Time Optimal Turn-Right Trajectory using Pontryagin's principle

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Abstract

Time Optimal Control is one of the most fundamental problem in Control Theory, however solutions are not necessarily straightforward, even in what appears to be very simple case. In this work, we derive the optimal control solution to complete a 90 degrees turn right in the minimum amount of time, for a 2D particle controlled in acceleration. The solution to this control problem is derived using Pontryagin's Maximum Principle.

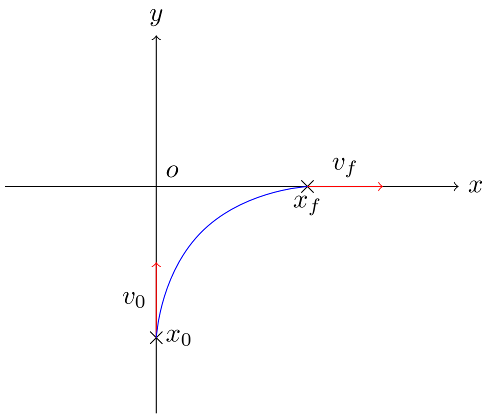

Figure 1: Optimal Turn-Right Trajectory

1 Introduction

Time-optimal control problems are widely used in robotics for a variety of different applications. Although it is a well studied and understood problem, the solution is not always straightforward, even in very simple cases.

In this work, we present how Pontryagin's Maximum Principle can be used to derive the time-optimal control solution to complete a 90 degrees turn right, for a 2D particle controlled in acceleration as shown in figure 1. We show that Pontryagin's Principle imposes constraints on the structure of the control, reducing the optimal control problem to an optimization problem on a set of constant parameters.

In the first section, we define the dynamical system, and the associated time-optimal control problem. In the second section, we introduce Pontryagin's Maximum Principle and derive the necessary and sufficient conditions on the optimal solution, yielding structure constraints on the control. Finally, we apply the general results derived in the previous section to solve our time-optimal turn right control problem.

2 Pontryagin's maximum principle for Time-Optimal control

2.1 Problem formulation

We consider 2D particle evolving in $\mathbb{R}^2$. Here $(x(t), y(t)) \in \mathbb{R}^2$ denotes the position of the particle at time t, $(\dot{x}(t), \dot{y}(t)) \in \mathbb{R}^2$ its velocity. The particle is controlled in acceleration such that $(\ddot{x}(t), \ddot{y}(t)) = (u_x(t), u_y(t))$ with $(u_x(t), u_y(t)) \in U$. The control set U is a compact and convex subset of $\mathbb{R}^2$, defined as $U = \{u \in \mathbb{R}^2 : \|u\|_2 \leq M\}$.

Let's define the state $\mathbf{x}(t) = (x(t), y(t), \dot{x}(t), \dot{y}(t))^T \in \mathbb{R}^4$ and control $\mathbf{u}(t) = (u_x(t), u_y(t))^T \in U$. We note $\mathcal{U}$ the admissible control set, corresponding to the set of piecewise continuous functions u defined on a $\mathbb{R}^+$ with values in the control set U. For a admissible control $u \in \mathcal{U}$, the dynamics of the system is described by the following equation:

$$\forall t \geq 0, \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{1}$$

Where:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{2}$$

Given an initial state $\mathbf{x}_o$ and a final state $\mathbf{x}_f$, the time-optimal control problem aims at finding the admissible control $u^* \in \mathcal{U}$ that would bring the system from the initial state x_0 to the final state x_f in the minimum amount of time.

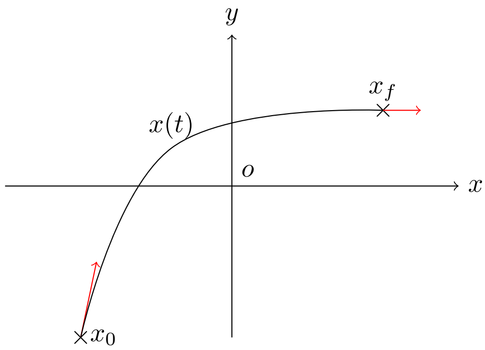

Figure 2: Time Optimal Trajectory

Formally, we can reformulate the problem as:

$$\begin{aligned} \underset{\mathbf{u} \in \mathcal{U}}{\text{minimize}} \quad & \int_0^{t_f} 1 dt \\ \text{subject to} \quad & \forall t \geq 0, \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ & \mathbf{x}(0) = \mathbf{x}_o \\ & \mathbf{x}(t_f) = \mathbf{x}_f \end{aligned} \tag{3}$$

