On Connections Between Association Schemes and Analyses of Polyhedral and Positive Semidefinite Lift-and-Project Relaxations

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Abstract

We explore some connections between association schemes and the analyses of the semidefinite programming (SDP) based convex relaxations of combinatorial optimization problems in the Lovász–Schrijver lift-and-project hierarchy. Our analysis of the relaxations of the stable set polytope leads to bounds on the clique and stability numbers of some regular graphs reminiscent of classical bounds by Delsarte and Hoffman, as well as the notion of deeply vertex-transitive graphs — highly symmetric graphs that we show arise naturally from some association schemes. We also study relaxations of the hypergraph matching problem, and determine exactly or provide bounds on the lift-and-project ranks of these relaxations. Our proofs for these results also inspire the study of the general hypermatching association scheme. While this scheme is generally non-commutative, we illustrate the usefulness of obtaining commutative subschemes from non-commutative schemes via contraction in this context.

1 Introduction

Association schemes provide a beautiful unifying framework for algebraic representations of symmetries of permutation groups. In the space of symmetric n-by-n matrices, $\mathbb{S}^n$, the optimization of a linear function subject to linear inequalities and equations on the matrix variable together with the positive semidefiniteness constraint, defines a canonical representation of semidefinite programming (SDP) problems. Let $\mathbb{S}^n_+$ denote the set of positive semidefinite matrices in the set of n-by-n symmetric matrices $\mathbb{S}^n$. Then, the automorphism group of $\mathbb{S}^n_+$ can be algebraically described as:

$$\mathrm{Aut}(\mathbb{S}^n_+) = \left\{ A \cdot A^\top \ : \ A \in \mathbb{R}^{n\times n},\ A \text{ is non-singular} \right\}.$$

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That is, an automorphism acts as $X \mapsto AXA^\top$. Clearly, conjugation by any n-by-n permutation matrix is in the automorphism group of $\mathbb{S}^n_+$. So, if the linear equations and inequalities of our SDP problem are invariant under the action of symmetries of a permutation group $\mathcal{G}$, then for every feasible solution $\bar{X}$ of our SDP and for every $\sigma \in \mathcal{G}$, we have $\sigma(\bar{X})$ feasible in the SDP (where the second usage of σ, by our abuse of notation, denotes the action of the permutation by the conjugation of the underlying permutation matrix, i.e., $\sigma : \mathbb{S}^n \to \mathbb{S}^n$). Therefore, using the convexity of the feasible regions of SDPs, we conclude

$$\frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \sigma(\bar{X}) \tag{1}$$

is feasible in our SDP. Similarly, if the objective function of our SDP is also invariant under the action of $\mathcal{G}$, we conclude that if $\bar{X}$ is an optimal solution of our SDP then so is the matrix given by (1).

In some cases, this invariance group $\mathcal{G}$ is so rich that the underlying SDP problems can be equivalently written as linear programming (LP) problems. One of the earliest applications of this idea (in the context of the intersection of combinatorial optimization, LP and SDP) appears in the 1970s (see [14, 32, 35]) as well as in the 1990s (see [29]). More recently, see [24, 16, 13] and the references therein.

Many, if not most problems in discrete mathematics (in particular, graph theory) are stated in a way that they already expose potential invariances under certain group actions. In others, we are sometimes able to impose such symmetries to simplify the analysis. Suitably constructed optimization problems formulating the underlying problem usually inherit such symmetries. When such optimization problems are intractable, one resorts to their convex relaxations. These convex relaxations typically inherit and sometimes even further enrich such symmetries.

Thus, tools in association schemes can be extremely helpful in analyzing matrix variables in semidefinite relaxations of which the underlying problem has rich symmetries. Conversely, the analyses of these matrix variables can lead to interesting observations for related association schemes. One of the goals of this paper is to highlight some of these connections.

In Section 2, we quickly review the basic definitions and facts we need from association schemes. We also introduce the hierarchy of semidefinite programming based convex relaxations generated by the LS_+ operator due to Lovász and Schrijver [33] (also known as the N_+ operator in the literature).

In Section 3, we provide some bounds on the clique and stability numbers of some regular graphs by utilizing algebraic graph theory techniques and the lift-and-project operator LS_+. The pursuit of when these bounds are tight leads to the notion of deeply vertex-transitive graphs, which are vertex-transitive graphs with additional symmetries and arise rather naturally from some association schemes. One of our main goals in Section 3 is to introduce our approach in an elementary way and in a well-known setting, where the analysis of a single step of LS_+ hierarchy is relatively simple, and relate our findings to existing results.

In Section 4, we delve into our techniques more deeply and analyze the behaviour of LS_+ hierarchy on a variety of integral packing and covering polyhedra arising from matchings in hypergraphs. In the process, we introduce the general association scheme of hypermatchings. While the scheme is generally non-commutative, the known properties of some of its commutative subschemes are useful in our proofs in this section.

Finally, in Section 5, we shift our focus from lift-and-project analysis to the aforementioned non-commutative hypermatching scheme. We uncover some interesting properties of this scheme, and explore the usefulness of obtaining commutative subschemes from non-commutative schemes via contracting associates.

2 Preliminaries

In this section, we introduce the necessary definitions and notation in association schemes and lift-and-project methods for our subsequent discussion. We refer the reader to [5, 3, 18, 20] for a more thorough treatment of association schemes, and to [1] for a comprehensive analysis of lift-and-project operators in combinatorial optimization.

2.1 Association schemes

Given a set of matrices $\mathcal{A}$, we let Span $\mathcal{A}$ denote the set of matrices that can be expressed as linear combinations of matrices in $\mathcal{A}$. Then an *association scheme* is defined as follows.

Definition 1. Let Ω be a finite set and $\mathcal{I}$ be a set of indices. A set $\mathcal{A} = \{A_i\}_{i \in \mathcal{I}}$ of $|\Omega|$-by-$|\Omega|$ $0,1$-matrices is an *association scheme* (or simply, a *scheme*) if

(A1) $A_i = I$, the identity matrix, for some $i \in \mathcal{I}$,

(A2) $B \in \mathcal{A} \Rightarrow B^\top \in \mathcal{A}$,

(A3) $\sum_{i \in \mathcal{I}} A_i = J$ where J is the all-ones matrix, and

(A4) $A_i A_j \in \text{Span } \mathcal{A}$ for all $i, j \in \mathcal{I}$.

Note that our definition of an association scheme is more general than that in some existing literature, which further requires that the matrices in an association scheme commute with each other [11].

Each non-identity matrix of a scheme is referred to as an *associate*. A scheme is *commutative* if $A_i A_j = A_j A_i$ for all $i, j \in \mathcal{I}$. Moreover, if $\mathcal{A}$ is commutative, then Span $\mathcal{A}$ is a commutative matrix algebra called the *Bose–Mesner algebra* of $\mathcal{A}$. A very useful property of commutative schemes is that the eigenspaces of the matrices in the scheme are aligned. That is, there exists an orthonormal set of eigenvectors $\{v_i\}_{i \in \Omega}$ that are eigenvectors for all matrices in Span $\mathcal{A}$. Therefore, the eigenvalues of any matrix in Span $\mathcal{A}$ can be obtained by taking the corresponding linear combination of the eigenvalues of matrices in $\mathcal{A}$.

We call a given scheme *symmetric* if all of its associates are symmetric matrices. It then follows from property (A4) that if $\mathcal{A}$ is symmetric, then $A_i A_j = A_j A_i$ for all $i, j \in \mathcal{I}$, thus a symmetric scheme is also commutative. Not all commutative schemes are symmetric, but the commutative schemes that occur in this work will be.

For any commutative scheme $\mathcal{A}$ in which each associate has up to q distinct eigenvalues, there is a set of projection matrices $\{E_i\}_{i=1}^q$ where each E_i corresponds to one of the q eigenspaces of matrices in $\mathcal{A}$. Then one can define the *P-matrix* of $\mathcal{A}$ to be the q-by-$|\mathcal{I}|$ matrix where $P[i,j]$ is the eigenvalue of A_j corresponding to the projection matrix E_i. (Notice that P is necessarily a square matrix, as $|\mathcal{I}| = q$ follows from the fact that the primitive idempotents are a dual basis for the Bose–Mesner algebra.) Characterizing $\{E_i\}_{i=1}^q$ and P allows us to analyze any matrix Y in the Bose–Mesner algebra of $\mathcal{A}$ in a unified way. This is particularly helpful in SDP problems where all feasible solutions lie in Span $\mathcal{A}$, as it could allow us to reduce the dimension and complexity of the SDP problem significantly, which is usually very helpful both in practice and in theoretical analysis of such SDP problems. In particular, a key consequence is that since the eigenspaces of all associates A_i of $\mathcal{A}$ are aligned, for every matrix $Y \in \text{Span } \mathcal{A}$, any cone inequality based on the Loewner order can be rewritten as a set of equivalent linear inequalities. For example, given $B \in \text{Span } \mathcal{A}$, define $b \in \mathbb{R}^{\mathcal{I}}$ such that $B = \sum_{i \in \mathcal{I}} b_i A_i$. Then the constraint $Y \succeq B$ holds if and only if $P(y - b) \geqslant 0$, where the variable vector $y \in \mathbb{R}^{\mathcal{I}}$ represents the matrix Y via $Y = \sum_{i \in \mathcal{I}} y_i A_i$.

One of the most ubiquitous association schemes is the Johnson scheme. Let $[p] := \{1, \ldots, p\}$, and let $[p]_q := \{S \subseteq [p] : |S| = q\}$. Given integers p, q, and i where $0 \leqslant i \leqslant \min\{q, p-q\}$, define the matrix $J_{p,q,i}$ whose rows and columns are indexed by elements in $[p]_q$, such that

$$J_{p,q,i}[S,T] := \begin{cases} 1 & \text{if } |S \cap T| = q - i; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $J_{p,q,0}$ is the $\binom{p}{q}$-by-$\binom{p}{q}$ identity matrix, while $J_{p,q,q}$ is the adjacency matrix of the Kneser graph of the q-subsets of $[p]$. Given fixed p and q, the *Johnson scheme* is the set of matrices $\mathcal{J}_{p,q} := \{J_{p,q,i} : i = 0, \ldots, \min\{q, p-q\}\}$. It is easy to check that $\mathcal{J}_{p,q}$ indeed satisfies (A1)-(A4), and is symmetric (and hence commutative). The eigenvalues of the associates in $\mathcal{J}_{p,q}$ are well known (see, for instance, [14, 20]).

Proposition 2. *The eigenvalues of $J_{p,q,i} \in \mathcal{J}_{p,q}$ are*

$$\sum_{h=i}^{q} (-1)^{h-i+j} \binom{h}{i} \binom{p-2h}{q-h} \binom{p-h-j}{h-j} \tag{2}$$

$$= \sum_{h=0}^{q} (-1)^{h} \binom{j}{h} \binom{q-j}{i-h} \binom{p-q-j}{i-h}, \tag{3}$$

for $j \in \{0, 1, \ldots, q\}$.

Another important association scheme is the Hamming scheme. Given integers $p, q \geqslant 1$ and $i \in \{0, 1, \ldots, q\}$, define the matrix $H_{p,q,i}$ whose rows and columns are indexed by elements in $\{0, 1, \ldots, p-1\}^q$, such that

$$H_{p,q,i}[S,T] := \begin{cases} 1 & \text{if } S, T \text{ differ at exactly } i \text{ positions;} \\ 0 & \text{otherwise.} \end{cases}$$

Then $H_{p,q,i}$ is a p^q-by-p^q matrix, and the *Hamming scheme* $\mathcal{H}_{p,q}$ is the set of matrices $\{H_{p,q,i} : i = 0, \ldots, q\}$. As with the Johnson scheme, the Hamming scheme is symmetric and commutative, with well-known eigenvalues (see, for instance, [10]).

Proposition 3. *The eigenvalues of $H_{p,q,i} \in \mathcal{H}_{p,q}$ are*

$$\sum_{h=0}^{i} (-1)^{h} (p-1)^{i-h} \binom{j}{r} \binom{q-j}{i-h}$$

for $j \in \{0, 1, \ldots, q\}$.

2.2 Lift-and-project methods and the LS_+ operator

Before we combine our knowledge of association schemes with the analyses of lift-and-project relaxations, let us first put the lift-and-project approach into perspective and introduce some necessary notation.

When faced with a difficult combinatorial optimization problem, one common approach is to model it as a $0,1$-integer program of the form

$$\max\left\{c^{\top} x : x \in P \cap \{0,1\}^n\right\},$$

where the set $P \subseteq [0,1]^n$ is convex and tractable (i.e., we can optimize any linear function over it in polynomial time). While integer programs are $\mathcal{NP}$-hard to solve in general, we could discard the integrality constraint, simply optimize $c^\top x$ over P, and efficiently obtain an approximate solution to the given problem. Furthermore, one can aim to improve upon the initial relaxation P. More precisely, given $P \subseteq [0,1]^n$, we define its *integer hull* to be

$$P_I := \operatorname{conv}\left\{P \cap \{0,1\}^n\right\}.$$

When $P \neq P_I$, we strive to derive from P another set P' where $P_I \subseteq P' \subset P$. Ideally, this tighter set P' is also tractable, and we can then optimize the same objective function $c^\top x$ over P' instead of P, and obtain a potentially better approximate solution.

One way to systematically generate such a tighter relaxation is via the *lift-and-project* approach. While there are many known algorithms that fall under this approach (see, among others, [36, 4, 28, 9, 2]), we will focus on the LS_+-operator due to Lovász and Schrijver [33]. Given $P \subseteq [0,1]^n$, define the homogenized cone of P to be

$$K(P) := \left\{ \begin{bmatrix} \lambda \\ \lambda x \end{bmatrix} \in \mathbb{R}^{n+1} : \lambda \geqslant 0, x \in P \right\}.$$

We index the new coordinate by 0. Also, let e_i be the i^{th} unit vector, and recall that $\mathbb{S}^k$ denotes the set of k-by-k symmetric matrices. Then we define

$$\begin{aligned} \mathrm{LS}_+(P) \quad := \quad & \Big\{ x \in \mathbb{R}^n : \exists Y \in \mathbb{S}^{n+1}, \\ & Ye_0 = \operatorname{diag}(Y) = \begin{bmatrix} 1 \\ x \end{bmatrix}, \\ & Ye_i, Y(e_0 - e_i) \in K(P),\ \forall i \in [n], \\ & Y \succeq 0 \Big\}. \end{aligned}$$

Intuitively, the operator LS_+ *lifts* a given n-dimensional set P to a collection of $(n+1)$-by-$(n+1)$ matrices, imposes some constraints, and then *projects* the set back down to $\mathbb{R}^n$. Among other properties, $\mathrm{LS}_+(P)$ satisfies

$$P_I \subseteq \mathrm{LS}_+(P) \subseteq P$$

for every $P \subseteq [0,1]^n$. To see the first containment, note that for every integral vector $x \in P$, $Y := \begin{bmatrix} 1 \\ x \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}^\top$ satisfies all conditions of LS_+ and thus certifies $x \in \mathrm{LS}_+(P)$. For the second containment, let $x \in \mathrm{LS}_+(P)$ with certificate matrix Y that satisfies $Ye_i, Y(e_0 - e_i) \in K(P)$. Then $Ye_0 \in K(P)$ (since the cone $K(P)$ is closed under vector addition), certifying $x \in P$.

Thus, compared to P, $\mathrm{LS}_+(P)$ contains exactly the same collection of integer solutions, while providing a tighter relaxation of P_I. Also, if P is tractable, so is $\mathrm{LS}_+(P)$, as optimizing a linear function over this set amounts to solving a semidefinite program whose number of variables and constraints depend polynomially on that of P.

Moreover, LS_+ can be applied iteratively to a set P to obtain yet tighter relaxations. If we let $\mathrm{LS}_+^k(P)$ denote the set obtained from k successive applications of LS_+ to P, then it holds in general that

$$P \supseteq \mathrm{LS}_+(P) \supseteq \mathrm{LS}_+^2(P) \supseteq \cdots \supseteq \mathrm{LS}_+^n(P) = P_I.$$

Thus, for every set $P \subseteq [0,1]^n$, LS_+ generates a hierarchy of progressively tighter relaxations of P_I, with the guarantee that the operator reaches P_I in at most n iterations. For a proof of these properties as well as other aspects of LS_+, the reader may refer to [33].

Given a set $P \subseteq [0,1]^n$, we define the LS_+-*rank* of P to be the smallest integer k where $\mathrm{LS}_+^k(P) = P_I$. The notion of lift-and-project rank gives us a measure of how far a given relaxation P is from its integer hull P_I with respect to the given lift-and-project operator. In particular, a relaxation having a high LS_+-rank could indicate that the underlying integer hull is difficult to solve for, and/or that the operator LS_+ is not well suited to tackle this particular problem.

