1. Show that
$$\frac{c_{j-\frac{1}{2}}u_{j-1} - (c_{j-\frac{1}{2}} + c_{j+\frac{1}{2}})u_j + c_{j+\frac{1}{2}}u_{j+1}}{\Delta x^2}$$
is a second order approximation to $\partial_x(c(x)\partial_x u(x))$.

2. Consider the θ-scheme
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \theta\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} + (1-\theta)\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$
for the PDE problem
$$u_t = u_{xx}, \tag{1a}$$
with the following initial and boundary conditions
$$u(x,0) = f(x), \mathbf{u_x(0,t) = u(0,t) + A}, u(1,t) = 0, \tag{1b}$$
where A is a constant.
Let $U^n = (u_0^n, \ldots, u_{N-1}^n)^T$, use the matrix-vector form to show how to update U^{n+1} from U^n.

3. Lax–Friedrichs scheme for $u_t + cu_x = 0$ is defined as
$$\frac{u_j^{n+1} - \frac{1}{2}(u_{j+1}^n + u_{j-1}^n)}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0. \tag{2}$$
   (a) Derive stability condition for the Lax-Friedrichs scheme with Fourier analysis.
   (b) Prove leading terms of the truncation error are $O(\Delta t) + O(\frac{\Delta x^2}{\Delta t})$.
   (c) Take $\Delta t = 0.5\Delta x/c$. Derive the modified equation for the Lax-Friedrichs scheme keeping $O(\Delta x)$ term.
   (d) Take $\Delta t = 0.5\Delta x/c$. Perform dispersion analysis for the Lax-Friedrichs scheme and explain the nature of the leading error is dispersive or dissipative.

4. The Lax-Friedrichs flux for $u_t + \partial_x(f(u)) = 0$ is defined as
$$\hat{F}(u_L, u_R) = \frac{1}{2}(fu_L + f(u_R)) - \alpha(u_R - u_L), \quad \alpha = \max_u |f'(u)|. \tag{3}$$
Consider a Burgers problem $u_t + \partial_x(\frac{1}{2}u^2) = 0$ with $\max_u |u| = \alpha$.
Write out the finite volume scheme with the Lax-Friedrichs flux for this problem.

5. Coding problem:

   Consider the problem:

$$u_t(x,y,t) = u_{xx}(x,y,t) + u_{yy}(x,y,t), (x,y) \in (0,\pi)\times(0,\pi), \tag{4a}$$

$$u(x,y,0) = \sin(x)\sin(y), u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0. \tag{4b}$$

   The exact solution for this problem is $u(x,y,t) = \exp(-2t)\sin(x)\sin(y)$.

   Write a code for the ADI scheme solving this equation.

   (Hint: one can mimic lec5_crank_nicolson_2d_version2.m and introduce a vector to save $u^{n+\frac{1}{2}}$ or directly write out how to u^{n+1} from u^n.)

   Let $T = 1$. Make an error plot $\max_j |u_j^M - u(x_j,T)|$ under the log-log scale with $N = 20, 40, 80, 160$ to show the error scales as $O(\Delta x^2)$.

   When submitting your homework, please also attach your code so that you can get as many as partial score as possible if something is not completely correct.