Lebanese International University

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Master Thesis In Control Of Pde's

DECAY RATES OF A TIMOSHENKO SYSTEM WITH KELVIN - VOIGT DAMPING

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Contents

Abstract

In this report, we first offer a quick review on sobelev spaces and spectral analysis. We also give an introduction on the basic results of semigroup theory that is beneficial in the study of existence of solutions of systems of PDEs and the types of stability attained by the solutions when certain controls are being applied to the system. We then study a Timoshenko system with a damping of type Kelvin - Voigt.The system describes the dynamics of some beams. The existence of the solution is proved for the system subject to two control terms as well as that subject to only one control. We offer a detailed explanation of the energy of the system decay that are sometimes exponential and sometimes polynomial depending on the considered hypothesis on the system with two dissipation sources. We also consider the same system but with only one dissipation source and prove that the strong stability still holds true. Generally, we proceed by proving certain estimates for the resolvent of an operator associated to our system.

Introduction

Control theory is the study of controlling the behaviour of an operator system to achieve a certain goal. The theory has many physical applications in many fields (for instance earthquake engineering, acoustics, fluid dynamics, system of beams, aeronautics, etc...). Different concepts are used in this theory of which we mention controllability, observability and stabilizability concepts. As for the last one, it's the discussion of inputs that lead the solution of the system to demolish as time increases and it's the main concept that we deal with in this work. More precisely, we discuss the effect of one locally damping source on the stability of a system of beam equations.

This report is divided into four chapters, which are as follows:

- In chapter one we recall some basic definitions and theorems in the field of spectral analysis as well as the fundamentals of the theory of P.D.Es as Sobolev spaces, sobolev embeddings and useful inequalities to be used later on.
- In chapter two, we introduce the basics of Semi-group theory and the infinitesimal generator of the so called strongly continuous semigroups. We also introduce the notion of maximal dissipative operators and recall the most important theorems from this theory that are used to prove the existence of solutions of evolution problems once the problem can be formulated into a Cauchy problem satisfying the needed conditions.
- Chapter three is devoted to the study of strong stability of the system of beams subject to one damping source. A similar system with two damping sources was previously studied in some works of X.Tian and Q.Zhang where they proved the strong stability of their system. Our system in this chapter just includes one damping source and we manage to prove the strong stability is achieved in this case. A more detailed introduction on the methods used and the discussed system is offered in the first pages of this chapter.
- The last chapter is devoted to the discussion of types of stability satisfied by the system with two damping sources. We explain therein the estimates satisfied by the corresponding resolvent when the damping sources satisfies certain hypothesis and conclude the types of stability using results of Borichev-Tomilov.

The main contribution in this work is the proof of the well-posedness of the considered problem with only one source of damping as well as proving the strong stability still holds in this case. The thesis offers as well detailed explanations for previous results and further perspectives that might be disscussed at the end of last chapter.

Chapter 1

Background on Spectral Analysis

1.1 Spectral Analysis

In this section, we offer a review on the basic notions of spectral analysis needed throughout this thesis. Throughout this section, E and F are two Banach spaces.

Definition 1.1.1. Let $\mathcal{D}(A)$ be a subspace of E. An unbounded linear operator is a linear mapping

$$A : \mathcal{D}(A) \subset E \longrightarrow F$$

defined on $\mathcal{D}(A)$. Moreover, A is bounded if there is a constant $m \geq 0$ such that :

$$\|Au\| \leq m\,\|u\|,\ \forall\, u \in \mathcal{D}(A).$$

Remark 1.1.2. If A is bounded and densely defined, then A can be extended to E. The set of linear bounded operators from E to F is denoted by $\mathcal{L}(E, F)$.

Definition 1.1.3. The linear operator A is said to be closed whenever its graph

$$G(A) = \{(x, Ax) : x \in \mathcal{D}(A)\}$$

is a closed subset in $E \times F$.

Equivalently,

A linear operator $A : \mathcal{D}(A) \subset E \longrightarrow F$ is closed if for every sequence (x_n) in $\mathcal{D}(A)$ converging to some x in E such that (Ax_n) conevreges to some y in F as $n \to \infty$, we have $x \in \mathcal{D}(A)$ and $Ax = y$.

Definition 1.1.4. A linear operator $A : E \longrightarrow F$ is called a **compact linear operator** if for any sequence $(x_n)_{n\geq 1}$ which is bounded in E, its corresponding sequence of images $(Ax_n)_{n\geq 1}$ admits a convergent subsequence.

Recall that if A is a bounded operator, then we can define the norm of A as,

$$\|A\| = \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|}.$$

Remark 1.1.5. Every compact operator A is bounded.
Indeed, if $\|A\| = \infty$, there is a sequence $(x_n)_{n\geq 1}$ such that $\|x_n\| \leq 1$ and $\|Ax_n\| \longrightarrow \infty$. Thus $(Ax_n)_{n\geq 1}$ does not admit a convergent subsequence. Hence, $\|A\|<\infty$ and A is bounded.

Definition 1.1.6. Let A be a compact linear operator from E to F . The resolvent set of A, denoted $\rho(A)$, is defined by $\rho(A) :=\{\lambda \in \mathbb{C}\,; A - \lambda I$ is invertible $\}$
The spectrum of A, $\sigma(A)$, is the complement of $\rho(A)$ in $\mathbb{C}$.

Definition 1.1.7. Let $\lambda \in \mathbb{C}$. λ is said to be an eigenvalue of A if $Ker(A - \lambda I) \neq \{0\}$. The point spectrum of A, denoted by $\sigma_p(A)$, is the set of all eigenvalues of A.

Definition 1.1.8. Let A be a linear operator from E to F, and let $\lambda \in \rho(A)$. The operator R(λ , A) = (A $-\lambda I)^{-1}$ is called a resolvent operator of A associated to λ.

It is useful to keep in mind that if A is a closed unbounded operator and $\lambda \in \rho(A)$ then the resolvent operator $(A - \lambda\, I)^{-1}$ is bounded.

Proposition 1.1.9. The spectrum of any bounded operator $A \in \mathcal{L}\;(E, F)$ is compact. More precisely, $\sigma(A) \subset \overline{B(0, ||A||)}$.

In what follows, we assume that E = H is a Hilbert space. We denote by $\mathcal{L}(H)$ the set of linear bounded operators from H into H.

Proposition 1.1.10. Let $A \in \mathcal{L}(H)$. There exists a unique operator $A^* \in \mathcal{L}(H)$ such that:

$$(A^*u, v) = (u, Av) \;\; \forall\, u, v \in \; H.$$

The operator A^* is called the adjoint of $A \in \mathcal{L}(H)$.

Definition 1.1.11. The adjoint of an unbounded operator $(A, \mathcal{D}(A))$ is the unbounded operator $(A^*, D(A^*))$ defined as follows,

- $D(A^*) = \{g \in H : \exists\, \eta_g \in H$ such that $\langle A\, f, g\rangle = \langle f, \eta_g\rangle,\ \forall f \in \mathcal{D}(A)\}$.
- $A^*\, g = \eta_g$

A is called self - adjoint if $A^* = A$ in the sense that $D(A^*) = \mathcal{D}(A)$ and $A^*\, f = A\, f \;\forall f \in \mathcal{D}(A)$.

Definition 1.1.12. An operator $A \in \mathcal{L}(H)$ is self-adjoint if $A^* = A$, that is

$$(Au_1, u_2){=}(u_1, Au_2)\;\forall\, u_1, u_2 \in \; H.$$

Definition 1.1.13. A linear unbounded operator A in H is said to have compact resolvent if there exists λ_0 in the set $\rho(A)$ such that R(λ_0 , A) = (A $-\lambda_0 I)^{-1}$ is compact.

Theorem 1.1.14. If $(A, \mathcal{D}(A))$ is an operator with compact resolvent, then either the spectrum of A is empty, or otherwise it consists of an infinite sequence $\{\lambda_n\}$ of eigenvalues with finite multiplicity such that,

$$|\lambda_1| \leq |\lambda_2| \leq ... |\lambda_n| \leq ..., \quad \lim_{n\to\infty} |\lambda_n| = \infty.$$

If moreover A is self - adjoint, then the sequence $\{\lambda_n\}$ is real and there exists an orthonormal sequence $\{f_n\}$ of associated eigenfunctions such that,

$$A\, f = \sum_{n\geq 1} \lambda_n \,\langle f, f_n\rangle\, f_n = \sum_{n\geq 1} \lambda_n\, P_{\lambda_n}\, f,\ \forall f \in \mathcal{D}(A).$$

Definition 1.1.15. The discrete spectrum of A, denoted $\sigma_d(A)$, is a subset of $\sigma(A)$ defined as follows,

$$\lambda \in \sigma_d(A) \iff \begin{cases} \lambda & \text{is an eigenvalue of finite multiplicity} \\ \lambda & \text{is isolated in } \sigma(A) \end{cases}$$

The essential spectrum denoted by $\sigma_{ess}(A)$, is the complement in $\sigma(A)$ of the discrete spectrum,

$$\sigma_{ess}(A) = \sigma(A) - \sigma_d(A).$$

Remark 1.1.16. The spectrum of a linear unbounded operator A on H with a compact resolvent is discrete in $\mathbb{C}$, i.e. $\sigma(A) = \sigma_d(A)$. If A is in addition self-adjoint, then $\sigma(A) \subset \mathbb{R}$.

1.2 Sobolev Spaces

In this section, we will introduce some notations of Sobolev spaces used later on and recall some results that we used in the next chapters.

Let $\Omega \subset \mathbb{R}^n$ be an open domain. We define the space of square - summable functions on Ω by:

$L^2(\Omega) = \{f : \Omega \longrightarrow \mathbb{R}\ ,\ \int_\Omega |f(x)|^2 dx\ < \infty\}$ endowed with the following inner product

$(f, g)_{L^2} = \int_\Omega f(x)g(x)dx$ and $\|f\|_{L^2} = \left(\int_\Omega |f(x)|^2 dx\right)^{\frac{1}{2}}$.

Let $u : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ in $\mathbb{N}^n$. Let us first recall the following notations:

We can represent a partial derivative of u of order α as:

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}, \quad |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n.$$

$C^m(\Omega) = \{u : \mathrm{D}^\alpha u$ are continuous on $\Omega :\ |\alpha| \leq m\ \}$, where $m \in \mathbb{N}$.

$C^\infty(\Omega) = \bigcap_{m \in \mathbb{N}} C^m(\Omega)$,

$C_c(\Omega) = \{u\ :\ u \in C(\Omega)\ :\ \mathrm{supp(u)} = \mathrm{K}\ \}$ where supp(u) denotes the support of u and K denotes a compact subset of Ω.

We also recall that the set of test functions $D(\Omega) = C_c^\infty(\Omega) = \{u \in C^\infty(\Omega);\ \mathrm{supp(u)} = \mathrm{K}\}$ is dense in $L^2(\Omega)$.

Motivation for the definition of weak derivative:

If u $\in C^1(\Omega)$, for any function $\phi \in C_c^\infty(\Omega)$ we have due to the classical integration by parts:

$$\int_\Omega u\phi'\, dx = -\int_\Omega u'\phi\, dx.$$

More generally, if $u \in C^m(\Omega)$, and $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is multi-index of order $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n = m$ then for all $\phi \in C_c^\infty$

$$\int_\Omega u\, D^\alpha \phi\, dx = (-1)^{|\alpha|} \int_\Omega \phi\, D^\alpha u\, dx \qquad (1)$$

Note that if $u \in C^m(\Omega)$ "$D^\alpha u$" is well defined, but if $u \notin C^m(\Omega)$, then "$D^\alpha u$" does not have a meaning in the classical sense, we also notice that the left hand side of (1) would make sense if $u \in L^1_{\text{loc}}$. We thus give a meaning to "$D^\alpha u$" by attributing this symbol to a locally summable function w if it exists in such a way that (1) holds once w replaces "$D^\alpha u$" in it.

Definition 1.2.1. Suppose $u, w \in L^1_{loc}(\Omega)$ and α is a multi-index. If

$$\int_\Omega u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_\Omega w \, \phi \, dx$$

for every test function $\phi \in C_c^\infty(\Omega)$, then w is the $\alpha^{th}-$ weak partial derivative of u denoted by $D^\alpha u$.

Example Let n=1 , $\Omega =]0,1[$. We define

$$u(x) = \begin{cases} \frac{x}{2} & 0 \le x < \frac{1}{2} \\ \frac{1-x}{2} & \frac{1}{2} \le x \le 1 \end{cases} \in C^0(\Omega) \quad \text{and} \quad v(x) = \begin{cases} \frac{1}{2} & 0 < x < \frac{1}{2} \\ \frac{-1}{2} & \frac{1}{2} < x < 1 \end{cases} \notin C^0(\Omega)$$

To show that $u' = v$ in the weak sense, prove that for any $\phi \in C_c^\infty$, we have

$$\int_0^1 u\phi' \, dx = -\int_0^1 v\phi dx.$$

Indeed, by dividing the integral and integrating by parts we obtain:

$$\begin{aligned} \int_0^1 u\phi' \, dx &= \int_0^{\frac{1}{2}} u\phi' \, dx + \int_{\frac{1}{2}}^1 u\phi' \, dx \\ &= -\int_0^{\frac{1}{2}} \frac{1}{2}\phi \, dx - \int_{\frac{1}{2}}^1 \big(-\frac{1}{2}\big)\phi \, dx = -\int_0^1 v\phi dx \end{aligned}$$

Definition 1.2.2. The Sobolev space $W^{m,p}(\Omega)$ consists of locally summable real valued functions on Ω whose weak derivative $D^\alpha u$ exists and belongs to $L^p(\Omega)$ $(1 \le p < \infty)$ for each multi-index α with $|\alpha| \le m$.

$$W^{m,p}(\Omega) = \{u \in L^1_{loc}(\Omega) \; : \; \forall \; \alpha \in \mathbb{N}^n, \, |\alpha| \le m \Longrightarrow D^\alpha u \in L^p(\Omega)\}$$

Remark 1.2.3. We denote by $W_0^{m,p}(\Omega)$ the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$

Remark 1.2.4. For p=2, we write $H^m(\Omega) = W^{m,2}(\Omega) = \{u(x) : D^\alpha u(x) \in L^2(\Omega) \, ; |\alpha| \le m\}$ which is a Hilbert space endowed with the inner product.

$$\big(u_1, u_2\big)_{H^m(\Omega)} = \int_\Omega \sum_{|\alpha| \le m} \big(D^\alpha u_1\big)\big(D^\alpha u_2\big) dx$$

Note that, $H^0(\Omega) = L^2(\Omega)$ and in particular if $\Omega \subset \mathbb{R}^2$ then :

$H^1(\Omega) = \{f\big(x,y\big) : f \in L^2(\Omega), f_x, f_y \in L^2(\Omega)\}$

$\big(f,g\big)_{H^1(\Omega)} = \iint_\Omega \big(fg + f_x g_x + f_y g_y\big) dxdy = \big(f,g\big)_{L^2(\Omega)} + \big(\nabla f, \nabla g\big)_{L^2(\Omega)}$

$\|f\|^2_{H^1(\Omega)} = \|f\|^2_{L^2(\Omega)} + \|\nabla f\|^2_{L^2(\Omega)}$

Traces

Now, if u is continuously defined on the closure of Ω, i.e $u \in C(\overline{\Omega})$, then clearly the values of u on $\partial\Omega$ are well defined. Here, we discuss the meaning of "boundary values" along $\partial\Omega$ assigned for functions with weaker regularity, more precisely for functions in $W^{1,p}(\Omega)$. We know that in such spaces the functions are not in general continuous and are mostly defined a.e. in Ω, thus we have to give a new sense of the boundary values of $u \in W^{1,p}(\Omega)$ restricted to $\partial\Omega$. Here we introduce a general notion of trace operator to tackle this problem.

For this section we consider $1 \leq p < \infty$ and a bounded domain Ω with C^1 boundary $\partial\Omega$.

Theorem 1.2.5. There is a bounded linear operator

$$T : W^{1,p}(\Omega) \longrightarrow L^p(\partial\Omega)$$

satisfying :

(i) T f = $f|_{\partial\Omega}$ if f $\in W^{1,p}(\Omega) \cap C(\overline{\Omega})$

and

(ii) $\|Tf\|_{L^p(\partial\Omega)} \leq C\|f\|_{W^{1,p}(\Omega)}$ for every $f \in W^{1,p}(\Omega)$, with C a constant depending solely on p and Ω .

Notation : We call Tf the trace of f on $\partial\Omega$.

Theorem 1.2.6. (Trace - zero functions)

Let f $\in W^{1,p}(\Omega)$. Then

$$f \in W_0^{1,p}(\Omega) \text{ if and only if } \mathrm{T\,f} = 0 \text{ on } \partial\Omega.$$

Thus,

$$W_0^{m,p}(\Omega) = \{f : f \in W^{m,p}(\Omega) \;\; and \;\; D^\alpha f = 0 \;\; on \;\; \partial\Omega, \forall\, |\alpha| \leq m-1\}$$

Remark that,

$$H_0^1(\Omega) = \{f : \; f \in H^1(\Omega) \;\; and \;\; f|_{\partial(\Omega)} = 0\} \;\; and \;\; \|f\|^2_{H_0^1(\Omega)} = \|\nabla f\|^2_{L^2(\Omega)}$$

Let H be a Hilbert space. Denote its norm by $\|\;\|$ and its inner product by $(\cdot,\cdot)$.

Definition 1.2.7. Let $B : H \times H \longrightarrow \mathbb{K}$. We say that B is a sesquilinear form on H if for all $\alpha \in \mathbb{K}$ and $u, v, h \in H$, we have

$$B(\alpha u + v, h) = \alpha B(u,h) + B(v,h) \;\; and \;\; B(u, \alpha v + h) = \overline{\alpha} B(u,v) + B(u,h).$$

Theorem 1.2.8. **(Lax - Milgram Theorem)**
Let

$$B : H \times H \longrightarrow \mathbb{C}$$

be a sesquilinear function (bilinear if $\mathbb{K} = \mathbb{R}$), such that there exist constants $\alpha, \beta > 0$ with

(i) $|B(u,v)| \leq \alpha \|u\| \|v\|$

and

(ii) $\left| Re\Big(B(u,u) \Big) \right| \geq \beta \|u\|^2,$

being valid for all $u, v \in H$. In addition, given a **bounded linear** function $f : H \longrightarrow \mathbb{R}$ on H. Then there exists a unique element u ∈ H such that

$$B(u,v) = \big\langle f, v \big\rangle_{H',H}$$

for any $v \in H$.

Now, we recall some well known inequalities. Let $\Omega \subset \mathbb{R}^n$ and $1 \leq p < n,$ the conjugate of p is a number

$$p^* := \frac{np}{n-p}.$$

Remark that $p^* > p$ and

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

Theorem 1.2.9. **(Poincare's Inequality)**
Suppose that $u \in W_0^{1,p}(\Omega)$ for $1 \leq p < n$. Then for any r satisfying $1 \leq r \leq p^*$, the following estimate holds

$$\|u\|_{L^r(\Omega)} \leq C \|Du\|_{L^p(\Omega)}$$

with C being a constant that depends solely on p , r , n and Ω.

Remark 1.2.10. Due to the above theorem on $W_0^{1,p}$, the norm $\|Du\|_{L^p(\Omega)}$ is equivalent to $\|u\|_{W^{1,p}(\Omega)}$ on the subspace $W_0^{1,p}$.

Lemma 1.2.11. Let $0 < p < \infty$ and $a, b \geq 0$. Then

$$(a+b)^p \leq 2^p(a^p + b^p)$$

Proof:

$$\begin{aligned}(a+b)^p &\leq (2\, max\{a,b\})^p \\ &= 2^p max\{a^p, b^p\} \\ &\leq 2^p(a^p + b^b).\end{aligned}$$

We recall the following inequality which is a generalization of Hardy's inequality.

