g_3 \to h_1, h_2$ transition requires me to know

$$U = |v\rangle\langle w| + |w\rangle\langle v| + |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2| + |w_1\rangle\langle w_1|.$$

Using the procedure above I can evaluate the vectors of interest

$$|v\rangle = \frac{\sqrt{p_{g_1}}\,|g_{11}\rangle + \sqrt{p_{g_2}}\,|g_{22}\rangle + \sqrt{p_{g_3}}\,|g_{33}\rangle}{N_g}$$

$$|v_1\rangle = \frac{\sqrt{p_{g_1}}\,|g_{11}\rangle - \frac{p_{g_1}}{\sqrt{p_{g_2}}}\,|g_{22}\rangle}{N_{g_1}}$$

$$|v_2\rangle = \frac{\sqrt{p_{g_1}}\,|g_{11}\rangle + \sqrt{p_{g_2}}\,|g_{22}\rangle - \frac{(p_{g_1}+p_{g_2})}{\sqrt{p_{g_3}}}\,|g_{33}\rangle}{N_{g_2}}$$

$$|w\rangle = \frac{\sqrt{p_{h_1}}\,|h_{11}\rangle + \sqrt{p_{h_2}}\,|h_{22}\rangle}{N_h}$$

$$|w_1\rangle = \frac{\sqrt{p_{h_2}}\,|h_{11}\rangle - \sqrt{p_{h_1}}\,|h_{22}\rangle}{N_h}$$

where N_g, N_{g_1}, N_{g_2}, N_h are normalisations. In fact I want to express the constraints in this basis. To evaluate the first term I use the above to find

$$|h_{11}\rangle = \frac{\sqrt{p_{h_1}}\,|w\rangle + \sqrt{p_{h_2}}\,|w_1\rangle}{N_h}$$

$$|h_{22}\rangle = \frac{\sqrt{p_{h_2}}\,|w\rangle - \sqrt{p_{h_1}}\,|w_1\rangle}{N_h}$$

which leads to

$$
\begin{aligned}
\mathrm{I} &= x_{h_1}\left|h_{11}\right\rangle\left\langle h_{11}\right| + x_{h_2}\left|h_{22}\right\rangle\left\langle h_{22}\right| \\
&= \frac{x_{h_1}}{N_h^2}\begin{array}{cc} & \begin{array}{cc}\langle w| & \langle w_1|\end{array} \\ \begin{array}{c}|w\rangle \\ |w_1\rangle\end{array} & \begin{bmatrix} p_{h_1} & \sqrt{p_{h_1}p_{h_2}} \\ \sqrt{p_{h_1}p_{h_2}} & p_{h_2}\end{bmatrix}\end{array} + \frac{x_{h_2}}{N_h^2}\begin{array}{cc} & \begin{array}{cc}\langle w| & \langle w_1|\end{array} \\ \begin{array}{c}|w\rangle \\ |w_1\rangle\end{array} & \begin{bmatrix} p_{h_2} & -\sqrt{p_{h_1}p_{h_2}} \\ -\sqrt{p_{h_1}p_{h_2}} & p_{h_1}\end{bmatrix}\end{array} \\
&= \frac{1}{N_h^2}\begin{array}{cc} & \begin{array}{cc}\langle w| & \langle w_1|\end{array} \\ \begin{array}{c}|w\rangle \\ |w_1\rangle\end{array} & \begin{bmatrix} p_{h_1}x_{h_1} + p_{h_2}x_{h_2} & \sqrt{p_{h_1}p_{h_2}}(x_{h_1} - x_{h_2}) \\ \sqrt{p_{h_1}p_{h_2}}(x_{h_1} - x_{h_2}) & p_{h_2}x_{h_1} + p_{h_1}x_{h_2}\end{bmatrix}\end{array}.
\end{aligned}
$$

(Remark: I had made a mistake in this term which was causing the matrix to sometimes become negative; after correction, the matrix seems to be positive for Mochon's f-function based construction) Evaluation of II is nearly trivial for identity can be expressed in any basis and that yields

$$
\begin{aligned}
\mathrm{II} &= x(|v\rangle\langle v| + |v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|) \\
&= \begin{array}{cc} & \begin{array}{ccc}\langle v| & \langle v_1| & \langle v_2|\end{array} \\ \begin{array}{c}|v\rangle \\ |v_1\rangle \\ |v_2\rangle\end{array} & \begin{bmatrix} x & & \\ & x & \\ & & x\end{bmatrix}\end{array}.
\end{aligned}
$$