2.2 Pontryagin's maximum principle

In this section, we derive the necessary condition on the optimal solution by applying Pontryagin Maximum Principle. Let's define the control Hamiltonian function H of the optimal control problem defined in (25):

$$H : \mathbb{R}^+, \mathbb{R}^+, \mathbb{R}^4, \mathbb{R}^4, \mathbb{R}^2 \to \mathbb{R}, \quad H(t, \lambda_0, \lambda, \mathbf{x}, \mathbf{u}) = \lambda_0 + \lambda^T(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}) \tag{4}$$

Applying *Pontryagin maximum principle* from [1], let $(\mathbf{x}^*, \mathbf{u}^*)$ be a controlled trajectory defined over $[0, t_f]$, with $u \in \mathcal{U}$ an admissible control. If $(\mathbf{x}^*, \mathbf{u}^*)$ is optimal, then there exist a constant $\lambda_0 \geq 0$ and a co-vector $\lambda : [0, t_f] \to \mathbb{R}^4$ called adjoint variable, such that the following conditions are satisfied:

$$\forall t \in [0, t_f], \quad H(t, \lambda_0, \lambda, \mathbf{x}^*(t), \mathbf{u}^*(t)) = 0 \tag{5}$$

$$\forall t \in [0, t_f], \quad \lambda(t) \neq 0 \quad \textit{and} \quad \dot{\lambda}^T(t) = -\lambda^T(t)\mathbf{A} \tag{6}$$

$$\forall t \in [0, t_f], \quad H(t, \lambda_0, \lambda, \mathbf{x}^*(t), \mathbf{u}^*(t)) = \min_{\mathbf{v} \in U} H(t, \lambda_0, \lambda, \mathbf{x}^*(t), \mathbf{v}) \tag{7}$$

Reference to the full derivation of Pontryagin's Maximum Principle for Linear Time-Invariant system can be found in appendix 5.1.

2.3 Sufficient conditions

Since the system defined in (1) and (2) is a Completely Controllable Linear Time-Invariant system (8), the conditions of Pontryagin's Maximum Principle are also sufficient. One can refer to refer to Appendix 5.3 for detailed references.

Let's define the Kalman matrix for the system defined in (1) and (2):

$$\begin{aligned}\mathbf{K} &= (\mathbf{B}, \mathbf{AB}, ..., \mathbf{A}^3\mathbf{B}) \\ &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},\end{aligned} \tag{8}$$

It is straightforward to see that $rank(\mathbf{K}) = 4$ (full rank), which means that the dynamical system is completely controllable.

Therefore, this result implies that if we find a control that satisfies the conditions from Pontryagin's Maximum Principle (5) - (6) - (7), then this control is the optimal control.

2.4 Control structure

Solving the differential equation (6), we have $\lambda^T(t) = \mu^T \exp(-\mathbf{A}t)$ with $\lambda(0) = \mu \in (\mathbb{R}^4)^*$ a constant. Let's define $\mu = (\mu_1, \mu_2, \mu_3, \mu_4)$, since $\exp(-\mathbf{A}t) = (\mathbf{I} - \mathbf{A}t)$ from (2), it yields:

$$\forall t \in [0, t_f], \quad \lambda(t) = (\mu_1, \mu_2, \mu_3 - \mu_1 t, \mu_4 - \mu_2 t)^T \tag{9}$$

Then, injecting (9) into (7) yields:

$$\begin{aligned}\forall t \in [0, t_f], \mathbf{u}^* &= \underset{\mathbf{v}\in U}{\arg\min} \quad \lambda^T \mathbf{B}\mathbf{v} \\ &= \underset{\mathbf{v}\in U}{\arg\min} \quad (\mu_3 - \mu_1 t, \mu_4 - \mu_2 t) \cdot (v_x, v_y)\end{aligned} \tag{10}$$

Proposition 1: The optimal control is defined by four variables $\mu_1, \mu_2, \mu_3, \mu_4$. Defining $b(t) = (\mu_3 - \mu_1 t, \mu_4 - \mu_1 t) \in \mathbb{R}^2$, the result from (10) states that if $b(t) \neq 0$ then $\mathbf{u}^*(t)$ is on the boundary of U.