Proposition 1.2.12. Let $g \in L^1_{loc}(0,1)$ and $T\,g(x) = \int\limits_0^x g(s)\,ds$. Let h_1 and h_2 be two positive functions on $]0,1[$. Suppose that

$$M = \sup_{x\in(0,1)} \Big(\int\limits_x^1 h_1(s)\,ds\Big)\Big(\int\limits_0^x [h_2(s)]^{-1}\,ds\Big) < \infty \tag{1.1}$$

Then there exists a constant $C \in [M, 2\,M]$ such that

$$\int\limits_0^1 h_1(x)\,|T\,g(x)|^2\,dx \le C \int\limits_0^1 h_2(x)\,|g(x)|^2\,dx \tag{1.2}$$

Theorem 1.2.13. (Young's Inequality)
Let $1 < p < \infty$ and $a, b \ge 0$. Then

$$ab \le \frac{a^p}{p} + \frac{b^{p^*}}{p^*}$$

with p^* being the conjugate of p.

In particular, for $\epsilon > 0$ and $p = 2$, we get

$$ab = \frac{1}{\sqrt{\epsilon}}\,a\,\sqrt{\epsilon}\,b \;\le\; \frac{1}{2\epsilon}\,a^2 + \frac{\epsilon}{2}\,b^2.$$

Definition 1.2.14. The space of functions $f \in C^j(\overline{\Omega})$ verifying

$$\|f\|_{C^{j,\lambda}(\overline{\Omega})} := \sum_{|\alpha|\le j} \|D^\alpha f\|_{C(\overline{\Omega})} + \sum_{|\alpha|\le j} \|D^\alpha u\|_{C^{0,\lambda}(\overline{\Omega})} < +\infty$$

is called the Holder space $C^{j,\lambda}(\overline{\Omega})$.

Proposition 1.2.15. Let $\Omega \subset \mathbb{R}^n$ lipschitz then,

$$W^{j+m,p}(\Omega) \subset C^{j,\lambda}(\overline{\Omega}), \quad 0 \le \lambda \le m - \frac{n}{p}.$$

So, for n = 1,

$$W^{0+m,2}(\Omega) \subset C^{0,\lambda}(\overline{\Omega}), \quad 0 \le \lambda \le m - \frac{1}{2}.$$

i.e,

$$H^m(\Omega) \subset C^{0,\lambda}(\overline{\Omega}), \quad 0 \le \lambda \le m - \frac{1}{2}.$$

thus,

$$H^1(\Omega) \subset C(\overline{\Omega}).$$

Chapter 2

Introduction to Semi-Group theory

In this chapter, we introduce the definition of C_0 semigroups, or strongly continuous semigroups of linear operators and then infinitesimal generators and study their properties.

2.1 Semigroups on Banach Spaces

Definition 2.1.1. Let E be a Banach space. A family $(T(t))_{t\geq 0} \subset \mathcal{L}(E)$ is a semigroup of bounded linear operators on E if

(i) $T(0) = Id_E$.

(ii) $T(t_1 + t_2) = T(t_1)T(t_2)$ for all $t_1, t_2 \geq 0$.

In addition, $(T(t))_{t\geq 0}$ is said to be uniformly continuous when

$$\lim_{t\to 0^+} \|T(t) - I\| = 0.$$

We now define the infinitesimal generator of T(t) by its domain

$$\mathcal{D}(A) = \{u \in E : \lim_{t\to 0} \frac{T(t)u - u}{t} \quad exists\}$$

and its expression for each $u \in \mathcal{D}(A)$,

$$Au = \lim_{t\to 0} \frac{T(t)u - u}{t} = \frac{d^+T(t)u}{dt}|_{t=0}.$$

Following the above definition, we notice that any uniformly continuous semigroup $(T(t))_{t\geq 0} \subset \mathcal{L}(E)$ obeys

$$\lim_{s\to t} \|T(s) - T(t)\| = 0$$

Theorem 2.1.2. Let A be a linear operator on E. Then A generates some uniformly continuous semigroup if and only if A is a bounded.

Note that Definition 2.1.1 ensures that the uniqueness of the infinitesimal generator of a semigroup T(t). The precedent theorem also ensures that if T(t) is uniformly continuous, then its infinitesimal generator is bounded and linear and that every bounded linear operator is a generator of a uniformly continuous semigroup. The following theorem guarantees the uniqueness of the semigroup generated by a certain linear bounded operator.

Theorem 2.1.3. Let $(T_1(t))_{t\geq 0} \subset \mathcal{L}(E)$ and $(T_2(t))_{t\geq 0} \subset \mathcal{L}(E)$ be two semigroups. Assume

$$\lim_{t\to 0^+} \frac{T_1(t)-I}{t} = A = \lim_{t\to 0^+} \frac{T_2(t)-I}{t}$$

then for all $t \geq 0,\ T_1(t) = T_2(t)$.

Corollary 2.1.4. If T(t) is a uniformly continuous semigroup in $\mathcal{L}(E)$. Then

a) There is $\omega \geq 0$ such that $\|T(t)\| \leq e^{\omega t}$.
b) There exists one and only one infinitesimal generator A of T(t).
c) $T(t)$ is differentiable with respect to t and

$$\frac{dT(t)}{dt} = AT(t) = T(t)A$$

Definition 2.1.5. A semigroup $(T(t))_{t\geq 0}$, of bounded linear operators on E is said to be a C_0 semigroup or a strongly continuous semigroup if

$$\lim_{t\to 0^+} T(t)u = u, \quad \forall\, u \in E.$$

Theorem 2.1.6. Every C_0 semigroup $(T(t))_{t\geq 0}$ satisfies

$$\|T(t)\| \leq Me^{\omega t}, \ \ \forall\ \ 0 \leq t < \infty,$$

for some $M \geq 1$ and $\omega \geq 0$.

Remark 2.1.7. If $\omega = 0$, we say (T(t)) is uniformly bounded and if $M = 1$ we call (T(t)) a C_0 semigroup of contractions.

Corollary 2.1.8. Let T(t) be a C_0 semigroup then for each $x \in X$, the function

$$T(.)u : t \longrightarrow T(t)u$$

is continuous from $\mathbb{R}^{+*}$ into E. That is, $T(t)u \in C^0(\mathbb{R}^+, E)$, for all u $\in E$

Theorem 2.1.9. Let T(t) be a C_0 semigroup of infinitesimal generator A. Then
a) for $u \in E$

$$\lim_{h\to 0} \frac{1}{h} \int_t^{t+h} T(s)u\, ds = T(t)u.$$

b) For $u \in E$, $\displaystyle\int_0^t T(s)u\, ds \in \mathcal{D}(A)$ and

$$A\Big(\int_0^t T(s)u\, ds\Big) = T(t)u - u$$

c) If $u \in \mathcal{D}(A)$, $T(t)u \in \mathcal{D}(A) \cap C^1(\mathbb{R}^+, X)$ and

$$\frac{d}{dt}T(t)u = AT(t)u = T(t)Au$$

d) For $u \in \mathcal{D}(A)$

$$T(t)u - T(s)u = \int_s^t T(m)Au\, dm = \int_s^t AT(m)u\, dm$$

Corollary 2.1.10. Every infinitesimal generator A of a C_0 semigroup (T(t)) is closed. In addition, its domain $\mathcal{D}(A)$ is dense in E.

Theorem 2.1.11. Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0 semigroups with the same infinitesimal generator. Then T = S.

The previous corollary shows that $\overline{\mathcal{D}(A)} = E$, where A is the infinitesimal generator of a C_0 semigroup. In the following theorem a stronger result is shown.

Theorem 2.1.12. Let T(t) be a strongly continuous semigroup and let A be its infinitesimal generator. Let $D(A^n)$ be the domain of A^n. Then $\overline{\bigcap_{n=1}^{\infty} D(A^n)} = E$.

Lemma 2.1.13. Let T(t) be a strongly continuous semigroup and let A be its infinitesimal generator such that $\|T(t)\| \leq M$ for $t \geq 0$. If $x \in D(A^2)$ then

$$\|Ax\|^2 \leq 4M^2\|A^2x\|\|x\|$$

Theorem 2.1.14. (Hille - Yosida)
Let A be a linear operator. Then A is the infinitesimal generator of a C_0 semigroup of contractions $(T(t))_{t\geq 0}$ if and only If

(i) A is a densely defined closed operator.

(ii) $\mathbb{R}^+ \subset \rho(A)$ and for any $\lambda > 0$

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}$$

We introduce here the Yosida approximation of A for each $\lambda > 0$:

$$A_\lambda = \lambda A(\lambda I - A)^{-1} = \lambda^2(\lambda I - A)^{-1} - \lambda I.$$

Lemma 2.1.15. Let A be a linear operator satisfying (i) and (ii) of the previous theorem. Then

$$\lim_{\lambda\to\infty} \lambda(\lambda I - A)^{-1}x = x \quad for\ x \in E$$

and

$$\lim_{\lambda\to\infty} A_\lambda x = Ax, \quad for \;\; x \in \mathcal{D}(A).$$

Corollary 2.1.16. Let (T(t)) be a srongly continuous semigroup of contractions generated by A. Then

$$T(t)x = \lim_{\lambda\to\infty} e^{tA_\lambda}x \quad for\ x \in E.$$

Corollary 2.1.17. Let T(t) be a srongly continuous semigroup of contractions generated by A. The resolvent set of A does not contain any complex number with non - positive real part i.e., $\{\lambda : Re\lambda > 0\} \subset \rho(A)$ and for any λ such that $Re\lambda > 0$, we have :

$$\|R(\lambda : A)\| \leq \frac{1}{Re\lambda}$$

Definition 2.1.18. Let $A : \mathcal{D}(A) \longrightarrow E$ be an unbounded linear operator . We say that A is dissipative if for all u $\in \mathcal{D}(A)$, and for all $\lambda > 0$, we have

$$\|u - \lambda Au\|_E \geq \|u\|_E.$$

We say that A is maximal dissipative, or simply m - dissipative if :

(i) A is dissipative .

(ii) $R(I - \lambda A) = E, \;\; \forall \, \lambda > 0$, that is for all f $\in E$, there exists $u \in \mathcal{D}(A)$ such that $u - \lambda \, Au = f$.

Theorem 2.1.19. Let A be a linear dissipative operator. The following properties are equivalent:

(i) A is m-dissipative

(ii) There exists $\lambda_0 > 0$ such that $R(I - \lambda_0 A) = E$

Proof. (i) implies (ii) is Evident. Let us show that (ii) implies (i). Let $\lambda > 0$ and f $\in E$.

We will find a solution of the equation : $u - \lambda Au = f, \quad u \in \mathcal{D}(A)$.

that is :

$$u - \lambda Au = f$$

multilying by $\lambda_0 > 0$, we get

$$\lambda_0 u - \lambda_0 \lambda Au = \lambda_0 f$$

thus,

$$\frac{\lambda_0}{\lambda} u - \lambda_0 Au = \frac{\lambda_0}{\lambda} f$$

which implies,

$$\frac{\lambda_0}{\lambda} u + u - \lambda_0 Au - u = \frac{\lambda_0}{\lambda} f$$

finally,

$$\begin{aligned} u - \lambda_0 Au &= \big(1 - \frac{\lambda_0}{\lambda}\big) u + \frac{\lambda_0}{\lambda} f \\ u &= \big(I - \lambda_0 A\big)^{-1} \big(1 - \frac{\lambda_0}{\lambda} u\big) + \big(I - \lambda_0 A\big)^{-1} \frac{\lambda_0}{\lambda} f \\ &= F(u) \end{aligned}$$

since, A is dissipative and $R(I - \lambda_0 A) = E$ i.e. $I - \lambda_0 A$ is injective and surjective respectively. $F : E \longrightarrow E$ is linear and the Banach equation $F(u) = u$ has a unique solution if and only if F is a contraction. We have :

$$\begin{aligned} \|\big(I - \lambda_0 A\big)^{-1}\|_{\mathcal{L}(E)} &= \sup_{\|g\|_E \leq 1} \frac{\|\big(I - \lambda_0 A\big)^{-1} g\|_E}{\|g\|_E} \\ &= \sup_{\|\big(I - \lambda_0 A\big) v\|_E \leq 1} \frac{\|v\|_E}{\|\big(I - \lambda_0 A\big) v\|_E} \\ &= \sup_{\|\big(I - \lambda_0 A\big) v\|_E = 1} \|v\|_E. \end{aligned}$$

and A is dissipative then

$$\|v\|_E \leq \|\big(I - \lambda_0 A\big) v\|_E = 1.$$

Hence,

$$\|\big(I - \lambda_0 A\big)^{-1}\|_{\mathcal{L}(E)} \leq 1$$

Choose $\lambda > 0$ such that

$$|1 - \frac{\lambda_0}{\lambda}| < 1 \Longleftrightarrow \lambda > \frac{\lambda_0}{2}$$

$$\begin{aligned} \|F(u) - F(v)\|_E &= \|\big(I - \lambda_0 A\big)^{-1}\big(1 - \frac{\lambda_0}{\lambda}\big)\big(u - v\big)\|_E \\ &\leq |1 - \frac{\lambda_0}{\lambda}|\, \|\big(I - \lambda_0 A\big)^{-1}\|_{\mathcal{L}(E)}\|u - v\|_E \\ &< k\|u - v\|_E \quad k < 1 \end{aligned}$$

This implies that F is a contraction and according to the Banach fixed point value theorem, there exists $u \in \mathcal{D}(A)$ such that

$$F(u) = u, \quad \forall\, \lambda > \frac{\lambda_0}{2}$$

We conclude that

$$R(I - \lambda A) = X, \quad \forall \lambda > \frac{\lambda_0}{2}$$

continue by induction to show that

$$R(I - \lambda A) = E, \quad \forall\, \lambda > \frac{\lambda_0}{2^n} \quad \forall\, n \geq 0$$

□

We define for each linear m-dissipative $A : \mathcal{D}(A) \longrightarrow E$ and for avery $u \in \mathcal{D}(A)$,

$$\|u\|_{\mathcal{D}(A)} := \|u\|_E + \|Au\|_E$$

We also define for each $\lambda > 0$,

$$\begin{aligned} J_\lambda : E &\longrightarrow E \\ u &\longmapsto J_\lambda(u) = \big(I - \lambda A\big)^{-1} u \quad \in \mathcal{D}(A). \end{aligned}$$

We know that $J_\lambda \in \mathcal{L}(E)$ and $\|J_\lambda\| \leq 1$. Let

$$A_\lambda = \frac{1}{\lambda}\big(J_\lambda - I\big)$$

Proposition 2.1.20. Let $A : \mathcal{D}(A) \longrightarrow E$ be linear m-dissipative. Then we have :

(i) $\lim\limits_{\lambda \to 0^+} \|J_\lambda u - u\|_E = 0, \quad for\, every$ u $\in \overline{\mathcal{D}(A)}$.

(ii) $\lim\limits_{\lambda \to 0^+} \|J_\lambda u - u\|_{\mathcal{D}(A)} = 0, \quad for\, every$ u $\in \mathcal{D}(A)$.

(iii) $\lim\limits_{\lambda \to 0^+} \|A_\lambda u - Au\|_E = 0, \quad for\, every$ u $\in \mathcal{D}(A)$.

2.2 Dissipative operators on Hilbert Spaces

In this section we prove some of the previously stated results when the cosidered space E is Hilbert. We, moreover, state some of the basic definitions of theorems that will be used in chapter 3 and 4 as we deal with Hilbert spaces.

Let H be a Hilbert Space equipped with the scalar product $(\, . \, , \, . \,)$

Proposition 2.2.1. Let $A : \mathcal{D}(A) \longrightarrow H$ be a linear operator A is dissipative if and only if $\big(Au, u\big) \leq 0$ for all $u \in \mathcal{D}(A)$

Proof. A is dissipative i.e.

$$\|u\|_H^2 \leq \|u - \lambda Au\|_H^2 \quad \forall\, u \in \mathcal{D}(A),\ \forall\, \lambda > 0$$

that is

$$(u, u) \leq (u - \lambda Au\,,\, u - \lambda Au)$$

which equivalently yields

$$(u, u) \leq (u, u) - 2\lambda(u, Au) + \lambda^2\|Au\|^2$$

which means

$$0 \leq -2\lambda(u, Au) + \lambda^2\|Au\|^2 \leq -2(Au, u) + \lambda\|Au\|^2 \quad \forall\lambda > 0$$

if and only if

$$(Au, u) \leq 0$$

□

Proposition 2.2.2. Let $A : \mathcal{D}(A) \longrightarrow H$ be linear and m-dissipative. Then the following statements hold

(a) $\overline{\mathcal{D}(A)} = H$
(b) A is closed
(c) For any $\lambda > 0$ the operator $\lambda I - A$ is invertible. In addition, $(\lambda I - A)^{-1}$ is bounded and verifies $\|(\lambda I - A)^{-1}\| \leq \dfrac{1}{\lambda}$.

Proof. (a) To prove $\overline{\mathcal{D}(A)} = H$, it is sufficient to show that $\mathcal{D}(A)^{\perp} = \{0\}$. Let u $\in \mathcal{D}(A)^{\perp}$, *then* $(u, v) = 0$ *for all* $v \in \mathcal{D}(A)$. Set $z = J_\lambda(u) \in \mathcal{D}(A)$. As, $(u, z) = 0$ then, $\big((I - \lambda A)z, z\big) = 0$ i.e. $(z, z) - \lambda(Az, z) = 0$ for all $\lambda > 0$.
Take $\lambda = 1$ to get $(z, z) = (Az, z) \leq 0$, then $z = 0$, hence $u = 0$.

(b) A is m-dissipative then, for all $f \in H$, there exists $u \in \mathcal{D}(A)$ satisfying $u - Au = f$. Moreover, as $(Au, u) \leq 0$, we get

$$\|u\|^2 \leq \|u\|^2 - (Au, u) = (f, u) \leq \|f\|\,\|u\|$$

which implies that $\|u\| \leq \|f\|$.

Now, we have $\|(I - A)^{-1} f\| = \|u\| \leq \|f\|$, which shows that $(I - A)^{-1}$ is a linear continuous operator in H with

$$\|(I - A)^{-1}\| \leq 1$$

Let $u_n \subset \mathcal{D}(A)$ with $u_n \to u$ *and* $Au_n \to f$. As $(I - A)^{-1}$ is continuous, we get

$$u_n = (I - A)^{-1}\,(I - A)u_n \longrightarrow (I - A)^{-1}\,(u - f)$$

It follows that, $u = (I - A)^{-1}\,(u - f)$. Thus $u \in \mathcal{D}(A)$ and $u - Au = u - f$.

(c) We proved in Theorem 2.1.19 that $R(\lambda I - A) = H, \forall\, \lambda > 0$ *i.e.* $\lambda I - A$ is onto $\forall\, \lambda > 0$. Moreover, since A is dissipative we can get $\lambda I - A$ is injective. We conclude that, the operator $\lambda I - A$ is an isomorphism from $\mathcal{D}(A)$ onto H. $\forall f \in H, \exists\, u \in \mathcal{D}(A)$ such $(\lambda I - A)u = f$ *i.e.* $\lambda u - Au = f$ Moreover, we have

$$\lambda\|u\|^2 \leq \lambda\|u\|^2 - (Au, u) = \big((\lambda\, I - A)u, u\big) = (f, u) \leq \|f\|\,\|u\|$$

which implies

$$\|u\| \leq \frac{1}{\lambda}\,\|f\|.$$

Therefore,

$$\|(\lambda I - A)^{-1} f\| = \|u\| \leq \frac{1}{\lambda}\,\|f\|.$$

□

The following theorem will be used later on to prove the existence of solutions of the evolution problem introduced.