To establish a lower bound on the LS_+-rank of a set, a standard approach is to show that there exists a point $\bar{x} \notin P_I$ that is contained $\text{LS}_+^k(P)$, which implies that the LS_+-rank of P is at least $k+1$. Verifying $\bar{x} \in \text{LS}_+^k(P)$ would require finding a *certificate matrix* Y that satisfies all conditions specified in the definition of LS_+. As we shall see, this is where symmetries in the given problem can be immensely useful. In particular, the task of establishing the positive semidefiniteness of Y could be significantly simplified by relating it to matrices from association schemes whose eigenvalues are known. For instance, Georgiou [17] used the known eigenvalues of the Johnson scheme when establishing a lower bound on the Lasserre-rank of a relaxation related to the max-cut problem. We shall see a few other examples of this application in this manuscript.

3 Bounding the clique and stability numbers of graphs

In this section, we focus on the SDP obtained from applying a single iteration of LS_+ to the standard LP relaxation of the stable set problem of graphs. Part of the goal of this section is to introduce the proof techniques that we will need subsequently in this relatively elementary and well-known setting. We will also highlight several points in our discussion that we will revisit in greater depth and complexity in Sections 4 and 5.

The structure of this section will be as follows. First, in Section 3.1, we look into the LS_+-relaxation of the stable set polytope of graphs, and prove our main result of the section (Proposition 4). Along the way, we will introduce the notion of deeply vertex-transitive graphs, which are graphs with very rich symmetries. We then show in Section 3.2 how deeply vertex-transitive graphs can be constructed from some association schemes. Finally, in Section 3.3, we relate Proposition 4 to some classic results, such as bounds on the clique and chromatic numbers due to Delsarte [14] and Hoffman [25].

3.1 LS_+-relaxations of the stable set polytope and deeply vertex-transitive graphs

Given a simple graph G, we define the *fractional stable set polytope* of G to be

$$\text{FRAC}(G) := \left\{ x \in [0,1]^{V(G)} : x_i + x_j \leqslant 1,\ \forall \{i,j\} \in E(G) \right\}.$$

We also define the *stable set polytope* to be $\text{STAB}(G) := \text{FRAC}(G)_I$, the convex hull of the integral vectors in $\text{FRAC}(G)$. Notice that a vector $x \in \{0,1\}^{V(G)}$ is contained in $\text{STAB}(G)$ if and only if it is the characteristic vector of a stable set in G. We let $\alpha(G)$ denote the stability number of G (i.e., the size of the largest stable set in G). We also let $\bar{e}$ be the all-ones vector (of appropriate dimensions). Then we see that

$$\alpha(G) = \max \left\{ \bar{e}^\top x : x \in \text{STAB}(G) \right\}. \tag{4}$$

Given a graph G, we define

$$\alpha_{\text{LS}_+}(G) := \max \left\{ \bar{e}^\top x : x \in \text{LS}_+(\text{FRAC}(G)) \right\}. \tag{5}$$

Since $\text{STAB}(G) \subseteq \text{LS}_+(\text{FRAC}(G))$ for every graph G, $\alpha(G) \leqslant \alpha_{\text{LS}_+}(G)$. Therefore, while it is $\mathcal{NP}$-hard to compute $\alpha(G)$ for a general graph, we can obtain an upper bound on $\alpha(G)$ by solving (5), which is a semidefinite program of manageable size. It is known that many classical families of inequalities that are valid for $\mathrm{STAB}(G)$ are also valid for $\mathrm{LS}_+(\mathrm{FRAC}(G))$, including (among others) clique, odd cycle, odd antihole, and wheel inequalities [33, 31]. Moreover, given a graph G, consider the theta body of G, defined as follows:

$$\begin{aligned}\mathrm{TH}(G) \ := \ \Big\{ & x \in \mathbb{R}^n : \exists Y \in \mathbb{S}^{n+1}, \\ & Ye_0 = \mathrm{diag}(Y) = \begin{bmatrix} 1 \\ x \end{bmatrix}, \\ & Y[i,j] = 0, \forall \{i,j\} \in E(G), \\ & Y \succeq 0. \Big\}\end{aligned}$$

(See [33] for a proof, as well as remarks on how the above is equivalent to the conventional definition of the theta body.) From their definitions, it is apparent that $\mathrm{LS}_+(\mathrm{FRAC}(G)) \subseteq \mathrm{TH}(G)$ for all graphs G. Thus, if we define $\theta(G) := \max\left\{\bar{e}^\top x : x \in \mathrm{TH}(G)\right\}$, it follows that $\alpha_{\mathrm{LS}_+}(G) \leqslant \theta(G)$ for all graphs.

There has also been recent interest [7, 8, 40] in classifying LS_+-*perfect graphs*, which are graphs G where $\mathrm{LS}_+(\mathrm{FRAC}(G)) = \mathrm{STAB}(G)$. Since the stable set polytope of a perfect graph is defined by only clique and non-negative inequalities, every perfect graph is also LS_+-perfect.

For our analysis of $\mathrm{LS}_+(\mathrm{FRAC}(G))$, we are particularly interested in graphs that have plenty of symmetries. Given a vertex $i \in V(G)$, we let $G \ominus i$ denote the subgraph of G induced by vertices that are neither i nor adjacent to i. (Equivalently, we obtain $G \ominus i$ from G by removing the closed neighborhood of i.) Then, we say that a graph G is *deeply vertex-transitive* if:

(i) G is vertex-transitive;

(ii) $G \ominus i$ is vertex-transitive for every $i \in V(G)$;

(iii) $\overline{G}$, the complement graph of G, contains a connected component that is not a complete graph.

Notice that if G is vertex-transitive, then the graphs $\{G \ominus i : i \in V(G)\}$ are all isomorphic. Thus, for G to satisfy (ii), it suffices to check if $G \ominus i$ is vertex-transitive for an arbitrary vertex i. Also, (iii) guarantees that the subgraph $G \ominus i$ would contain at least one edge for every vertex i.

Finally, given a graph G that is k-regular with $n \geqslant 4$ vertices, we define the quantity

$$\Delta(G) := \begin{cases} \frac{2n-3k}{n-3} & \text{if } G \text{ contains a triangle;} \\ \frac{n-3k+2+\sqrt{(n-3k+2)^2+4(n-3)(n-2k)}}{2(n-3)} & \text{otherwise.} \end{cases}$$

Notice that, whenever $n \geqslant 4$ and $k \geqslant 1$,

$$\begin{aligned} & \frac{n-3k+2+\sqrt{(n-3k+2)^2+4(n-3)(n-2k)}}{2(n-3)} \\ \leqslant \ & \frac{n-3k+2+\sqrt{(n-3k+2)^2}+\sqrt{4(n-3)(n-2k)}}{2(n-3)} \\ \leqslant \ & \frac{n-3k+2+(n-3k+2)+2(n-k-\frac{3}{2})}{2(n-3)} \\ \leqslant \ & \frac{2n-3k}{n-3}. \end{aligned}$$

Thus, $\Delta(G) \leqslant \frac{2n-3k}{n-3} \leqslant 2$ for all regular graphs G on 4 or more vertices. The following is the main result of this section.

Proposition 4. *Suppose G is a k-regular graph on $n \geqslant 4$ vertices, and let λ_2 be the second largest eigenvalue of G. Define $a := \max\{1, \lambda_2, \Delta(G)\}$. Then*

$$\alpha_{\mathrm{LS}_+}(G) \geqslant \frac{n-k+a}{a+1}. \tag{6}$$

Moreover, if G is deeply vertex-transitive, then equality holds in (6)*.*

Proof. We first prove the inequality for all k-regular graphs. For convenience, let $\beta_1 := \frac{2n-3k}{n-3}$ and $\beta_2 := \frac{n-3k+2+\sqrt{(n-3k+2)^2+4(n-3)(n-2k)}}{2(n-3)}$ throughout this proof. First, we will show that the given bound applies for $a := \max\{1, \lambda_2, \beta_1\}$ for all graphs, and then mention how, in the case of triangle-free graphs, we might be able to use a smaller a and potentially obtain a better lower bound for $\alpha_{\mathrm{LS}_+}(G)$.

Let $A(\overline{G})$ be the adjacency matrix of the complement of G. Note that the largest and smallest eigenvalues of $A(\overline{G})$ are $n-k-1$ and $-1-\lambda_2$, respectively. Also, let $d := \frac{n(a+1)^2}{n-k+a}$, and we show that the certificate matrix $Y := \frac{1}{d}\begin{bmatrix} d & (a+1)\bar{e}^\top \\ (a+1)\bar{e} & (a+1)I + A(\overline{G}) \end{bmatrix}$ satisfies all conditions of LS_+. First, for every $i \in V(G)$,

$$(Ye_i)[j] = \begin{cases} \frac{a+1}{d} & \text{if } j = 0 \text{ or } j = i; \\ 0 & \text{if } j \in V(G) \text{ is adjacent to } i; \\ \frac{1}{d} & \text{otherwise.} \end{cases}$$

Since $a \geqslant 1$, the above vector must belong to $K(\mathrm{FRAC}(G))$. Likewise, notice that $Ye_i \geqslant \begin{bmatrix} \frac{a+1}{d} \\ 0 \end{bmatrix}$ for all i. Hence,

$$Ye_0 - Ye_i \leqslant \begin{bmatrix} 1 \\ \frac{a+1}{d}\bar{e} \end{bmatrix} - \begin{bmatrix} \frac{a+1}{d} \\ 0 \end{bmatrix} = \frac{1}{d}\begin{bmatrix} d-(a+1) \\ (a+1)\bar{e} \end{bmatrix}.$$

Now, observe that $a \geqslant \frac{2n-3k}{n-3} = \beta_1$ if and only if $d \geqslant 3(a+1)$. Thus,

$$Ye_0 - Ye_i = \frac{1}{d}\begin{bmatrix} d-(a+1) \\ (a+1)\bar{e} \end{bmatrix} \leqslant \frac{d-(a+1)}{d}\begin{bmatrix} 1 \\ \frac{1}{2}\bar{e} \end{bmatrix},$$

and so $Ye_0 - Ye_i \in K(\mathrm{FRAC}(G))$ for every $i \in V(G)$.

Finally, notice that the minimum eigenvalue of $(a+1)I + A(\overline{G})$ is $(a+1) + (-1-\lambda_2)$, which is nonnegative since $a \geqslant \lambda_2$. Thus, using the Schur complement, we see that $Y \succeq 0$ if and only if the eigenvalue of

$$(a+1)I + A(\overline{G}) - \frac{1}{d}(a+1)\bar{e}(a+1)\bar{e}^\top = (a+1)I + A(\overline{G}) - \frac{n-k-a}{n}J$$

corresponding to $\bar{e}$ is non-negative. Indeed, one can check that this eigenvalue is $2a$, which is positive.

Since all conditions of LS_+ are met, we conclude that $\frac{a+1}{d}\bar{e} \in \mathrm{LS}_+(G)$. Therefore,

$$\alpha_{\mathrm{LS}_+}(G) \geqslant \bar{e}^\top\left(\frac{a+1}{d}\bar{e}\right) = \frac{n-k+a}{a+1}$$

for $a := \max\{1, \lambda_2, \beta_1\}$ for all k-regular graphs.

We now consider the case where G does not contain a triangle. Let us take a closer look at the vector $Ye_0 - Ye_i$. Observe that

$$(Ye_0 - Ye_i)[j] = \begin{cases} 1 - \frac{a+1}{d} & \text{if } j = 0; \\ 0 & \text{if } j = i; \\ \frac{a+1}{d} & \text{if } j \in V(G) \text{ is adjacent to } i; \\ \frac{a}{d} & \text{otherwise.} \end{cases}$$

Since G does not have a triangle, there are no edges $\{j_1, j_2\} \in E(G)$ where both j_1, j_2 are adjacent to i. Thus, to ensure $Ye_0 - Ye_i \in K(\mathrm{FRAC}(G))$, it suffices to have

$$\frac{a+1}{d} + \frac{a}{d} \leqslant 1 - \frac{a+1}{d},$$

or equivalently $d \geqslant 3a + 2$. This holds if $a \geqslant \frac{n-3k+2+\sqrt{(n-3k+2)^2+4(n-3)(n+2k)}}{2(n-3)} = \beta_2$. This proves the bound for graphs without triangles.

We next prove the reverse inequality for deeply vertex-transitive graphs, and again first establish the case where G contains a triangle. Notice that if G is deeply vertex-transitive, then $\lambda_2 > -1$. To see that, let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of G. If $\lambda_2 \leqslant -1$, then $\sum_{i=2}^{n} \lambda_i \leqslant (n-1)\lambda_2 \leqslant -(n-1)$. On the other hand, it is true for all graphs that $\lambda_1 \leqslant n-1$ and the n eigenvalues sum to zero. This implies that $\lambda_1 = n-1$ and $\lambda_2 = \cdots = \lambda_n = -1$ in this case. Thus G is a complete graph, which is not deeply vertex-transitive since its complement does not contain a component that is not complete. Hence, we will assume that $\lambda_2 > -1$ for the rest of the proof.

Next, let $\bar{x} \in \mathrm{LS}_+(\mathrm{FRAC}(G))$ where $\bar{e}^\top \bar{x} = \alpha_{\mathrm{LS}_+}(G)$. Then there must be a matrix $\bar{M}$ such that $Y = \begin{bmatrix} 1 & \bar{x}^\top \\ \bar{x} & \bar{M} \end{bmatrix}$ satisfies all conditions imposed by LS_+. Now, let P_σ be a permutation matrix corresponding to $\sigma \in \mathrm{Aut}(G)$, the automorphism group of G. Consider the matrix

$$Y' := \frac{1}{|\operatorname{Aut}(G)|} \sum_{\sigma \in \operatorname{Aut}(G)} \begin{bmatrix} 1 & (P_\sigma \bar{x})^\top \\ P_\sigma \bar{x} & P_\sigma \bar{M} P_\sigma^\top \end{bmatrix}.$$

Since $\sigma \in \mathrm{Aut}(G)$, every matrix in the sum above also satisfies all conditions for LS_+. This implies that Y' is also a certificate matrix (since the set of certificate matrices is a convex set), and that $Y'e_0 \in K(\mathrm{LS}_+(\mathrm{FRAC}(G)))$. Also, since G is vertex-transitive, it must be the case that $Ye_0 = \begin{pmatrix} 1 \\ b\bar{e} \end{pmatrix}$ for some constant b. Moreover, since $\bar{e}^\top \bar{x} = \bar{e}^\top (P_\sigma \bar{x})$ for every permutation σ, $\bar{e}^\top (b\bar{e}) = \bar{e}^\top \bar{x} = \alpha_{\mathrm{LS}_+}(G)$.

Furthermore, since G is deeply vertex-transitive, for every $i \in V(G)$ and $j_1, j_2 \in V(G \ominus i)$, there exists $\sigma \in \mathrm{Aut}(G)$ where $\sigma(i) = i$ and $\sigma(j_1) = j_2$. Hence, it follows that $Y'[i, j_1] = Y'[i, j_2]$ for all $j_1, j_2 \in V(G \ominus i)$. Since Y' is symmetric, we deduce that $Y'[i, j]$ must be constant over all distinct $i, j \in V(G)$ where $\{i, j\} \notin E(G)$. Therefore, we see that $Y' = \begin{bmatrix} 1 & b\bar{e}^\top \\ b\bar{e} & bI + cA(\overline{G}) \end{bmatrix}$ for some real numbers b and c. Now consider some of the restrictions LS_+ imposes on Y' (and hence b and c). First, $bI + cA(\overline{G}) \succeq 0$ implies that

$$c \leqslant \frac{b}{\lambda_2 + 1}. \tag{7}$$

Note that we applied the assumption $\lambda_2 > -1$ here. Likewise, $bI + cA(\overline{G}) - (b\bar{e})(b\bar{e}^\top) \succeq 0$ implies

$$b + c(n-k-1) - b^2 n \geqslant 0. \tag{8}$$

Next, since $\overline{G}$ contains a component that is not the complete graph, there exists vertex i in this component that is adjacent to vertices j_1, j_2 where $\{j_1, j_2\} \notin E(\overline{G})$. This means that the subgraph $(G \ominus i)$ contains at least one edge $\{j_1, j_2\}$, and so the condition $Ye_i \in K(\mathrm{FRAC}(G))$ imposes that $Ye_i[j_1] + Ye_i[j_2] \leqslant Ye_i[0]$, which implies that

$$c \leqslant \frac{b}{2}. \tag{9}$$

Likewise, since G is vertex-transitive and contains a triangle by assumption, for every vertex $i \in V(G)$ there exist two vertices j_1, j_2 that are adjacent to i while $\{j_1, j_2\} \in E(G)$. Then since $(Ye_0 - Ye_i)[j_1] = (Ye_0 - Ye_i)[j_2] = b$ and $(Ye_0 - Ye_i)[0] = 1 - b$, the constraint

$$(Ye_0 - Ye_i)[j_1] + (Ye_0 - Ye_i)[j_2] \leqslant (Ye_0 - Ye_i)[0]$$

implies

$$b \leqslant \frac{1}{3}. \tag{10}$$

Now, if we define $a' := \max\{\lambda_2, 1\}$, then (7) and (9) hold if and only if $c \leqslant \frac{b}{a'+1}$. Combining this with (10) and (8) yields $b \leqslant \min\left\{\frac{1}{n}\left(\frac{n-k+a'}{a'+1}\right), \frac{1}{3}\right\}$. Since $\frac{1}{n}\left(\frac{n-k+a'}{a'+1}\right) \leqslant \frac{1}{3}$ if and only if $a' \geqslant \beta_1(G)$, this finishes the proof for graphs that contain a triangle.