For the last term

$$
\mathrm{III} = \underbrace{x_{g_1}U\left|g_{11}\right\rangle\left\langle g_{11}\right|U^\dagger}_{\text{(i)}} + \underbrace{x_{g_2}U\left|g_{22}\right\rangle\left\langle g_{22}\right|U^\dagger}_{\text{(ii)}} + \underbrace{x_{g_3}U\left|g_{33}\right\rangle\left\langle g_{33}\right|U^\dagger}_{\text{(iii)}}
$$

I evaluate

$$
\begin{aligned}
U\left|g_{11}\right\rangle &= \frac{\sqrt{p_{g_1}}}{N_g}|w\rangle + \frac{\sqrt{p_{g_1}}}{N_{g_1}}|v_1\rangle + \frac{\sqrt{p_{g_1}}}{N_{g_2}}|v_2\rangle \\
U\left|g_{22}\right\rangle &= \frac{\sqrt{p_{g_2}}}{N_g}|w\rangle + \frac{\left(-\frac{p_{g_1}}{\sqrt{p_{g_2}}}\right)}{N_{g_1}}|v_1\rangle + \frac{\sqrt{p_{g_2}}}{N_{g_2}}|v_2\rangle \\
U\left|g_{33}\right\rangle &= \frac{\sqrt{p_{g_3}}}{N_g}|w\rangle + 0\,|v_1\rangle + \frac{\left(-\frac{p_{g_1}+g_{g_2}}{\sqrt{p_{g_3}}}\right)}{N_{g_2}}|v_2\rangle.
\end{aligned}
$$

I must now find each sub term, starting with the most regular

$$
\text{(i)} = x_{g_1}p_{g_1}\begin{array}{cc} & \begin{array}{ccc}\langle v_1| & \langle v_2| & \langle w|\end{array} \\ \begin{array}{c}|v_1\rangle \\ |v_2\rangle \\ |w\rangle\end{array} & \begin{bmatrix} \frac{1}{N_{g_1}^2} & \frac{1}{N_{g_1}N_{g_2}} & \frac{1}{N_{g_1}N_g} \\ \frac{1}{N_{g_2}N_{g_1}} & \frac{1}{N_{g_2}^2} & \frac{1}{N_{g_2}N_g} \\ \frac{1}{N_gN_{g_1}} & \frac{1}{N_gN_{g_2}} & \frac{1}{N_g^2}\end{bmatrix}\end{array}.
$$

For the second term, I re-write $U\left|g_{22}\right\rangle = \sqrt{p_{g_2}}\left(\frac{1}{N_g}|w\rangle - \frac{1}{N'_{g_1}}|v_1\rangle + \frac{1}{N_{g_2}}|v_2\rangle\right)$ where I have defined

$$
N'_{g_1} = \frac{p_{g_2}}{p_{g_1}}N_{g_1}
$$

to obtain

$$
(ii) = x_{g_2}p_{g_2}\begin{array}{cc} & \begin{array}{ccc}\langle v_1| & \langle v_2| & \langle w|\end{array} \\ \begin{array}{c}|v_1\rangle \\ |v_2\rangle \\ |w\rangle\end{array} & \begin{bmatrix} \frac{1}{N'^2_{g_1}} & -\frac{1}{N'_{g_1}N_{g_2}} & -\frac{1}{N'_{g_1}N_g} \\ -\frac{1}{N_{g_2}N'_{g_1}} & \frac{1}{N_{g_2}^2} & \frac{1}{N_{g_2}N_g} \\ -\frac{1}{N_gN'_{g_1}} & \frac{1}{N_gN_{g_2}} & \frac{1}{N_g^2}\end{bmatrix}\end{array}
$$

and finally $U\left|g_{33}\right\rangle = \sqrt{p_{g_3}}\left(\frac{1}{N_g}|w\rangle + 0\,|v_1\rangle - \frac{1}{N'_{g_2}}|v_2\rangle\right)$ with