Theorem 2.2.3. (Lumer Phillips Theorem)
The linear operator $A : \mathcal{D}(A) \longrightarrow H$ generates a C^0 semigroup of contractions $(T(t))_{t\geq 0}$ on H if and only if A is m-dissipative .

2.3 Stability of C_0 - semigroups $(T(t))_{t\geq 0}$ on a Hilbert space H.

Definition 2.3.1.

(i) We say that $(T(t))_{t\geq 0}$ *is strongly stable if and only if* $\lim_{t\to\infty} \|T(t)u\|_H = 0.$

(ii) We say that $(T(t))_{t\geq 0}$ *is weakly stable if and only if* $T(t)u \rightharpoonup 0$ *in* H *that is* $\lim_{t\to\infty} \big(T(t)u, v\big)_H = 0.\ \forall\, v \in H.$

(iii) We say that $(T(t))_{t\geq 0}$ *is uniformly stable if and only if* $\lim_{t\to\infty} \|T(t)u\|_{\mathcal{L}(\mathcal{H})} = 0$
i.e. there exists $M \geq 1,\ \omega > 0$ *such that* $\|T(t)u\|_H \leq Me^{-\omega t}\,\|u\|_H$ *for all* $t \geq 0.$

(iv) We say that $(T(t))_{t\geq 0}$ *is polynomially stable if and only if* $\|T(t)(a - A)^{-\alpha}\| \leq c\,t^{-\beta}$ *for all t >0 and some* $a, \alpha, \beta > 0$ *where,* A *is the infinitesimal generator of* $T(t)$.

Proposition 2.3.2. (Arendt)
Let $T(t)_{t\geq 0}$ be a C^0 semigroup of contractions of linear operators on H generated by A. If $\sigma(A) \cap i\,\mathbb{R}$ is countable and $\sigma_d(A) \cap i\mathbb{R} = \phi$, then $T(t)$ **is strongly stable.**

Theorem 2.3.3. (Huang-Pruss)
Let $T(t)$ be a C^0 semigroup of contractions of linear operators on H with infinitesimal generator A. Then T(t) is exponentially stable if and only if :

(i) $i\mathbb{R} \subset \rho(A)$

(ii) $\overline{\lim_{|\beta|\to\infty}} \|\big(i\beta I - A\big)^{-1}\|_{\mathcal{L}(H)} < \infty$

Remark 2.3.4. In practice to show that A is not exponentially stable, it's enough to show that there exists a sequence α_n and ω_n such that $\|\omega_n\| = 1$ and $\|(i\alpha_n - A)\omega_n\| \to 0$ as n $\to \infty$. This is due to the fact that their existence yields the following :

$$\overline{\lim_{|\beta|\to\infty}} \|\left(i\beta I - A\right)^{-1}\|_{\mathcal{L}(\mathcal{H})} = \overline{\lim_{|\beta|\to\infty}} \sup_{u\in H^*} \frac{\|\left(i\beta I - A\right)^{-1} u\|}{\|u\|} =$$

$$\overline{\lim_{|\beta|\to\infty}} \sup_{v\in\mathcal{D}(A)^*} \frac{\|v\|}{\|\left(i\beta I - A\right)v\|} \geq \frac{\|\omega_n\|}{\|\left(i\alpha_n I - A\right)\omega_n\|} = \frac{1}{\|\left(i\alpha_n I - A\right)\omega_n\|} \to \infty \; as \; n \to \infty$$

In this case we say that the resolvent is not uniformly bounded on the imaginary axis.

Theorem 2.3.5. (Kato-Rellich)

If A is m-dissipative in a Hilbert space H, then $\forall\, u_0 \in H$, the problem

$$\frac{d}{dt}u(t) = Au(t) \quad \forall t > 0, \quad u(0) = u_0$$

has a unique solution $u \in C^0(\mathbb{R}^+, H)$.

If moreover $u_0 \in \mathcal{D}(A)$, this solution $\in W^{1,\infty}(\mathbb{R}^+, H) \bigcap L^\infty(\mathbb{R}^+, \mathcal{D}(A))$.

Remark

Uniform stability $\Longrightarrow$ Polynomial stability $\Longrightarrow$ Strong stability $\Longrightarrow$ Weak stability .
The converse is true in finite dimensional space .

Chapter 3

Well - posedness and strong stability of a Timoshenko system

Consider the following system with a locally distributed internal damping

$$\begin{cases} \rho_1\omega_{tt} - [k(\omega_x + \phi) + D_1(\omega_{xt} + \phi_t)]_x = 0 & in\ (0, L) \times \mathbb{R}^+ \\ \rho_2\phi_{tt} - (\mu\phi_x + D_2\phi_{xt})_x + k(\omega_x + \phi) + D_1(\omega_{xt} + \phi_t) = 0 & in\ (0, L) \times \mathbb{R}^+ \end{cases} \tag{3.1}$$

with initial conditions

$$\omega(x, 0) = \omega_0(x),\ \omega_t(x, 0) = \omega_1(x), \phi(x, 0) = \phi_0(x),\ \phi_t(x, 0) = \phi_1(x) \qquad in\ (0, L) \tag{3.2}$$

and boundry conditions

$$\omega(0, t) = \omega(L, t) = \phi(0, t) = \phi(L, t) = 0 \qquad in\ \ \mathbb{R}^+ \tag{3.3}$$

The system is constituted of two coupled hyperbolic equations, while the damping in the system is of type Kelvin - Voigt. The function ω designates the transverse displacement while the function ϕ designates the rotation of the neutral axis due to bending. In addition, $\rho_1 = \rho A$ and $\rho_2 = \rho I$, where ρ denotes the density. A denotes the cross-sectional area, and I the second moment of area of the cross-sectional area. L is the length of the beam and $k = nGA$, where n is the transverse shear factor and G is the modulus of rigidity. $\mu = EI$, where E is the Young's modulus. Moreover, $D_1(.)$ and $D_2(.)$ are two continuous functions defined from [0 , L] into $\mathbb{R}^+ \cup \{0\}$ assuming that there exists $0 \leq \ell_2 \leq \ell_1 < L$ and $\delta_i(.) \in \mathrm{C}([\ell_i, L])$ such that $\delta_i(.) > 0$ on (ℓ_i, L) and

$$D_i(x) = \begin{cases} 0 & if\ x \in [0, \ell_i) \\ \delta_i(x) & if\ x \in [\ell_i, L] \end{cases} \tag{3.4}$$

for i = 1 , 2 .
We define the corresponding energy of the system by

$$E(t) = \frac{1}{2}\int_0^L \left(k|\omega_x + \phi|^2 + \mu|\phi_x|^2 + \rho_1|\omega_t|^2 + \rho_2|\phi_t|^2\right)dx \tag{3.5}$$

and prove that it dissipates as follows: :

$$\frac{d}{dt}E(t) = -\int_0^L [D_1(x)|\omega_{xt} + \phi_t|^2 + D_2(x)|\phi_{xt}|^2]dx. \tag{3.6}$$

System (3.1) - (3.3) along with Kelvin - Voigt damping applied to the whole interval $[0, L]$ as well as that applied to a proper part of it while D_1 and D_2 belong to $C([0, L])$ has been studied by X.Tian and Q.Zhang. The study of this system starts by proving that the solution exists and decays to zero as times increases. It also shows that the associated semigroup is analytic if the applied damping is global. The analysis of the system when the damping is applied locally leads to exponential and polynomial stability following appropriate conditions considered for both D_1 and D_2. As expected, stronger regularity conditions on the damping coefficients yields better decay rates, thus resulting in stronger stability and faster decay. The methods used in the discussion of the exponential decay is based on the theorem of Pruss [6] and Huang [2] stated in the previous chapter, while the methods used in discussing polynomial decay rate rely on Borichev and Tomilov [1]. As usual, the inequalities of Hardy's and Poincare's type are essential throughout the study. In this chapter, we investigate the strong stability of the system where we eliminate D_1 and show that the result still holds true.

We are here interested in studying the effect of taking only one source of dissipation instead of two sources of dissipation on the strong stability of the system.

More precisely, we study the following Timoshenko system with local distributed Kelvin - Voigt damping:

$$\begin{cases} \rho_1\omega_{tt} - [k(\omega_x + \phi)]_x = 0 & in\ (0, L) \times \mathbb{R}^+ \\ \rho_2\phi_{tt} - (\mu\phi_x + D\phi_{xt})_x + k(\omega_x + \phi) = 0 & in\ (0, L) \times \mathbb{R}^+ \end{cases} \tag{3.7}$$

with initial conditions

$$\omega(x, 0) = \omega_0(x),\ \omega_t(x, 0) = \omega_1(x),\ \phi(x, 0) = \phi_0(x),\ \phi_t(x, 0) = \phi_1(x) \qquad in\ (0, L) \tag{3.8}$$

and boundry conditions

$$\omega(0, t) = \omega(L, t) = \phi(0, t) = \phi(L, t) = 0 \qquad in\ \mathbb{R}^+. \tag{3.9}$$

$\omega, \phi, \rho_1, \rho_2, k$ and μ are the same as introduced before,the only difference is that $D_1 = 0$ and we are only left with the dissipation D_2 denoted now by D. Thus, $D(.) : [0, L] \longrightarrow \mathbb{R}^+ \cup \{0\}$ is continuous such that there exists $0 \leq \ell < L$ and a function $\delta(.) \in \mathrm{C}([\ell, L])$ satisfying $\delta(.) > 0$ on (ℓ, L) and

$$D(x) = \begin{cases} 0 & if\ x \in [0, \ell) \\ \delta(x) & if\ x \in [\ell, L] \end{cases} \tag{3.10}$$

The energy of system (3.7) is thus defined as (3.5)

Indeed,

$$\rho_1\omega_{tt} - [k(\omega_x + \phi)]_x = 0$$

Multiplying by ω_t and integrating over [0 , L], we get

$$\int_0^L \rho_1\omega_{tt}\omega_t\, dx - \int_0^L [k(\omega_x + \phi)]_x\, \omega_t\, dx = 0.$$

Now, applying Green's formula,

$$\int_0^L \rho_1\omega_{tt}\omega_t\, dx + \int_0^L [k(\omega_x + \phi)]\, \omega_{xt}\, dx = 0 \tag{3.11}$$

because,

$$\omega_t(x,t) = \lim_{h\to 0} \frac{\omega(x,t+h) - \omega(x,t)}{h}$$

so,

$$\omega \Big|_0^L = 0 \quad \Longrightarrow \quad \omega_t \Big|_0^L = 0$$

Similarly,

$$\rho_2 \phi_{tt} - (\mu\phi_x + D\phi_{xt})_x + k(\omega_x + \phi) = 0$$

Again, multiply by ϕ_t, integrate over [0 , L] and use Green's formula to get,

$$\int_0^L \rho_2 \phi_{tt}\phi_t \, dx + \int_0^L (\mu\phi_x + D\phi_{xt})\phi_{xt} \, dx + \int_0^L k(\omega_x + \phi)\phi_t \, dx = 0 \tag{3.12}$$

(3.11) + (3.12) implies,

$$\int_0^L [\rho_1\omega_{tt}\omega_t + \rho_2\phi_{tt}\phi_t + \mu\phi_x\phi_{xt} + k(\omega_x+\phi)(\omega_{xt}+\phi_t)]dx + \int_0^L D\,\phi_{xt}\phi_{xt}dx = 0$$

i.e.,

$$\int_0^L \frac{d}{dt}\frac{1}{2}\left[\rho_1|\omega_t|^2 + \rho_2|\phi_t|^2 + \mu|\phi_x|^2 + k|\omega_x+\phi|^2\right]dx = -\int_0^L D\,|\phi_{xt}|^2 dx.$$

So, we define the energy of the system by:

$$E(t) = \frac{1}{2}\int_0^L \left(k|\omega_x+\phi|^2 + \mu|\phi_x|^2 + \rho_1|\omega_t|^2 + \rho_2|\phi_t|^2\right)dx \tag{3.13}$$

which thus dissipates according to:

$$\frac{d}{dt}E(t) = -\int_0^L [D(x)|\phi_{xt}|^2]dx. \tag{3.14}$$

From now on, $\|.\|$ and $(.,.)$ denotes $\|.\|_{L^2(0,L)}$ and $(.,.)_{L^2(0,L)}$, respectively. In order to discuss the

existence of a solution of our system we write it as an evolution problem. We first introduce the following Hilbert spaces.

- $H = L^2(0,L) \times L^2(0,L), \quad \|(v,\psi)\|_H^2 = \int_0^L \left(\rho_1|v|^2 + \rho_2|\psi|^2\right)dx, \ \forall\,(v,\psi) \in H,$

- $V = H_0^1(0,L) \times H_0^1(0,L), \quad \|(\omega,\phi)\|_V^2 = \int_0^L \left(k|\omega'+\phi|^2 + \mu|\phi'|^2\right)dx, \ \forall\,(\omega,\phi) \in V.$

The energy space is defined by $\mathcal{H} = V \times H$ equipped with :

$$\|(\omega,\phi,v,\psi)\|_{\mathcal{H}}^2 = \|(\omega,\phi)\|_V^2 + \|(v,\psi)\|_H^2, \ \forall\,(\omega,\phi,v,\psi) \in \mathcal{H}.$$

Define the following linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$,

$$\mathcal{A}\begin{pmatrix}\omega\\ \phi\\ v\\ \psi\end{pmatrix} = \begin{pmatrix}\omega_t\\ \phi_t\\ v_t\\ \psi_t\end{pmatrix} = \begin{pmatrix}\omega_t\\ \phi_t\\ \omega_{tt}\\ \phi_{tt}\end{pmatrix} = \begin{pmatrix} v \\ \\ \psi \\ \\ \rho_1^{-1}T' \\ \\ \rho_2^{-1}(R'-T)\end{pmatrix} \in \mathcal{H}$$

with

$$D(\mathcal{A}) = \{(\omega, \phi, v, \psi) \in \mathcal{H} \mid \quad (v, \psi) \in V, \quad T', R' \in L^2(0, L)\},$$

where $\quad T = k(\omega' + \phi) + D_1\,(v' + \psi) = k(\omega' + \phi) \quad$ and $\quad R = \mu\phi' + D_2\psi' = \mu\phi' + D\,\psi'$.

We rewrite system (3.7) $-$ (3.9) in the form of the following evolution equation

$$U_t = \mathcal{A}U, \; U(0) = U_0 = (\omega_0, \phi_0, \omega_1, \phi_1) \in \mathcal{H}$$

3.1 Well - posedness

We proceed first in proving the well - posedness of the problem. To show that our system (3.7) $-$ (3.9) admits a solution it is enough to show that $\mathcal{A}$ is m - dissipative, thus we need to check two conditions $\; Ker(\lambda I - \mathcal{A}) = \{0\}$ and $R(\lambda I - \mathcal{A}) = \mathcal{H}$ for some $\lambda > 0$.

Theorem 3.1.1. Assume the coefficient function D is continuous, nonnegative and satisfies (3.10). Then, $\mathcal{A}$ generates a C_0 - semigroup of contractions on $\mathcal{H}$.

Proof. Let $U = \begin{pmatrix} \omega \\ \phi \\ v \\ \psi \end{pmatrix} \in D(\mathcal{A})$. We have, $\mathcal{A}U = \begin{pmatrix} v \\ \psi \\ \rho_1^{-1}T' \\ \rho_2^{-1}(R' - T) \end{pmatrix}$

$$(\mathcal{A}U, U)_{\mathcal{H}} = \left(\begin{pmatrix} v \\ \psi \\ \rho_1^{-1}T' \\ \rho_2^{-1}(R' - T) \end{pmatrix}, \begin{pmatrix} \omega \\ \phi \\ v \\ \psi \end{pmatrix} \right)_{\mathcal{H}} = \left(\begin{pmatrix} v \\ \psi \end{pmatrix}, \begin{pmatrix} \omega \\ \phi \end{pmatrix} \right)_V + \left(\begin{pmatrix} \rho_1^{-1}T' \\ \rho_2^{-1}(R' - T) \end{pmatrix}, \begin{pmatrix} v \\ \psi \end{pmatrix} \right)_H$$

$$= \int_0^L [k(v' + \psi)\overline{(\omega' + \phi)} + \mu\psi'\overline{\phi'}]dx + \int_0^L [\rho_1\rho_1^{-1}T'\overline{v} + \rho_2\rho_2^{-1}(R' - T)\overline{\psi}]dx$$

substitute T and R then use Green's formula

$$= \int_0^L kv'\overline{(\omega' + \phi)}dx + \int_0^L k\psi\overline{(\omega' + \phi)}dx + \int_0^L \mu\psi'\overline{\phi'}dx$$
$$- \int_0^L [k(\psi' + \phi) + D_1\,(v' + \psi)]\overline{v'}dx - \int_0^L (\mu\phi' + D_2\psi')\overline{\psi'}dx - \int_0^L [k(\omega' + \phi) + D_1\,(v' + \psi)]\overline{\psi}dx$$

because, $\quad (v = \omega_t$ and $\omega_t \Big|_0^L = 0\;) \quad \& \quad (\;\psi = \phi_t$ and $\phi_t \Big|_0^L = 0\;)$.

We know that for any $z \in \mathbb{C}$ we have : $z - \overline{z} = 2iIm(z)$ then,

$$Re(\mathcal{A}U, U)_{\mathcal{H}} = -\int_0^L \left[D_1\,|v' + \psi|^2 + D_2\,|\psi'|^2\right]dx = -\int_0^L D\,|\psi'|^2 dx \tag{3.15}$$

Therefore, $\mathcal{A}$ is dissipative .

To show that $\mathcal{A}$ is maximal we will check the surjectivity of $\mathcal{A}$. In what follows we will show that $\mathcal{A}$ is bijective. Let $f = (f_1, f_2, g_1, g_2) \in \mathcal{H}$ and let us prove that there exists a unique $U = (\omega, \phi, v, \psi) \in D(\mathcal{A})$ such that $\mathcal{A}U = f$. Thus, $v = f_1$, $\psi = f_2$ and

$$[k(\omega' + \phi)]' = \rho_1 g_1 = G_1, \tag{3.16}$$

$$(\mu\phi' + D\psi')' - k(\omega' + \phi) = \rho_2 g_2 = G_2. \tag{3.17}$$

Multiply (3.16)[resp.(3.17)] by $\overline{W} \in \mathrm{H}_0^1$ (resp. by $\overline{\Phi} \in H_0^1$), integrate in (0 , L), and use Green's formula to obtain :

$$-\int_0^L k(\omega' + \phi)\overline{W'}\,dx = \int_0^L G_1\overline{W}\,dx$$

$$-\int_0^L (\mu\phi' + D\,\psi')\overline{\Phi'}\,dx - \int_0^L k(\omega' + \phi)\overline{\Phi}\,dx = \int_0^L G_2\overline{\Phi}\,dx$$

Add to get :

$$\int_0^L [k(\omega' + \phi)(\overline{W' + \Phi}) + \mu\phi'\overline{\Phi'}]dx = -\int_0^L [D\,f_2'\overline{\Phi'} + G_1\overline{W} + G_2\overline{\Phi}]dx \tag{3.18}$$

Now, change the form into : $a\Big((\omega, \phi), (W, \Phi)\Big) = L(W, \Phi)$ and use Lax - Miligram lemma to prove (3.18) admits a unique solution $(\omega, \phi) \in V$.

$$a\Big((\omega, \phi), (W, \Phi)\Big) = \int_0^L [k(\omega' + \phi)(\overline{W' + \Phi}) + \mu\phi'\overline{\Phi'}]dx$$

$$L(W, \Phi) = -\int_0^L [G_1\overline{W} + G_2\overline{\Phi} + D\,f_2'\overline{\Phi'}]dx$$

Indeed, we have :

$$\begin{aligned}
|L(W, \Phi)| &\leq \|G_1\|_{L^2}\|W\|_{L^2} + \|G_2\|_{L^2}\|\Phi\|_{L^2} + \|D\|_{L^2}\|f_2'\|_{L^2}\|\Phi'\|_{L^2} \\
&\leq c_1\|W\|_{L^2} + c_2\|\Phi\|_{L^2} + c_3\|\Phi'\|_{L^2} \\
&\leq c(\|W'\|_{L^2} + \|\Phi'\|_{L^2}) \qquad \text{(poincare's inequality)}
\end{aligned}$$

where, $\mathrm{c} = max\{c_1, c_2, c_3\}$

$$\begin{aligned}
&\leq c(\|W\|_{H_0^1} + \|\Phi\|_{H_0^1}) \\
&= c\,\|(W, \Phi)\|_{(H_0^1)^2}
\end{aligned}$$

Hence, L(. , .) is continuous over V.

$$\begin{aligned}
\left|a\Big((\omega, \phi), (W, \Phi)\Big)\right| &= \left|\int_0^L [k(\omega' + \phi)(\overline{W' + \Phi}) + \mu\phi'\overline{\Phi'}]dx\right| \\
&= \left|\Big((\omega, \phi), (W, \Phi)\Big)_V\right| \\
&\leq \|(\omega, \phi)\|_V\|(W, \Phi)\|_V
\end{aligned}$$

Hence, a(. , .) is continuous over V$\times V$.

$$\begin{aligned}\left|a\Big((\omega,\phi),(\omega,\phi)\Big)\right| &= \int_0^L k|\omega'+\phi|^2+\mu\,|\phi'|^2dx\\ &= \|(\omega,\phi)\|_V^2\end{aligned}$$

Hence, a(. , .) is coercive.