Finally, if G does not contain a triangle, then instead of imposing $3b \leqslant 1$ as in (10), the condition $Ye_0 - Ye_i \in K(\mathrm{FRAC}(G))$ would impose the weaker inequality $3b - c \leqslant 1$. Combined with (7), (9), and (8), we obtain that $b \leqslant \frac{n-k+a'}{a'+1}$ where $a' \geqslant \max\{1, \lambda_2, \beta_2\}$. This finishes the proof. □

Given a regular graph G, if we know that $\alpha(G) < \frac{n-k+a}{a+1}$, then Proposition 4 implies that $\alpha(G) < \alpha_{\mathrm{LS}_+}(G)$, and consequently $\mathrm{LS}_+(\mathrm{FRAC}(G)) \neq \mathrm{STAB}(G)$. On the other hand, given a deeply vertex-transitive graph G, Proposition 4 determines $\alpha_{\mathrm{LS}_+}(G)$ and implies that $\alpha(G) \leqslant \lfloor\frac{n-k+a}{a+1}\rfloor$. We also remark that the same ingredients used in the proof of Proposition 4 can be used to prove a similar (but weaker) bound for $\theta(G)$:

Proposition 5. *Suppose G is a k-regular, non-complete graph on n vertices, and let λ_2 be the second largest eigenvalue of G. Then*

$$\theta(G) \geqslant \frac{n-k+\lambda_2}{\lambda_2+1}. \tag{11}$$

Moreover, if G is deeply vertex-transitive, then equality holds in (11).

Thus, it follows from Propositions 4 and 5 that $\alpha_{\mathrm{LS}_+}(G) < \theta(G)$ in some situations when $\lambda_2 < 1$, as we shall see in Example 6.

Example 6. Let G be the odd antihole with $n = 2\ell + 1$ vertices, for some integer $\ell \geqslant 2$. Then G is deeply vertex-transitive (notice that $G \ominus i$ is the complete graph on 2 vertices for every vertex i), with $\lambda_2 = -1 + 2\cos\left(\frac{\pi}{2\ell+1}\right) < 1$, and $\Delta(G) \leqslant 1$ for all $\ell \geqslant 2$. In this case we have $\theta(G) = 1 + \sec\left(\frac{\pi}{2\ell+1}\right) > 2$, while $\alpha_{\mathrm{LS}_+}(G) = 2 = \alpha(G)$.

Next, we provide another example showing that the bound in Proposition 4 can indeed be not tight when G is not deeply vertex-transitive.

Example 7. Let G be the 7-cycle (which obviously does not contain a triangle). Then G is 2-regular with $\lambda_2 = 2\cos\left(\frac{2\pi}{7}\right) \approx 1.247$, and $\Delta(G) = \frac{3+\sqrt{57}}{8} \approx 1.319$. Thus, $a = \frac{3+\sqrt{57}}{8}$, and Proposition 4 implies that $\alpha_{\mathrm{LS}_+}(G) \geqslant 2.72$, which is not tight as $\alpha(G) = \alpha_{\mathrm{LS}_+}(G) = 3$.

In this case, while it is true that $\frac{3}{7}\bar{e} \in \mathrm{LS}_+(\mathrm{FRAC}(G))$, we cannot use the symmetry reduction in the proof of Proposition 4 to deduce that $\frac{3}{7}\bar{e}$ has a certificate matrix with constant non-edge entries, due to the fact that $G \ominus i$ is not vertex-transitive for any vertex i. In fact, one can check that the unique matrix that certifies $\frac{3}{7}\bar{e} \in \mathrm{LS}_+(\mathrm{FRAC}(G))$ is

$$Y = \frac{1}{7}\begin{bmatrix} 7 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 0 & 2 & 1 & 1 & 2 & 0 \\ 3 & 0 & 3 & 0 & 2 & 1 & 1 & 2 \\ 3 & 2 & 0 & 3 & 0 & 2 & 1 & 1 \\ 3 & 1 & 2 & 0 & 3 & 0 & 2 & 1 \\ 3 & 1 & 1 & 2 & 0 & 3 & 0 & 2 \\ 3 & 2 & 1 & 1 & 2 & 0 & 3 & 0 \\ 3 & 0 & 2 & 1 & 1 & 2 & 0 & 3 \end{bmatrix}.$$

We also remark that deeply vertex-transitivity is incomparable with arc-transitivity, a well-studied property that also characterizes graphs with rich symmetries (see, for instance, [23, Chapter 4]). For example, odd antiholes are deeply vertex-transitive but not arc-transitive (or even edge-transitive), while the opposite is true for odd cycles of length at least 7.

3.2 Constructing deeply vertex-transitive graphs from association schemes

Next, we describe how deeply vertex-transitive graphs arise from some association schemes. Given two vertices $i, j \in V(G)$, their distance $\delta(i,j)$ is the number of edges in the shortest path joining i and j in G, and we define the diameter of a connected graph G to be $\delta(G) := \max\{\delta(i,j) : i, j \in V(G)\}$. Then a graph G is *distance-regular* if, given integers d_1, d_2, the quantity

$$|\{j \in V(G) : \delta(i_1, j) = d_1, \delta(i_2, j) = d_2\}|$$

is invariant under the choice of $i_1, i_2 \in V(G)$ as long as $\delta(i_1, i_2)$ is fixed. Now, let G be a connected and non-complete graph (so $\delta(G)$ is finite and at least 2) that is also distance-regular. Given $d \in [\delta(G)]$, let $G^{(d)}$ denote the graph with the same vertex set as G, and two vertices in $G^{(d)}$ are joined by an edge if they are exactly at distance d from each other in G. Then we have the following.

Lemma 8. *Suppose G is connected, non-complete, distance-regular and vertex-transitive. If $G^{(d)}$ contains a connected component that is not a complete graph, then $\overline{G^{(d)}}$ is deeply vertex-transitive for every $d \in [\delta(G)]$.*

Proof. Since G is vertex-transitive, $G^{(d)}$ is vertex-transitive for all d, and so is its complement $\overline{G^{(d)}}$. Next, consider a vertex $i \in V(G)$ and let $H := \overline{G^{(d)}} \ominus i$. Notice that H consists of the vertices in G that are exactly distance d from i in G. Now given $j_1, j_2 \in V(H)$, there exists $\sigma \in \mathrm{Aut}(G)$ where $\sigma(j_1) = j_2$ (since G is vertex transitive). Since any automorphism of a graph preserves distances between vertices, we see that $\{\sigma(j) : j \in V(H)\} = V(H)$, and we obtain an automorphism on H by restricting σ to the vertices of H. Thus, $\overline{G^{(d)}} \ominus i$ must be vertex-transitive for all i.

Thus, with the additional assumptions that G is connected, non-complete, and that $G^{(d)}$ (which is the complement of $\overline{G^{(d)}}$) contains a non-complete component, we obtain that $\overline{G^{(d)}}$ is deeply vertex-transitive. □

Let $A(G)$ denote the adjacency matrix of a given graph G, and conversely given a symmetric $0,1$-matrix A we let $G(A)$ denote the undirected graph whose adjacency matrix is A. It is well known that, given a distance-regular graph G, the set of matrices

$$\mathcal{A} := \left\{I, A\left(G^{(1)}\right), \ldots, A\left(G^{(\delta(G))}\right)\right\}$$

is a commutative association scheme. In this case, we say that the scheme $\mathcal{A}$ is *metric* with respect to the distance-regular graph $G^{(1)}$. Thus, Lemma 8 gives us a way to generate families of deeply vertex-transitive graphs based on metric association schemes. We will now look at a few such examples.

First, the Johnson scheme $\mathcal{J}_{p,q}$ is metric with respect to $G(J_{p,q,1})$ (see, for instance, [18, Section 2.3]). Thus, we can use Proposition 4 to verify if optimizing over $\mathrm{LS}_+(\mathrm{FRAC}(G))$ gives the correct stability number of some deeply vertex-transitive graphs related to the Johnson scheme.

Proposition 9. *Given integers $p \geqslant 4$ and $q \geqslant 2$,*

(i) Let $G := \overline{G(J_{p,q,1})}$. Then

$$\alpha_{\mathrm{LS}_+}(G) = p - q + 1 = \alpha(G).$$

(ii) Further suppose that $p \geqslant 2q$ (so $G(J_{p,q,q})$ is not an empty graph), and let $G := \overline{G(J_{p,q,q})}$. Then

$$\alpha_{\mathrm{LS}_+}(G) = \frac{p}{q},$$

which is equal to $\alpha(G) = \lfloor \frac{p}{q} \rfloor$ if and only if $q|p$.

Proof. For (i), we see that G has $n = \binom{p}{q}$ vertices, each with degree $k = \binom{p}{q} - q(p-q) - 1$. Also, with $p \geqslant 4$ and $q \geqslant 2$, $\overline{G}$ indeed has a non-complete component, and thus G must be deeply vertex-transitive. From Proposition 2, one obtains the minimum eigenvalue of $J_{p,q,1}$ is $-q$, occurring when $j = q$. Thus, the second largest eigenvalue of G is $\lambda_2 = q - 1 \geqslant 1$. Also, observe that

$$\Delta(G) \leqslant \frac{2n - 3k}{n-3} = \frac{3q(p-q) - 3 - \binom{p}{q}}{\binom{p}{q} - 3} \leqslant \frac{3(\frac{p}{2})(p - \frac{p}{2}) - 3 - \binom{p}{2}}{\binom{p}{2} - 3} \leqslant 1$$

for all $p \geqslant 4$. Thus, $a = \lambda_2$, and we obtain from Proposition 4 that

$$\alpha_{\mathrm{LS}_+}(G) = \frac{n - k + \lambda_2}{\lambda_2 + 1} = p - q + 1.$$

Since $\alpha(G) \leqslant \alpha_{\mathrm{LS}_+}(G)$ in general, it only remains to show that $\alpha(G) \geqslant p - q + 1$. If we let $S_j := \{[q-1] \cup \{j\}\}$, then it is easy to check that $S_q, S_{q+1}, \ldots, S_p$ form a stable set in G as any two of these sets have $q-1$ elements in common. Thus, (i) follows.

The proof of (ii) is similar. In this case, $n = \binom{p}{q}$, $k = \binom{p}{q} - \binom{p-q}{q} - 1$, and $\lambda_2 = \binom{p-q-1}{q-1} - 1$. Notice that $\lambda_2 \geqslant 2$ for all $p \geqslant 4$ and $q \geqslant 2$, while $\Delta(G) \leqslant 2$ for all graphs G. Thus, again, G is deeply vertex-transitive and $a = \lambda_2$, so Proposition 4 implies that

$$\alpha_{\mathrm{LS}_+}(G) = \frac{n - k + \lambda_2}{\lambda_2 + 1} = \frac{p}{q}.$$

On the other hand, a stable set in G corresponds to a collection of disjoint q-subsets of $[p]$, and so $\alpha(G) = \lfloor \frac{p}{q} \rfloor$. This finishes our proof. $\square$

Proposition 9(ii) implies that when q does not divide p, the LS_+-rank of $\mathrm{FRAC}(\overline{G(J_{p,q,q})})$ is at least 2. We will revisit these polytopes from a different perspective and determine their exact LS_+-rank when we study matchings in hypergraphs in Section 4.

3.3 Relating Proposition 4 to classic bounds on clique and chromatic numbers

Next, we relate Proposition 4 to some well-known bounds on the clique and chromatic numbers of graphs. For ease of comparison, let us restate Proposition 4 in terms of the clique number of a graph. If we let $\omega(G)$ be the size of the largest clique in G, then Proposition 4 readily implies the following:

Corollary 10. *Suppose G is a graph whose complement $\overline{G}$ is deeply vertex-transitive. If we let λ_1 and λ_n be the maximum and minimum eigenvalues of G respectively, then*

$$\omega(G) \leqslant \left\lfloor 1 - \frac{\lambda_1}{\min\left\{\lambda_n, -2, -1-\Delta(\overline{G})\right\}} \right\rfloor.$$

Proof. Let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of G, and likewise $\overline{\lambda}_1 \geqslant \overline{\lambda}_2 \geqslant \cdots \geqslant \overline{\lambda}_n$ be the eigenvalues of $\overline{G}$. Observe that $\lambda_1 = n - \overline{\lambda}_1 - 1$ and $\lambda_n = -1 - \overline{\lambda}_2$. Also, let $\overline{a} := \max\left\{1, \overline{\lambda}_2, \Delta(\overline{G})\right\}$ and $a' := \min\left\{\lambda_n, -2, -1-\Delta(\overline{G})\right\}$, then $a' = -1 - \overline{a}$ regardless of the values of $\overline{\lambda}_2$ and $\Delta(\overline{G})$. Next, applying Proposition 4 to $\overline{G}$, we obtain

$$\alpha(\overline{G}) \leqslant \alpha_{\mathrm{LS}_+}(\overline{G}) = \frac{n - \overline{\lambda}_1 + \overline{a}}{\overline{a}+1} = 1 - \frac{\lambda_1}{a'}.$$

Since it is obvious that $\omega(G)$ is an integer and is equal to $\alpha(\overline{G})$ for every graph, the claim follows. □

Notice that $\overline{G}$ is deeply vertex-transitive if and only if G is vertex-transitive, and the subgraph induced by the vertices adjacent to any fixed vertex $i \in V(G)$ is also vertex-transitive and not a clique. We next relate Corollary 10 to some well-known results. First, the following is due to Delsarte [14], which establishes a similar upper bound on the clique number for a different family of graphs.

Proposition 11. *Let G be a graph whose adjacency matrix A is an associate in a commutative association scheme. Then*

$$\omega(G) \leqslant \left\lfloor 1 - \frac{\lambda_1}{\lambda_n} \right\rfloor.$$

Thus, Corollary 10 could provide a tighter upper bound than Delsarte's in cases where $\lambda_n > \min\left\{-2, -1-\Delta(\overline{G})\right\}$. (See, for instance, [12] for more on the rather restrictive families of graphs where $\lambda_n > -2$.) Corollary 10 also covers some graphs whose adjacency matrix does not belong to an association scheme. One uninteresting class of such examples is vertex-transitive graphs whose girth is at least 4. Here, the subgraph induced by any neighborhood of a vertex is an empty graph, which is vertex-transitive. However, in this case it is obvious that $\omega(G) = 2$. Another example is the icosahedron, where $\lambda_1 = 5, \lambda_n = -\sqrt{5}$, $\Delta(\overline{G}) = \frac{2}{3}$, and Corollary 10 does give a tight bound for the clique number for the graph.

In addition to Delsarte's result, Hoffman [25] has a similar bound on $\chi(G)$, the chromatic number of a graph.

Proposition 12. *Let G be a graph. Then*

$$\chi(G) \geqslant \left\lceil 1 - \frac{\lambda_1}{\lambda_n} \right\rceil.$$

Since $\chi(G) = \omega(G)$ for perfect graphs, combining Hoffman's bound and Corollary 10 implies that if G is perfect and deeply vertex-transitive, then $\lambda_n \leqslant -2, \omega(G) = 1 - \frac{\lambda_1}{\lambda_n}$, and that λ_n must divide λ_1. More recently, Godsil et al. [22, Lemma 5.2] showed that $\theta(\overline{G}) = 1 - \frac{\lambda_1}{\lambda_n}$ if G is 1-homogeneous, thus implying that $\omega(G) \leqslant \lfloor 1 - \frac{\lambda_1}{\lambda_n} \rfloor$ for these graphs. 1-homogeneous graphs contain graphs that are both vertex-transitive and edge-transitive, and can be shown to be incomparable with deeply vertex transitive graphs using the same odd antihole and odd cycle examples mentioned earlier.