Proof of Proposition 1 is straightforward via contradiction, and is given in Appendix section 5.2.

With $b(t) = (\mu_3 - \mu_1 t, \mu_4 - \mu_1 t)$ and $\mathbf{u}^*(t) \in U = \{u \in \mathbb{R}^2 : \|u\|_2 \leq M\}$, since $\mathbf{u}^*(t)$ is on the boundary of U and from (10), for all $t \in [0, t_f]$ such that $b(t) \neq 0$, we have $\mathbf{u}^*(t) = -M\frac{b(t)}{\|b(t)\|_2}$. This can be written using the expression of $b(t)$:

$$\begin{aligned}u_x &= -M\frac{\mu_3 - \mu_1 t}{\sqrt{(\mu_3 - \mu_1 t)^2 + (\mu_4 - \mu_2 t)^2}} \\ u_y &= -M\frac{\mu_4 - \mu_2 t}{\sqrt{(\mu_3 - \mu_1 t)^2 + (\mu_4 - \mu_2 t)^2}}\end{aligned} \tag{11}$$

Geometrically, we can see $b(t) = (\mu_3 - \mu_1 t, \mu_4 - \mu_1 t)$ for $t \in [0, t_f]$ as a point moving along a support line. The optimal control is driven by the motion of this point along the line, as it corresponds to the intersection of $-b(t)$ and the control set U as shown in figure 3.

It is interesting to note that if the line $L = \{b(t) \in \mathbb{R}^2, t \in \mathbb{R}\}$ intersects the origin, then the control is bang-bang. Also, one can notice that the a set of parameters $\mu_1, \mu_2, \mu_3, \mu_4$ defining the optimal control is not unique. It is easy to understand that any line parallel to $b(t)$ (on the same side with the respect to the origin) could drive the same optimal solution.

2.5 Final time expression

In this section, we present how the final time t_f can be expressed in the general case, as a function of $\mu_1, \mu_2, \mu_3, \mu_4$. From the result of Pontrygin Maximum Principle (5), and with the structure $u(t)$ defined

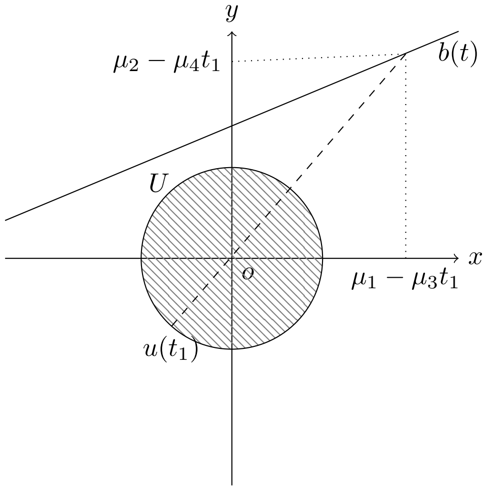

Figure 3: Optimal Control Structure

(11), we have for almost every t in $[0, t_f]$:

$$0 = \lambda_0 + \mu_1 \dot{x}(t) + \mu_2 \dot{y}(t) - \sqrt{(\mu_1 t - \mu_3)^2 + (\mu_2 t - \mu_4)^2} \tag{12}$$

We can evaluate for $t = 0$ and $t = tf$, with the initial and final condition $\dot{x}(0) = v_{x_0}$, $\dot{y}(0) = v_{y_0}$ and $\dot{x}(t_f) = v_{x_f}$, $\dot{y}(t_f) = v_{y_f}$:

$$0 = \lambda_0 + \mu_1 v_{x_0} + \mu_2 v_{y_0} - \sqrt{\mu_3^2 + \mu_4^2} \tag{13}$$

$$0 = \lambda_0 + \mu_1 v_{x_f} + \mu_2 v_{y_f} - \sqrt{(\mu_1 t_f - \mu_3)^2 + (\mu_2 t_f - \mu_4)^2} \tag{14}$$

