1. Consider the forward Euler method with central difference spatial discretization for $u_t = u_{xx}, \quad x \in [0,1]$, $u(x,0) = u_0(x)$, $u(0,t) = a(t)$ and $u(1,t) = b(t)$.:
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{1}{\Delta x^2}\left(u_{j-1} - 2u_j + u_{j+1}\right), \quad 1 \le j \le N-1, \tag{1}$$
and $u_0^n = a(t_n)$, $u_N^n = b(t_n)$.
Prove that if $\frac{\Delta t}{\Delta x^2} \le \frac{1}{2}$, then
$$u_{\min} \le u_j^n \le u_{\max},$$
where
$$u_{\min} := \min\{u_j^0,\ 0 \le j \le N;\ u_0^m,\ 0 \le m \le n;\ u_N^m,\ 0 \le m \le n\},$$
and
$$u_{\max} := \max\{u_j^0,\ 0 \le j \le N;\ u_0^m,\ 0 \le m \le n;\ u_N^m,\ 0 \le m \le n\}.$$
In other words, the maximum/minimum value of u_{jk}^n must be achieved either on the initial time or on the boundary.

2. Let $\{\phi_j,\ j = 1, \ldots N\}$ be a basis for the finite element space. Prove the mass matrix $M \in \mathbb{R}^{N\times N}$, whose (j,k)-th element $M_{jk} = (\phi_j, \phi_k) = \int_\Omega \phi_j \phi_k dx$, is a symmetric positive definite matrix.

3. Consider the following problem
$$-u_{xx} + u = f, u(0) = 1, u(1) = 0. \tag{2}$$
   (a) Write out the variational problem corresponding to the equation above.
   (b) Write out the finite element formulation for this problem using piecewise linear basis functions.

4. Coding problem:

   Consider $N = 39, 79, 159, 319$. Let $\Delta x = \frac{1}{N+1}$, i.e. $\Delta x = \frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}$. Given a mesh partition, $I_j = [x_{j-1}, x_j]$ with $j = 1, \ldots, N+1$ and
$$x_j = \begin{cases} j\Delta x, & j \text{ is an even number,} \\ (j - \frac{1}{2})\Delta x, & j \text{ is an odd number.} \end{cases}$$

Use piecewise linear finite element method (FEM) to solve

$$-u_{xx} = 2, u(0) = 0 \text{ and } u(1) = 0. \tag{3}$$

The exact solution for this problem is $u(x, y, t) = x(1 - x)$.

(a) Write out the matrix-vector form for the FEM scheme. (The answer should only have Δx terms. Replace Δx_j with their exact value.

(b) Write a code for the FEM solver. Make an error plot for $\max_{j=1,\ldots,N} |u(x_j) - u_{\text{FEM}}(x_j)|$.

Hint: if you are using Matlab or Julia, remember the first index in an array is 1.