Finally, we conclude this section by considering another example that highlights an idea we will discuss further in Sections 4 and 5. Given an integer $\ell \geqslant 2$, consider the Hamming scheme $\mathcal{H}_{2,2\ell+1}$, and define the graph

$$G_\ell := G(H_{2,2\ell+1,\ell} + H_{2,2\ell+1,\ell+1}).$$

In other words, G_ℓ has vertex set $\{0,1\}^{2\ell+1}$, and two vertices are joined by an edge if their corresponding binary strings differ by ℓ or $\ell+1$ positions. Then we have the following:

Proposition 13. *For every $\ell \geqslant 2$,*

$$\omega(G_\ell) \leqslant 2\ell + 2. \tag{12}$$

Proof. We provide two proofs to this claim. First, it is not hard to check that $\overline{G_\ell}$ is deeply vertex-transitive, so Corollary 10 applies. Observe that G_ℓ is $2\binom{2\ell+1}{\ell}$-regular, so $\lambda_1 = 2\binom{2\ell+1}{\ell}$. For the least eigenvalue, it follows from [10, Proposition 2.2] that $\lambda_n = \frac{-4}{\ell+1}\binom{2\ell-1}{\ell}$ (notice that $\lambda_n \leqslant -2$ for all $\ell \geqslant 2$). Also, one can check that $\Delta(\overline{G_\ell}) < 1$ for all $\ell \geqslant 2$. Thus, we obtain that

$$\omega(G_\ell) \leqslant 1 - \frac{2\binom{\ell+1}{\ell}}{\frac{-4}{\ell+1}\binom{2\ell-1}{\ell}} = 2\ell + 2.$$

We now present a second proof that utilizes Delsarte's bound (Proposition 11). While G_ℓ is not the graph of a single associate in the Hamming scheme, we can define an alternative association scheme of which $A(G_\ell)$ is an associate. For each $j \in [\ell]$, we define

$$B_j := H_{2,2\ell+1,2\ell+1-j} + H_{2,2\ell+1,j}.$$

Then one can check that

$$\mathcal{H}' := \{I, B_1, B_2, \ldots, B_\ell\}$$

is indeed a commutative association scheme. We mention the details of verifying (A4) (as (A1) to (A3) are relatively straightforward to check). Let $P := H_{2,2\ell+1,2\ell+1}$ for convenience. Notice that P is a permutation matrix that satisfies $P^2 = I$, and that

$$H_{2,2\ell+1,2\ell+1-j} = PH_{2,2\ell+1,j} = H_{2,2\ell+1,j}P$$

for every $j \in \{0, \ldots, 2\ell+1\}$. Now, given $B_i, B_j \in \mathcal{H}'$,

$$B_iB_j = (P+I)H_{2,2\ell+1,i}(P+I)H_{2,2\ell+1,j} = (P+I)H_{2,2\ell+1,i}H_{2,2\ell+1,j}.$$

Since $H_{2,2\ell+1,i}H_{2,2\ell+1,j} \in \text{Span } \mathcal{H}_{2,2\ell+1}$, and that $B_i = (P+I)H_{2,2\ell+1,i}$ for every $i \in [\ell]$, it follows that $(P+I)H_{2,2\ell+1,i}H_{2,2\ell+1,j} \in \text{Span } \mathcal{H}'$.

Finally, since B_ℓ is the adjacency matrix of the graph G_ℓ, Proposition 11 applies and we again obtain the bound (12). $\square$

When is the bound in (12) tight? First, we see that when ℓ is odd, (12) is tight if and only if there exists a $(2\ell+2)$-by-$(2\ell+2)$ Hadamard matrix. Given a $(2\ell+2)$-by-$(2\ell+2)$ Hadamard matrix H, removing one row from H yields $2\ell+2$ column vectors that are binary, with pairwise distance ℓ or $\ell+1$. Conversely, given a clique C in G_ℓ of size $2\ell+2$, we may assume (since G_ℓ is vertex-transitive) that $C = \{0\} \cup S_\ell \cup S_{\ell+1}$, where every vector in S_ℓ has ℓ ones, and every vector in $S_{\ell+1}$ has $\ell+1$ ones. Then we define the set of vectors $C' \subseteq \{0,1\}^{2\ell+2}$ where

$$C' = \{0\} \cup \left\{ \begin{bmatrix} v \\ 1 \end{bmatrix} : v \in S_\ell \right\} \cup \left\{ \begin{bmatrix} v \\ 0 \end{bmatrix} : v \in S_{\ell+1} \right\}.$$

Since ℓ is odd, the vectors in C' have pairwise distance $\ell+1$, and one can construct a Hadamard matrix from C'. This shows that the bound in Proposition 13 is tight for infinitely many values of ℓ. On the other hand, the bound is not tight for $\ell = 2$ as one can check that $\omega(G_2) = 5$. It would be interesting to determine the values of ℓ for which the bound in (12) is tight.

Also, the second proof of Proposition 13 demonstrates the usefulness of finding commutative subschemes within a given association scheme. Given an association scheme $\mathcal{A}$, we say that a set of matrices $\mathcal{A}'$ is a *subscheme* of $\mathcal{A}$ if every matrix in $\mathcal{A}'$ is the sum of a subset of matrices (not necessarily plural) in $\mathcal{A}$, and that $\mathcal{A}'$ is an association scheme in its own right. Thus, in the proof above, $\mathcal{H}'$ is a subscheme of $\mathcal{H}_{2,2\ell+1}$. Given a commutative association scheme, it is easy to check if a certain contraction of its associates lead to a subscheme (see, for instance, [18, Section 4.2]). We shall see in the next section that the situation is more complicated when the initial scheme is not commutative.

4 Lift-and-project ranks for hypermatching polytopes

In this section, we study LS_+-relaxations related to matchings in hypergraphs. More elaboratively, given a q-uniform hypergraph G and an integer $r \geqslant 1$, let $E_r(G)$ denote the set of matchings in G of size r. That is, $S \in E_r(G)$ if $S = \{S_1, \ldots, S_r\}$ where $S_1, \ldots, S_r \in E(G)$ and are mutually disjoint. Consider the following optimization problem: Given a graph G, what is the maximum number of disjoint r-matchings in G such that their union is also a matching in G? Notice that $E_1(G) = E(G)$ and so when $r = 1$ this problem reduces to the classical matching problem of finding the largest subset of hyperedges that are mutually disjoint. Given a set of hyperedges S and vertex i, we also say that S *saturates* i if i is contained in at least one hyperedge in S. Next, we define the polytope

$$\mathrm{MT}_r(G) := \left\{ x \in [0,1]^{E_r(G)} : \sum_{\substack{S \in E_r(G) \\ S \text{ saturates } i}} x_S \leqslant 1, \ \forall i \in V(G) \right\}.$$

Then each integral vector in $\mathrm{MT}_r(G)$ corresponds to a set of r-matchings in G where no vertex is saturated by more than one matching in this set.

Consider $G = K_p^q$, the complete q-uniform hypergraph on p vertices. That is, G is the graph where $V(G) = [p]$ and $E(G) = [p]_q$. In this case, since each r-matching saturates qr vertices, it is apparent that one can choose up to $\lfloor \frac{p}{qr} \rfloor$ disjoint r-matchings. Thus, we obtain that

$$\max\left\{ \bar{e}^\top x : x \in \mathrm{MT}_r(G)_I \right\} = \left\lfloor \frac{p}{qr} \right\rfloor.$$

Next, we compute the optimal value of the linear program

$$\max\left\{ \bar{e}^\top x : x \in \mathrm{MT}_r(G) \right\}. \tag{13}$$

Let $[p]_q^r$ denote $E_r(K_p^q)$ for convenience. Notice that

$$\left|[p]_q^r\right| = \frac{1}{r!}\binom{p}{q}\binom{p-q}{q}\cdots\binom{p-(r-1)q}{q} = \frac{p!}{r!(q!)^r(p-qr)!}.$$

Also, every fixed vertex in $[p]$ is saturated by exactly $\binom{p-1}{qr-1}\left|[qr]_q^r\right|$ distinct r-matchings. Thus, we see that the optimal value of (13) is attained by the solution

$$\bar{x} := \left(\binom{p-1}{qr-1}\left|[qr]_q^r\right|\right)^{-1}\bar{e},$$

giving an optimal value of

$$\bar{e}^\top\bar{x} = \frac{\left|[p]_q^r\right|}{\binom{p-1}{qr-1}\left|[qr]_q^r\right|} = \frac{p}{qr}. \tag{14}$$

Therefore, $\mathrm{MT}_r(G) \neq \mathrm{MT}_r(G)_I$ when qr does not divide p, and one could apply LS_+ to $\mathrm{MT}_r(G)$ to obtain better relaxations of $\mathrm{MT}_r(G)_I$. This leads naturally to the question of determining the LS_+-rank of $\mathrm{MT}_r(G)$ when p is not a multiple of qr.

For the case $q = 2$ and $r = 1$, the given problem reduces to finding a maximum matching in ordinary graphs, which is well known to be solvable in polynomial time [15]. Strikingly, it was shown [39] that for every positive integer p, the LS_+-rank of $\mathrm{MT}(K_{2p+1})$ is p, providing what was then the first known family of instances where LS_+ requires exponential effort to return the integer hull of a given set. In the lower-bound analysis therein, the authors explicitly described the eigenvalues and eigenvectors of their certificate matrix, which is closely related to matrices in Span $\mathcal{J}_{2p+1,2}$ (also see, [6]).

Herein, we generalize their result to r-matchings in hypergraphs. Our main result of this section is the following.

Theorem 14. *Given positive integers p, q, r where $p > qr$ and qr does not divide p, the LS_+-rank of $\mathrm{MT}_r(K_p^q)$ is $\lfloor \frac{p}{qr} \rfloor$.*

We do note that, if we let $\nu(G)$ be the size of the largest matching in a graph G, then the size of the largest r-matching is simply $\lfloor \nu(G)/r \rfloor$. Thus, the maximum r-matching problem may seem to be an unnecessary generalization of the maximum matching problem. However, as we shall see, the introduction of the additional parameter r allows us to work with a very rich association scheme based on matchings in hypergraphs, and analyzing the LS_+-certificate matrices in this more general contexts make the tools presented in our proofs more readily translatable to future analyses of other semidefinite relaxations.

This section is structured as follows: We first introduce the hypergraph matching scheme in Section 4.1. While this scheme is not commutative in general, we will point out a number of its commutative subschemes that are easier to work with. We then provide the proof of Theorem 14 in Section 4.2 while pointing out some immediate consequences of the result. Finally, in Section 4.3 we analyze the LS_+-rank of the b-matching polytope, which gives another example of using known eigenvalues from familiar association schemes to help analyze lift-and-project relaxations.

4.1 The hypergraph matching scheme

Consider the complete hypergraph $G = K_p^q$. Given a permutation $\sigma : [p] \to [p]$ and a set $W \subseteq [p]$, we let $\sigma(W) := \{\sigma(j) : j \in W\}$ for convenience. Next, we define the equivalence relation on $[p]_q^r \times [p]_q^r$ as follows:

Definition 15. Given $S, S', T, T' \in [p]_q^r$, define the relation $\sim$ where $(S,T) \sim (S',T')$ if there exists a permutation $\sigma : [p] \to [p]$ such that

(I1) for every hyperedge $S_i \in S$, the hyperedge $\sigma(S_i) \in S'$, and

(I2) for every hyperedge $T_i \in T$, the hyperedge $\sigma(T_i) \in T'$.

For instance, under this relation, there are 10 equivalence classes in the cases where $q = 2, r = 2$, and $p \geqslant 8$ as illustrated in Figure 1. Now let $X_0, \ldots, X_m \subseteq [p]_q^r \times [p]_q^r$ denote these equivalence classes, and define $|[p]_q^r|$-by-$|[p]_q^r|$ matrices $M_{p,q,r,i}$ where

$$M_{p,q,r,i}[S,T] := \begin{cases} 1 & \text{if } (S,T) \in X_i, \\ 0 & \text{otherwise.} \end{cases}$$

for every $i \in \{0, 1, \ldots, m\}$. Then the (p,q,r)*-hypermatching scheme* is defined to be $\mathcal{M}_{p,q,r} := \{M_{p,q,r,i}\}_{i=0}^m$. To see that this is indeed an association scheme, consider the action of the symmetric group $\mathcal{S}_p$ on a matching $T = \{T_i\}_{i=1}^r \in [p]_q^r$ defined such that, given $\sigma \in \mathcal{S}_p$,

$$\sigma \cdot T = \{\sigma(T_i)\}_{i=1}^r . \tag{15}$$

We can further extend this action to pairs of r-matchings by defining that, given $S, T \in [p]_q^r$,

$$\sigma \cdot (S,T) = (\sigma \cdot S, \sigma \cdot T) . \tag{16}$$

Observe that there is a one-to-one correspondence between the orbits of this action on $[p]_q^r \times [p]_q^r$ and the aforementioned equivalence classes. In particular, the elements in the orbit associated with the isomorphism class X_i are precisely the indices of the non-zero entries of the matrix $M_{p,q,r,i}$. Thus, it follows that $\mathcal{M}_{p,q,r}$ is a *Schurian coherent configuration* (see [11], for example), which assures that the properties (A1), (A3), and (A4) hold. Furthermore, notice that the group action defined above is transitive on $[p]_q^r$ (and thus only has one orbit), and therefore (A2) holds as well. Hence, $\mathcal{M}_{p,q,r}$ is indeed an association scheme.

While we are unaware of previous literature that studies the scheme $\mathcal{M}_{p,q,r}$ in its full generality (which is not commutative), there are some notable choices of p, q, r where $\mathcal{M}_{p,q,r}$ specializes to familiar and commutative schemes. For instance, when $r = 1$, each matching has exactly one hyperedge and thus can simply be seen as a q-subset of $[p]$, and so $\mathcal{M}_{p,q,1} = \mathcal{J}_{p,q}$ for all p and q. Another case where $\mathcal{M}_{p,q,r}$ reduces to the Johnson scheme is when $q = 1$, where we obtain $\mathcal{M}_{p,1,r} = \mathcal{J}_{p,r}$. In the cases when $p = qr$ (i.e., each matching is a partition of the p vertices into r subsets of size q), Godsil and Meagher [19] showed that $\mathcal{M}_{qr,q,r}$ is a commutative scheme if and only if $q = 2$, or $r = 2$, or $(q,r) \in \{(3,3), (3,4), (4,3), (5,3)\}$. They also showed that $\mathcal{M}_{p,q,r}$ is commutative when $q = 2$ and $p = 2r + 1$, as well as when $r = 2$ and $p = 2q + 1$.

As mentioned previously, the commutativity of a scheme $\mathcal{A}$ is a very desirable property that allows us to have a much better handle on the eigenvalues of the matrices in Span $\mathcal{A}$. Hence, given a non-commutative scheme, it can be helpful to instead work with subschemes of it that are commutative. While there are various notions of subschemes in the existing literature (see, for instance, [3, 18]), we will focus on obtaining subschemes by *contraction*. More precisely, given a scheme $\mathcal{A}$ and $S \subseteq \mathcal{A}$ a subset of the associates, we define the contraction of S in $\mathcal{A}$ to be the collection of matrices:

$$\mathcal{A}' := (\mathcal{A} \setminus S) \cup \left\{ \sum_{A_i \in S} A_i \right\}.$$

That is, we remove matrices in S from $\mathcal{A}$ and replace them by a single matrix that is the sum of all matrices in S. If $\mathcal{A}'$ satisfies the properties (A1)-(A4) and thus is an association scheme in its own right, then we call $\mathcal{A}'$ a *subscheme* of $\mathcal{A}$. Note that the trivial scheme with just one associate (i.e., $\mathcal{A} = \{I, J - I\}$) is a commutative subscheme of every non-trivial scheme defined on the same ground set.

While $\mathcal{M}_{p,q,r}$ is not necessarily commutative, we point out that it must have at least one non-trivial commutative subscheme as long as $p > qr$. Observe that, given two matchings $S, T \in [p]_q^r$, $S \cup T$ saturates at least qr and up to $2qr$ vertices. Now, for every $i \in \{qr, qr+1, \ldots, 2qr\}$, define the matrix $B_i \in \mathbb{R}^{[p]_q^r \times [p]_q^r}$ such that

$$B_i[S,T] = \begin{cases} 1 & \text{if } S \cup T \text{ saturates } i \text{ vertices;} \\ 0 & \text{otherwise.} \end{cases}$$

Also, given a pair of symmetric association schemes $\mathcal{A}_1 = \{A_i^{(1)}\}_{i \in \mathcal{I}_1}$ and $\mathcal{A}_2 = \{A_i^{(2)}\}_{i \in \mathcal{I}_2}$ on ground sets Ω_1 and Ω_2 respectively, we define their *wreath product* $\mathcal{A}_1 \wr \mathcal{A}_2$ to be the set consisting of the matrices:

$$(A_i^{(1)} \otimes J_{|\Omega_2|}) \text{ for all } i \in \mathcal{I}_1 \text{ where } A_i \neq I \quad \text{and} \quad (I_{|\Omega_1|} \otimes A_i^{(2)}) \text{ for all } i \in \mathcal{I}_2.$$

It is known that the wreath product of two symmetric schemes must also be a symmetric scheme (see, for example, [3] for a proof). Then we have the following:

Proposition 16. *Given positive integers p, q, r where $p > qr$,*

(i) The set of matrices

$$\tilde{\mathcal{M}}_{p,q,r} = \{I, B_{qr} - I\} \cup \{B_i\}_{i=qr+1}^{\min\{p,2qr\}}$$

is a commutative subscheme of $\mathcal{M}_{p,q,r}$.

