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# Lab 01: Linear regression



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**How to do your homework**

- You will work directly on this notebook; the word **TODO** indicates the parts you need to do.
- You can discuss the ideas as well as refer to the documents, but *the code and work must be yours*.

**How to submit your homework**

- Before submitting, save this file as `<ID>.jl`. For example, if your ID is 123456, then your file will be `123456.jl`. And export to PDF with name `123456.pdf` then submit zipped source code and pdf into `123456.zip` onto Moodle.

> Note
>
> **Note that you will get 0 point for the wrong submit**.

**Content of the assignment**:

- Linear regression
- Polinomial linear regression

# 1. The hypothesis set

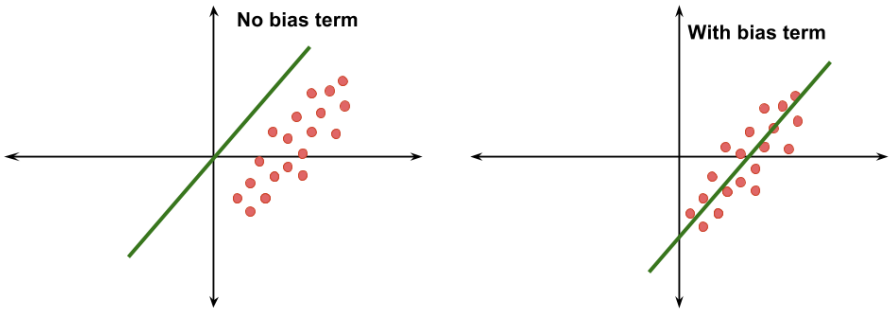
- Linear regression is a linear model, e.g. a model that assumes a linear relationship between the input variables (x) and the single output variable (y). More specifically, that y can be calculated from a linear combination of the input variables (x).
- Generally, a linear model will make predictions by calculating a weighted sum of the input features (independent variables).

$$\hat{y} = w_0 + \sum_{i=1}^{d} w_i x_i$$

  - $\hat{y}$ is the predicted value.
  - $d$ is the number of features.
  - $x_i$ is the $i^{th}$ feature value.
  - $w_j$ is the $j^{th}$ model parameter (including the bias term $w_0$ and the feature weights $w_1, w_2, \dots w_d$)
- You can rewrite the first equation in the matrix form as following:

$$\hat{y} = h_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}^T \cdot \mathbf{x}$$

  - $\mathbf{w}$ is the model **parameter vector** (including the bias term $w_0$ and the feature weights $w_1, w_2, \dots w_n$).
  - $\mathbf{w}^T$ is a transpose of $\mathbf{w}$ (a row vector insteade of column vector).
  - $\mathbf{x}$ is the instance's **feature vector**, *containing* $x_0$ to $x_d$, with $x_0$ *always equal to* 1.
  - $\mathbf{w}^T \cdot \mathbf{x}$ is the dot product of $\mathbf{w}^T$ and $\mathbf{x}$.
  - $h_{\mathbf{w}}$ is the hypothesis function, using the parameters $\mathbf{w}$.



# 2. Performance measure and the learning goal

- Before we start to train the model, we need to determine how good the model fits the training data. There are a couple of ways to determine the level of quality, but we are going to use the most popular one and that is the **MSE** (Mean Square Error). We need to find the value for $\mathbf{w}$ that will minimize the MSE:

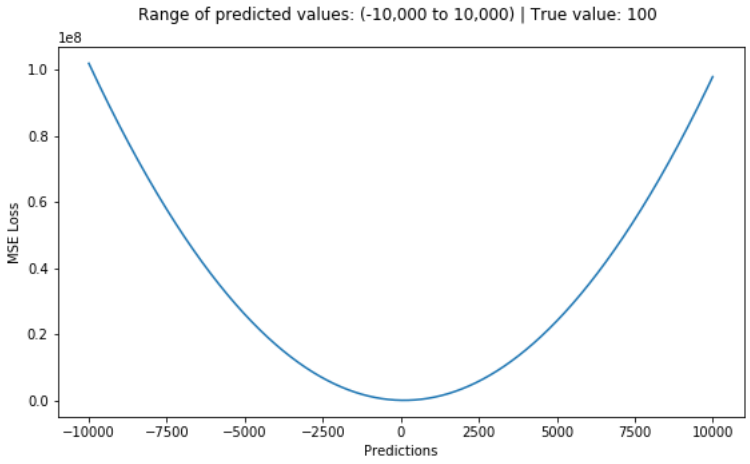
$$\mathbf{w} = \arg\min MSE_{\mathcal{D}_{train}}$$

- MSE on the train set $\mathcal{D}_{train}$ denoted as $(\mathbf{X}, \mathbf{y})$ including n samples $\{(\mathbf{x}_1, y_1), \dots (\mathbf{x}_n, y_n)\}$

$$MSE(X, h_{\mathbf{w}}) = \frac{1}{m} \sum_{i=1}^{m} \left(\mathbf{w}^T \cdot \mathbf{x}_i - y_i\right)^2$$

$$MSE(X, h_{\mathbf{w}}) = \frac{1}{m} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$$

- Example below is a plot of an MSE function where the true target value is 100, and the predicted values range between -10,000 to 10,000. The MSE loss (Y-axis) reaches its minimum value at prediction (X-axis) = 100. The range is 0 to ∞.