$$
N'_{g_2} = \frac{p_{g_3}}{p_{g_1} + p_{g_2}}
$$

to get

$$
\text{(iii)} = x_{g_3}p_{g_3}\begin{array}{cc} & \begin{array}{ccc}\langle v_1| & \langle v_2| & \langle w|\end{array} \\ \begin{array}{c}|v_1\rangle \\ |v_2\rangle \\ |w\rangle\end{array} & \begin{bmatrix} & & \\ & \frac{1}{N'^2_{g_2}} & -\frac{1}{N'_{g_2}N_g} \\ & -\frac{1}{N_gN'_{g_2}} & \frac{1}{N_g^2}\end{bmatrix}\end{array}.
$$

Now I can combine all of these into a single matrix and try to obtain some simpler constraints.

$$M \overset{\text{def}}{=} \left[\begin{array}{c|ccccc}
 & \langle v| & \langle v_1| & \langle v_2| & \langle w| & \langle w_1| \\ \hline
|v\rangle & x & & & & \\
|v_1\rangle & & x-\frac{x_{g_1}p_{g_1}}{N_{8_1}^2}-\frac{x_{g_2}p_{g_2}}{N_{g_1}'^2} & -\frac{x_{g_1}p_{g_1}}{N_{g_1}N_{g_2}}+\frac{x_{g_2}p_{g_2}}{N_{g_1}'N_{g_2}} & -\frac{x_{g_1}p_{g_1}}{N_{g_1}N_g}+\frac{x_{g_2}p_{g_2}}{N_{g_1}'N_g} & \\
|v_2\rangle & & -\frac{x_{g_1}p_{g_1}}{N_{g_2}N_{g_1}}+\frac{x_{g_2}p_{g_2}}{N_{g_2}N_{g_1}'} & x-\frac{x_{g_1}p_{g_1}}{N_{g_2}^2}-\frac{x_{g_2}p_{g_2}}{N_{g_2}^2}-\frac{x_{g_3}p_{g_3}}{N_{g_2}'^2} & -\frac{x_{g_1}p_{g_1}}{N_{g_2}N_g}-\frac{x_{g_2}p_{g_2}}{N_{g_2}N_g}+\frac{x_{g_3}p_{g_3}}{N_{g_2}'N_g} & \\
|w\rangle & & -\frac{x_{g_1}p_{g_1}}{N_gN_{g_1}}+\frac{x_{g_2}p_{g_2}}{N_gN_{g_1}'} & -\frac{x_{g_1}p_{g_1}}{N_gN_{g_2}}-\frac{x_{g_2}p_{g_2}}{N_gN_{g_2}}+\frac{x_{g_3}p_{g_3}}{N_gN_{g_2}'} & \frac{p_{h_1}x_{h_1}+p_{h_2}x_{h_2}}{N_h^2}-\frac{1}{N_g^2}\sum_i x_{g_i}p_{g_i} & \frac{\sqrt{p_{h_1}p_{h_2}}}{N_h^2}(x_{h_1}-x_{h_2}) \\
|w_1\rangle & & & & \frac{\sqrt{p_{h_1}p_{h_2}}}{N_h^2}(x_{h_1}-x_{h_2}) & \frac{p_{h_2}x_{h_1}+p_{h_1}x_{h_2}}{N_h^2}
\end{array}\right] \geq 0.$$

Despite this appearing to be a complicated expression, I can conclude that it will always be so that larger the x looser will be the constraint. To show this and to simplify this calculation, note that M can be split into a scalar condition, $x \geq 0$ (from the $|v\rangle\langle v|$ part) and a sub-matrix which I choose to write as

$$\begin{array}{c|c|c}
 & \langle v_1| \quad \langle v_2| & \langle w| \quad \langle w_1| \\ \hline
\begin{matrix}|v_1\rangle \\ |v_2\rangle\end{matrix} & C & B^T \\ \hline
\begin{matrix}|w\rangle \\ |w_1\rangle\end{matrix} & B & A
\end{array} \geq 0.$$