(ii) If $\mathcal{M}_{qr,q,r}$ is a symmetric scheme, let $\{M_1, \ldots, M_d\} \subset \mathcal{M}_{p,q,r}$ be the associates corresponding to isomorphism classes where the union of two matchings saturates exactly qr vertices. (Notice it then follows that $B_{qr} = I + \sum_{i=1}^d M_i$.) Then

$$\overline{\mathcal{M}}_{p,q,r} = \{I\} \cup \{M_i\}_{i=1}^d \cup \{B_i\}_{i=qr+1}^{\min\{p,2qr\}}$$

is a commutative subscheme of $\mathcal{M}_{p,q,r}$.

Proof. (i) We show that $\tilde{\mathcal{M}}_{p,q,r}$ is the wreath product of two simple schemes. Let $\mathcal{K} = \{I, J - I\}$ be the trivial association scheme defined on the ground set $[qr]_q^r$. Then we claim that

$$\tilde{M}_{p,q,r} \cong \mathcal{J}_{p,qr} \wr \mathcal{K} = \{J_{p,qr,i} \otimes J\}_{i=1}^{qr} \cup \{I, I \otimes (J - I)\}.$$

To see this, first notice that each element of the ground set of the scheme $\mathcal{J}_{p,qr}$ is a subset of $[p]$ of size qr. Given such a set $W \in [p]_{qr}$ with the elements of W being $w_1, w_2, \ldots, w_{qr}$ listed in ascending order, we can consider W as the function from $[qr]$ to $[p]$ where $W(i) = w_i$ for every i. Also, given any matching $S \in [qr]_q^r$, each ordered pair (W, S) naturally corresponds to a matching in $[p]_q^r$, obtained from applying W to all vertices in S in the same way as described in (15).

Now, let $W, W' \in [p]_{qr}$ and $S, S' \in [qr]_q^r$. Consider the two matchings $T = (W, S), T' = (W', S') \in [p]_q^r$. Notice that for every integer $i, qr \leqslant i \leqslant \min\{p, qr\}$,

$$(J_{p,qr,i-qr} \otimes J)[T, T'] = 1$$

if and only if $J_{p,qr,i-qr}(W, W') = 1$ (i.e., $|W \cup W'| = i$) and $J[S, S'] = 1$ (which is true since J is the matrix of all ones). This happens if and only if $T \cup T'$ saturates exactly i vertices, which is the case exactly when $B_i[T, T'] = 1$.

Next, observe that

$$(I \otimes (J - I))[T, T'] = 1$$

if and only if $I[W,W'] = 1$ (i.e., $W = W'$), and $(J - I)[S,S'] = 1$ (i.e., $S \neq S'$). This is equivalent to saying that T, T' are distinct matchings that saturate exactly the same qr vertices, and hence $(B_{qr} - I)[T,T'] = 1$. This proves that the scheme $\tilde{\mathcal{M}}_{p,q,r}$ is equivalent to $\mathcal{J}_{p,qr} \wr \mathcal{K}$.

(ii) In the case when $\mathcal{M}_{qr,q,r}$ is a symmetric scheme itself, one can show that $\overline{\mathcal{M}}_{p,q,r} = \mathcal{J}_{p,qr} \wr \mathcal{M}_{qr,q,r}$ using essentially the same argument for (i), which implies that $\overline{\mathcal{M}}_{p.q.r}$ is a commutative subscheme of $\mathcal{M}_{p,q,r}$. □

The above result shows that when $p > qr$ and $\mathcal{M}_{qr,q,r}$ is a symmetric and non-trivial scheme (e.g., when $q = 2$ and $r \geqslant 3$), then there are at least two distinct commutative subschemes of $\mathcal{M}_{p,q,r}$. In Section 5, we will return to the question of which contractions of $\mathcal{M}_{p,q,r}$ lead to commutative subschemes.

For now, we will focus on the subscheme $\tilde{\mathcal{M}}_{p,q,r}$. Since it is simply the wreath product of the Johnson scheme and the trivial scheme, we can easily obtain the eigenvalues of any matrix $M \in \text{Span } \tilde{\mathcal{M}}_{p,q,r}$ as long as we can express M as a linear combination of matrices in $\tilde{\mathcal{M}}_{p,q,r}$, which is very easy if we know the entries of M. This will be useful in our analyses of lift-and-project relaxations subsequently in this section.

On the other hand, while Span $\overline{\mathcal{M}}_{p,q,r}$ gives a broader set of matrices than Span $\tilde{\mathcal{M}}_{p,q,r}$ and still possesses the aligned-eigenspaces property, we have less of a grip on the eigenvalues of the matrices therein as the eigenvalues for the associates in $\mathcal{M}_{qr,q,r}$ are less well understood, even in the cases when it is indeed a commutative scheme.

4.2 Packing matchings in hypergraphs

Having introduced the hypermatching scheme $\mathcal{M}_{p,q,r}$ and discussed some of its commutative subschemes, we are now ready to prove Theorem 14. Again, the case where $q = 2$ and $r = 1$ were first shown in [39]. Our proof uses many similar ideas as theirs, as well as our knowledge of the eigenvalues of matrices in $\tilde{\mathcal{M}}_{p,q,r}$.

Proof of Theorem 14. For convenience, let $G := K_p^q$ and $P := \text{MT}_r(G)$ throughout this proof. We first prove the lower bound of the rank. Let $\alpha_0 := \left(\binom{p-1}{qr-1}|[qr]_q^r|\right)^{-1}$. Notice that when $qr < p < 2qr$, $\alpha_0 \bar{e} \in P \setminus P_I$ (as explained in (14) when we computed the optimal value of (13)) and so LS_+-rank of P is at least 1. Thus, for the rest of our lower-bound argument, we may assume that $p \geqslant 2qr$.

Next, we show that $\alpha_0 \bar{e} \in \text{LS}_+^{\ell}(P)$ for all integers $\ell < \lfloor \frac{p}{qr} \rfloor$. Note that $\alpha_0 \bar{e} \notin P_I$ if qr does not divide p, so the claim above would imply that P has LS_+-rank at least $\lfloor \frac{p}{qr} \rfloor$.

We prove our claim by induction on ℓ. The base case $\ell = 0$ is immediate as $\alpha_0 \bar{e} \in \text{LS}_+^0(P) = P$ for all p, q, and r.

For the inductive step, let $\alpha_1 := \left(\binom{p-qr-1}{qr-1}|[qr]_q^r|\right)^{-1}$, and so $\alpha_1 \bar{e} \in \text{LS}_+^{\ell-1}\left(\text{MT}_r(K_{p-qr}^q)\right)$ by the inductive hypothesis. Also, given a set of edges $C \subseteq E(G)$, we let $\text{sat}(C) \subseteq V(G)$ be the set of vertices saturated by the set C. Define the certificate matrix

$$Y := \begin{bmatrix} 1 & \alpha_0 \bar{e}^\top \\ \alpha_0 \bar{e} & Y' \end{bmatrix}$$

where the $|[p]_q^r|$-by-$|[p]_q^r|$ matrix Y' has entries

$$Y'[S,T] = \begin{cases} \alpha_0 & \text{if } S = T; \\ \alpha_0 \alpha_1 & \text{if } \text{sat}(S) \cap \text{sat}(T) = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

We show that Y satisfies all conditions imposed by LS_+. First, it is apparent that Y is symmetric, and $Ye_0 = \mathrm{diag}(Y) = \begin{bmatrix} 1 \\ \alpha_0 \bar{e} \end{bmatrix}$.

Next, given $S \in [p]_q^r$, define the set

$$F := \left\{ x \in [0,1]^{[p]_q^r} : x_S = 1, x_T = 0 \text{ for all } T \text{ where } \mathrm{sat}(S) \cap \mathrm{sat}(T) \neq \emptyset \right\}.$$

By the inductive hypothesis $\alpha_1 \bar{e} \in \mathrm{LS}_+^{\ell-1}\left(\mathrm{MT}_r(K_{p-qr}^q)\right)$. Moreover, observe that the projection of P onto the coordinates not restricted to 0 or 1 in F is exactly $\mathrm{MT}_r(K_{p-qr}^q)$. Since LS_+ satisfies the general property that $\mathrm{LS}_+(P \cap F) \subseteq \mathrm{LS}_+(P) \cap F$ for every face F of the unit hypercube, if we define vector $w_S \in \mathbb{R}^{[p]_q^r}$ where

$$w_S[T] = \begin{cases} 1 & \text{if } S = T; \\ \alpha_1 & \text{if } \mathrm{sat}(S) \cap \mathrm{sat}(T) = \emptyset; \\ 0 & \text{otherwise,} \end{cases}$$

it then follows that $w_S \in \mathrm{LS}_+^{\ell-1}(P)$. Thus, $Ye_S = \alpha_0 \begin{bmatrix} 1 \\ w_S \end{bmatrix} \in K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$. Next, we show that $Y(e_0 - e_S) \in K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$. We claim that, for every matching $S \in [p]_q^r$,

$$\sum_{T \in [p]_q^r} \frac{|\,\mathrm{sat}(S) \cap \mathrm{sat}(T)|}{qr} Ye_T = Ye_0. \tag{17}$$

Notice that the coefficient of Ye_S in the left hand side of (17) is $\frac{|\,\mathrm{sat}(S) \cap \mathrm{sat}(S)|}{qr} = \frac{qr}{qr} = 1$. Thus, the above implies that $Y(e_0 - e_S)$ can be expressed as a non-negative linear combination of vectors in $\left\{ Ye_T : T \in [p]_q^r \right\}$. Since we have shown that Ye_T is contained in the convex cone $K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$ for every $T \in [p]_q^r$, it follows from (17) that $Y(e_0 - e_S) \in K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$ as well.

Now, to prove (17), observe that given a fixed $S \in [p]_q^r$ and for every integer $i \in \{0, 1, \ldots, q\}$,

$$\sum_{\substack{T \in [p]_q^r \\ |\,\mathrm{sat}(T) \cap \mathrm{sat}(S)| = i}} (Ye_T)[W] = \begin{cases} \binom{qr}{q}\binom{p-qr}{qr-q} |[qr]_q^r| \alpha_0 & \text{if } W = 0; \\ \alpha_0 + \binom{qr-i}{q}\binom{p-2qr+i}{qr-q} |[qr]_q^r| \alpha_0 \alpha_1 & \text{if } |\,\mathrm{sat}(W) \cap \mathrm{sat}(S)| = i; \\ \binom{qr-j}{q}\binom{p-2qr+j}{qr-q} |[qr]_q^r| \alpha_0 \alpha_1 & \text{if } |\,\mathrm{sat}(r) \cap \mathrm{sat}(i)| = j \neq i. \end{cases}$$

Thus, for every $S, W \in [p]^r_q$ where $|\operatorname{sat}(W) \cap \operatorname{sat}(S)| = j$,

$$\begin{aligned}
& \sum_{T \in [p]^r_q} \frac{|\operatorname{sat}(S) \cap \operatorname{sat}(T)|}{qr} \left(Y e_T\right)[W] \\
= \;& \sum_{i=0}^{q} \sum_{\substack{T \in [p]^r_q \\ |\operatorname{sat}(T) \cap \operatorname{sat}(S)| = i}} \frac{i}{qr} \left(Y e_S\right)[W] \\
= \;& \frac{j}{qr}\alpha_0 + \sum_{q=0}^{k} \frac{q}{qr} \binom{qr-j}{q} \binom{p-2qr+j}{qr-q} |[qr]^r_q| \alpha_0 \alpha_1 \\
= \;& \frac{j}{qr}\alpha_0 + \frac{qr-j}{qr} \sum_{i=0}^{q} \binom{qr-j-1}{q-1} \binom{p-2qr+j}{qr-q} |[qr]^r_q| \alpha_0 \alpha_1 \\
= \;& \frac{j}{qr}\alpha_0 + \frac{qr-j}{qr} \binom{p-qr-1}{qr-1} |[qr]^r_q| \alpha_0 \alpha_1 \\
= \;& \frac{j}{qr}\alpha_0 + \frac{qr-j}{qr}\alpha_0 \\
= \;& \alpha_0 = \left(Y e_0\right)[W].
\end{aligned}$$

By a similar argument, one can show that

$$\sum_{T \in [p]^r_q} \frac{|\operatorname{sat}(S) \cap \operatorname{sat}(T)|}{qr} \left(Y e_T\right)[0] = 1 = \left(Y e_0\right)[0],$$

which completes the proof of (17).

Finally, we show that $Y \succeq 0$. When $p \geqslant 2qr$, we have

$$Y = \begin{bmatrix} \frac{qr}{p}\bar{e}^\top \\ I \end{bmatrix} Y' \begin{bmatrix} \frac{qr}{p}\bar{e} & I \end{bmatrix}.$$

Thus, to show that $Y \succeq 0$, it suffices to prove that $Y' \succeq 0$. Observe that

$$Y' = \alpha_0 \left(I + \alpha_1 B_{2qr}\right),$$

where $B_{2qr} \in \tilde{\mathcal{M}}_{p,q,r}$ was defined before Proposition 16. We also showed in the proof of Proposition 16 that B_{2qr} has the same eigenvalues as $J_{p,qr,qr} \otimes J$. From Proposition 2, $J_{p,qr,qr}$ has eigenvalues $(-1)^j \binom{p-qr-j}{qr-j}$ for $j = 0, \ldots, qr$. Also, J here is the $|[qr]^r_q|$-by-$|[qr]^r_q|$ matrix of all-ones and thus has eigenvalues $|[qr]^r_q|$ and 0. Hence, the eigenvalues of Y' are

$$\alpha_0 \left(1 + (-1)^j \binom{p-qr-j}{qr-j} |[qr]^r_q| \alpha_1\right) = \alpha_0 \left(1 + (-1)^j \frac{\binom{p-qr-j}{qr-j}}{\binom{p-qr-1}{qr-1}}\right),$$

which are non-negative for all $j \in \{0, 1, \ldots, qr\}$. Thus, Y' is positive semidefinite, and so is Y. This establishes that $\alpha_0 \bar{e} \in \mathrm{LS}^\ell_+(P)$ for all $\ell < \frac{p}{qr}$, and thus shows that P has LS_+-rank at least $\lfloor \frac{p}{qr} \rfloor$.

We next turn to prove the upper bound on the rank of P. By [33, Lemma 1.5], if an inequality is valid for $\{x \in P : x_S = 1\}$ for all $S \in [p]^r_q$, then it is valid for $\mathrm{LS}_+(P)$. Thus, it follows that $P = \mathrm{MT}_r(K^q_p)$ has LS_+-rank at most 1 if $p < 2qr$. By the same rationale, the lemma implies that if $\mathrm{MT}_r(K^q_p)$ has LS_+-rank ℓ, then $\mathrm{MT}_r(K^q_{p+qr})$ has LS_+-rank at most $\ell + 1$. Thus, we see that $\mathrm{MT}_r(K^q_p)$ has LS_+-rank at most $\lfloor \frac{p}{qr} \rfloor$. □

In general, one of the greatest challenges in establishing lower-bound results for semidefinite lift-and-project relaxations is to verify the positive semidefiniteness of a given family of certificate matrices. In the case of the proof of Theorem 14, this task was made relatively simple by observing Y has a full-rank symmetric minor Y' and that Y' is a simple linear combination of associates in $\tilde{\mathcal{M}}_{p,q,r}$.

An immediate implication of the proof of Theorem 14 is the following integrality gap result on $\mathrm{MT}_r(K_p^q)$

Corollary 17. *Let $P = \mathrm{MT}_r(K_p^q)$ where $p > qr$ and qr does not divide p. Then*

$$\frac{\max\left\{\bar{e}^\top x : x \in \mathrm{LS}_+^\ell(P)\right\}}{\max\left\{\bar{e}^\top x : x \in P_I\right\}} = \frac{p/qr}{\lfloor p/qr \rfloor} = 1 + \frac{(p \bmod qr)}{p - (p \bmod qr)}$$

for all $\ell \in \left\{0, 1, \ldots, \lfloor \frac{p}{qr} \rfloor - 1\right\}$.