By subtracting (13) to (14), we have:

$$\mu_1 (v_{x_f} - v_{x_0}) + \mu_2 (v_{y_f} - v_{y_0}) + \sqrt{\mu_3^2 + \mu_4^2} = \sqrt{(\mu_1 t_f - \mu_3)^2 + (\mu_2 t_f - \mu_4)^2} \tag{15}$$

Taking the square and rearranging (15) yields:

$$(\mu_1^2 + \mu_2^2) t_f^2 - 2(\mu_1 \mu_3 + \mu_2 \mu_4) t_f + \mu_3^2 + \mu_4^2 - \left(\mu_1 (v_{x_f} - v_{x_0}) + \mu_2 (v_{y_f} - v_{y_0}) + \sqrt{\mu_3^2 + \mu_4^2} \right)^2 = 0 \tag{16}$$

Equation (16) is a second degree polynomial expression in t_f, which can be solved to find the expression of tf as a function of $\mu_1, \mu_2, \mu_3, \mu_4$.

3 Case study

3.1 Problem statement

Let the initial and final position be $(0, -L)$ and $(L, 0)$, and the initial and final velocity $(0, V)$ and $(V, 0)$, such that $\mathbf{x}_0 = (0, -L, 0, V)$ and $\mathbf{x}_f = (L, 0, V, 0)$, with $L, V \in \mathbb{R}^+$. We want to find the control that brings the system from $\mathbf{x}_0$ to $\mathbf{x}_f$ in the minimum amount of time. This case study is illustrated in the figure 1.

3.2 Symmetries

According to the initial and final condition, the problem has a spatial and temporal symmetries, as shown figure 5.

Given the symmetries of the problem:

$$\forall t \in [0, t_f], \quad u_x(t) = -u_y(t_f - t) \tag{17}$$

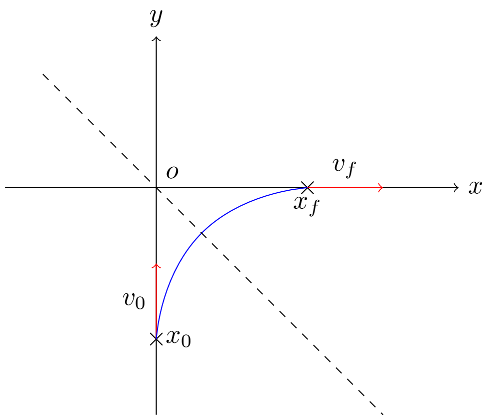

Figure 4: Symmetrie in Optimal Turn-Right Trajectory

Case 1: $b(t) \neq 0, \forall t \in \mathbb{R}$

From the control structure defined in (10), and evaluating the relationship between u_x and u_y defined in (17) for $t = 0$, $t = t_f$ and $t = t_f/2$, we have:

$$\begin{aligned} -\frac{\mu_3}{\sqrt{\mu_3^2+\mu_4^2}} &= \frac{\mu_4-\mu_2 t_f}{\sqrt{(\mu_3-\mu_1 t_f)^2+(\mu_4-\mu_2 t_f)^2}} \\ \frac{\mu_4}{\sqrt{\mu_3^2+\mu_4^2}} &= -\frac{\mu_3-\mu_1 t_f}{\sqrt{(\mu_3-\mu_1 t_f)^2+(\mu_4-\mu_2 t_f)^2}} \\ \mu_1\frac{t_f}{2}-\mu_3 &= -(\mu_2\frac{t_f}{2}-\mu_4) \end{aligned} \tag{18}$$

Rearranging the (18), we have:

$$\begin{aligned} 0 &= (\mu_2-\mu_1)(\mu_4+\mu_3)^2 \\ t_f &= \frac{2(\mu_3+\mu_4)}{\mu_1+\mu_2} \end{aligned} \tag{19}$$