Now since $\begin{bmatrix} C & B^T \\ B & A \end{bmatrix} \geq 0 \iff \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \geq 0 \iff C \geq 0,\ A - BC^{-1}B^T \geq 0,\ (\mathbb{I} - CC^{-1})B^T = 0$ using Shur's Complement condition for positivity where C^{-1} is supposed to be the generalised inverse. Since x is in our hands, we can take it to be sufficiently large so that $C > 0$ and thereby make sure that $\mathbb{I} - CC^{-1} = 0$. Evidently then, the only condition of interest is

$$A - BC^{-1}B^T \geq 0.$$

I can do even better than this actually. Note that if $C > 0$ then $C^{-1} > 0$ and that the second term is of the form

$$\underbrace{\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}}_{C^{-1}}\underbrace{\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}}_{B^T} = \begin{bmatrix} \begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} & 0 \\ 0 & 0 \end{bmatrix} \geq 0$$

because $C^{-1} > 0$. I can therefore write my constraint equation as

$$A \geq BC^{-1}B^T \geq 0$$

and note that $A \geq 0$ is a necessary condition. This also becomes a sufficient condition in the limit that $x \to \infty$ because $C^{-1} \to 0$ in that case. We have thereby reduced the analysis to simply checking if

$$\begin{bmatrix} \frac{p_{h_1}x_{h_1}+p_{h_2}x_{h_2}}{N_h^2} - \frac{1}{N_g^2}\sum_i x_{g_i}p_{g_i} & \frac{\sqrt{p_{h_1}p_{h_2}}}{N_h^2}(x_{h_1}-x_{h_2}) \\ \frac{\sqrt{p_{h_1}p_{h_2}}}{N_h^2}(x_{h_1}-x_{h_2}) & \frac{p_{h_2}x_{h_1}+p_{h_1}x_{h_2}}{N_h^2} \end{bmatrix} \geq 0.$$

This being a 2×2 matrix can be checked for positivity by the trace and determinant method. Another possibility is the use of Schur's Complement conditions again. Here, however, I intend to use a more general technique (similar to the one used in the split analysis). Let me introduce

$$\langle x_g \rangle \overset{\text{def}}{=} \frac{1}{N_g^2}\sum_i x_{g_i}p_{g_i},\ \left\langle \frac{1}{x_h} \right\rangle \overset{\text{def}}{=} \frac{1}{N_h^2}\sum_i \frac{p_{h_i}}{x_{h_i}}$$

and recall/note that term (I) and one element from term (III) constitute matrix A, which can also be written as

$$\begin{aligned} A &= x_{h_1}|h_{11}\rangle\langle h_{11}| + x_{h_2}|h_{22}\rangle\langle h_{22}| - \langle x_g\rangle|w\rangle\langle w| \\ &= \begin{array}{c|cc} & \langle h_{11}| & \langle h_{22}| \\ \hline |h_{11}\rangle & x_{h_1} & \\ |h_{22}\rangle & & x_{h_2} \end{array} - \langle x_g\rangle|w\rangle\langle w| \end{aligned}$$

Note that this now has the exact same form as that of the split constraint with $x_{g_1} \to \langle x_g\rangle$. I use the same $F - M \geq 0 \iff \mathbb{I} - \sqrt{F}^{-1}M\sqrt{F}^{-1} \geq 0$ for $F > 0$ technique to obtain $\mathbb{I} \geq \langle x_g\rangle|w''\rangle\langle w''|$ where $|w''\rangle = \frac{\sqrt{\frac{p_{h_1}}{x_{h_1}}}|h_{11}\rangle + \sqrt{\frac{p_{h_2}}{x_{h_2}}}|h_{22}\rangle}{N_h}$. Normalising this one gets $|w'\rangle = \frac{|w''\rangle}{\sqrt{\left\langle\frac{1}{x_h}\right\rangle}}$ which entails $\mathbb{I} \geq \langle x_g\rangle\left\langle\frac{1}{x_h}\right\rangle|w'\rangle\langle w'|$ and that leads us to the final condition

$$\frac{1}{\langle x_g\rangle} \geq \left\langle \frac{1}{x_h} \right\rangle.$$

In fact all the techniques used in reaching this result can be extended to the $m \to n$ transition case as well and so the aforesaid result should hold in general.