Proof. First of all, it is obvious that $\max\left\{\bar{e}^\top x : x \in P_I\right\} = \lfloor \frac{p}{qr} \rfloor$ and $\max\left\{\bar{e}^\top x : x \in P\right\} = \frac{p}{qr}$. This establishes the above integrality gap for $\ell = 0$. Next, as shown in Theorem 14, $\left(\binom{p-1}{qr-1}|[qr]_q^r|\right)^{-1} \bar{e} \in \mathrm{LS}_+^\ell(P)$ for all $\ell < \lfloor \frac{p}{qr} \rfloor$. Thus, the corresponding integrality gap for $\mathrm{LS}_+^{\lfloor p/qr \rfloor - 1}(P)$ is greater than or equal to that of P. Since $\mathrm{LS}_+^{\ell+1}(P) \subseteq \mathrm{LS}_+^\ell(P)$ for all ℓ, the integrality gap must be a non-increasing function of ℓ. This shows that the gap is identical for all values of $\ell \in \left\{0, \ldots, \lfloor \frac{p}{qr} \rfloor - 1\right\}$, and our claim follows. □

Next, recall from Proposition 9 that the LS_+-rank of the fractional stable set polytope of the graph $\overline{G(J_{p,q,q})}$ is at least 2 when $p \geqslant 2q$ and is not a multiple of q. With Theorem 14, we can now determine the exact rank of this set.

Corollary 18. *Given positive integers p, q where $p \geqslant 2q \geqslant 2$ and q does not divide p, the LS_+-rank of $\mathrm{FRAC}(\overline{G(J_{p,q,q})})$ is $\lfloor \frac{p}{q} \rfloor$.*

Proof. Notice that $\overline{G(J_{p,q,q})}$ is the line graph of K_p^q, and so there is a natural one-to-one correspondence between matchings in K_p^q and stable sets in $\overline{G(J_{p,q,q})}$. Therefore, we know that

$$\mathrm{MT}_1(K_p^q)_I = \mathrm{STAB}(\overline{G(J_{p,q,q})}).$$

Moreover, it is obvious from their definitions that

$$\mathrm{MT}_1(K_p^q) \subseteq \mathrm{FRAC}(\overline{G(J_{p,q,q})}).$$

Since LS_+ preserves containment, this implies that the LS_+-rank of $\mathrm{FRAC}(\overline{G(J_{p,q,q})})$ is at least that of $\mathrm{MT}_1(K_p^q)$. On the other hand, the same rank upper-bound argument in the proof of Theorem 14 also applies for $\mathrm{FRAC}(\overline{G(J_{p,q,q})})$. Thus, we conclude that when $q \nmid p$, the LS_+-rank of $\mathrm{FRAC}(\overline{G(J_{p,q,q})})$ is exactly $\lfloor \frac{p}{q} \rfloor$. □

Finally, we remark that our lower-bound analysis in the proof of Theorem 14 also applies to the covering variant of the same problem. If we define the r-matching covering polytope to be

$$\mathrm{MT}_r^C(G) := \left\{ x \in [0,1]^{E_r(G)} : \sum_{\substack{S \in E_r(G) \\ S \text{ saturates } i}} x_S \geqslant 1, \ \forall i \in V(G) \right\},$$

then each integral vector in $\mathrm{MT}_r^C(G)$ gives a set of r-matchings in G whose union form an edge cover. Then we have the following result:

Corollary 19. *Let* $P := \mathrm{MT}_r^C(K_p^q)$ *where* $p > 2qr$ *and* qr *does not divide* p*. Then the* LS_+*-rank of* P *is at least* $\lfloor \frac{p}{qr} \rfloor$*.*

Proof. Following the notation used in the proof of Theorem 14, notice that the fractional vector $\bar{x} = \alpha_0 \bar{e}$ used therein is also contained in P, as it satisfies each of the p vertex-incidence constraints of P with equality. Hence, one can use the same certificate matrix Y and induction process to show that $\bar{x} \in \mathrm{LS}_+^\ell(P)$, for all $\ell < \lfloor \frac{p}{qr} \rfloor$. When qr does not divide p, it is easy to see that

$$\max\left\{\bar{e}^\top x : x \in P_I\right\} = \left\lceil \frac{p}{qr} \right\rceil > \frac{p}{qr} = \bar{e}^\top \bar{x}.$$

This shows that P has LS_+-rank at least $\lfloor \frac{p}{qr} \rfloor$. □

4.3 The *b*-hypermatching problem

Next, we turn to a different generalization of the classical matching problem, and study its corresponding LS_+-relaxations. Given a q-uniform hypergraph G and a positive integer b, we say that $S \subseteq E(G)$ is a *b-matching* if every vertex has degree at most b in the subgraph of G with edge set S. The maximum b-matching problem is to find the largest b-matching in a given graph. Note that this problem reduces to the maximum matching problem when $b = 1$. A natural polyhedral relaxation of this problem is

$$\mathrm{bMT}(G) := \left\{ x \in [0,1]^{E(G)} : \sum_{S \in E(G), S \ni i} x_S \leqslant b,\ \forall i \in V(G) \right\}.$$

Then there is a one-to-one correspondence between the integral vectors in $\mathrm{bMT}(G)$ and the b-matchings of G.

For the complete graph $G = K_p^q$ and any integer $b < p$, it is easy to see that there exists a b-regular subgraph in G if and only if bp is a multiple of q, in which case an optimal b-matching would contain exactly $\frac{bp}{q}$ hyperedges. On the other hand, if $q \nmid bp$, then the largest b-matching has size $\lfloor \frac{bp}{q} \rfloor$. Now since $b\binom{p-1}{q-1}^{-1}\bar{e} \in \mathrm{bMT}(G)$, $\max\left\{\bar{e}^\top x : x \in \mathrm{bMT}(G)\right\} \geqslant \frac{bp}{q}$, and so $\mathrm{bMT}(G) \neq (\mathrm{bMT}(G))_I$ when q does not divide bp.

Recently, Kurpisz et al. [27] also studied lift-and-project relaxations of the b-matching problem on hypergraphs. Therein, their focus is on the Las operator (due to Lasserre [28] and yields tighter relaxations than LS_+ in general), applied to a relaxation that is $\mathrm{bMT}(K_p^q)$ with an additional linear constraint. They proved a Las-rank upper bound of $\max\left\{b, \frac{1}{2}\lfloor \frac{bp}{q} \rfloor\right\}$ for that relaxation, and that the bound is tight in the cases where $b = 1, q = 2$ and $4 | p - 1$. Their results are incomparable with Theorem 20 below.

Herein, we focus on the LS_+-rank of $\mathrm{bMT}(K_p^q)$. First, we remark that the ideas in establishing the upper bound in the proof of Theorem 14 can be used to show that $\mathrm{bMT}(K_p^q)$ has LS_+-rank at most $\lfloor \frac{bp}{q} \rfloor$. Next, we establish a lower bound for the LS_+-rank below, showing that this is another case where LS_+ is not efficient at computing the integer hull of the given relaxation. Once again, a key component of our proof involves the known eigenvalues of the Johnson scheme.

Theorem 20. *Let* b, p, q *be positive integers where* q *does not divide* bp *and* $q^2 \geqslant b$*. Then the* LS_+*-rank of* $\mathrm{bMT}(K_p^q)$ *is at least* $\lfloor \frac{p-b-q+1}{2q} \rfloor + 1$ *for all* $p \geqslant b + q - 1$*.*

Proof. For convenience, let $P := \mathrm{bMT}(K_p^q), \alpha_0 := b\binom{p-1}{q-1}^{-1}$, and $\ell := \lfloor \frac{p-b-q+1}{2q} \rfloor$. By induction on ℓ, we shall show that $\alpha_0 \bar{e} \in \mathrm{LS}_+^\ell(P)$. Since $\alpha_0 \bar{e} \notin P_I$ when q does not divide bp, the claim above would imply that P has LS_+-rank at least $\ell + 1$.

For the base case $\ell = 0$, since $p \geqslant b+q-1$, $b\binom{p-1}{q-1}^{-1} \leqslant 1$, and so $b\binom{p-1}{q-1}^{-1}\bar{e} \in P = \mathrm{LS}_+^0(P)$.

Next, for the inductive step, we aim to show that $\alpha_0 \bar{e} \in \mathrm{LS}_+^{\ell}(P)$ follows from the inductive hypothesis $b\binom{p-2q-1}{q-1}^{-1}\bar{e} \in \mathrm{LS}_+^{\ell-1}\left(\mathrm{bMT}(K_{p-2q}^q)\right)$. Define $\alpha_1 := \frac{b-1}{q(p-q)}$, $\alpha_2 := \frac{b-1}{q\binom{p-q}{q-1}}$, and $\alpha_3 := \frac{bp-2bq+q}{q\binom{p-q}{q}}$, and the certificate matrix

$$Y = \begin{bmatrix} 1 & \alpha_0 \bar{e}^{\top} \\ \alpha_0 \bar{e} & Y' \end{bmatrix}$$

where Y' is the $|[p]_q|$-by-$|[p]_q|$ matrix with entries

$$Y'[i,j] = \begin{cases} \alpha_0 & \text{if } i = j; \\ \alpha_0\alpha_1 & \text{if } |i \cap j| = q-1; \\ \alpha_0\alpha_2 & \text{if } |i \cap j| = 1; \\ \alpha_0\alpha_3 & \text{if } i \cap j = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

We show that Y satisfies all conditions imposed by LS_+. First, it is apparent that $Y = Y^{\top}$ and $Ye_0 = \mathrm{diag}(Y) = \begin{bmatrix} 1 \\ \alpha_0 \bar{e} \end{bmatrix}$. Next, we show that $Ye_i \in K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$ for every edge i. Given two disjoint edges i, j, define

$$\mathcal{S}_{i,j} := \left\{\{v_1, v_2\} : v_1 \in i, v_2 \in j\right\}.$$

That is, $\mathcal{S}_{i,j}$ consists of the q^2 2-vertex sets where one vertex belongs to i and the other belongs to j. Then, given i, j and $S \subseteq \mathcal{S}_{i,j}$ where $|S| = b-1$, we construct a vector $w_{i,j,S} \in \mathbb{R}^{[p]_q}$ as follows:

$$w_{i,j,S}[h] = \begin{cases} 1 & \text{if } h = i \text{ or } h = j; \\ 1 & \text{if } h = (i \setminus s) \cup (s \cap j) \text{ for some } s \in S; \\ 1 & \text{if } h = (j \setminus s) \cup (s \cap i) \text{ for some } s \in S; \\ b\binom{p-2q-1}{q-1}^{-1} & \text{if } h \text{ is disjoint from } i \cup j; \\ 0 & \text{otherwise.} \end{cases}$$

Notice that the hyperedges receiving 1's in $w_{i,j,S}$ form a b-regular subgraph on the vertices $i \cup j$. Also, choosing S requires $b - 1 \leqslant q^2$, which is implied by the assumption $b \leqslant q^2$.

Then we know by the inductive hypothesis that $w_{i,j,S} \in \mathrm{LS}_+^{\ell-1}(P)$. Next, consider the vector

$$z := \frac{1}{\binom{p-q}{q}\binom{q^2}{b-1}} \sum_{\substack{j \in E(G) \\ j \cap i = \emptyset}} \sum_{\substack{S \subseteq \mathcal{S}_{i,j} \\ |S| = b-1}} w_{i,j,S}.$$

Since $w_{i,j,S}[i] = 1$ for all j, S included in the sum above, $z[i] = 1$. Likewise, for all $h \in E(G)$ where $2 \leqslant |h \cap i| \leqslant q-2$, $w_{i,j,S}[h] = 0$. Hence, $z[h] = 0$ for these edges as well. Next, by symmetry of the underlying complete graph G, we know that $z[h] = z[h']$ if there is an automorphism on G that maps vertices in i to itself while mapping h to h'. Thus, there must exist constants $\beta_1, \beta_2, \beta_3$ such that

$$z[h] = \begin{cases} \beta_1 & \text{if } |h \cap i| = q-1; \\ \beta_2 & \text{if } |h \cap i| = 1; \\ \beta_3 & \text{if } |h \cap i| = 0. \end{cases}$$

Now, notice that $\sum_{h \in E(G), |h \cap i| = q-1} w_{i,j,S} = b-1$ for all j and S. Thus,

$$\beta_1 = \frac{b-1}{|\{h \in E(G) : |h \cap i| = q-1\}|} = \frac{b-1}{q(p-q)} = \alpha_1.$$

One can likewise show that $\beta_2 = \alpha_2$ and $\beta_3 = \alpha_3$, which shows that $Ye_i = \alpha_0 \begin{bmatrix} 1 \\ z \end{bmatrix}$. Since $z \in \mathrm{LS}_+^{\ell-1}(P)$ (due to it being a convex combination of $w_{i,j,S}$'s, points inside the convex set $\mathrm{LS}_+^{\ell-1}(P)$), we obtain that $Ye_i \in K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$.

We next show that $Y(e_0 - e_i) \in K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$ with a similar argument. Given disjoint edges i, j and $S \subseteq \mathcal{S}_{i,j}$ where $|S| = b$, define $\overline{w}_{i,j,S} \in \mathbb{R}^{[p]_q}$ such that

$$\overline{w}_{i,j,S}[h] = \begin{cases} 1 & \text{if } h = (i \setminus s) \cup (s \cap j) \text{ for some } s \in S; \\ 1 & \text{if } h = (j \setminus s) \cup (s \cap i) \text{ for some } s \in S; \\ b\binom{p-2q-1}{q-1}^{-1} & \text{if } h \text{ is disjoint from } i \cup j; \\ 0 & \text{otherwise.} \end{cases}$$

(This construction is where we need $q^2 \geqslant b$.) Next, if we define

$$\overline{z} := \frac{1}{\binom{p-q}{q}\binom{q^2}{b}} \sum_{\substack{j \in E(G) \\ j \cap i = \emptyset}} \sum_{\substack{S \subseteq \mathcal{S}_{i,j} \\ |S| = b}} \overline{w}_{i,j,S},$$

one can apply a similar argument to the above and deduce that

$$\overline{z}[h] = \begin{cases} 0 & \text{if } h = i; \\ 0 & \text{if } 2 \leqslant |h \cap i| \leqslant q-2; \\ \frac{b}{q(p-q)} = \frac{\alpha_0}{1-\alpha_0}(1-\alpha_1) & \text{if } |h \cap i| = q-1; \\ \frac{b}{q\binom{p-q}{q-1}} = \frac{\alpha_0}{1-\alpha_0}(1-\alpha_2) & \text{if } |h \cap i| = 1; \\ \frac{bp-2bq}{q\binom{p-q}{q}} = \frac{\alpha_0}{1-\alpha_0}(1-\alpha_3) & \text{if } |h \cap i| = 0. \end{cases}$$

Thus, $Y(e_0 - e_i) = (1-\alpha_0)\begin{bmatrix} 1 \\ \overline{z} \end{bmatrix} \in K\left(\mathrm{LS}_+^{\ell-1}(P)\right)$.

Finally, we show that $Y \succeq 0$. Observe that

$$Y = \begin{bmatrix} \frac{q}{bp}\bar{e}^\top \\ I \end{bmatrix} Y' \begin{bmatrix} \frac{q}{bp}\bar{e} & I \end{bmatrix}.$$

Thus, to show that $Y \succeq 0$, it suffices to prove that $Y' \succeq 0$. Now, notice that $Y' \in \operatorname{Span} \mathcal{J}_{p,q}$. In fact,

$$Y' = \alpha_0 \left(I + \alpha_1 J_{p,q,1} + \alpha_2 J_{p,q,q-1} + \alpha_3 J_{p,q,q}\right).$$

Applying Proposition 2 (using formula (2) for the case $i = q-1$ and (3) for the case $i = 1$), one obtains that the eigenvalues of Y' are

$$\begin{aligned} & \alpha_0 \Bigg(1 + \alpha_1((q-j)(p-q-j) - j) \\ & +\alpha_2 \left((-1)^j (p-2q+2)) \binom{p-q+1-j}{q-1-j} + (-1)^{j+1} q \binom{p-q-j}{q-j} \right) \\ & +\alpha_3 (-1)^j \binom{p-q-j}{q-j} \Bigg) \\ = \;& \alpha_0 \left(1 + (-1)^j \frac{\binom{p-q-j}{q-j}}{\binom{p-q-1}{q-1}} \right) \left(1 + \frac{(b-1)((q-j)(p-q-j) - j)}{q(p-q)} \right). \end{aligned} \tag{18}$$

Notice that (18) is non-negative for all $p \geqslant 2q$ and for all $j < q$. When $j = q$, (18) is non-negative when

$$\left(1 + \frac{(b-1)((q-q)(p-q-q)-q)}{q(p-q)}\right) \geqslant 0 \iff p \geqslant q+b-1,$$

which is an assumption in the hypothesis. This finishes our proof. $\square$

We remark that the lower bound given in Theorem 20 is not always tight. For instance, when $b = 1$, the theorem gives $\text{bMT}(K_p^q)$ has LS_+-rank at least $\lfloor \frac{p-q}{2q} \rfloor + 1$, while Theorem 14 (specialized to $r = 1$) gives a better rank lower bound of $\lfloor \frac{p}{q} \rfloor$. It is possible one can improve the bound in Theorem 20 (and/or weaken the assumption $b \leqslant q^2$) by using a different certificate matrix, perhaps by involving more associates in the Johnson scheme. Of course, this could potentially lead to a more challenging analysis of its eigenvalues.