Therefore, we have $\mu_1 = \mu_2$ since otherwise $tf = 0$. The structure of the control, and the symmetry, we have:

$$\begin{aligned} \mu_1 &= \mu_2 \\ t_f &= \frac{(\mu_3+\mu_4)}{\mu_1} \end{aligned} \tag{20}$$

Case 2: Let's assume $\exists t_d \in \mathbb{R}, \quad b(t_d) = 0$

Since $b(t) = (\mu_3 - \mu_1 t, \mu_4 - \mu_2 t)$, there is a unique t_d such that $b(t_d) = 0$, otherwise $\mu_1 = \mu_2 = \mu_3 = \mu_4 = 0$, which is contradict (6). So $b(t_d) = 0$ means:

$$\begin{aligned} \mu_1 t_d &= \mu_3 \\ \mu_2 t_d &= \mu_4 \end{aligned} \tag{21}$$

From (11) and (21), we have:

$$\begin{aligned} u_x(t) &= \frac{\mu_1}{\sqrt{\mu_1^2+\mu_2^2}} sgn(t-td) \\ u_y(t) &= \frac{\mu_1}{\sqrt{\mu_2^2+\mu_2^2}} sgn(t-td) \end{aligned} \tag{22}$$

<u>Subcase 2.a:</u> $td \in]0, tf[$
Then $sgn(t-td)$ changes sign over $]0, tf[$, and $sgn(t-td) = -1$ for $t \in]0; t_d[$ and $sgn(t-td) = 1$ for $t \in]td; t_f[$. From (17) and (22), we have:

$$\mu_1 = \mu_2 \tag{23}$$

<u>Subcase 2.b:</u> $td \in \mathbb{R}\backslash]0, tf[$
Then $sgn(t-td)$ does not change sign over $]0, tf[$, and $sgn(t-td) = 1$ or $sgn(t-td) = -1$ for $t \in]0; t_f[$. From (17) and (22), we have:

$$\mu_1 = -\mu_2 \tag{24}$$

3.3 Solution

In theory it is possible to reduce the number of variables in the optimization problem from the expression of the final time in section 2.5 and from the symmetries section 3.2. However, in practice we found that the optimizer performs better if all the variables are left in the optimization problem.

Formally, we can reformulate the optimization problem as:

$$\begin{aligned}
&\underset{(\mu_1,\mu_2,\mu_3,\mu_4,t_f)\in\mathbb{R}^{4*}\times\mathbb{R}^{+*}}{\text{minimize}} && \|\mathbf{x}(t_f)-\mathbf{x}_f\|_2 \\
&\text{subject to} && \forall t\geq 0, \dot{\mathbf{x}}(t)=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t) \\
& && \mathbf{x}(0)=\mathbf{x}_o \\
& && \text{For almost all } t\in[0,t_f]: \\
& && \mathbf{u}(t)=\begin{bmatrix} -M\frac{\mu_3-\mu_1 t}{\sqrt{(\mu_3-\mu_1 t)^2+(\mu_4-\mu_2 t)^2}} \\ -M\frac{\mu_4-\mu_2 t}{\sqrt{(\mu_3-\mu_1 t)^2+(\mu_4-\mu_2 t)^2}} \end{bmatrix},
\end{aligned} \tag{25}$$

The time-optimal trajectories obtained for different initial velocities are shown in Figure (5).

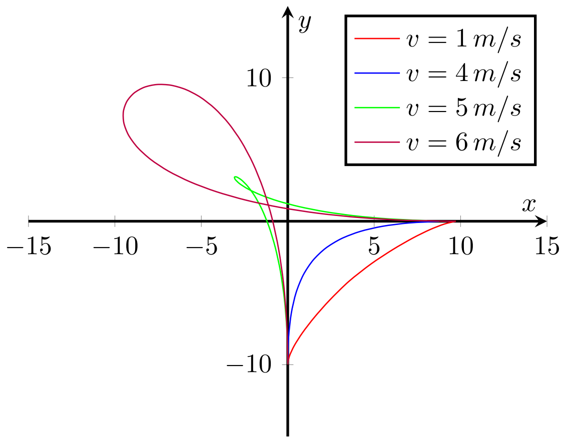

Figure 5: Optimal Turn Right Trajectories for different initial and final velocities

The control parameters and final time for the trajectories shwown in Figure (5) are given in Table (1).