B Mochon's Assignments

Lemma (Mochon's Denominator)**.** $\sum_{i=1}^{n} \frac{1}{\prod_{j\neq i}(x_j - x_i)} = 0$ *for* $n \geq 2$.

Proof. I will prove this by induction (following Mochon's proof, just optimised for dummies instead of space). For $n = 2$

$$\frac{1}{(x_2 - x_1)} + \frac{1}{(x_1 - x_2)} = 0.$$

Now I show that if the result holds for $n-1$ and it would also hold for n which would complete the inductive proof. I start with noting that

$$\frac{1}{(x_n - x_i)(x_1 - x_i)} = \frac{1}{x_n - x_1}\left[\frac{1}{x_1 - x_i} - \frac{1}{x_n - x_i}\right].$$

This is useful because it helps breaking the product into a sum. My strategy would be to pull off one common term so that I can apply the result to the remaining $n-1$ terms. The expression of interest is

$$\sum_{i=1}^{n} \frac{1}{\prod_{j\neq i}(x_j - x_i)} = \frac{1}{\prod_{j\neq 1}(x_j - x_1)} + \sum_{i=2}^{n-1} \frac{1}{\prod_{j\neq i}(x_j - x_i)} + \frac{1}{\prod_{j\neq n}(x_j - x_n)}$$

where notice that the ith term in the sum (of the second term) can be written as

$$\frac{1}{(x_n - x_i)(x_1 - x_i)\prod_{j\neq i,1,n}(x_j - x_i)} = \frac{1}{x_n - x_1}\left[\frac{1}{\prod_{j\neq i,n}(x_j - x_i)} - \frac{1}{\prod_{j\neq 1,i}(x_j - x_i)}\right].$$

The first term can be written as

$$\frac{1}{(x_n - x_1)\prod_{j\neq 1,n}(x_j - x_1)}$$

while the last can be written as

$$\frac{-1}{(x_n - x_1)\prod_{j\neq n,1}(x_j - x_n)}.$$

Putting all these together, I get

$$\begin{aligned}
\sum_{i=1}^{n} \frac{1}{\prod_{j\neq i}(x_j - x_i)} &= \frac{1}{(x_n - x_1)}\left[\underbrace{\frac{1}{\prod_{j\neq 1,n}(x_j - x_1)} + \sum_{i=2}^{n-1} \frac{1}{\prod_{j\neq i,n}(x_j - x_i)}} - \underbrace{\sum_{i=2}^{n-1} \frac{1}{\prod_{j\neq 1,i}(x_j - x_i)} - \frac{1}{\prod_{j\neq 1,n}(x_j - x_n)}}\right] \\
&= \frac{1}{(x_n - x_1)}\left[\sum_{i=1}^{n-1} \frac{1}{\prod_{j\neq i,n}(x_j - x_i)} - \sum_{i=2}^{n} \frac{1}{\prod_{j\neq 1,i}(x_j - x_i)}\right]
\end{aligned}$$

where both sums disappear if the result holds for $n-1$. This completes the proof. □

Lemma (Mochon's f-assignment Lemma)**.** $\sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j\neq i}(x_j - x_i)} = 0$ *where* $f(x_i)$ *is of order* $k \leq n-2$.

Proof. Again I do this by induction on k. For $k = 0$ the result holds by the previous result. I assume it holds for order $k-1$ and show using this that it will also hold for order k (this proof is also Mochon's). Let $g(x_i)$ be a polynomial of order $k-1$ s.t.

$$\sum_{i=1}^{n} \frac{f(x_i)}{\prod_{j\neq i}(x_j - x_i)} = \sum_{i=1}^{n} \frac{(x_1 - x_i)(x_2 - x_i)\dots(x_k - x_i) - g(x_i)}{\prod_{j\neq i}(x_j - x_i)}.$$

Notice that the first part of the sum will disappear for all $1 \leq i \leq k$ because of the numerator. Consequently I can write the aforesaid as

$$\begin{aligned}
&= \sum_{i=k+1}^{n} \frac{(x_1 - x_i)(x_2 - x_i)\dots(x_k - x_i)}{\prod_{j\neq i}(x_j - x_i)} - \sum_{i=1}^{n} \frac{g(x_i)}{\prod_{j\neq i}(x_j - x_i)} \\
&= \sum_{i=k+1}^{n} \frac{1}{\prod_{j\neq i,1,2\dots,k}(x_j - x_i)} \\
&= 0
\end{aligned}$$

where in the first step, the second term becomes zero by assuming the result holds for $k-1$ and in the second step the sum disappears because of the previous result (Mochon's Denominator). Note that $k \leq n-2$ for the aforesaid argument to work because otherwise the last step would become invalid. □

Lemma. $\sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{j\neq i}(x_j - x_i)} = (-1)^{n-1}$ *for* $n \geq 2$.