We thus proved that U $=(\omega,\phi,v,\psi)\in D(\mathcal{A})$ and is a unique solution of $\mathcal{A}U=f$. Hence $\mathcal{A}$ is invertible. Therefore, $0\in\rho(\mathcal{A})$. This is enough to show that there exists $\lambda>0$ small enough such that $R(\lambda I-\mathcal{A})=\mathcal{H}$ [5]. □

3.2 Strong stability

Notice that $D(\mathcal{A})$ is not compactly embedded in $\mathcal{H}$ and thus the resolvent $\mathcal{A}^{-1}$ is not compact. Consequently, many methods that hold for operators with compact resolvent such as LaSalle's principle or spectral decomposition cannot be applied here. In what follows we use the Arendt - Batty criteria stated in the previous chapter in order to show that our C_0 semigroup of contractions is strongly stable. More precisely, we will show that $\mathcal{A}$ does not admit pure imaginary eigenvalues and that $\sigma(\mathrm{A})\cap i\mathbb{R}$ is countable.

Theorem 3.2.1. Assume that D satisfies the conditions of the previous theorem. Then, for all $0\neq\lambda\in\mathbb{R}$, $i\lambda\notin\sigma_d(\mathcal{A})$

Proof. Let us prove that $i\lambda I-\mathcal{A}$ injective. Let $U=(\omega,\phi,v,\psi)\in D(\mathcal{A})$ and suppose that $U\in Ker(i\lambda I-\mathcal{A})$ i.e. $\mathcal{A}U=i\lambda U$, that gives $Re(\mathcal{A}U,U)=Re(i\lambda U,U)=Re(i\lambda\|U\|^2)=0$.

Now we have,

$$Re(\mathcal{A}U,U)=-\int_\ell^L\delta\,|\psi'|^2dx=0,$$

implies,

$\psi'=0$ in $[\ell,L]$ then, ψ is constant, and as $\psi(L)=0$, we get $\psi=0$ in $[\ell,L]$.

$$i\lambda U=\mathcal{A}U$$

Equivalently,

$$\begin{cases} i\lambda\phi=\psi\\ i\lambda\omega=v\\ i\lambda v=\rho_1^{-1}k(\omega'+\phi)'\\ i\lambda\psi=\rho_2^{-1}[(\mu\phi'+D\psi')'-k(\omega'+\phi)]\end{cases}\implies\begin{cases}\phi=0\quad in\,[\ell,L]\\ i\lambda\omega=v\\ -\lambda^2\omega=\rho_1^{-1}k\,\omega''\;\; in\;[\ell,L]\\ \lambda^2\phi=\rho_2^{-1}k\,\omega'\;\; in\;[\ell,L]\end{cases}$$

$\phi=0$ in $[\ell,L]$, implies $\omega'=0$ in $[\ell,L]$, that yields to $v=0$ in $[\ell,L]$.

Now, as we have

$$D=0\;in\,[0,\ell]\;\;and\;\;\psi'=0\;in\,[\ell,L]$$

we obtain,

$$\begin{cases}\lambda^2\omega+\rho_1^{-1}k(\omega'+\phi)'=0 & in\;[0,L]\\ \\ \lambda^2\phi+\rho_2^{-1}[\mu\phi''-k(\omega'+\phi)]=0 & in\;[0,L]\end{cases}\tag{3.19}$$

By the 1st equation of (3.19) we have,

$$\lambda^2\,\omega + \rho_1^{-1}\,k\,(\omega' + \phi)' = 0$$

which implies,

$$\lambda^2\,\omega + \rho_1^{-1}\,k\,\omega'' + \rho_1^{-1}\,k\,\phi' = 0$$

deriving with respect to x gives,

$$\lambda^2\,\omega' + \rho_1^{-1}\,k\,\omega^{(3)} + \rho_1^{-1}\,k\,\phi^{(2)} = 0 \tag{3.20}$$

2nd equation of (3.19) implies,

$$\omega' = \big(\frac{\lambda^2\,\rho_2}{k} - 1\big)\,\phi + \frac{\mu}{k}\,\phi^{(2)}$$

thus, substituting ω' in (3.19) implies,

$$\frac{\mu}{\lambda^2\,\rho_1}\,\phi^{(4)} + \big(\frac{\mu}{k} + \frac{\rho_2}{\rho_1}\big)\,\phi^{(2)} + \big(\frac{\lambda^2\,\rho_2}{k} - 1\big)\,\phi = 0 \tag{3.21}$$

Now, due to Holder's embeddings and (3.19) we get that the solution ϕ is of calss $C^\infty([0,L])$ and of the form $\phi = c_1\,u_1 + c_2\,u_2 + c_3\,u_3 + c_4\,u_4$ with, u_1, u_2, u_3 and u_4 are linearly independent, then, $c_1 = c_2 = c_3 = c_4 = 0$ and thus $\phi = 0$ in [0 , L].
Hence, using the 2nd equation of (3.19) we get, $\omega = 0$. Therefore, $\omega = v = \phi = \psi = 0$ in $[0,L]$. Hence, $Ker(i\lambda I - \mathcal{A}) = \{0\}$ i.e $i\lambda \notin \sigma_d(\mathcal{A})$ for all $0 \neq \lambda \in \mathbb{R}$ i.e. $\sigma_d(A) \cap i\mathbb{R} = \emptyset$. □

Theorem 3.2.2. Let D be a continuous, nonnegative function that satisfies (3.10). Then, for all $0 \neq \lambda \in \mathbb{R}$, $i\lambda I - \mathcal{A}$ is surjective.

Proof. Let $G = (f_1, f_2, g_1, g_2) \in \mathcal{H}$, and solve the equation :

$$i\lambda U - \mathcal{A}U = G, \qquad U = (\omega, \phi, v, \psi)\ \in D(\mathcal{A}). \tag{3.22}$$

More precisely we have :

$$\begin{cases} v = i\lambda\omega - f_1 \\ \psi = i\lambda\phi - f_2 \\ \lambda^2\omega + \rho_1^{-1}[k(\omega' + \phi)'] = G_1 \\ \lambda^2\phi + \rho_2^{-1}[((\mu + i\lambda\,D)\phi')' - k(\omega' + \phi)] = G_2 \end{cases} \tag{3.23}$$

with

$$\begin{aligned} G_1 &= -(g_1 + i\lambda f_1) \in H^{-1}(0,L) \\ G_2 &= -(g_2 + i\lambda f_2) + \rho_2^{-1}(Df_2')' \in H^{-1}(0,L) \end{aligned}$$

Define

$$\begin{aligned} A : H_0^1 \times H_0^1 &\longrightarrow H^{-1} \times H^{-1} \\ (\omega,\phi) &\longrightarrow A(\omega,\phi) \end{aligned}$$

where, $A(\omega,\phi) = -\big(\rho_1^{-1}k(\omega'+\phi)'\,,\ \rho_2^{-1}[((\mu + i\lambda\,D)\phi')' - k(\omega'+\phi)]\big)$ and $\mathcal{D}(A) = V$, so the last two equations of (3.23) can be written as,

$$\lambda^2\,(\omega,\phi) - A\,(\omega,\phi) = (G_1, G_2)$$

Let us show that A is bijective. Indeed, for all $(F_1, F_2) \in H^{-1}(\Omega) \times H^{-1}(\Omega)$, we have

$$\begin{aligned} \left\langle A \begin{pmatrix} \omega \\ \phi \end{pmatrix}, \begin{pmatrix} \omega \\ \phi \end{pmatrix} \right\rangle_{H^{-1}, H_0^1} &= \left\langle \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \begin{pmatrix} \omega \\ \phi \end{pmatrix} \right\rangle_{H^{-1}, H_0^1} \\ \left\langle \begin{pmatrix} -\rho_1^{-1} k(\omega' + \phi)' \\ -\rho_2^{-1}[((\mu + i\lambda\, D)\phi')' - k(\omega' + \phi)] \end{pmatrix}, \begin{pmatrix} \omega \\ \phi \end{pmatrix} \right\rangle_{H^{-1}, H_0^1} &= \left\langle \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \begin{pmatrix} \omega \\ \phi \end{pmatrix} \right\rangle_{H^{-1}, H_0^1} \end{aligned} \tag{3.24}$$

We know that in the sense of distribution,

$$\langle \omega', \phi \rangle = -\langle \omega, \phi' \rangle_{L^2, L^2} = -\int_0^L \omega\, \overline{\phi'} dx$$

hence, we can directly write (3.24) as

$$\int_0^L \rho_1^{-1} k(\omega' + \phi)\overline{\omega'}\, dx + \int_0^L \rho_2^{-1}(\mu + i\lambda\, D)\phi'\, \overline{\phi'} dx + \int_0^L k\rho_2^{-1}(\omega' + \phi)\overline{\phi}\, dx$$

$$= \langle F_1, \omega \rangle_{H^{-1}, H_0^1} + \langle F_2, \phi \rangle_{H^{-1}, H_0^1}$$

Let,

$$-k(\omega' + \phi)' = \rho_1 F_1 \tag{3.25}$$

$$-[((\mu + i\lambda\, D)\phi')' - k(\omega' + \phi)] = \rho_2 F_2 \tag{3.26}$$

Proceeding exactly as in theorem 3.1.1, we obtain :

$$\int_0^L k(\omega' + \phi)(\overline{W' + \Phi})\, dx + \int_0^L (\mu + i\lambda\, D)\phi'\, \overline{\Phi'} dx = \int_0^L \rho_1\, F_1\, \overline{W}\, dx + \int_0^L \rho_2\, F_2\, \overline{\Phi}\, dx$$

by Lax - Milgram, we show that A is an isomorphism from V onto $H^{-1}(0, L) \times H^{-1}(0, L)$. Indeed, let

$$a\Big((\omega, \phi), (W, \Phi)\Big) = \int_0^L k(\omega' + \phi)(\overline{W' + \Phi})\, dx + \int_0^L (\mu + i\lambda\, D)\phi'\, \overline{\Phi'} dx$$

and

$$L(W, \Phi) = \langle F_1, W \rangle + \langle F_2, \Phi \rangle$$

From the definition of L, we have $L(.,.)$ is linear and continuous over V, and moreover

$$\Big| a\Big((\omega, \phi), (W, \Phi)\Big) \Big| \leq |k| \|\omega' + \phi\|_{L^2} \|W' + \Phi\|_{L^2} + |\mu + i\,\lambda\, D| \|\phi'\|_{L^2} \|\Phi'\|_{L^2}$$

which by poincare's inequality gives

$$\leq c_1 \|\omega' + \phi'\|_{L^2} \|W' + \Phi'\|_{L^2} + c_2 \|\phi'\|_{L^2} \|\Phi'\|_{L^2}$$

$$\leq c\big(\|\omega'\|_{L^2} + \|\phi'\|_{L^2}\big)\big(\|W'\|_{L^2} + \|\Phi'\|_{L^2}\big)$$

where, $\mathrm{c} = max\{c_1, c_2\}$

$$= c\, \|(\omega, \phi)\|_{(H_0^1)^2} \|(W, \Phi)\|_{(H_0^1)^2}$$

Hence, a(. , .) is continuous over V×V

$$
\begin{aligned}
Re\left(a\Big((\omega,\phi),(\omega,\phi)\Big)\right) &= Re\left(\int_0^L k|\omega'+\phi|^2+(\mu+i\lambda D)\,|\phi'|^2\,dx\right)\\
&= \int_0^L k|\omega'+\phi|^2+\mu\,|\phi'|^2\,dx\\
&= \|(\omega,\phi)\|_V^2.
\end{aligned}
$$

Therefore, a(. , .) is coercive.

Thus, A is invertible.

We transform last two equations of (3.23) into:

$$\lambda^2 A^{-1}(\omega,\phi)-(\omega,\phi)=A^{-1}\,(G_1,G_2) \tag{3.27}$$

$A^{-1}: H^{-1}\times H^{-1}\longrightarrow H_0^1\times H_0^1$ is continuous and $H_0^1\times H_0^1$ is compactly embedded in $L^2\times L^2$ then, $A^{-1}: H^{-1}\times H^{-1}\longrightarrow L^2\times L^2$ is compact. Therefore due to Fredholm's alternative $\forall\ \lambda\neq 0$, $\frac{1}{\lambda^2}\in\sigma_d(A^{-1})$ *or* $\frac{1}{\lambda^2}\in\rho(A^{-1})$, so to prove $\frac{1}{\lambda^2}\in\rho(A^{-1})$ it is enough to prove that $\frac{1}{\lambda^2}\notin\sigma_d(A^{-1})$ i.e $\lambda^2A^{-1}-I$ is injective. But, $\lambda^2A^{-1}(\omega,\phi)-(\omega,\phi)=0$ if and only if $\lambda^2(\omega,\phi)-A\,(\omega,\phi)=0$. It follows that

$$\begin{cases}\lambda^2\omega+\rho_1^{-1}[k(\omega'+\phi)']=0\\ \lambda^2\phi+\rho_2^{-1}[((\mu+i\lambda\,D)\phi')'-k(\omega'+\phi)]=0\end{cases} \tag{3.28}$$

Once again, let U $=\begin{pmatrix}\omega\\ \phi\\ v\\ \psi\end{pmatrix}$ with, $v=i\lambda\omega$ and $\psi=i\lambda\phi$, then,

(3.28) is equivalent to

$$i\lambda U-AU=0$$

i.e.,

$$i\lambda\begin{pmatrix}\omega\\ \phi\\ v\\ \psi\end{pmatrix}=A\begin{pmatrix}\omega\\ \phi\\ i\lambda\omega\\ i\lambda\phi\end{pmatrix}$$

By (3.15) we have that

$$Re(AU,U)_{\mathcal{H}}=-\int_0^L D\,|\psi'|^2dx$$

as $\psi=i\lambda\phi$, we get

$$Re(AU,U)=-\int_0^L \lambda^2\,D\,|\phi'|^2dx$$

but,

$$Re(AU,U)=Re(i\lambda U,U)=Re(i\lambda\|U\|^2)=0$$

then,

$$-\int_0^L\lambda^2D\,|\phi'|^2dx=0\ \textit{leads to},\ \ \phi'=0\ \textit{hence},\ \ \phi=0\ \textit{in}\,[\ell,L]$$

By (3.28) we get $\omega' = 0$ which gives $\omega = 0$ in $[\ell, L]$ Hence, $\omega = \phi = 0$ in $[\ell, L]$. Now, over $[0, \ell]$ we have, $D = 0$, then (3.28) will be transformed to (3.19) in theorem 3.2.1 that implies, $\omega = \phi = 0$ in $[0, L]$, therefore, $Ker(\lambda^2 A^{-1} - I) = \{0\}$. Since A^{-1} is compact, and using Fredholm's alternative we deduce that (3.28) possess only one solution $(\omega, \phi) \in V$. $Ker(\lambda^2 A^{-1} - I) = \{0\}$ i.e. $\dfrac{1}{\lambda^2} \notin \sigma_d(A^{-1})$ then, $\dfrac{1}{\lambda^2} \in \rho(A^{-1})$ hence, $\lambda^2 A^{-1} - I$ is surjective. Thus, there exists (ω, ϕ) satisfying the last two equations of (3.23). Hence, (3.27) gives (3.23) which is equivalent to (3.22) that yields to $i\lambda I - \mathcal{A}$ is surjective, i.e. $R(i\lambda I - \mathcal{A}) = \mathcal{H}$. As $Ker(i\lambda I - \mathcal{A}) = \{0\}$, we coclude by closed graph theorem that no pure imaginary $i\,\lambda$ is in $\sigma(\mathcal{A})$. □

As a result of the previous theorem, we conclude that if $D \geq 0$ is continuous and satisfies (3.20), then $\mathcal{A}$ generates a strongly continuous semigroup. Thus $U(t) = e^{t\,\mathcal{A}} U_0$ is a solution of our evolution problem. We have proved the existence of solution for $(3.18) - (3.19)$ and in addition we proved that $i\mathbb{R} \subset \rho(\mathcal{A})$ which will benefit us in the discussion of both Strong stability and the proof of exponential stability.

Corollary 3.2.3. If D is continuous, non-negative and satisfies 3.10. Then the system is strongly stable.

Proposition 3.2.4. Assume coefficient functions $D_1 \geq 0$ and $D_2 \geq 0$ are continuous verifying (3.4). Then $\mathcal{A}$ generates a strongly continuous semigroup of contractions $e^{t\mathcal{A}}$ on $\mathcal{H}$, and $i\mathbb{R} \subset \rho(\mathcal{A})$.

Proof. (See [9]) □

Chapter 4

Stability type and regularity of the solution.

In this chapter we discuss the type of stability of system (3.1) with two dissipation sources D_1 and D_2 as defined in (3.4). We first introduce the following hypothesis on D_1 and D_2 :

(H1) $\delta(.)$ is continuous on $[\ell, L]$ and $c_1 = \sup\limits_{x \in (\ell,L)} \big[\delta(x) \int\limits_x^L \frac{ds}{\delta(s)}\big] < \infty$

(H2) $\delta(.)$ is continuously differentiable on $[\ell, L]$ and $c_2 = \sup\limits_{x \in (\ell,L)} \big(\int\limits_\ell^x \left|\delta'(s)\right|^2 ds\big)\big(\int\limits_x^L \frac{ds}{\delta(s)}\big) < \infty$

(H3) $\delta(.)$ is continuously differentiable on $[\ell, L]$ and $\exists\, c_3 > 0$ such that

$$\int\limits_\ell^x \frac{\left|\delta'(s)\right|^2}{\delta(s)} ds \le c_3 \left|\delta'(x)\right|, \qquad \forall\, x \in (\ell, L).$$

(H4) $D(.)$ is continuously differentiable on $[0, L]$ and $\exists\, c_4$ such that $D(.) \ge c_4$.