Also, for the same reason why Theorem 14 implies Corollary 19, the proof of Theorem 20 can be easily adapted to show that the same LS_+-rank lower bound applies for the covering variant of the b-matching problem.

Corollary 21. *Let b, p, q be positive integers where q does not divide bp and $q^2 \geqslant b$, and let $G = K_p^q$. Then the LS_+-rank of*

$$\text{bMT}^C(G) := \left\{ x \in [0,1]^{E(G)} : \sum_{S \in E(G), S \ni i} x_S \geqslant b, \ \forall i \in V(G) \right\}$$

is at least $\lfloor \frac{p-b-q+1}{2q} \rfloor + 1$ for all $p \geqslant b+q-1$.

5 More on the hypermatching scheme $\mathcal{M}_{p,q,r}$

After working with the simple subscheme $\tilde{\mathcal{M}}_{p,q,r}$ in Section 4, we now look into the full scheme $\mathcal{M}_{p,q,r}$ and try to gain a better understanding of it. In Section 5.1, we discuss some combinatorial characterizations of the associates of $\mathcal{M}_{p,q,r}$, and in particular enumerate the associates in $\mathcal{M}_{p,2,r}$ via counting a certain type of integer partitions. This will in turn help us study which contractions of $\mathcal{M}_{p,2,r}$ result in symmetric subschemes (Section 5.2).

5.1 Characterizing associates in $\mathcal{M}_{p,q,r}$

Herein, we look into characterizing associates in $\mathcal{M}_{p,q,r}$ via some familiar combinatorial objects. For convenience, let $a_{p,q,r}$ be the number of equivalence classes in the relation defined in Definition 15. We first focus on ordinary graphs (i.e., the case $q = 2$) and map associates in $\mathcal{M}_{p,2,r}$ to a specific type of integer partitions, before returning to discuss the case for arbitrary q later in this subsection.

When $q = 2$ and $p = 2r$, it is known that there is a one-to-one correspondence between the isomorphism classes and even partitions of $2r$ (i.e., the number ways to write $2r$ as the sum of a non-increasing sequence of even positive integers). The following result extends this to general values of p.

Proposition 22. *For all positive integers p, r where $p \geqslant 2r$, $a_{p,2,r}$ is equal to the number of partitions of $2r$ with four types of parts $\{\ell^+, \ell^-, \overline{\ell}, \ell' \geqslant 1\}$, such that*

(P1) the parts of the types ℓ^+, ℓ^- are all odd, and the parts of the types $\overline{\ell}, \ell'$ are all even;

(P2) the number of parts of type ℓ^+ is equal to that of type ℓ^-; and

(P3) there are at most $p-2r$ total number of parts from the types $\ell^+, \ell^-, \overline{\ell}$ combined.

Proof. We construct a bijection between the equivalence classes and the set of partitions described in our claim. For each equivalence class X_i, take any element $(S,T) \in X_i$, and consider the components in the subgraph formed by the edges in $S \cup T$. Notice that since S, T is each a matching, every vertex in this subgraph has degree at most 2. Now, for each component that contains ℓ edges, we assign it to a part as follows:

- ℓ^+ if the component is a path of odd length, with $\frac{\ell+1}{2}$ edges coming from S.
- ℓ^- if the component is a path of odd length, with $\frac{\ell-1}{2}$ edges coming from S.
- $\overline{\ell}$ if the component is a path of even length.
- ℓ' if the component is a cycle of length 2ℓ. This includes the case of 2-cycles, which occurs when the component consists of two overlapping edges, one from S and one from T.

Figure 1 illustrates the correspondence between the partitions and equivalence classes for the case $r = 2$.

If we do that for each component in $S \cup T$, we obtain parts that add up to $2r$ (since the value of each part is equal to the number of edges in the corresponding component), and the partition satisfies (P1) by construction. Next, (P2) holds since $|S| = |T| = r$ and the number of odd paths with one more edge from S must be equal to the number of odd paths with one more edge from T. Also, notice that a component corresponding to parts $\ell^+, \ell^-, \overline{\ell}$ saturates $\ell+1$ vertices, while a component corresponding to ℓ' has exactly ℓ vertices. Thus, the total number of vertices saturated by all components is $2r$ plus the number of non-primed parts in the partition. Therefore, these components do not occupy more than p vertices if and only if the number of the non-primed parts is no more than $p-2r$, satisfying (P3).

Note that the construction of the partition is reversible — given a partition of $2r$ with the aforementioned four kinds of parts and the given conditions, we can uniquely recover the types of components in the graph with the edges $S \cup T$, and thus the equivalence class X_i. This finishes our proof. $\square$

Observe that, when $p = 2r$, (P3) assures that the corresponding partitions all have only primed parts, and so $a_{2r,2,r}$ is indeed the number of even partitions of $2r$. As p increases from $2r$ to $4r$, so does $a_{p,2,r}$. However, notice that $a_{p,2,r}$ is constant for all $p \geqslant 4r$. In fact, it follows from Proposition 22 that

$$a_{p,2,r} = [x^{2r}y^r] \prod_{i \geqslant 1} \left(\frac{1}{(1-x^{2i-1}y^i)(1-x^{2i-1}y^{i-1})(1-x^{2i}y^i)^2} \right) \tag{19}$$

for all $p \geqslant 4r$. Here, the degree of x counts the total number of edges in a component, the degree of y counts the number of edges in the component that belong to S, and the generating functions $\frac{1}{1-x^{2i-1}y^i}, \frac{1}{1-x^{2i-1}y^{i-1}}, \frac{1}{1-x^{2i}y^i}$, and $\frac{1}{1-x^{2i}y^i}$ correspond to parts of types $\ell^+, \ell^-, \overline{\ell}$, and ℓ' respectively. Using (19), we determine that the first few terms of the sequence $\{a_{4r,2,r}\}_{r \geqslant 0}$ are:

$$1, 3, 10, 27, 69, 161, 361, 767, 1578, 3134, 6064, 11432, 21105, 38175, 67863, \ldots$$

This sequence was previously unreported to the Online Encyclopedia of Integer Sequences (OEIS) [37], now sequence A316587 therein.

We next describe an alternative approach to characterizing the associates of the hyper-matching scheme using equivalence classes of meet tables. While this approach boosts the

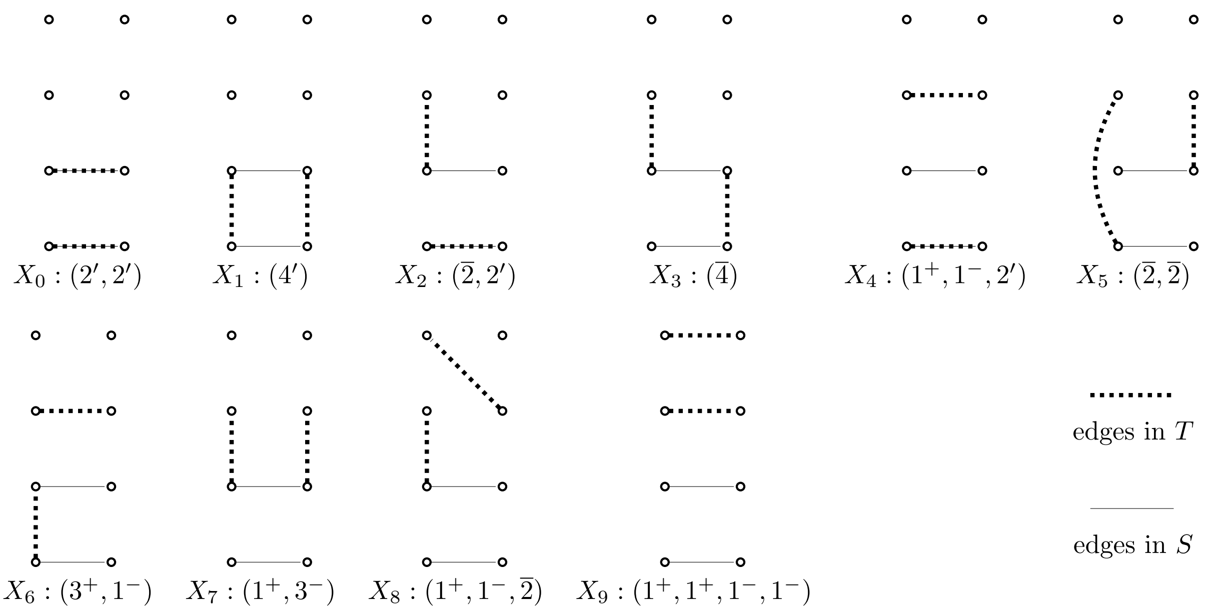

Figure 1: The bijection between integer partitions and non-isomorphic unions of two matchings in $[p]_2^2$.

advantage of applying for all values of q (unlike the integer partitions approach above that is restricted to the case $q = 2$), enumerating the equivalence classes of meet tables is seemingly difficult.

Given matchings $S, S' \in [p]_q^r$ where $S = \{S_1, \ldots, S_r\}$ and $S' = \{S'_1, \ldots, S'_r\}$, define the *meet table* of S and S' to be the r-by-r matrix $M_{S,S'}$ where

$$M_{S,S'}[i, j] = |S_i \cap S'_j|$$

for all $i, j \in [r]$. Notice that the matrix $M_{S,S'}$ must satisfy the following properties:

(T1) Every entry of the matrix is an integer between 0 and q.

(T2) The entries in every row and every column sum to no more than q.

(T3) The r^2 entries of the matrix sum to at least $2qr - p$.

Conversely, given an r-by-r matrix T that satisfies properties (T1)-(T3), one can find $S, S' \in [p]_q^r$ such that $T = M_{S,S'}$. Next, we say that two meet tables T, T' are related if there exist r-by-r permutation matrices P, P' such that $T' = PTP'^{\top}$. In other words, T, T' are related if one matrix can be obtained from the other by permuting rows and columns. Then it is not hard to see that given matchings $S_1, S'_1, S_2, S'_2 \in [p]_q^r$, (S_1, S'_1) and (S_2, S'_2) belong to the same equivalence class (as defined in Definition 15) if and only if $M_{S_1,S'_1} \sim M_{S_2,S'_2}$. For instance, the 10 isomorphism classes of $\mathcal{M}_{p,2,2}$ illustrated in Figure 1 correspond to the following meet tables:

$$X_0 : \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad X_1 : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad X_2 : \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_3 : \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad X_4 : \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix},$$
$$X_5 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_6 : \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_7 : \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad X_8 : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad X_9 : \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the problem of counting the number of associates in $\mathcal{M}_{p,q,r}$ can be solved by enumerating the equivalence classes of meet tables that satisfy (T1)-(T3).

We do so for the special case of $r = 2$. Recall that $a_{p,q,r}$ denotes the number of equivalence classes in $\mathcal{M}_{p,q,r}$. Then we have the following:

Proposition 23. *Given positive integers p, q where $p \geqslant 4q$,*

$$a_{p,q,2} = \begin{cases} \frac{1}{24}(q^4 + 6q^3 + 20q^2 + 36q + 24) & \text{if } q \text{ is even;} \\ \frac{1}{24}(q^4 + 6q^3 + 20q^2 + 30q + 15) & \text{if } q \text{ is odd.} \end{cases}$$

Proof. We count the number of equivalence classes of 2-by-2 meet tables M that satisfy properties (T1)-(T3) by cases. First, if M has at least one zero entry, then it is related to one of the following:

- $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $q \geqslant a \geqslant b \geqslant 0$. This gives $\binom{q+2}{2}$ possibilities.
- $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ where $a \geqslant b \geqslant 1$ and $a + b \leqslant q$. This gives $2\left(\frac{q^2}{4}\right)$ possibilities when q is even, and $2\left(\frac{q^2-1}{4}\right)$ possibilities when q is odd.
- $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ where $a, b, c \geqslant 1, a + b \leqslant q$, and $a + c \leqslant q$. For each fixed $a \in [q-1]$ there are $q - a$ choices for each of b and c. Thus, this gives

  $$\sum_{a=1}^{q-1}(q-a)^2 = \frac{q(q-1)(2q-1)}{6}$$

  possibilities.

Now suppose $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \geqslant 1$. Note that $M' := \begin{bmatrix} a-1 & b-1 \\ c-1 & d-1 \end{bmatrix}$ would be a meet table for some equivalence class in $\mathcal{M}_{p-4,q-2,2}$. Moreover, the correspondence is bijective. So this gives $a_{p-4,q-2,2}$ possibilities.

Thus, when q is even, the total number of equivalence classes is

$$\begin{aligned} a_{p,q,2} &= \binom{q+2}{2} + 2\left(\frac{q^2}{4}\right) + \frac{q(q-1)(2q-1)}{6} + a_{p-4,q-2,2} \\ &= \frac{1}{6}(2q^3 + 3q^2 + 10q + 6) + a_{p-4,q-2,2}. \end{aligned}$$

Likewise, when q is odd, we obtain $a_{p,q,2} = \frac{1}{6}(2q^3+3q^2+10q+3)+a_{p-4,q-2,2}$. Using $a_{p,0,2} = 1$ for all $p \geqslant 0$ and $a_{p,1,2} = 3$ for all $p \geqslant 4$, we obtain the formulas as claimed by solving a simple recurrence for each parity of q. □

Using Proposition 23, we determine that the first few terms of the sequence $\{a_{4q,q,2}\}_{q\geqslant 0}$ are:

$$1, 3, 10, 22, 47, 85, 148, 236, 365, 535, 766, 1058, 1435, 1897, 2472, 3160, 3993,$$

This sequence was also not previously reported to the OEIS, now sequence A336529 therein.

While the proof of Proposition 23 is elementary, it is also rather ad hoc and does not seem easily extendable to obtain a formula for $a_{p,q,r}$ for general r, which may require a more sophisticated approach.

5.2 Symmetric subschemes of $\mathcal{M}_{p,q,r}$

As seen in the analyses of lift-and-project relaxations in Section 4, it is much easier to work with a commutative scheme where the eigenspaces of the associates are aligned. Moreover, many lift-and-project operators (including LS_+) require its certificate matrices to be symmetric. This naturally raises the question of when $\mathcal{M}_{p,q,r}$ is indeed a symmetric scheme (which would imply that it is also a commutative scheme), and also, which contractions of associates in $\mathcal{M}_{p,q,r}$ would lead to symmetric subschemes.

We have already seen that when $r = 1$, $\mathcal{M}_{p,q,r}$ reduces to the Johnson scheme $\mathcal{J}_{p,q}$, which is obviously symmetric and commutative. For $q = 2$ and arbitrary r, it is known that $\mathcal{M}_{p,2,r}$ is a commutative scheme if and only if $p \in \{2r, 2r+1\}$ [19]. We provide an elementary proof of this below:

Proposition 24. *Suppose $r \geqslant 2$ is a fixed integer. Then $\mathcal{M}_{p,2,r}$ is a commutative scheme if and only if $p \in \{2r, 2r+1\}$.*

Proof. First, suppose $p \in \{2r, 2r+1\}$. In this case, given (S,T) in any equivalence class X_i, the components in $S \cup T$ consist of at most one path (which must be even), with the rest all being even cycles. Then we see that (S,T) and (T,S) belong to the same equivalence class, which implies that $\mathcal{M}_{p,2,r}$ is a symmetric (and hence commutative) scheme in these cases.

Now suppose $p \geqslant 2r+2$. let $M, M' \in \mathcal{M}_{p,2,r}$ be the associates corresponding respectively to the equivalence classes $X : (\overline{2}, 2'^{(r-1)})$ and $X' : (1^+, 1^-, 2'^{(r-1)})$ (where the superscripts denote multiplicities). Now consider the matchings

$$\begin{aligned} S &:= \{\{2i-1, 2i\} : i \in [r]\}, \\ T &:= \{1,3\} \cup \{\{2i-1, 2i\} : i \in \{3, 4, \ldots, r+1\}\}. \end{aligned}$$

Then $(MM')[S,T] = 0$, and $(M'M)[S,T] = 2$. Since $MM' \neq M'M$, $\mathcal{M}_{p,2,r}$ is not commutative. ☐

While $\mathcal{M}_{p,q,r}$ is not symmetric in general, we have seen in Section 4 that we can obtain symmetric subschemes of it (such as $\tilde{\mathcal{M}}_{p,q,r}$ and $\overline{\mathcal{M}}_{p,q,r}$) by contracting associates. Herein, we investigate the possibility of obtaining other symmetric subschemes of $\mathcal{M}_{p,q,r}$.

Given a scheme that is not symmetric, a reasonable first attempt might be to take every matrix B in the scheme that is not symmetric, and contract $\{B, B^\top\}$. While this preserves the properties (A1)-(A3), (A4) may no longer hold. For an example, the proof of Proposition 24 is based on two symmetric associates whose product is not symmetric. Thus, any set of symmetric matrices containing these two associates would fail the spanning condition (A4).