Proof. Let me call $d(n) = \sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{j\neq i}(x_j - x_i)}$ because I will do this inductively. I can then write

$$d(2) = \frac{x_1}{x_2 - x_1} + \frac{x_2}{x_1 - x_2} = \frac{x_1(x_1 - x_2) + x_2(x_2 - x_1)}{(x_2 - x_1)(x_1 - x_2)} = -1.$$

I assume the result holds for $d(n)$ and write

$$\begin{aligned}
d(n+1) &= \sum_{i=1}^{n+1} \frac{x_i^n}{\prod_{j\neq i}(x_j - x_i)} \\
&= \sum_{i=1}^{n+1} \frac{-(x_{n+1} - x_i)(x_i^{n-1}) + x_{n+1}x_i^{n-1}}{\prod_{j\neq i}(x_j - x_i)} \\
&= -\sum_{i=1}^{n+1} (x_{n+1} - x_i) \frac{x_i^{n-1}}{\prod_{j\neq i}(x_j - x_i)} + x_{n+1} \underbrace{\sum_{i=1}^{n+1} \frac{x_i^{n-1}}{\prod_{j\neq i}(x_j - x_i)}}_{=0 \text{ (Mochon's Denominator)}} \\
&= -\sum_{i=1}^{n} \cancel{\frac{(x_{n+1} - x_i)}{(x_{n+1} - x_i)}} \frac{x_i^{n-1}}{\prod_{j\neq i, n+1}(x_j - x_i)} + \cancelto{0}{(x_{n+1} - x_{n+1})} \frac{x_{n+1}^{n-1}}{\prod_{j\neq n+1}(x_j - x_{n+1})} \\
&= -d(n).
\end{aligned}$$

$\square$

Proposition. $\langle x_h \rangle - \langle x_g \rangle = \frac{1}{N_h^2} = \frac{1}{N_g^2}$ *for a Mochon's TDPG assignment with* $k = n-2$ *and coefficient of* x^{n-2} ± 1 *in* $f(x)$*. As above here* $\langle x_h \rangle = \frac{1}{N_h^2} \sum p_{h_i} x_{h_i}$ *and* $\langle x_g \rangle = \frac{1}{N_g^2} \sum p_{g_i} x_{g_i}$.

Proof. Note, to start with, that the coefficient of x^{n-2} being ± 1 is not an artificial requirement because for killing $n-2$ points $f(x)$ will have the form

$$f(x) = (x_{k_1} - x)(x_{k_2} - x) \dots (x_{k_{n-2}} - x) = (-1)^{n-2} x^{n-2} + \tilde{f}(x)$$

where $\tilde{f}$ is a polynomial of order $n-2$. Observe that

$$\begin{aligned}
N_h^2 (\langle x_h \rangle - \langle x_g \rangle) &= \sum_{i=1}^{n} p(x_i) x_i = -\sum_{i=1}^{n} \frac{x_i f(x_i)}{\prod_{j\neq i}(x_j - x_i)} \\
&= -\sum_{i=1}^{n} \frac{x_i (-1)^{n-2} x_i^{n-2}}{\prod_{j\neq i}(x_j - x_i)} - \sum_{i} \frac{\tilde{f}(x_i)}{\prod_{j\neq i}(x_j - x_i)} \\
&= -(-1)^{n-2} \sum_{i=1}^{n} \frac{x_i^{n-1}}{\prod_{j\neq i}(x_j - x_i)} \\
&= 1
\end{aligned}$$

where the second term in the second step vanishes because of Mochon's f-assignment Lemma and the last step follows from the previous result. $\square$