Initial / Final velocity	μ_1	μ_2	μ_3	μ_4	$t_f(s)$
1 m/s	12.14651638	12.14655487	40.60633328	35.56975848	6.27143287
4 m/s	5.6374956	5.63744983	26.10085666	1.90806396	4.96833966
5 m/s	-1.2390784	-1.23910072	-4.14367549	-9.3599676	10.89799504
6 m/s	-0.62099344	-0.62103989	-3.3249513	-6.01708324	15.04294854

Table 1: Optimal Trajectories results

As expected, the solution found by the optimizer respects the conditions given by section 2.5 and section 3.2, and one can verify that equations from these sections hold on the results presented in Table (1).

4 Conclusion

In this work we have presented a method to find the time-optimal control for a turn right trajectory problem, based on the application of Pontryagin's Maximum Principle. Applying this principle to the general form of time-optimal control problem yields a parametrization of the control function, which allow to reduce this control problem to an optimization problem over a set of constant parameters (control parameters).

The general results derived from section 2 are applied to the Time-Optimal Turn-Right Trajectory problem. The symmetries exposed in this special case enable to further constraint the control structure of the solution. It is interesting to note that the time-optimal solution is not necessarily the shortest one, as shown in the results obtained in section 3.3. Furthermore, one can apply the general results introduced in section 2 to other time-optimal control problems with different initial and final conditions.

5 Appendix

5.1 Full detail of the Pontryagin's Maximum Principle

For the derivation of the Maximum Principle applied to a Linear Time-Invariant system, we invite the reader to refer to Section 2.2.2 in [1] for the general formulation of Pontryagin's Maximum Principle, and to section 2.5 in [1] for its derivation in the Linear-Time Invariant system case.

5.2 Proof of Proposition 1

By contradiction, if there is $t_1 \in [0, t_f]$ such that $b(t_1) \neq 0$ and $\mathbf{u}^*(t_1)$ is not on the boundary of U. Let's express in polar coordinates $\mathbf{u}^*(t_1) = \|\mathbf{u}^*(t_1)\| cos(\theta_{u^*}(t_1))$ and $b(t_1) = \|b(t_1)\| cos(\theta_b(t_1))$. Then, (7) corresponds to:

$$\|b(t_1)\| \|u^*(t_1)\| cos(\theta_b(t_1) - \theta_{u^*}(t_1)) = \arg\min_{\mathbf{v} \in U} \quad \|b(t_1)\| \|v\| cos(\theta_b(t_1) - \theta_v) \tag{26}$$

Since U is a closed set, then there exists $\epsilon > 0$ such that $w = (\|u^*(t_1)\| + \epsilon) cos(\theta_{u^*}(t_1)) \in U$, contradicting (26)

5.3 Detail on Necessary and Sufficient Condition for Time-Optimality

One can refer to Section 3.5 of *Geometric Optimal Control: Theory, Methods and Examples* [1], for proof of the following propositions:

Proposition 3.5.1: A completely controllable linear time-invariant system $\Sigma : \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad u \in U$, for which 0 is an interior point of the control set U, $0 \in int(U)$, is small-time locally controllable from the origin.

Corollary 3.5.1: Let $\Sigma : \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}, \quad u \in U$ be small-time locally controllable from the origin. Then an admissible control $u \in \mathcal{U}$ is time-optimal if and only if there exists a nontrivial solution $\lambda : [0, T] \to (\mathbb{R})^*$ to the adjoint equation $\dot{\lambda} = -\lambda \mathbf{A}$ such that

$$\lambda(t)\mathbf{Bu} = \min_{\mathbf{v} \in U} \quad \lambda^T \mathbf{Bv} \quad \text{almost everywhere on } [0; T]$$

In other words, for completely controllable linear time-invariant systems, the conditions of Pontryagin's Maximum Principle are both necessary and sufficient for time optimality.

References

[1] H. Schattler and U. Ledzewicz, *Geometric Optimal Control: Theory, Methods and Examples*. Interdisciplinary Applied Mathematics, Springer New York, 2012.