So, which are the contractions of $\mathcal{M}_{p,q,r}$ that do result in symmetric subschemes? In the special case of $q = r = 2$, let $X_0, X_1, \ldots, X_9$ denote the equivalence classes corresponding to the partitions as in Figure 1. Note that $X_0 : (2', 2')$ corresponds to the identity matrix, and we have a scheme with 9 associates (when $p \geqslant 8$). For convenience, in this section, we will refer to the matrices in $\mathcal{M}_{p,2,2}$ as $M_0, \ldots, M_9$ (instead of $M_{p,2,2,0}, \ldots, M_{p,2,2,9}$), when the value of p is clear from the context.

Recall the B_i matrices defined before Proposition 16 that correspond to contracting associates based on the number of vertices the union of the matchings saturate. Then we have

$$\begin{aligned} B_4 &= M_0 + M_1, \\ B_5 &= M_2 + M_3, \\ B_6 &= M_4 + M_5 + M_6 + M_7, \\ B_7 &= M_8 \\ B_8 &= M_9. \end{aligned}$$

Also, we know from Proposition 16 that

$$\tilde{\mathcal{M}}_{p,2,2} = \overline{\mathcal{M}}_{p,2,2} = \{I, M_1, B_5, B_6, B_7, B_8\}$$

is a 5-associate symmetric subscheme of $\mathcal{M}_{p,2,2}$ for all $p \geqslant 8$. Of course, there is also the 1-associate trivial subscheme. To investigate if there are any other contractions of $\mathcal{M}_{p,2,2}$ that also result in symmetric subschemes, we look into how the associates of $\mathcal{M}_{p,2,2}$ interact with each other. As shown in the proof of Proposition 24, not all pairs of these matrices in $\mathcal{M}_{p,2,2}$ commute when $p \geqslant 6$. In fact, some of these matrices commute for some values of p but not others.

Table 1 shows the commutativity data for the scheme $\{M_i\}_{i=0}^{9}$ for up to $p = 15$. A checkmark ($\checkmark$) indicates that the matrices commute for all $p \leqslant 15$. A number indicates that those two matrices M_i, M_j only commute for that specific value of p, among values of $p \leqslant 15$ for which both M_i, M_j are non-zero. For example, $X_4 : (1^+, 1^-, 2')$ and $X_8 : (1^+, 1^-, \overline{2})$ correspond to matching unions that saturate 6 and 7 vertices respectively, and thus M_4, M_8 are both non-zero only when $p \geqslant 7$. Now the entry "9" in the table means that M_4, M_8 commute when $p = 9$, and do not commute for any $p \in \{7, 8, 10, 11, 12, 13, 14, 15\}$. Finally, a blank entry indicates the matrices do not commute for any $p \leqslant 15$ for which they are both non-zero.

M_iM_j commute?	$(2',2')$	$(4')$	$(\overline{2},2')$	$(\overline{4})$	$(1^+,1^-,2')$	$(\overline{2},\overline{2})$	$(3^+,1^-)$	$(1^+,3^-)$	$(1^+,1^-,\overline{2})$	$(1^+,1^+,1^+,1^-)$
$(2',2')$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$(4')$	✓	✓	✓	✓					✓	✓
$(\overline{2},2')$	✓	✓	✓	✓					✓	✓
$(\overline{4})$	✓	✓	✓	✓	6	6	6	6	✓	✓
$(1^+,1^-,2')$	✓			6	✓	6			9	
$(\overline{2},\overline{2})$	✓			6	6	✓			9	
$(3^+,1^-)$	✓			6			✓		9	
$(1^+,3^-)$	✓			6				✓	9	
$(1^+,1^-,\overline{2})$	✓	✓	✓	✓	9	9	9	9	✓	✓
$(1^+,1^+,1^-,1^-)$	✓	✓	✓	✓					✓	✓

Table 1: Commutativity data for matrices in $\mathcal{M}_{p,2,2}$ for $p \leqslant 15$.

We have also exhaustively tested all possible contractions of $\mathcal{M}_{p,2,2}$ for $p \leqslant 15$ to see which contractions result in symmetric, commutative subschemes, and found the following:

Proposition 25. *The following is an exhaustive list of all symmetric (and thus commutative) subschemes of $\mathcal{M}_{p,2,2}$, for $6 \leqslant p \leqslant 15$.*

(i) The following are symmetric subschemes of $\mathcal{M}_{p,2,2}$ for all p where $6 \leqslant p \leqslant 15$,

- *the scheme made up of the non-zero matrices in the set* $\tilde{\mathcal{M}}_{p,2,2} = \{I, M_1, B_5, B_6, B_7, B_8\}$*;*
- *the trivial scheme* $\{I, J - I\}$*;*

– *the 2-associate scheme* $\{I, M_1, J - M_1 - I\}$.

(ii) *The following are sets of matrices that are only symmetric subschemes for certain values of* p*:*

p	Symmetric Subschemes
6	$\{I, M_4, J - M_4 - I\}$ $\{I, M_2 + M_5, J - M_2 - M_5 - I\}$ $\{I, M_1 + M_2 + M_6 + M_7, M_3 + M_5, M_4\}$ $\{I, M_1 + M_3 + M_4, M_2 + M_5, M_6 + M_7\}$ $\{I, M_1 + M_4, M_2 + M_5, M_3, M_6 + M_7\}$
7	$\{I, M_1, B_5 + B_7, B_6\}$
8	$\{I, M_1 + B_8, B_5 + B_6 + B_7\}$ $\{I, M_1, B_5 + B_6 + B_7, B_8\}$ $\{I, M_1 + B_8, B_5 + B_7, B_6\}$ $\{I, M_1, B_5 + B_7, B_6, B_8\}$
9	$\{I, M_1 + M_2 + M_6 + M_7 + M_9, M_3 + M_4 + M_8\}$ $\{I, M_1, B_5 + B_8, B_6 + B_7\}$
11	$\{I, M_1, B_5 + B_8, B_6 + B_7\}$
12	$\{I, M_1, B_5 + B_7, B_6 + B_8\}$

The three subschemes that work for all values of p we checked are no surprises: The trivial scheme and $\tilde{\mathcal{M}}_{p,2,2}$ are expected, and the third scheme $\{I, M_1, J - M_1 - I\}$ is in fact the wreath product $\mathcal{K}_1 \wr \mathcal{K}_2$, where $\mathcal{K}_1, \mathcal{K}_2$ are trivial schemes on ground sets of sizes $\binom{p}{4}$ and 3, respectively. This can be shown using the same argument as in the proof of Proposition 16, while noting that $\mathcal{K}_2$ is equivalent to the perfect matching scheme $\mathcal{M}_{4,2,2}$. Thus, we see that these three would be symmetric subschemes of $\mathcal{M}_{p,2,2}$ for all $p \geqslant 6$.

For Proposition 25(ii), the B_i notation is used when the given subscheme can be resulted from contracting associates in $\tilde{\mathcal{M}}_{p,2,2}$. Notice that for $p \neq 6, 9$ (in which Table 1 showed there are some "coincidental" commutativity between associates that are not present in other values of p), all subschemes we obtained are contractions of $\tilde{\mathcal{M}}_{p,2,2}$. It would be interesting to know if it is indeed true that, for all $p \neq 6, 9$, all symmetric subschemes of $\mathcal{M}_{p,2,2}$ are in fact subschemes of $\tilde{\mathcal{M}}_{p,2,2}$ (or $\tilde{\mathcal{M}}_{p,2,2}$ itself).

We finish this section by proving a result that shows that, if a certain contraction of associates in $\mathcal{M}_{p,q,r}$ produces a symmetric subscheme for all values of p up to a certain point, then it is assured that this contraction would yield a symmetric subscheme of $\mathcal{M}_{p,q,r}$ for all p. As we have seen in Section 5.1, with fixed q, r, the number of associates in $\mathcal{M}_{p,q,r}$ increases as p increases from qr to $2qr$, and remains constant for all $p \geqslant 2qr$. For the compactness of stating our results, for the rest of the section, we let $\mathcal{I}_{q,r}$ denote the collection of isomorphism classes in $\mathcal{M}_{2qr,q,r}$, and we will think of $\mathcal{M}_{p,q,r}$ as a set of $|\mathcal{I}_{q,r}|$ matrices, some of which are all zeros when $p < 2qr$.

We first need the following lemma.

Lemma 26. *Let* $C_1, C_2 \subseteq \mathcal{I}_{q,r}$ *be subsets of isomorphism classes in* $\mathcal{M}_{p,q,r}$*, and define matrices* $M_{p,1} := \sum_{i \in C_1} M_{p,q,r,i}$ *and* $M_{p,2} := \sum_{i \in C_2} M_{p,q,r,i}$*. Also let* $S_1, S_1', S_2, S_2' \in [p]_q^r$*. If*

$$(M_{p,1} M_{p,2})[S_1, S_1'] = (M_{p,1} M_{p,2})[S_2, S_2'] \tag{20}$$

holds for all $p \leqslant 3qr$*, then* (20) *holds for all integers* p*.*

Proof. Given $S, S' \in [p]_q^r$, and a set of vertices $U \subseteq [p]$, define $f^U(S, S')$ to be the set of matchings $T \in [p]_q^r$ where

- (S, T) belongs to an isomorphism class in C_1;
- (T, S') belongs to an isomorphism class in C_2;
- The vertices saturated by S, S' and T are all contained in U.

Notice that $(M_{p,1}M_{p,2})[S, S'] = |f^{[p]}(S, S')|$. Thus, the hypothesis that (20) holds for all $p \leqslant 3qr$ can be restated as

$$\left|f^U(S_1, S_1')\right| = \left|f^U(S_2, S_2')\right|, \quad \forall U \subseteq [p], |U| \leqslant 3qr.$$

Now notice that, for arbitrary $p' \geqslant 3qr$,

$$(M_{p',1}M_{p',2})[S_1, S_1'] = \left|f^{[p']}(S_1, S_1')\right| = \left|\bigcup_{U \subseteq [p'], |U| = 3qr} f^U(S_1, S_1')\right|.$$

The last equality follows since the union of any 3 matchings $S_1, S_1', T \in [p]_q^r$ saturates at most $3qr$ vertices, so every matching in $f^{[p']}(S_1, S_1')$ is accounted for in the union. Next, one can apply the principle of inclusion-exclusion to express $\left|\bigcup_{U \subseteq [p'], |U| = 3qr} f^U(S_1, S_1')\right|$ as a linear combination of $\left|f^W(S_1, S_1')\right|$'s where W is an intersection of sets of size $3qr$ (and thus has size no more than $3qr$). Thus, we obtain integers b_W's such that

$$\left|\bigcup_{U \subseteq [p'], |U| = 3qr} f^U(S_1, S_1')\right| = \sum_{W \subseteq [p'], |W| \leqslant 3qr} b_W \left|f^W(S_1, S_1')\right|.$$

Notice that the coefficients b_W only depend on p', q, and r, and not S_1, S_1'. Thus, by the same rationale we obtain that

$$(M_{p',1}M_{p',2})[S_2, S_2'] = \sum_{W \subseteq [p'], |W| \leqslant 3qr} b_W \left|f^W(S_2, S_2')\right|.$$

By our hypothesis, $\left|f^W(S_1, S_1')\right| = \left|f^W(S_2, S_2')\right|$ for all W of size no more than $3qr$. Thus, we conclude that (20) indeed holds for all $p' > 3qr$. □

Finally, let $X_0 \in \mathcal{I}_{q,r}$ denote the isomorphism class that corresponds to the identity matrix. Then we have the following:

Proposition 27. *Let $C_1, \ldots, C_m$ be a partition of the non-identity isomorphism classes $\mathcal{I}_{q,r} \setminus \{X_0\}$. Define matrices*

$$B_{p,i} := \sum_{j \in C_i} M_{p,q,r,j}$$

for every $i \in [m]$ and $p \geqslant qr$. If

$$\mathcal{B}_p := \{I\} \cup \{B_{p,i} : i \in [m]\}$$

is a symmetric subscheme of $\mathcal{M}_{p,q,r}$ for all $p \leqslant 3qr$, then it is in fact a symmetric subscheme of $\mathcal{M}_{p,q,r}$ for all p.

Proof. For convenience, let $B_{p,0} := I$ throughout this proof. It is clear that $\mathcal{B}_p$ satisfies (A1) and (A3) in Definition 1. We next prove that it also satisfies (A4). By hypothesis, we have

$$B_{p,i}B_{p,j} \in \operatorname{Span} \mathcal{B}_p \tag{21}$$

for all $i, j \in \{0, \ldots, m\}$ and $p \leqslant 3qr$. Now suppose for a contradiction that there is an integer $p' > 3qr$ where (21) fails. Since $\mathcal{M}_{p',q,r}$ is a scheme and thus satisfies (A4), we know that $B_{p',i}B_{p',j} \in \operatorname{Span} \mathcal{M}_{p',q,r}$. Thus, there must exist an index ℓ and $S_1, S_1', S_2, S_2' \in [p]_q^r$ where

$$B_{p',\ell}[S_1, S_1'] = B_{p',\ell}[S_2, S_2'] \quad \text{and} \quad (B_{p',i}B_{p',j})[S_1, S_1'] \neq (B_{p',i}B_{p',j})[S_2, S_2'].$$

However, by our hypothesis, for all $p \leqslant 3qr$ we have $B_{p,i}B_{p,j} \in \operatorname{Span} \mathcal{B}_p$, and thus $B_{p,\ell}[S_1, S_1'] = B_{p,\ell}[S_2, S_2']$, which implies that $(B_{p,i}B_{p,j})[S_1, S_1'] = (B_{p,i}B_{p,j})[S_2, S_2']$. Then by Lemma 26, it must be the case that $(B_{p',i}B_{p',j})[S_1, S_1'] = (B_{p',i}B_{p',j})[S_2, S_2']$ as well. Thus, $B_{p',i}B_{p',j} \in \operatorname{Span} \mathcal{B}_{p'}$.

Next, we show that for all $p' > 3qr$, $B_{p',i}$ is a symmetric matrix. By assumption,

$$(B_{p,0}B_{p,i})[S, S'] = (B_{p,0}B_{p,i})[S', S] \tag{22}$$

for all $p \leqslant 3qr$. Thus, applying Lemma 26 again, we obtain $B_{p',i} = B_{p',i}^{\top}$ for all $p' > 3qr$ as well. It then follows that (A2) holds as well. This finishes the proof. □

In the case of $q = r = 2$, Proposition 27 simply tells us that the three subschemes listed in Proposition 25(i) are indeed subschemes of $\mathcal{M}_{p,2,2}$ for all p, which we have already discussed. It would be interesting to see if some version of the converse of Proposition 27 is true — that if a certain contraction fails to yield a symmetric subscheme for enough small values of p, then we can guarantee that it would also fail to do so for large p.

6 Concluding remarks

Throughout this paper, we have pointed out some connections between association schemes and the analyses of semidefinite programs, as illustrated mainly by studying the lift-and-project relaxations of several classical problems in combinatorial optimization. In particular, we saw that the process of verifying the positive semidefiniteness of a certificate matrix could be simplified if said matrix is related to a commutative association scheme whose eigenvalues are known.

We comment that, since the hypermatching packing problem considered in Section 4.2 only concerns vertex saturation, two matchings are essentially interchangeable in the problem if they saturate the exact same set of vertices. Thus, instead of considering a matching of r hyperedges in K_p^q, one could have worked with a single hyperedge of size qr. Then we would be working with the simpler scheme $\mathcal{M}_{p,qr,1} = \mathcal{J}_{p,qr}$ instead. One of the reasons why we based our discussion on the more general framework of $\mathcal{M}_{p,q,r}$ is that this allows easier adaptation to study other combinatorial optimization problems where such a reduction may not be possible or suitable.

Finally, the approach of using association schemes to help analyze lift-and-project relaxations could also benefit from a better understanding of the underlying schemes, as we attempted to do for $\mathcal{M}_{p,q,r}$ in Section 5. For instance, for the perfect matching scheme $\mathcal{M}_{2r,2,r}$, it is known that its eigenvalues can be determined using zonal polynomials [34]. More recently, Srinivasan [38] showed that the eigenvalues of $\mathcal{M}_{2r,2,r}$ can be computed by recursively solving systems of linear equations that involve the central characters of the symmetric groups $\{\mathcal{S}_{2i} : i \in [r]\}$. Also lately, there has been interest in studying the eigenvalues of the perfect matching derangement graph [21, 30, 26], whose adjacency matrix is the sum of a subset of associates in $\mathcal{M}_{2r,2,r}$. However, we still do not have explicit and tractable combinatorial descriptions of the eigenvalues of the scheme $\mathcal{M}_{2r,2,r}$ in general, and a breakthrough on this front could give us a better handle on the eigenvalues of the matrices in Span $\overline{\mathcal{M}}_{p,2,r}$, a broader class of potential certificate matrices than those in Span $\tilde{\mathcal{M}}_{p,2,r}$.

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