Lemma 4.0.1. Assume that $0 < \delta(.) \in C^1([\ell, L])$ and that $y(.) \in L^2(\ell, L)$, satisfies $\delta^{\frac{1}{2}}\, y' \in L^2(\ell, L)$ *and* $y(L) = 0$. Let

$$\sigma(.) = \begin{cases} \delta^{\frac{1}{2}}(.) & if\ \delta(.)\ satisfies\ (H1); \\ \delta'(.) & if\ \delta(.) \in C^1([\ell, L])\ satisfies\ (H2); \\ \delta'(.)\, \delta^{-\frac{1}{2}} & if\ \delta(.) \in C^1([\ell, L])\ satisfies\ (H3); \end{cases} \tag{4.1}$$

Then, there exists $C > 0$ such that :

$$\|\sigma\, y\|_{L^2(\ell,L)} \le C\, \|\delta^{\frac{1}{2}}\, y'\|_{L^2(\ell,L)}. \tag{4.2}$$

Proof. Using Cauchy-Schwarz inequality, we get

$$\left|y(x)\right| \le \int\limits_x^L \left|y'(s)\right| \delta^{\frac{1}{2}}(s)\, \delta^{-\frac{1}{2}}(s)\, ds \le \Big(\int\limits_x^L \frac{ds}{\delta(s)}\Big)^{\frac{1}{2}} \Big(\int\limits_x^L \delta(s) \left|y'(s)\right|^2 ds\Big)^{\frac{1}{2}} \tag{4.3}$$

Thus, substituting (H1) in (4.3) gives

$$\|\delta^{\frac{1}{2}}\, y\|^2_{L^2(\ell,L)} \le c_1 \int\limits_\ell^L \int\limits_x^L \delta(s) \left|y'(s)\right|^2 ds\, dx \le c_1 (L - \ell)\, \|\delta^{\frac{1}{2}}\, y'\|^2_{L^2(\ell,L)} \tag{4.4}$$

If $\delta(.)$ satisfy $(H2)$, take

$h_1(s) = |\delta'(L - s(L-\ell))|^2$, $h_2(s) = \delta(L - s(L-\ell))$ and $g(s) = y'(L - s(L-\ell))$ for $0 < s < 1$.

Take,

$$r = L - s(L-\ell),$$

then

$$\ell < r < L$$

and

$$(1.1) \iff \sup_{x \in (\ell,L)} \Big(\int_{\ell}^{x} \left|\delta'(s)\right|^2 ds \Big) \Big(\int_{x}^{L} \frac{ds}{\delta(s)} \Big) < \infty.$$

thus,

$$(1.2) \iff \int_{\ell}^{L} |\delta'(t)|^2 \, |y(t)|^2 \, dt \leq C \int_{\ell}^{L} \delta(t) \, |y'(t)|^2 \, dt.$$

Therefore,

$$\|\delta' \, y\|_{L^2(\ell,L)} \leq C \, \|\delta^{\frac{1}{2}} \, y'\|_{L^2(\ell,L)} \text{ (see [4], [7])}.$$

If $\delta(.)$ satisfy $(H3)$. Let $\Phi(x) = \displaystyle\int_{\ell}^{x} \frac{\left|\delta'(s)\right|^2}{\delta(s)} ds$. Then, $(H3)$ yields to :

$$\frac{\left|\Phi(x)\right|^2}{\delta(x)} \leq c_3^2 \, \frac{\left|\delta'(x)\right|^2}{\delta(x)} = c_3^2 \, \Phi'(x).$$

Now,

$$\begin{aligned}
\|\delta' \, \delta^{-\frac{1}{2}} \, y\|^2_{L^2(\ell,L)} &= \int_{\ell}^{L} \frac{\left|\delta'(s)\right|^2}{\delta(s)} \left|y(s)\right|^2 ds \\
&= \int_{\ell}^{L} \Phi'(x) \left|y(x)\right|^2 dx \\
&= -2 \, Re \int_{\ell}^{L} \Phi(x) \, y(x) \, \overline{y'(x)} \, dx \\
&= -2 \, Re \int_{\ell}^{L} \Phi(x) \, y(x) \, \delta^{-\frac{1}{2}} \, \delta^{\frac{1}{2}} \, \overline{y'(x)} \, dx \\
&\leq \frac{1}{2 \, c_3^2} \int_{\ell}^{L} \frac{\left|\Phi(x)\right|^2}{\delta(x)} \left|y(x)\right|^2 dx + \frac{c_3^2}{2} \int_{\ell}^{L} \delta(x) \left|y'(x)\right|^2 dx \\
&\leq \frac{1}{2} \int_{\ell}^{L} \Phi'(x) \left|y(x)\right|^2 dx + \frac{c_3^2}{2} \int_{\ell}^{L} \delta(x) \left|y'(x)\right|^2 dx
\end{aligned}$$

Therefore,

$$\|\delta' \, \delta^{-\frac{1}{2}} \, y\|^2_{L^2(\ell,L)} \leq C \, \|\delta^{\frac{1}{2}} \, y'\|^2_{L^2(\ell,L)} \text{ (see [8])}.$$

$\square$

The following theorem offers the final result on what type of stability is followed in each case.

Theorem 4.0.2. Assume coefficient functions $D_1(.)$ and $D_2(.) \in C[0, L]$, nonnegative, and satisfy (3.4). Then:

(i) If $\delta_1(.)$ and $\delta_2(.)$ satisfy (H1) and $\exists\, C > 0$ such that

$$\delta_1(x) \leq C\, \delta_2(x), \quad \forall\, x \in [\ell_1, L]. \tag{4.5}$$

Then, $e^{t\mathcal{A}}$ is polynomially stable of decay rate 1.

(ii) If $\delta_1(.)$ and $\delta_2(.)$ satisfy (H2), (4.5) and $\delta_1(\ell_1) = \delta_2(\ell_2) = 0$. Then, $e^{t\mathcal{A}}$ is polynomially stable of decay rate $\dfrac{3}{2}$.

(iii) If $\delta_1(.)$ and $\delta_2(.)$ satisfy (H3), (4.5) and $\delta_1(\ell_1) = \delta_2(\ell_2) = 0$. Then, $e^{t\mathcal{A}}$ is exponentially stable.

(iv) If $D_1(.)$ and $D_2(.)$ satisfy (H4). Then, $e^{t\mathcal{A}}$ is analytic.

We shall explain here the main results concerned with the types of full-filled stability. Let us first recall the following lemma.

Lemma 4.0.3. Let $A : \mathcal{D}(A) \subset H \longrightarrow H$ be a generator of a bounded C_0 - semigroup $e^{t\,A}$ on H. Suppose

$$i\lambda \in \rho(A), \quad \forall\, \lambda \in \mathbb{R}. \tag{4.6}$$

Then :

(i) $e^{t\,A}$ decays polynomially of order $\dfrac{1}{\gamma}$ if and only if

$$\overline{\lim_{\lambda \in \mathbb{R}, |\lambda| \to \infty}}\; |\lambda|^{-\gamma}\, \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{4.7}$$

(ii) $e^{t\,A}$ decays exponentially if and only if

$$\overline{\lim_{\lambda \in \mathbb{R}, |\lambda| \to \infty}}\; \|(i\lambda I - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{4.8}$$

(iii) Semigroup $e^{t\,A}$ is analytic if and only if

$$\overline{\lim_{\lambda \in \mathbb{R}, |\lambda| \to \infty}}\; |\lambda|\, \|(i\lambda - A)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty \tag{4.9}$$

In the following theorems, we intend to prove some estimations on the resolvent in order to conclude what type of stability is obeyed in the considered cases. Indeed we will combine the results of theorem 4.0.4 and lemma 4.0.3 to prove this.

Theorem 4.0.4. *Let*

$$\gamma = \begin{cases} 1 & if \;\; \delta_i \;\; satisfies\,(H1) \; and \; (4.5), \\ \\ \frac{2}{3} & if \;\; \delta_i \;\; satisfies\,(H2), \; (4.5) \; and \; \delta_i(\ell_i) = 0, \\ \\ 0 & if \;\; \delta_i \;\; satisfies\,(H3), \; (4.5) \; and \; \delta_i(\ell_i) = 0, \\ \\ -1 & if \;\; \delta_i \;\; satisfies\,(H4), \end{cases} \tag{4.10}$$

for $i = 1, 2$. Then

$$\overline{\lim_{\lambda \in \mathbb{R}, |\lambda| \to \infty}} \; |\lambda|^{-\gamma} \, \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \; < \; \infty \tag{4.11}$$

We will prove theorem 4.0.4 by contradiction. Suppose for some given γ, (4.11) is not true.Then, as

$$\overline{\lim_{|\lambda| \to \infty}} \; |\lambda|^{-\gamma} \, \|(i\lambda I - \mathcal{A}^{-1}\|_{\mathcal{L}(\mathcal{H})} = \overline{\lim_{|\lambda| \to \infty}} \; |\lambda|^{-\gamma} \sup_{u \in \mathcal{H}} \frac{\|\left(i\lambda I - \mathcal{A}\right)^{-1} u\|}{\|u\|}$$
$$= \overline{\lim_{|\lambda| \to \infty}} \; |\lambda|^{-\gamma} \sup_{v \in \mathcal{D}(A)} \frac{\|v\|}{\|\left(i\lambda I - \mathcal{A}\right) v\|} \; \geq \; \frac{\|U_n\|}{\lambda_n^{\gamma} \, \|\left(i\lambda_n I - \mathcal{A}\right) U_n\|} \; \to \infty \text{ as n} \to \infty$$

i.e., there exists $\{(\lambda_n, U_n) \;\; : \; n \; \geq \; 1\} \; \subset \; \mathbb{R} \times D(\mathcal{A})$ with λ_n large enough, say $\lambda_n > 1$, and

$$\begin{cases} \lim\limits_{n \to \infty} \lambda_n \; = \infty \\ \\ \|U_n\|^2_{\mathcal{H}} = \|(\omega_n, \phi_n)\|^2_V + \|(v_n, \psi_n)\|^2_H = 1, \quad n \geq 1, \end{cases} \tag{4.12}$$

such that

$$\lim_{n \to \infty} \lambda_n^{\gamma} \, \|i\lambda_n \, U_n - \mathcal{A} \, U_n\|_{\mathcal{H}} = 0 \tag{4.13}$$

i.e.,

$$(f_{1n} \, , \, f_{2n}) \doteq \lambda_n^{\gamma} \, (i\lambda_n \, \omega_n - v_n \;\; , \;\; i\lambda_n \, \phi_n - \psi_n) = o(1) \; in \; V, \tag{4.14}$$
$$(g_{1n} \, , \, g_{2n}) \doteq \lambda_n^{\gamma} \, (i\lambda_n \, v_n - \rho_1^{-1} \, T_n' \;\; , \;\; i\lambda_n \, \psi_n - \rho_2^{-1}(R_n' - T_n)) = o(1) \; in \; H, \tag{4.15}$$

where $o(1) \to 0$ in $\mathbb{R}$ as $n \to \infty$, and

$$T_n = k \, (\omega_n' + \phi_n) + D_1 \, (v_n' + \psi_n) \; , \; R_n = \mu \, \phi_n' + D_2 \, \psi_n'.$$

Furthermore, it follows from (4.13) that :

$$\lim_{n \to \infty} \lambda_n^{\gamma} \, Re \, ((i\lambda_n \, I - \mathcal{A}) U_n \, , \, U_n)_{\mathcal{H}} = 0.$$

Indeed,

$$\left| Re \, ((i\lambda_n \, I - \mathcal{A}) U_n \, , \, U_n) \right| \; \leq \; \left| ((i\lambda_n \, I - \mathcal{A}) \, U_n \;\; , \;\; U_n) \right| \; \leq \; \|(i\lambda_n \, I - \mathcal{A}) \, U_n\| \, \|U_n\|$$

but

$0 \le \left|\lambda_n^\gamma \, Re((i\lambda_n \, I - \mathcal{A}) \, U_n \, , \, U_n)\right| \le \lambda_n^\gamma \|(i\lambda_n \, I - \mathcal{A}) \, U_n\| \to 0$

Thus, due to (3.15) and (4.14), as

$$\lim_{n\to\infty} \lambda_n^\gamma \, Re \left((i\lambda_n \, I - \mathcal{A})U_n \, , \, U_n\right)_{\mathcal{H}} = 0.$$

then,

$$\lambda_n^\gamma \, Re(i\lambda_n \, U_n \, , \, U_n) + \lambda_n^\gamma \, Re(-\mathcal{A} \, U_n \, , \, U_n) = o(1)$$

i.e.,

$$\lambda_n \, Re\left(i\lambda_n \, \|U_n\|^2\right) + \lambda_n^\gamma \int_0^L (D_1 \, |v_n' + \psi_n|^2 + D_2 \, |\psi_n'|^2) dx = o(1)$$

hence,

$$\lambda_n^\gamma \, \|D_1^{\frac{1}{2}} \, (v_n' + \psi_n)\|^2 + \lambda_n^\gamma \|D_2^{\frac{1}{2}} \, \psi_n'\|^2 = o(1)$$

therefore,

$$\lambda_n^{\frac{\gamma}{2}} \, \|D_2^{\frac{1}{2}} \, \psi_n'\| = o(1) \;\; and \;\; \lambda_n^{\frac{\gamma}{2}} \, \|D_1^{\frac{1}{2}} \, (v_n' + \psi_n)\| = o(1)$$

Using (4.14) we substitute

$$\psi_n = i\lambda_n \phi + o(1) \;\; in \;\; \lambda_n^{\frac{\gamma}{2}} \, \|D_2^{\frac{1}{2}} \, \psi_n'\| = o(1)$$

to get

$$\lambda_n^{1+\frac{\gamma}{2}} \, \|D_2^{\frac{1}{2}} \, \phi_n'\| = o(1)$$

and

$$v_n' = i \, \lambda_n \, \omega_n' + o(1) \;\; in \;\; \lambda_n^{\frac{\gamma}{2}} \, \|D_1^{\frac{1}{2}} \, (v_n' + \psi_n)\| = o(1)$$

to get

$$\lambda_n^{1+\frac{\gamma}{2}} \, \|D_1^{\frac{1}{2}} \, (\omega_n' + \phi_n)\| = o(1)$$

Therefore,

$$\lambda_n^{\frac{\gamma}{2}} \, \|D_2^{\frac{1}{2}} \, \psi_n'\| = \lambda_n^{\frac{\gamma}{2}} \, \|D_1^{\frac{1}{2}} \, (v_n' + \psi_n)\| = o(1) \tag{4.16}$$

$$\lambda_n^{1+\frac{\gamma}{2}} \, \|D_2^{\frac{1}{2}} \, \phi_n'\| = \lambda_n^{1+\frac{\gamma}{2}} \, \|D_1^{\frac{1}{2}} \, (\omega_n' + \phi_n)\| = o(1) \tag{4.17}$$

Particularly, when $\gamma \ge 0$, we also have

$$\|T_n\| = k \, \|\omega_n' + \phi_n\| + o(1), \tag{4.18}$$

$$\|R_n\| = \mu \, \|\phi_n'\| + o(1). \tag{4.19}$$

For the sake of clarity, we will sub-divide our proofs into several lemmas and combine the result.

Lemma 4.0.5. Suppose that $\gamma \geq 0$. Let $p \in C^1([0,L])$ be real - valued. For the sequence $\{(\lambda_n, \omega_n, \phi_n, v_n, \psi_n)\}$ satisfying (4.12) - (4.13), we have

$$\rho_1 \int_0^L p\, v_n\, \overline{\psi_n}\, dx = k \int_0^L p\,(\omega_n' + \phi_n)\, \overline{\phi_n'}\, dx \ + o(1) \tag{4.20}$$

and

$$\begin{aligned} &\rho_1\, k \int_0^L p'\, |v_n|^2\, dx \ - \ p(L)\, |T_n(L)|^2 + p(0)\, |T_n(0)|^2 \ + \ k^2 \int_0^L p'\, |\omega_n' + \phi_n|^2\, dx \\ &+ \rho_2\, k \int_0^L p'\, |\psi_n|^2\, dx \ - \ k\, \mu^{-1}\, p(L)\, |R_n(L)|^2 + k\, \mu^{-1}\, p(0)\, |R_n(0)|^2 \ + \ k\, \mu \int_0^L p'\, |\phi_n'|^2\, dx \\ &\leq \ C\ \lambda_n \left(\|D_1^{\frac{1}{2}}\, v_n\|\, \|D_1^{\frac{1}{2}}(v_n' + \psi_n)\| + \|D_2^{\frac{1}{2}}\, \psi_n\|\, \|D_2^{\frac{1}{2}}\, \psi_n'\| \right) + o(1), \ \ C \ > \ 0. \end{aligned} \tag{4.21}$$

Proof. First, consider the inner product of (4.15) with $\lambda_n^{-\gamma}\,(p\,\phi_n, 0)$ on H, then:

$$\left\langle \left(i\lambda_n\, v_n - \rho_1^{-1} T_n'\ ,\ i\lambda_n\, \psi_n - \rho_2^{-1}(R_n' - T_n)\right)\ ,\ \left(p\,\phi_n\ ,\ 0\right) \right\rangle_H = o(1)$$

thus,

$$\int_0^L \rho_1 (i\lambda_n\, v_n - \rho_1^{-1} T_n')\, (\overline{p\,\phi_n})\, dx = o(1)$$

hence,

$$\rho_1 \int_0^L i\lambda_n v_n\, p\, \overline{\phi_n}\, dx - \int_0^L T_n'\, p\overline{\phi_n} = o(1).$$

by Green's formula we get,

$$\rho_1 \int_0^L i\lambda_n\, \overline{\phi_n}\, p\, v_n\, dx + \int_0^L T_n (p\, \overline{\phi_n})'\, dx = o(1).$$

Using (4.14) we have

$$\lambda_n^{\gamma}\, (i\lambda_n\, \phi_n - \psi_n) \to 0 \ \ where \ \ \gamma \geq 0$$

but

$$\lambda_n^{-\gamma} \longrightarrow 0 \ \ since \ \ \lambda_n \longrightarrow \infty \ \ and \ \ \gamma \ \geq \ 0$$

then, we get

$$\lambda_n^{-\gamma}\, \lambda_n^{\gamma} (i\lambda_n\, \phi_n - \psi_n) \to 0$$

which implies,

$$i\lambda_n\, \phi_n - \psi_n \to 0 \ \ i.e. \ \ i\lambda_n\, \phi_n = \psi_n + f_{2n} \ \ where \ \ f_{2n} \to 0 \ \ as \ \ n \ \to \infty.$$

Moreover, we have:

$$
\Big|\int_0^L p\,v_n\,\overline{f_{2n}}\Big| \leq \max_{x\in(0,L)} |p(x)|\,\|v_n\|_{L^2}\,\|\overline{f_{2n}}\|_{L^2} \leq \max_{x\in(0,L)} |p(x)|\,\|(v_n,\psi_n)\|_H\,\|\overline{f'_{2n}}\|_{L^2}
$$
$$
\leq \max_{x\in(0,L)} |p(x)|\,\|U_n\|\,\|(f_{1n},f_{2n})\|_V \;\to 0 \;\; as \;\; n\to\infty.
$$

Hence, as

$$
i\lambda_n\,\overline{\phi_n} = \overline{-\,i\lambda_n\,\phi_n} = \overline{-\psi_n - f_{2n}}.
$$

we get

$$
\rho_1\int_0^L i\lambda_n\,\overline{\phi_n}\,p\,v_n\,dx+\int_0^L T_n(p\,\overline{\phi_n})'\,dx \;=\; \rho_1\int_0^L -\overline{\psi_n}\,p\,v_n\,dx+\int_0^L p'\,T_n\,\overline{\phi_n}\,dx+\int_0^L T_n\,p\,\overline{\phi'_n}\,dx = o(1).
$$