- To find the value of $\mathbf{w}$ that minimizes the MSE cost function the most common way (*we have known since high school*) is to solve the derivative (gradient) equation.

$$\hat{\mathbf{w}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

  - $\hat{\mathbf{w}}$ is the value of $\mathbf{w}$ that minimizes the cost function
- In order to reduce the complexity of this approach, I will provide you another version of computing the optimal $\mathbf{w}$:

$$\mathbf{X}\mathbf{w} = \mathbf{y}$$

$$\Leftrightarrow \mathbf{X}^T\mathbf{X}\mathbf{w} = \mathbf{X}^T\mathbf{y}$$

$$\Leftrightarrow \mathbf{w} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

# 3. Implementation

## Import library

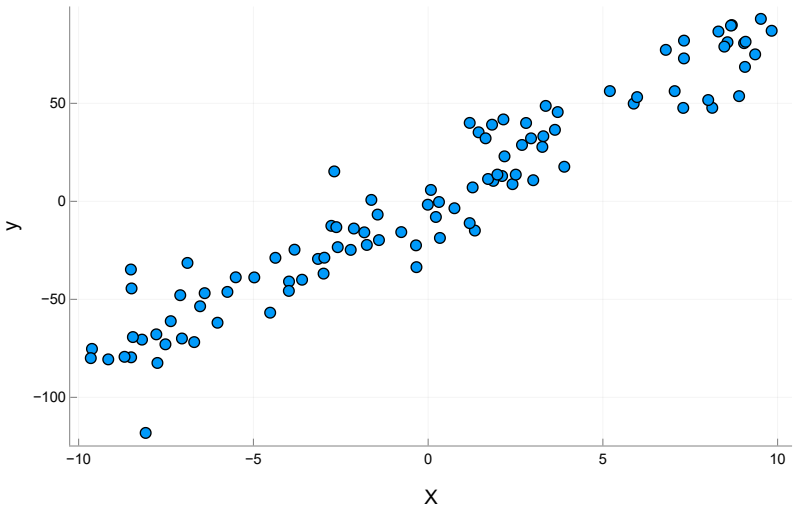
```
PlotlyBackend()
```

## Create data

```
([-8.18791, 2.11583, -1.8267, -2.76941, -3.60762, -9.61942, 1.337, 0.337482, 1.8647,  mo
```

```
begin
    function create_data(f, num_data=100)
        # f is the function that is used to generate data
        X = Base.rand(Distributions.Uniform(-10,10), num_data)
        y = f.(X);
        return X, y
    end

    a = Base.rand(Distributions.Uniform(-10,10), 1)[1]
    f(x) = a*x + Base.rand(Distributions.Normal(1,15), 1)[1]
    X, y = create_data(f)
end
```

# Visualize data




**TODO**:

Your observations about data:

- Biến độc lập x và biến phụ thuộc y có quan hệ tuyến tính cao.
- Các điểm trên đồ thị có độ phân tán thấp, phân bố theo hình dạng tuyến tính

# Training function

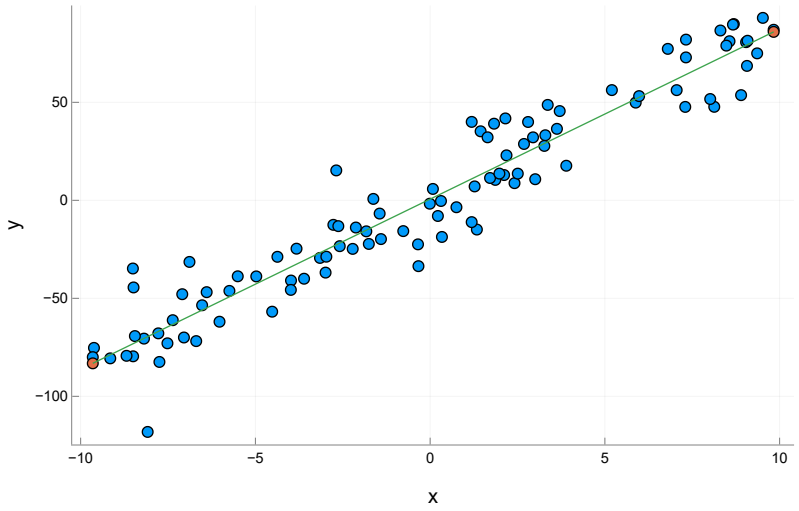
```
train_linear_regression (generic function with 1 method)
```

```
function train_linear_regression(X, y)
    """
    Trains Linear Regression on the dataset (X, y).

    Parameters
    ----------
    X : Matrix, shape (n, d + 1)
        The matrix of input vectors (each row corresponds to an input vector);
        the first column of this matrix is all ones (corresponding to x_0).
    y : Matrix, shape (n, 1)
        The vector of outputs.

    Returns
    -------
    w : Matrix, shape (d + 1, 1)
        The vector of parameters of Linear Regression after training.
    """
    # TODO: compute your weight

    w = inv(X'*X)*(X'*y)

    return w
end
```

```
begin
    # Construct one_added_X
    # TODO: First column of one_added_X is all ones (corresponding to x_0).
    column_ones = ones(size(X)[1])
    one_added_X = hcat(column_ones, X)

    println("size of one_added_X = ", Base.size(one_added_X))
    println("size of y = ", Base.size(y))
end
```

```
size of one_added_X = (100, 2)
size of y = (100,)
```