Therefore, replacing T_n we get,

$$
-\rho_1\int_0^L p\,v_n\overline{\psi_n}\,dx+\int_0^L p'\,T_n\,\overline{\phi_n}\,dx+k\int_0^L p\,(\omega'_n+\phi_n)\overline{\phi'_n}\,dx+\int_0^L p\,D_1\,(v'_n+\psi_n)\,\overline{\phi'_n}\,dx = o(1). \quad (4.22)
$$

It follows from (4.16) and Cauchy-Schwarz inequality that

$$
\left|\int_0^L p\,D_1\,(v'_n+\psi_n)\,\overline{\phi'_n}\,dx\right| \leq \left[\max_{x\in(0,L)} |p(x)|\,D_1^{\frac{1}{2}}(x)\right]\lambda_n^{-\frac{\gamma}{2}}\,\lambda_n^{\frac{\gamma}{2}}\|D_1^{\frac{1}{2}}\,(v'_n+\psi_n)\|\,\|\phi'_n\| = o(1). \quad (4.23)
$$

Notice that, by (4.18),

$$
\|T_n\| = k\|\omega'_n+\phi_n\|+o(1) \leq \|(\omega_n,\phi_n)\|_V + c_1 \leq \|U_n\|+c_1 = 1+C
$$

for some positive constants c_1 and C

and

$$
\lim_{n\to\infty}\|\phi_n\|_{L^2(0,L)} = 0.
$$

Indeed, (4.14) implies ,

$$
\|(f_{1n},f_{2n})\|_V^2 = o(1),
$$

i.e.

$$
|\lambda_n^\gamma|^2\int_0^L \big(k|i\,\lambda_n\,\omega'_n - v'_n + i\,\lambda_n\,\phi_n - \psi_n|^2 + \mu|(i\,\lambda_n\,\phi_n-\psi_n)'|^2\big)\,dx \longrightarrow 0,
$$

then, as

$$
\lambda_n^\gamma\,(i\,\lambda_n\,\phi_n-\psi_n) = f_{2n}
$$

we get,

$$
\int_0^L \mu\,|f'_{2n}|^2 \;\longrightarrow 0
$$

so,

$$
\|f_{2n}\| \longrightarrow 0 \;\; in \;\; H_0^1
$$

thus, by poincare's inequality we get,

$$
f_{2n} \longrightarrow 0 \;\; in \;\; L^2
$$

so,

$$\|\lambda_n^{\gamma}\,(i\,\lambda_n\,\phi_n-\psi_n)\|_{L^2}=\|f_{2n}\|_{L^2}\longrightarrow 0$$

but as,

$$\lambda_n\to\infty\ \ and\ \ \gamma\geq 0,$$

we get

$$\lambda_n^{-\gamma}\to 0,$$

hence,

$$\|(i\,\lambda_n\,\phi_n-\psi_n)\|_{L^2}\longrightarrow 0.$$

Now, using triangular inequality we get,

$$\|i\,\lambda_n\,\phi_n\|_{L^2}\leq\|\lambda_n^{\gamma}\,(i\,\lambda_n\,\phi_n-\psi_n)\|_{L^2}+\|\psi_n\|_{L^2}$$

but, (4.12) implies,

$$|\lambda_n|\,\|\phi_n\|_{L^2}\leq o(1)+\mathcal{O}(1)$$

then,

$$\|\phi_n\|_{L^2}\leq\frac{o(1)}{|\lambda_n|}+\frac{\mathcal{O}(1)}{|\lambda_n|}\longrightarrow 0\ \ as\ \ n\ \to\ \infty.$$

that yields to

$$\Big|\int_0^L p'\,T_n\,\overline{\phi_n}\,dx\Big|\leq\max_{x\in(0,L)}|p'(x)|\,\|T_n\|\|\phi_n\|\leq\max_{x\in(0,L)}|p'(x)|\,C\,\|\phi_n\|\ \to 0$$

Consequently,

$$\int_0^L p'\,T_n\,\overline{\phi_n}\,dx=o(1)\tag{4.24}$$

Next, let $c\in\mathbb{R}$. Consider the inner product of (4.15) with $\lambda_n^{-\gamma}\,(p\,T_n\,,\,c\,p\,R_n)$ on H, then

$$\Big\langle\big(i\lambda_n\,v_n-\rho_1^{-1}\,T_n'\,,\,i\lambda_n\,\psi_n-\rho_2^{-1}\,(R_n'-T_n)\big)\,,\,\big(p\,T_n\,,\,c\,p\,R_n\big)\Big\rangle_H=o(1)$$

i.e.,

$$\int_0^L\rho_1\,[\,i\lambda_n\,v_n-\rho_1^{-1}\,T_n']\,[\overline{p\,T_n}\,]+\rho_2\,[\,i\lambda_n\,\psi_n-\rho_2^{-1}\,(R_n'-T_n)\,]\,[\overline{c\,p\,R_n}]\,dx=o(1)$$

then,

$$i\lambda_n\,\rho_1\int_0^L p\,v_n\,\overline{T_n}dx-\int_0^L T_n'\,p\,\overline{T_n}dx+\rho_2\,c\int_0^L i\lambda_n\,p\,\psi_n\,\overline{R_n}dx-c\int_0^L p\,(R_n'-T_n)\,\overline{R_n}dx=o(1)$$

that is,

$$i\lambda_n\,\rho_1\int_0^L p\,v_n\,\overline{T_n}\,dx-\int_0^L T_n'\,p\,\overline{T_n}\,dx+c\int_0^L\Big[\rho_2\,i\lambda_n\,p\,\psi_n\,\overline{R_n}-p\,R_n'\,\overline{R_n}+p\,T_n\,\overline{R_n}\Big]\,dx=o(1)$$

Now, we have,

$$-\int_0^L T_n'\,p\,\overline{T_n}\,dx=\int_0^L T_n\,p'\,\overline{T_n}\,dx+\int_0^L T_n\,p\,\overline{T_n'}\,dx-\Big[\,p\,|T_n|^2\,\Big]_0^L$$

i.e.

$$-\int_0^L T_n'\,p\,\overline{T_n}\,dx-\int_0^L T_n\,p\,\overline{T_n'}\,dx=\int_0^L T_n\,p'\,\overline{T_n}\,dx-\Big[\,p\,|T_n|^2\,\Big]_0^L$$

that gives,

$$-2\,Re\int_0^L T_n'\,p\,\overline{T_n}\,dx=\int_0^L T_n\,p'\,\overline{T_n}\,dx-p(L)\,|T_n(L)|^2+p(0)\,|T_n(0)|^2$$

then,

$$Re\int_0^L T_n'\,p\,\overline{T_n}\,dx=\frac{1}{2}\Big[p(L)\,|T_n(L)|^2-p(0)\,|T_n(0)|^2-\int_0^L p'\,|T_n|^2\Big] \tag{4.25}$$

Therefore,

$$\begin{aligned}&Re\left[i\lambda_n\,\rho_1\int_0^L p\,v_n\,\overline{T_n}\,dx\,\right]-\frac{p(L)}{2}\,|T_n(L)|^2+\frac{p(0)}{2}\,|T_n(0)|^2+\frac{1}{2}\int_0^L|T_n|^2\,p'\,dx\\&+\rho_2\,Re\left[i\lambda_n\int_0^L c\,p\,\psi_n\,\overline{R_n}\,dx\,\right]-\frac{c\,p(L)}{2}\,|R_n(L)|^2+\frac{c\,p(0)}{2}\,|R_n(0)|^2\\&+\frac{c}{2}\int_0^L|R_n|^2\,p'\,dx+Re\int_0^L c\,p\,T_n\,\overline{R_n}\,dx=o(1)\end{aligned} \tag{4.26}$$

Substituting (4.18) – (4.19) in (4.26), then applying Cauchy - Schwarz inequality gives

$$\begin{aligned}&Re\left[i\lambda_n\,\rho_1\int_0^L p\,v_n\,\overline{T_n}\,dx\,\right]-\frac{p(L)}{2}\,|T_n(L)|^2+\frac{p(0)}{2}\,|T_n(0)|^2+\frac{k^2}{2}\int_0^L|\omega_n'+\phi_n|^2\,p'\,dx\\&+\rho_2\,Re\left[i\lambda_n\int_0^L c\,p\,\psi_n\,\overline{R_n}\,dx\,\right]-\frac{c\,p(L)}{2}\,|R_n(L)|^2+\frac{c\,p(0)}{2}\,|R_n(0)|^2\\&+\frac{c\,\mu^2}{2}\int_0^L|\phi_n'|^2\,p'\,dx+c\,k\,\mu\,Re\int_0^L p\,(\omega_n'+\phi_n)\,\overline{\phi_n'}=o(1)\end{aligned} \tag{4.27}$$

Using (4.14), we have :

$$-i\lambda_n\,\overline{\omega_n'}=\overline{v_n'}+o(1)\;\; and \;\;-i\lambda_n\,\overline{\phi_n'}=\overline{\psi_n'}+o(1)$$

so, we get

$$\begin{aligned}&\mathrm{Re}\left[\mathrm{i}\lambda_n\,\rho_1\int_0^L p\,v_n\,\overline{T_n}\,dx+i\lambda_n\,\rho_2\int_0^L c\,p\,\psi_n\,\overline{R_n}dx\right]\\&=Re\left[i\lambda_n\,\rho_1\int_0^L p\,v_n\left(k\,(\overline{\omega_n'}+\overline{\phi_n})+D_1\,(\overline{v_n'+\psi_n})\right)dx\right]+Re\left[i\lambda_n\,\rho_2\int_0^L c\,p\,\psi_n\,(\,\mu\,\overline{\phi_n'}+D_2\,\overline{\psi_n'}\,)dx\right]\\&=Re\left[i\lambda_n\,\rho_1\int_0^L p\,k\,v_n\,(\overline{\omega_n'}+\overline{\phi_n})dx\right]+Re\left[i\lambda_n\,\rho_1\int_0^L p\,v_n\,D_1\,(\overline{v_n'+\psi_n})dx\right]\\&+Re\left[i\lambda_n\,\rho_2\int_0^L c\,p\,\psi_n\,\mu\,\overline{\phi_n'}\right]+Re\left[i\lambda_n\,\rho_2\int_0^L c\,p\,\psi_n\,D_2\,\overline{\psi_n'}dx\right]\\&=Re\left[-\rho_1\,k\int_0^L p\,v_n\,\overline{v_n'}\right]+Re\left[-\rho_1\,k\int_0^L p\,v_n\,\overline{\psi_n}\right]+Re\left[-\rho_2\,\mu\,c\int_0^L p\,\psi_n\,\overline{\psi_n'}\right]+\\&Re\left[i\lambda_n\,\rho_1\int_0^L p\,v_n\,D_1\,(\overline{v_n'+\psi_n})dx+i\lambda_n\,\rho_2\int_0^L c\,p\,\psi_n\,D_2\,\overline{\psi_n'}dx\right]\end{aligned}$$

$$= \text{-}\rho_1\, k\, Re \int_0^L p\, v_n\, \overline{v_n'} - \rho_1\, k\, Re \int_0^L p\, v_n\, \overline{\psi_n} - \rho_2\, \mu\, c\, Re \int_0^L p\, \psi_n\, \overline{\psi_n'}$$

$$+ Re\Big[i\lambda_n\, \rho_1 \int_0^L p\, v_n\, D_1\, (\overline{v_n' + \psi_n})dx + i\lambda_n\, \rho_2 \int_0^L c\, p\, \psi_n\, D_2\, \overline{\psi_n'}dx\Big]$$

but,

$$(v_n, \psi_n) \in V = H_0^1 \times H_0^1$$

then,

$$v_n(L) = v_n(0) = \psi_n(L) = \psi_n(0) = 0$$

and using (4.25) we conclude,

$$Re \int_0^L v_n\, p\, \overline{v_n'} = \frac{1}{2}\Big[p(L)\,|v_n(L)|^2 - p(0)\,|v_n(0)|^2 - \int_0^L p'\,|v_n|^2\Big] = -\frac{1}{2}\int_0^L p'\,|v_n|^2$$

$$Re \int_0^L \psi_n\, p\, \overline{\psi_n'} = \frac{1}{2}\Big[p(L)\,|\psi_n(L)|^2 - p(0)\,|\psi_n(0)|^2 - \int_0^L p'\,|\psi_n|^2\Big] = -\frac{1}{2}\int_0^L p'\,|\psi_n|^2.$$

Hence,

$$\text{Re}\Big[\text{i}\lambda_n\, \rho_1 \int_0^L p\, v_n\, \overline{T_n} + i\lambda_n\, \rho_2 \int_0^L c\, p\, \psi_n\, \overline{R_n}dx\Big]$$

$$= -\rho_1\, k\, Re \int_0^L p\, v_n\, \overline{v_n'} - \rho_1\, k\, Re \int_0^L p\, v_n\, \overline{\psi_n} - \rho_2\, \mu\, c\, Re \int_0^L p\, \psi_n\, \overline{\psi_n'}$$

$$+ Re\Big[i\lambda_n\, \rho_1 \int_0^L p\, v_n\, D_1\, (\overline{v_n' + \psi_n})dx + i\lambda_n\, \rho_2 \int_0^L c\, p\, \psi_n\, D_2\, \overline{\psi_n'}dx\Big]$$

$$= \frac{\rho_1\, k}{2} \int_0^L p'\,|v_n|^2 dx - \rho_1\, k\, Re \int_0^L p\, v_n\, \overline{\psi_n}dx + \frac{c\,\rho_2\,\mu}{2} \int_0^L p'\,|\psi_n|^2 dx$$

$$+ Re\Big[i\lambda_n\, \rho_1 \int_0^L p\, v_n\, D_1\, (\overline{v_n' + \psi_n})dx + i\lambda_n\, \rho_2 \int_0^L c\, p\, \psi_n\, D_2\, \overline{\psi_n'}dx\Big] + o(1)$$

Therefore ,

$$\begin{aligned} &Re\Big[i\lambda_n\, \rho_1 \int_0^L p\, v_n\, \overline{T_n} + i\lambda_n\, \rho_2 \int_0^L c\, p\, \psi_n\, \overline{R_n}dx\Big] \\ &= \frac{\rho_1\, k}{2} \int_0^L p'\,|v_n|^2 dx - \rho_1\, k\, Re \int_0^L p\, v_n\, \overline{\psi_n}dx + \frac{c\,\rho_2\,\mu}{2} \int_0^L p'\,|\psi_n|^2 dx \\ &+ Re\Big[i\lambda_n\, \rho_1 \int_0^L p\, v_n\, D_1\, (\overline{v_n' + \psi_n})dx + i\lambda_n\, c\,\rho_2 \int_0^L p\, D\, \psi_n\, \overline{\psi_n'}dx\Big] + o(1) \end{aligned} \tag{4.28}$$

Now, by taking $c = k\mu^{-1}$ and substitute (4.20) and (4.28) in (4.27), we get

$$\rho_1\, k \int_0^L p'\,|v_n|^2 dx - p(L)\,|T_n(L)|^2 + p(0)\,|T_n(0)|^2 + k^2 \int_0^L p'|\omega_n' + \phi_n|^2 dx$$
$$+ \rho_2\, k \int_0^L p'|\psi_n|^2 dx - k\mu^{-1}\, p(L)\,|R_n(L)|^2 + k\mu^{-1}\, p(0)\,|R_n(0)|^2 + k\,\mu \int_0^L p'|\phi_n'|^2 dx$$
$$+ 2\,Re\left[i\lambda_n\, \rho_1 \int_0^L p\, v_n\, D_1\,(\overline{v_n' + \psi_n}) dx + i\lambda_n\, k\, \rho_2\, \mu^{-1} \int_0^L p\, D_2\, \psi_n\, \overline{\psi_n'} dx \right] = o(1).$$

We finally, conclude (4.21) by combining the above estimation and the Cauchy - Schwarz inequality. □

Lemma 4.0.6. Assume $\gamma \geq 0,\ \delta_i \in C^1([\ell_i, L])$ and $\delta_i(\ell_i) = 0,\ i = 1, 2$. Then, one has

$$\lambda_n^2 \int_0^L \left(\rho_1\, D_1\, |v_n|^2 + \rho_2\, D_2\, |\psi_n|^2\right) dx$$
$$\leq 2\,k \left| \int_{\ell_1}^L \delta_1'\,(v_n' + \psi_n)\,\overline{v_n}\, dx \right| + 2\,\mu \left| \int_{\ell_2}^L \delta_2'\, \psi_n'\, \overline{\psi_n}\, dx \right| + \mathcal{O}(1) \tag{4.29}$$

Proof. Consider the inner product of (4.15) with $i\lambda_n^{1-\gamma}\,(D_1\, v_n, 0)$ on H, we have

$$(\, g_{1n}\,,\; g_{2n},\; i\lambda_n^{1-\gamma}\, D_1\, v_n\,, 0\,)_H = \int_0^L \rho_1\, g_{1n}\, \overline{i\lambda_n^{1-\gamma}\, D_1\, v_n} \;=\; -\int_0^L \rho_1\, \left(i\lambda_n\, v_n \;-\; \rho_1^{-1}\, T_n'\right)\left(i\lambda_n\, D_1\, \overline{v_n}\right)$$
$$= \rho_1\, \lambda_n^2 \int_0^L D_1\, |v_n|^2 \;+\; i\lambda_n \int_0^L T_n'\, D_1\, \overline{v_n}$$

Since $\delta_1(\ell_1) = 0$, we get

$$i\,\lambda_n \int_0^L T_n'\, D_1\, \overline{v_n}\, dx = -i\,\lambda_n \int_{\ell_1}^L T_n\, (\delta_1\, \overline{v_n})'\, dx = -i\,\lambda_n \int_{\ell_1}^L (k(\omega_n' + \psi_n) + \delta_1(v_n' + \psi_n))\,(\delta_1\, \overline{v_n})'\, dx$$

due to (4.14) we obtain :

$$i\lambda_n\, \omega_n' = v_n' + \lambda_n^{-\gamma}\, f_{1n}' \;\; and \;\; i\lambda_n\, \phi_n = \psi_n + \lambda_n^{-\gamma}\, f_{2n}.$$

then,

$$i\,\lambda_n \int_0^L T_n'\, D_1\, \overline{v_n}\, dx = \int_{\ell_1}^L \left[-k(v_n' + \lambda_n^{-\gamma}\, f_{1n}' + \psi_n + \lambda_n^{-\gamma}\, f_{2n}) - i\,\lambda_n\, \delta_1(v_n' + \psi_n)\right] \left[\delta_1\, \overline{v_n}\right]'\, dx$$
$$= -\int_{\ell_1}^L (k + i\,\lambda_n\, \delta_1)\,(v_n' + \psi_n)\,(\delta_1\, \overline{v_n})'\, dx + o(1)$$

where,

$$\left| \int_{\ell_1}^L k\,\lambda_n^{-\gamma}\,(f_{1n}' + f_{2n})(\delta_1\, \overline{v_n})' \right| \leq k\,\lambda_n^{-\gamma}\, \|(f_{1n}, f_{2n})\|_V\, \|\delta_1\|_{L^2}\, \|U_n\|_{\mathcal{H}} = o(1)$$

but

$$|\rho_1 \int_0^L g_{1n}\, \overline{i\lambda_n^{1-\gamma}\, D_1\, v_n} dx| \leq \rho_1 \int_0^L |g_{1n}\, \lambda_n^{1-\gamma}\, D_1\, v_n dx| = \rho_1 \int_0^L |\,\lambda_n\, D_1^{\frac{1}{2}}\, v_n\, \lambda_n^{-\gamma}\, D_1^{\frac{1}{2}} g_{1n} dx|$$

$$\leq \frac{1}{2\epsilon} \int_0^L |\lambda_n\, D_1^{\frac{1}{2}}\, v_n|^2 \;+\; \frac{\epsilon}{2} \int_0^L |\lambda_n^{-\gamma}\, D_1^{\frac{1}{2}}\, g_{1n}|^2 \;\leq\; \frac{\lambda_n^2}{2\epsilon} \int_0^L D_1\, |v_n|^2 + \frac{\epsilon}{2}\lambda_n^{-2\,\gamma} \int_0^L D_1\, |g_{1n}|^2$$

$$= \frac{1}{4}\rho_1 \lambda_n^2 \int_0^L D_1 |v_n|^2 + \rho_1^{-1}\, \lambda_n^{-2\,\gamma} \int_0^L D_1 |g_{1n}|^2 \;=\; \frac{1}{4}\rho_1 \lambda_n^2 \int_0^L D_1 |v_n|^2 + o(1) \tag{4.30}$$

Consequently,

$$Re(i\,\lambda_n \int_0^L T_n'\, D_1\, \overline{v_n}\,) = -Re \int_{\ell_1}^L (k + i\,\lambda_n\,\delta_1)\,(v_n' + \psi_n)\,(\delta_1\,\overline{v_n})'\,dx + o(1)$$

and

$$Re\big(\rho_1\,\lambda_n^2 \int_0^L D_1\,|v_n|^2 \;+\; i\lambda_n \int_0^L T_n'\, D_1\, \overline{v_n}\big) = Re\big(\int_0^L \rho_1\, g_{1n}\, \overline{i\lambda_n^{1-\gamma}\, D_1\, v_n}\big)$$

$$\leq |\rho_1 \int_0^L g_{1n}\, \overline{i\lambda_n^{1-\gamma}\, D_1\, v_n} dx| \leq \frac{1}{4}\rho_1 \lambda_n^2 \int_0^L D_1 |v_n|^2 + o(1)$$

yields

$$\frac{3\rho_1}{4}\,\lambda_n^2 \int_0^L D_1 |v_n|^2 - Re \int_{\ell_1}^L (k + i\,\lambda_n\,\delta_1)\,(v_n' + \psi_n)\,(\delta_1\,\overline{v_n})'\,dx = o(1) \tag{4.31}$$

Due to (4.12) and (4.16) we deduce that

$$\begin{aligned}\int_{\ell_1}^L \delta_1\,(v_n' + \psi_n)\,\overline{v_n'} dx &= \int_{\ell_1}^L \delta_1\,(v_n' + \psi_n)\,\overline{(v_n' + \psi_n - \psi_n)} dx \\ &\leq \int_{\ell_1}^L \delta_1\,|v_n' + \psi_n|^2 dx + \int_{\ell_1}^L \delta_1\,(v_n' + \psi_n)\,\overline{\psi_n} dx \\ &= o(1)\end{aligned}$$

thus (4.31) implies

$$\frac{3\rho_1}{4}\,\lambda_n^2 \int_0^L D_1 |v_n|^2 - Re \int_{\ell_1}^L (k + i\,\lambda_n\,\delta_1)\,(v_n' + \psi_n)\,(\delta_1'\,\overline{v_n} + \delta_1\,\overline{v_n}')\,dx = o(1)$$

i.e.,

$$\begin{aligned}\frac{3\rho_1}{4}\,\lambda_n^2 \int_0^L D_1 |v_n|^2 - Re \int_{\ell_1}^L \Big[&(k + i\,\lambda_n\,\delta_1)\,(v_n' + \psi_n)\,(\delta_1'\,\overline{v_n}) \\ &+ k\,(v_n' + \psi_n)(\delta_1\,\overline{v_n'}) + i\,\lambda_n\,\delta_1^2\,|v_n'|^2 + i\,\lambda_n\,\delta_1^2\,\psi_n\,\overline{v_n'}\Big] dx = o(1)\end{aligned}$$

therefore,

$$\frac{3\rho_1}{4}\,\lambda_n^2\int_0^L D_1|v_n|^2 - Re\int_{\ell_1}^L\Big[\delta_1'(k+i\,\lambda_n\,\delta_1)\,(v_n'+\psi_n)\,\overline{v_n}dx + i\,\lambda_n\,\delta_1^2\,\psi_n\,\overline{v_n'}\Big]dx = o(1). \qquad (4.32)$$

Now we consider, by taking the inner product of (4.15) with $i\lambda_n^{1-\gamma}\,(0, D_2\,\psi_n)$ on H, we get

$$(g_{1n}\,,\,g_{2n},\,0\,,\,i\lambda_n^{1-\gamma}\,D_2\,\psi_n)_H = \int_0^L \rho_2\,g_{2n}\,\overline{i\lambda_n^{1-\gamma}\,D_2\,\psi_n} = -\int_0^L \rho_2\,\big[i\lambda_n\,\psi_n - \rho_2^{-1}(R_n'-T_n)\big]\,\big[i\lambda_n\,D_2\,\overline{\psi_n}\,\big]$$

$$= -\int_0^L \rho_2\left(-\lambda_n^2\,|\psi_n|^2\,D_2 - i\lambda_n\,\rho_2^{-1}(R_n'-T_n)\,D_2\,\overline{\psi_n}\right) = \rho_2\,\lambda_n^2\int_0^L|\psi_n|^2\,D_2 + i\lambda_n\int_0^L(R_n'-T_n)\,D_2\,\overline{\psi_n}$$

Moreover, we have

$$i\lambda_n\int_0^L(R_n' - T_n)\,D_2\,\overline{\psi_n} = i\lambda_n\int_0^L R_n'\,D_2\,\overline{\psi_n} - i\lambda_n\int_0^L T_n\,D_2\,\overline{\psi_n}$$

$$= -i\lambda_n\int_0^L R_n\,(D_2\,\overline{\psi_n})' - i\lambda_n\int_0^L T_n\,D_2\,\overline{\psi_n} = -i\lambda_n\int_0^L R_n(D_2'\,\overline{\psi_n} + D_2\,\overline{\psi_n'}) - i\lambda_n\int_0^L T_n\,D_2\,\overline{\psi_n}$$

$$= -i\lambda_n\int_0^L(\mu\,\phi_n' + D_2\psi_n')\,(D_2'\,\overline{\psi_n}) - i\lambda_n\,(\mu\,\phi_n' + D_2\psi_n')\,(D_2\,\overline{\psi_n'})$$

$$-\ i\lambda_n\int_0^L\big(k(\omega_n'+\phi_n) + D_1(v_n'+\psi_n)\big)\,D_2\,\overline{\psi_n}$$

$$= -i\lambda_n\Big[\int_0^L\mu\,\phi_n'\,D_2'\,\overline{\psi_n} + \int_0^L D_2\,\psi_n'\,D_2'\,\overline{\psi_n} + \int_0^L\mu\,\phi_n'\,D_2\,\overline{\psi_n'} + \int_0^L D_2\,\psi_n'\,D_2\,\overline{\psi_n'}$$

$$+\int_0^L k\,\omega_n'\,D_2\,\overline{\psi_n} + \int_0^L k\phi_n D_2\overline{\psi_n} + \int_0^L D_1\,D_2\,v_n'\,\overline{\psi_n} + \int_0^L D_1\,D_2\,|\psi_n|^2\Big]$$

Due to (4.14) and $\delta(\ell)=0$ we have, $i\lambda_n\,\omega_n = v_n + \lambda_n^{-\gamma}\,f_{1n}$ and $i\lambda_n\,\phi_n = \psi_n + \lambda_n^{-\gamma}\,f_{2n}$. Hence,

$$i\lambda_n\int_{\ell_2}^L(R_n' - T_n)\,\delta\,\overline{\psi_n} =$$

$$-\Big[\mu\int_{\ell_2}^L(\psi_n' + \lambda_n^{-\gamma}\,f_{2n}')\,\delta_2'\,\overline{\psi_n} + \mu\int_{\ell_2}^L(\psi_n' + \lambda_n^{-\gamma}\,f_{2n}')\,\delta_2\,\overline{\psi_n'} + i\lambda_n\int_{\ell_2}^L\delta_2\,\delta_2'\psi_n'\,\overline{\psi_n} + i\lambda_n\int_{\ell}^L\delta_2\,|\psi_n'|^2$$

$$+\,k\int_{\ell_2}^L\delta_2\,(v_n' + \lambda_n^{-\gamma}\,f_{1n}')\,\overline{\psi_n} + k\int_{\ell_2}^L(\psi_n + \lambda_n^{-\gamma}\,f_{2n})\,\delta_2\overline{\psi_n} + \int_{\ell_2}^L i\,\lambda_n\,\delta_1\,\delta_2\,v_n'\,\overline{\psi_n} + \int_{\ell_2}^L i\,\lambda_n\,\delta_1\,\delta_2\,|\psi_n|^2\Big]$$

Notice that,

$$\|(f_{1n},f_{2n})\|_V = o(1)$$

implies,

$$f_{1n} = f_{2n} = \lambda_n^{-\gamma}\,f_{1n}' = \lambda_n^{-\gamma}\,f_{2n}' = o(1).$$

Hence, we get

$$i\lambda_n \int_0^L (R_n' - T_n)\, D\,\overline{\psi_n} =$$

$$-\Big[\mu \int_{\ell_2}^L \delta_2'\, \psi_n'\, \overline{\psi_n} + \mu \int_{\ell_2}^L \delta_2\, |\psi_n'|^2 + i\lambda_n \int_{\ell_2}^L \delta_2\, \delta_2' \psi_n'\, \overline{\psi_n} + i\lambda_n \int_{\ell_2}^L \delta_2\, |\psi_n'|^2 + k \int_{\ell_2}^L \delta_2\, v_n'\, \overline{\psi_n} + k \int_{\ell_2}^L \delta_2\, |\psi_n|^2$$
$$+ \int_{\ell_2}^L i\,\lambda_n \delta_1\, \delta_2\, v_n'\, \overline{\psi_n} + \int_{\ell_2}^L i\,\lambda_n\, \delta_1\, \delta_2\, |\psi_n|^2\Big]$$

It follows from (4.16) that :

$$Re(\, i\lambda_n \int_0^L (R_n' - T_n)\, D\,\overline{\psi_n}\,) =$$

$$-\,Re \int_{\ell_2}^L \Big[\delta_2'\,(\mu + i\,\lambda_n\,\delta_2)\psi_n'\,\overline{\psi_n}dx + \delta_2\,(k + i\,\lambda_n\,\delta_1)\,v_n'\,\overline{\psi_n}dx + k\,\delta_2\,|\psi_n|^2 dx\Big]$$

Now, for any $\epsilon > 0$ as

$$|\rho_2 \int_0^L g_{2n}\, \overline{i\lambda_n^{1-\gamma}\, D_2\,\psi_n dx}| \le \rho_2 \int_0^L |g_{2n}\,\lambda_n^{1-\gamma}\, D_2\,\psi_n dx| = \rho_2 \int_0^L |\,\lambda_n\, D_2^{\frac{1}{2}}\,\psi_n\,\lambda_n^{-\gamma}\, D_2^{\frac{1}{2}} g_{2n} dx|$$

$$\le \frac{1}{2\epsilon}\int_0^L |\lambda_n\, D_2^{\frac{1}{2}}\,\psi_n|^2 + \frac{\epsilon}{2}\int_0^L |\lambda_n^{-\gamma}\, D_2^{\frac{1}{2}}\, g_{2n}|^2 \le \frac{\lambda_n^2}{2\epsilon}\int_0^L D_2\,|\psi_n|^2 + \frac{\epsilon}{2}\lambda_n^{-2\gamma}\int_0^L D_2\,|g_{2n}|^2$$

$$= \frac{1}{4}\rho_2\lambda_n^2 \int_0^L D_2\,|\psi_n|^2 + \rho_2^{-1}\,\lambda_n^{-2\gamma}\int_0^L D_2\,|g_{2n}|^2 = \frac{1}{4}\rho_2\lambda_n^2 \int_0^L D_2\,|\psi_n|^2 + o(1) \tag{4.33}$$

and due to (4.15) as $\|(g_{1n}, g_{2n})\|_H = o(1)$ we get, $\int_0^L D_2\,|g_{2n}|^2 = o(1)$

Consequently, by substitution we get :

$$Re\,\big(\rho_2\,\lambda_n^2 \int_0^L |\psi_n|^2\, D_2 + i\lambda_n \int_0^L (R_n' - T_n)\, D_2\,\overline{\psi_n}\big) = Re\big(\rho_2 \int_0^L g_{2n}\,\overline{i\lambda_n^{1-\gamma}\, D_2\,\psi_n dx}\big)$$
$$\le \frac{1}{4}\rho_2\lambda_n^2 \int_0^L D_2\,|\psi_n|^2 + o(1)$$

thus, we conclude that

$$\frac{3}{4}\rho_2\,\lambda_n^2 \int_0^L D_2\,|\psi_n|^2 - Re\int_{\ell_2}^L \Big[\,\delta_2'\,(\mu + i\,\lambda_n\,\delta_2)\psi_n'\,\overline{\psi_n}dx$$
$$+ \delta_2\,(k + i\,\lambda_n\,\delta_1)\,v_n'\,\overline{\psi_n}dx + k\,\delta_2\,|\psi_n|^2 dx\Big] = o(1) \tag{4.34}$$

Adding (4.32) and (4.34) gives

$$
\begin{aligned}
&\frac{3}{4}\,\lambda_n^2 \int\limits_0^L \left(\rho_1\, D_1\, |v_n|^2 + \rho_2\, D_2\, |\psi_n|^2\right) \\
&= Re \int_{\ell_1}^{L} \left[\delta_1'(k + i\,\lambda_n\,\delta_1)\,(v_n' + \psi_n)\,\overline{v_n} + i\,\lambda_n\,\delta_1^2\,\psi_n\,\overline{v_n'}\right] dx \\
&+ Re \int_{\ell_2}^{L} \left[\delta_2'\,(\mu + i\,\lambda_n\,\delta_2)\psi_n'\,\overline{\psi_n} + \delta_2\,(k + i\,\lambda_n\,\delta_1)\,v_n'\,\overline{\psi_n} + k\,\delta_2\,|\psi_n|^2\right] dx + o(1).
\end{aligned}
\tag{4.35}
$$

Let us find an estimation of the R.H.S of (4.35). Let $\epsilon > 0$. Then, by (4.16) we have

$$
\begin{aligned}
\left| i\lambda_n \int\limits_{\ell_1}^{L} \delta_1\,\delta_1'\,(v_n' + \psi_n\,\overline{v_n} dx \right| &\leq \int\limits_{\ell_1}^{L} \left| \frac{\sqrt{\rho_1}}{\sqrt{2}} \lambda_n\,\delta_1^{\frac{1}{2}}\,\overline{v_n}\,\frac{\sqrt{2}}{\sqrt{\rho_1}} \delta_1^{\frac{1}{2}}\,\delta_1'\,(v_n' + \psi_n) \right| \\
&\leq \frac{1}{2\epsilon} \int\limits_{\ell_1}^{L} \lambda_n^2\,\delta_1\,|v_n|^2 dx \,+\, \frac{\epsilon}{2} \int\limits_{\ell_1}^{L} \delta_1\,\delta_1'^{\,2}\,|v_n' + \psi_n|^2 dx \\
&\leq \frac{\rho_1}{2 \times 2} \int\limits_{\ell_1}^{L} \lambda_n^2\,\delta_1\,|v_n|^2 \,+\, \frac{2}{2 \times \rho_1} \max_{x \in [\ell_1, L]} [\delta_1'(x)]^2 \int_{\ell_1}^{L} \delta_1\,|v_n' + \psi_n|^2 \\
&\leq \frac{\rho_1}{4} \int\limits_{\ell_1}^{L} \lambda_n^2\,\delta_1\,|v_n|^2 \,+\, \rho_1^{-1} \max_{x \in [\ell_1, L]} [\delta_1'(x)]^2 \int_{\ell_1}^{L} \delta_1\,|v_n' + \psi_n|^2 \\
&\leq \frac{\rho_1}{4}\,\lambda_n^2 \int\limits_{\ell_1}^{L} \delta_1\,|v_n|^2 + o(1)
\end{aligned}
\tag{4.36}
$$

and

$$
\begin{aligned}
Re\left[i\,\lambda_n \int\limits_{\ell_1}^{L} \delta_1^2\,\psi_n\,\overline{v_n'} dx \right] &= Re\left[i\,\lambda_n \int\limits_{\ell_1}^{L} \delta_1^2\,\psi_n\,\overline{(v_n' + \psi_n)} dx \right] \\
&\leq \int\limits_{\ell_1}^{L} \left| i\,\lambda_n\,\delta_1^{\frac{1}{2}}\,\psi_n\,\delta_1^{\frac{3}{2}}\,\overline{(v_n' + \psi_n)} dx \right| \\
&\leq \frac{1}{2\epsilon} \int\limits_{\ell_1}^{L} \lambda_n^2\,\delta_1\,|\psi_n|^2 dx \,+\, \frac{\epsilon}{2} \int\limits_{\ell_1}^{L} \delta_1\,\delta_1^2\,|v_n' + \psi_n|^2 dx \\
&\leq \frac{\rho_2}{16}\,\lambda_n^2 \int\limits_{\ell_2}^{L} \delta_2\,|\psi_n|^2 + o(1)
\end{aligned}
\tag{4.37}
$$

where we used $\delta_1 \leq C\,\delta_2$ for some $C > 0$ on $[\ell_1, L]$ in the last inequality of (4.37)

Similarly, due to (4.14) and (4.16), we get

$$\begin{aligned}
\left| i\lambda_n \int_{\ell_2}^{L} \delta_2\, \delta_2'\, \psi_n'\, \overline{\psi_n} \right| &\leq \int_{\ell_2}^{L} \left| \frac{\sqrt{\rho_2}}{\sqrt{8}} \lambda_n\, \delta_2^{\frac{1}{2}}\, \psi_n\, \frac{\sqrt{8}}{\sqrt{\rho_2}} \delta_2^{\frac{1}{2}}\, \delta_2'\, \psi_n' \right| \leq \frac{1}{2\epsilon} \int_{\ell_2}^{L} \lambda_n^2\, \delta_2\, |\psi_n|^2 + \frac{\epsilon}{2} \int_{\ell_2}^{L} \delta_2\, {\delta_2'}^2 |\psi_n'|^2 \\
&\leq \frac{\rho_2}{2\times 8} \int_{\ell_2}^{L} \lambda_n^2\, \delta_2\, |\psi_n|^2 + \frac{8}{2\times\rho_2} \max_{x\in[\ell_2,L]} [\delta'(x)]^2 \int_{\ell_2}^{L} \delta_2\, |\psi_n'|^2 \\
&\leq \frac{\rho_2}{16} \int_{\ell_2}^{L} \lambda_n^2\, \delta_2\, |\psi_n|^2 + 4\,\rho_2^{-1} \max_{x\in[\ell_2,L]} |\delta_2'(x)|^2 \int_{\ell_2}^{L} \delta_2\, |\psi_n'|^2 \\
&\leq \frac{\rho_2}{16} \int_{\ell_2}^{L} \lambda_n^2\, \delta_2\, |\psi_n|^2 + o(1) \qquad (4.38)
\end{aligned}$$

$$\begin{aligned}
Re\left[i\lambda_n \int_{\ell_2}^{L} \delta_2\, \delta_1\, v_n'\, \overline{\psi_n} \right] &= Re\left[i\,\lambda_n \int_{\ell_2}^{L} \delta_2^{\frac{1}{2}}\, \overline{\psi_n}\, \delta_2^{\frac{1}{2}}\, \delta_1 (v_n' + \psi_n) \right] \\
&\leq \int_{\ell_2}^{L} \left| \lambda_n\, \delta_2^{\frac{1}{2}}\, \overline{\psi_n}\, \delta_2^{\frac{1}{2}}\, \delta_1\, (v_n' + \psi_n) \right| \\
&\leq \frac{1}{2\,\epsilon} \int_{\ell_2}^{L} \lambda_n^2\, \delta_2\, |\psi_n|^2 + \frac{\epsilon}{2} \int_{\ell_2}^{L} \delta_2\, \delta_1\, |v_n' + \psi_n|^2 \\
&\leq \frac{\rho_2}{16}\, \lambda_n^2 \int_{\ell_2}^{L} \delta_2\, |\psi_n|^2 + o(1) \qquad (4.39)
\end{aligned}$$

$$\begin{aligned}
\left| k \int_{\ell_2}^{L} \delta_2\, v_n'\, \overline{\psi_n} \right| &\leq \int_{\ell_2}^{L} \left| k^{\frac{1}{2}}\, \delta_2^{\frac{1}{2}}\, \psi_n\, k^{\frac{1}{2}}\, \delta_2^{\frac{1}{2}}\, (i\lambda_n\, \omega_n' - \frac{f_1'}{\lambda_n^{\gamma}}) \right| \leq \int_{\ell_2}^{L} \left| k^{\frac{1}{2}}\, \delta_2^{\frac{1}{2}}\, \psi_n \lambda_n\, k^{\frac{1}{2}}\, \delta_2^{\frac{1}{2}}\, \omega_n' \right| + o(1) \\
&\leq \frac{\rho_2}{2\times 8} \int_{\ell_2}^{L} k\, \delta_2\, |\psi_n|^2 \lambda_n^2 + \frac{8}{2\rho_2} \int_{\ell_2}^{L} k\, \delta_2\, |\omega_n'|^2 + o(1) \\
&\leq \frac{k\,\rho_2}{16} \lambda_n^2 \int_{\ell_2}^{L} \delta_2\, |\psi_n|^2 + 4\,k\,\rho_2^{-1} \max_{x\in[\ell_2,L]} \delta_2(x) \int_{\ell_2}^{L} |\omega_n'|^2 + o(1) \\
&\leq \frac{\rho_2}{16} \lambda_n^2 \int_{\ell_2}^{L} \delta_2\, |\psi_n|^2 + 4\,\rho_2^{-1}\, \|U_n\|_{\mathcal{H}} \max_{x\in[\ell_2,L]} \delta_2(x) \\
&\leq \frac{\rho_2}{16} \lambda_n^2 \int_{\ell_2}^{L} \delta_2\, |\psi_n|^2 + \mathcal{O}(1) \qquad (4.40)
\end{aligned}$$

Finally, substitute (4.12) and (4.36) – (4.40) in (4.35), we deduce (4.29). □

4.1 Polynomial stability

4.1.1 Polynomial stability of order 1

Lemma 4.1.1. If δ_1 and δ_2 satisfy $(H1)$ and (4.5). Then, (4.11) valid for $\gamma = 1$.