## Train our model and visualize

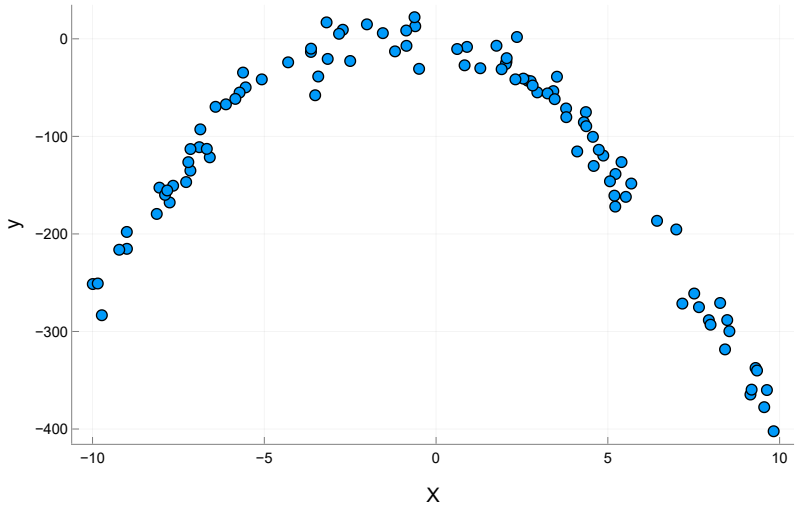



# 4. Polinomial regression

- Observe the following dataset. Can we use linear regression model to fit this data?

  **TODO**: Linear Regression là phương pháp để huấn luyện dữ liệu theo đường thẳng, phù hợp để dự đoán với những dữ liệu có biến độc lập và biến phụ thuộc có quan hệ tuyến tính. Với đồ thị ở dưới, các mẫu phân tán theo đường cong, việc sử dụng Linear Regression để dự đoán có thể dẫn đến những sai sót do dữ liệu có biến độc lập và biến phụ thuộc không có quan hệ tuyến tính.

```
([3.78467, -5.07375, -7.15163, 3.7907, -0.862131, 7.94025, 4.36112, 2.04819, -8.99893,
```



# Abstract the problem

- For this kind of datasets, you have to **change the hypothesis set**. For example, in this case, we assume that our data is in the parabol form. Therefore, the hypothesis set will be $\hat{y} = ax^2 + bx + c$. Another assumption: $\hat{y} = ax^3 + bx^2 + cx + d$.
- In general, we have the polinomial regression form:

$$\hat{y} = w_0 + \sum_{i=1}^{d} w_i x_i^i$$

- Re-write the equation above:

$$\hat{y} = h_{\mathbf{w}}(\mathbf{x}) = \mathbf{w}^T \cdot \mathbf{x}, \text{ where } \mathbf{x} = \begin{bmatrix} x_0 = 1 \\ x_1^1 \\ \vdots \\ x_d^d \end{bmatrix} \in \mathbb{R}^{d+1}$$

- To solve this problem, we have to find $\mathbf{w}$ such that:

$$\min_{\mathbf{w}} MSE(X, h_{\mathbf{w}}) = \min_{\mathbf{w}} \frac{1}{m} \|\mathbf{X}\mathbf{w} - \mathbf{y}\|^2$$

- Recall the solution of MSE in section 2:

$$\hat{\mathbf{w}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{y}$$

- Now, it's time for coding!

# Solve the problem in code

**Step 1**: Create polynomial feature

$$\mathbf{X} = \{\mathbf{x}_i = (x_0 = 1, x_1^1, x_2^2, \ldots, x_d^d)\}_{i=1}^n$$

```
# TODO: Create X

begin
    function poly_features(X, K)
        # X: inputs of size N x 1
        # K: degree of the polynomial
        X_poly=X;
        column_ones = ones(size(X)[1])
        X_poly = hcat(column_ones,X_poly)
        if K >= 2
            for i in range(2, K, step=1)
                X_poly=hcat(X_poly, X.^(i))
            end
        end
        return X_poly
    end

    # assume that our data is in form of a parabol
    X = poly_features(X_, 2)
    print(size(X))
end
```

```
(100, 3)
```

**Step 2**: train our model and find $\mathbf{w}$

```
w =  [-0.729915, -7.37431, -3.39144]
```

```
# TODO: train our model

w = inv(X'*X)*(X'*y_)
```

**Step 3**: Visualize our result