Proof. Due to Lemma 4.0.1 and (4.16), there exists $C > 0$ such that

$$\begin{aligned}
\lambda_n^{\frac{1}{2}} \|\delta_1^{\frac{1}{2}} v_n\|_{L^2(\ell_1,L)} &\leq \lambda_n^{\frac{1}{2}} C \|\delta_1^{\frac{1}{2}} v_n'\|_{L^2(\ell_1,L)} \\
&\leq \lambda_n^{\frac{1}{2}} C \|\delta_1^{\frac{1}{2}} (v_n' + \psi_n)\|_{L^2(\ell_1,L)} + \lambda_n^{\frac{1}{2}} C \|\delta_1^{\frac{1}{2}} \psi_n\|_{L^2(\ell_1,L)} \\
&\leq \lambda_n^{\frac{1}{2}} C \|\delta_1^{\frac{1}{2}} (v_n' + \psi_n)\|_{L^2(\ell_1,L)} + \lambda_n^{\frac{1}{2}} C^2 \|\delta_2^{\frac{1}{2}} \psi_n\|_{L^2(\ell_2,L)} \\
&= o(1)
\end{aligned} \tag{4.41}$$

By (4.16) and (4.41), we get

$$\lambda_n \|D_1^{\frac{1}{2}} v_n\| \, \|D_1^{\frac{1}{2}} (v_n' + \psi_n)\| = o(1) \tag{4.42}$$

Moreover, Lemma 4.0.1 and (4.16), there exists $C > 0$ such that

$$\lambda_n^{\frac{1}{2}} \|\delta_2^{\frac{1}{2}} \psi_n\|_{L^2(\ell_2,L)} \leq \lambda_n^{\frac{1}{2}} C \|\delta_2^{\frac{1}{2}} \psi_n'\|_{L^2(\ell_2,L)} = o(1) \tag{4.43}$$

Thus

$$\lambda_n \|D_2^{\frac{1}{2}} \psi_n\| \, \|D_2^{\frac{1}{2}} \psi_n'\| = o(1). \tag{4.44}$$

Substituting (4.44) and (4.42) in (4.21) gives

$$\begin{aligned}
&\rho_1 k \int_0^L p' |v_n|^2 \, dx - p(L) |T_n(L)|^2 + p(0) |T_n(0)|^2 + k^2 \int_0^L p' |\omega_n' + \phi_n|^2 \, dx + \rho_2 k \int_0^L p' |\psi_n|^2 \, dx \\
&- k \mu^{-1} p(L) |R_n(L)|^2 + k \mu^{-1} p(0) |R_n(0)|^2 + k \mu \int_0^L p' |\phi_n'|^2 \, dx = o(1)
\end{aligned} \tag{4.45}$$

Setting $p(x) = \int_0^x D_1(s) ds$, and using (4.45) we deduce that

$$\begin{aligned}
&\rho_1 k \int_0^L D_1 |v_n|^2 \, dx - \int_0^L D_1(s) ds \, |T_n(L)|^2 + k^2 \int_0^L D_1 |\omega_n' + \phi_n|^2 \, dx + \rho_2 k \int_0^L D_1 |\psi_n|^2 \, dx \\
&- k \mu^{-1} \int_0^L D_1(s) ds \, |R_n(L)|^2 + k \mu \int_0^L D_1 |\phi_n'|^2 \, dx = o(1)
\end{aligned} \tag{4.46}$$

Since there exists $C > 0$ such that $D_1(x) \leq C \, D_2(x)$ for all $x \in [0, L]$. Then, by (4.17), (4.41), (4.43) and (4.46),

$$\left|T_n(L)\right|, \left|R_n(L)\right| = o(1). \tag{4.47}$$

Substituting (4.47) in (4.45) gives

$$\rho_1 k \int_0^L p' |v_n|^2 dx + p(0) |T_n(0)|^2 + k^2 \int_0^L p' |\omega_n' + \phi_n|^2 dx + \rho_2 k \int_0^L p' |\psi_n|^2 dx$$
$$+ k \mu^{-1} p(0) |R_n(0)|^2 + k \mu \int_0^L p' |\phi_n'|^2 dx = o(1) \tag{4.48}$$

Setting $p = x$ in (4.48) we obtain

$$\rho_1 k \int_0^L |v_n|^2 dx + k^2 \int_0^L |\omega_n' + \phi_n|^2 dx + \rho_2 k \int_0^L |\psi_n|^2 dx + k \mu \int_0^L |\phi_n'|^2 dx = o(1) \tag{4.49}$$

Hence $\lim\limits_{n\to\infty} \|U_n\|_{\mathcal{H}} = 0$ from (4.49) which contradicts $\|U_n\|_{\mathcal{H}} = 1$. □

4.1.2 Polynomial stability of order $\frac{3}{2}$

Lemma 4.1.2. Assume δ_i satisfies (H2), (4.5) and $\delta_i(\ell_i) = 0$ for i = 1, 2. Then, (4.11) is true for $\gamma = \dfrac{2}{3}$.

Proof. Due to Lemma 4.0.1, (4.12) and (4.14),

$$\begin{aligned}
&\left| \int_{\ell_1}^L \delta_1' (v_n' + \psi_n) \overline{v_n} dx \right| \\
&\leq C \|v_n' + \psi_n\|_{L^2(\ell_1,L)} \|\delta_1^{\frac{1}{2}} v_n'\|_{L^2(\ell_1,L)} \\
&\leq C \lambda_n \|\omega_n' + \phi_n\|_{L^2(\ell_1,L)} \left[\|\delta_1^{\frac{1}{2}} (v_n' + \psi_n)\|_{L^2(\ell_1,L)} + \|\delta_2^{\frac{1}{2}} \psi_n\|_{L^2(\ell_2,L)} \right] \\
&\leq C \lambda_n \|\omega_n' + \phi_n\|_{L^2(\ell_1,L)} \|\delta_1^{\frac{1}{2}} (v_n' + \psi_n)\|_{L^2(\ell_1,L)} + C \lambda_n \|\omega_n' + \phi_n\|_{L^2(\ell_1,L)} \|\delta_2^{\frac{1}{2}} \psi_n\|_{L^2(\ell_2,L)} \\
&= C \lambda_n \|\omega_n' + \phi_n\|_{L^2(\ell_1,L)} \|\delta_1^{\frac{1}{2}} (v_n' + \psi_n)\|_{L^2(\ell_1,L)} + C \|\omega_n' + \phi_n\|_{L^2(\ell_1,L)} \lambda_n \|\delta_2^{\frac{1}{2}} \psi_n\|_{L^2(\ell_2,L)} \\
&\leq C \lambda_n \|\omega_n' + \phi_n\|_{L^2(\ell_1,L)} \|\delta_1^{\frac{1}{2}} (v_n' + \psi_n)\|_{L^2(\ell_1,L)} + \frac{k}{\rho_2} \|\omega_n' + \phi_n\|^2_{L^2(\ell_1,L)} + \frac{1}{4} \frac{\rho_2}{k} \lambda_n^2 \|\delta_2^{\frac{1}{2}} \psi_n\|^2_{L^2(\ell_2,L)} \\
&\leq C \lambda_n \|\omega_n' + \phi_n\|_{L^2(\ell_1,L)} \|\delta_1^{\frac{1}{2}} (v_n' + \psi_n)\|_{L^2(\ell_1,L)} + \frac{1}{4} \frac{\rho_2}{k} \lambda_n^2 \|\delta_2^{\frac{1}{2}} \psi_n\|^2_{L^2(\ell_2,L)} + \mathcal{O}(1)
\end{aligned} \tag{4.50}$$

Using (4.12), (4.16) and (4.50) we deduce that

$$\lambda_n^{-\frac{2}{3}} \left| \int_{\ell_1}^L \delta_1' (v_n' + \psi_n) \overline{v_n} dx \right| \leq \frac{1}{4} \frac{\rho_2}{k} \lambda_n^{\frac{4}{3}} \|\delta_2^{\frac{1}{2}} \psi_n\|^2_{L^2(\ell_2,L)} + o(1). \tag{4.51}$$

Similarly, by using Lemma (4.0.1) and (4.12), we have

$$\begin{aligned}
\left|\int_{\ell_2}^{L} \delta_2'\,\psi_n'\,\overline{\psi_n}dx\right| &\leq \|\delta_2'\,\psi_n\|_{L^2(\ell_2,L)}\,\|\psi_n'\|_{L^2(\ell_2,L)} \\
&\leq C\,\|\delta_2^{\frac{1}{2}}\,\psi_n'\|_{L^2(\ell_2,L)}\,\|\psi_n'\|_{L^2(\ell_2,L)} \\
&\leq C\,\|\delta_2^{\frac{1}{2}}\,\psi_n'\|_{L^2(\ell_2,L)}\,\|\psi_n'\|_{L^2(\ell_2,L)} \\
&\leq \lambda_n^{-\frac{1}{3}}\,\lambda_n^{\frac{1}{3}}\,C\,\|\delta_2^{\frac{1}{2}}\,\psi_n'\|_{L^2(\ell_2,L)}\,\|\psi_n'\|_{L^2(\ell_2,L)} \\
&= o(1)
\end{aligned} \tag{4.52}$$

thus,

$$\lambda_n^{-\frac{2}{3}}\left|\int_{\ell_2}^{L} \delta_2'\,\psi_n'\,\overline{\psi_n}dx\right| = o(1) \tag{4.53}$$

Therefore, using (4.29) and (4.51) $-$ (4.53) we deduce

$$\lambda_n^{\frac{4}{3}}\int_0^L(\,\rho_1\,D_1\,|v_n|^2+\rho_2\,D_2\,|\psi_n|^2\,)dx = o(1) \tag{4.54}$$

then,

$$\lambda_n^{\frac{4}{3}}\int_0^L \rho_1\,D_1\,|v_n|^2\,dx = \lambda_n^{\frac{4}{3}}\int_0^L \rho_2\,D_2\,|\psi_n|^2\,dx = o(1)$$

multiplying by $\lambda_n^{-\frac{2}{3}}$ we get,

$$\lambda_n^{\frac{2}{3}}\|D_1^{\frac{1}{2}}\,v_n\| = \lambda_n^{\frac{2}{3}}\|D_2^{\frac{1}{2}}\,\psi_n\| = o(1)$$

Hence, (4.42) and (4.44) follows from (4.16), (4.54) and assuming $\gamma = \dfrac{2}{3}$. Then, similarly to Lemma 4.1.1 we get $\lim\limits_{n\to\infty}\|U_n\|_{\mathcal{H}} = 0$, which leads to a contradiction. $\square$

4.2 Exponential stability and Analyticity of the semigroup.

4.2.1 Exponential stability

Lemma 4.2.1. Assume δ_i satisfies (H3), (4.5) and $\delta_i(\ell_i) = 0$ for i = 1, 2. Then, (4.11) valid for $\gamma = 0$.

Proof. By Lemma 4.0.1 and (4.16) that

$$\begin{aligned}
\left|\int_{\ell_1}^{L} \delta_1'\,(v_n'+\psi_n)\,\overline{v_n}dx\right| &= \left|\int_{\ell_1}^{L} \delta_1^{-\frac{1}{2}}\delta_1'\,(v_n'+\psi_n)\,\delta_1^{\frac{1}{2}}\,\overline{v_n}dx\right| \\
&\leq C\,\|\delta_1^{\frac{1}{2}}\,(v_n'+\psi_n)\|_{L^2(\ell_1,L)}\,\|\delta_1^{\frac{1}{2}}\,v_n\|_{L^2(\ell_1,L)} \\
&= o(1).
\end{aligned} \tag{4.55}$$

Similarly, it follows from Lemma 4.0.1 and (4.16) that

$$\begin{aligned}\left|\int_{\ell_2}^{L} \delta_2' \, \psi_n' \, \overline{\psi_n} dx\right| &= \left|\int_{\ell_2}^{L} \delta_2' \, \delta_2^{-\frac{1}{2}} \, \overline{\psi_n} \, \delta_2^{\frac{1}{2}} \, \psi_n' dx\right| \\ &\leq C \, \|\delta_2^{\frac{1}{2}} \, \psi_n'\|_{L^2(\ell_2,L)} \|\delta_2^{\frac{1}{2}} \, \psi_n'\|_{L^2(\ell_2,L)} \\ &= o(1). \end{aligned} \tag{4.56}$$

Substituting (4.55)-(4.56) in (4.29) gives

$$\lambda_n^2 \int_0^L (\, \rho_1 \, D_1 \, |v_n|^2 + \rho_2 \, D_2 \, |\psi_n|^2 \,) dx = o(1) \tag{4.57}$$

Finally, (4.16) and (4.57) yields (4.42) and (4.44). □

4.2.2 Analyticity of semigroup

Lemma 4.2.2. Assume D_1 and D_2 satisfy (H4). Then, (4.11) is valid for $\gamma = -1$.

Proof. For $\gamma = -1$, (H4) and (4.17) gives

$$\|\omega_n' + \phi_n\|, \; \|\phi_n'\| = o(1). \tag{4.58}$$

Now, considering the inner product of (4.15) with $i(v_n, \psi_n)$, in H we get

$$i \, (g_{1n}, g_{2n}, v_n, \psi_n)_H \; = \; i \, \lambda_n^{\gamma} \, (i \, \lambda_n \, v_n - \rho_1^{-1} \, T_n' \;, \; \; i \, \lambda_n \, \psi_n - \rho_2^{-1} \, (R_n' - T_n) \;, \; v_n \;, \; \psi_n)_H \; = \; o(1)$$

i.e.,

$$i \, \lambda_n^{-1} \left[\int_0^L \rho_1 (i \, \lambda_n \, v_n - \rho_1^{-1} \, T_n') \, \overline{v_n} \, dx + \int_0^L \rho_2 (i \, \lambda_n \, \psi_n - \rho_2^{-1} \, (R_n' - T_n)) \overline{\psi_n} \, dx \right] \; = \; o(1)$$

i.e.,

$$i \, \lambda_n^{-1} \int_0^L \left[i \, \lambda_n \, \rho_1 \, |v_n|^2 \; - \; T_n' \, \overline{v_n} \right] dx \; + \; i \, \lambda_n^{-1} \int_0^L \left[i \, \lambda_n \, \rho_2 \, |\psi_n|^2 \; - \; (R_n' - T_n) \, \overline{\psi_n} \right] dx = o(1)$$

i.e.,

$$-\rho_1 \int_0^L |v_n|^2 dx \; - \rho_2 \int_0^L |\psi_n|^2 dx \; - i \, \lambda_n^{-1} \int_0^L T_n' \, \overline{v_n} dx \; - i \, \lambda_n^{-1} \int_0^L (R_n' - T_n) \, \overline{\psi_n} dx \; = o(1)$$

i.e.,

$$-\rho_1 \, \|v_n\|^2 - \rho_2 \, \|\psi_n\|^2 \; - i \, \lambda_n^{-1} (T_n', v_n) \; - i \, \lambda_n^{-1} \, (R_n', \psi_n) \; + i \, \lambda_n^{-1} \, (T_n, \psi_n) \; = o(1)$$

i.e.,

$$\rho_1 \, \|v_n\|^2 + \rho_2 \, \|\psi_n\|^2 - i \lambda_n^{-1} \left[(T_n, v_n') + (R_n, \psi_n') + (T_n, \psi_n) \right] = o(1) \tag{4.59}$$

It follows from (4.12), (4.16) and (4.58) that

$$\lambda_n^{-1} \, (T_n, v_n') \; \leq \; k \|\omega_n' + \phi_n\| \, \|\omega_n'\| + \lambda_n^{-1} \, (D_1 (v_n' + \psi_n), v_n') + o(1) = o(1). \tag{4.60}$$

Similarly,

$$\lambda_n^{-1} \, (R_n, \psi_n'), \; \lambda_n^{-1} \, (T_n, \psi_n) = o(1) \tag{4.61}$$

Substituting (4.60) - (4.61) in (4.59) yields $\|v_n\|, \, \|\psi_n\| = o(1)$. Combining these with (4.58), we reach that $\|U_n\|_{\mathcal{H}} = o(1)$. □

Remark 4.2.3. The system (3.1) − (3.3) disscussed here is only a $1-D$ system on $[0, L]$, the local damping is applied on a part of the interval which contains a boundary point more precisely it contained L and $\omega(L) = \phi(L) = 0$. The results in that case were explained in details in this chapter. The inequality (4.2) played a role in estimating the higher-order terms on the damping sub-interval. We also proved in the previous chapter that even one damping is enough to get the strong stability and we believe that it's easy to prove that, in this case, the solution has slightly weaker decay rates than those mentioned in the last chapter. It would be interesting as well to study the case where the damping is fully internal and not just applied on a neighborhood of one of the endpoints. Also another interesting problem is to discuss the dissipation of the system in higher dimensions when $\Omega \subset \mathbb{R}^n$ and $n > 1$, the study of course will be then more complicated and will require other methods to discuss the stability.

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