Derivation of (3.47) on P. 66 e.g.

W.t.s.

$$X\hat{\beta}^{ridge} = \sum_{j=1}^{p} u_j \frac{d_j^2}{d_j^2+\lambda} u_j^T y$$

We will use the SVD of X: $X = UDV^T$ (3.45)

$$\Rightarrow X\hat{\beta}^{ridge} = X(X^TX+\lambda\mathbb{I})^{-1}X^Ty$$
$$= UDV^T((UDV^T)^T(UDV^T)+\lambda\mathbb{I})^{-1}(UDV^T)^T y$$
$$= UDV^T(VDU^TUDV^T+\lambda\mathbb{I})^{-1}VDU^Ty$$

using:

$D = D^T$ and for orthogonal matrix U: $U^TU = \mathbb{I}$

V: $V^TV = \mathbb{I}$

$$\Rightarrow UDV^T(VD^2V^T+\lambda\mathbb{I})^{-1}VDU^Ty$$
$$= UDV^T(VD^2V^T+\lambda VV^T)^{-1}VDU^Ty$$
$$= UDV^T(V(D^2+\lambda\mathbb{I})V^T)^{-1}VDU^Ty$$
$$= UDV^T(V^T)^{-1}(D^2+\lambda\mathbb{I})^{-1}V^{-1}VDU^Ty$$

using:

$(AB)^{-1} = B^{-1}A^{-1}$ for non-singular matrices A, B

$$= UD(D^2+\lambda\mathbb{I})^{-1}DU^Ty$$

and since the middle parts are diagonal this gives us that

$$X\hat{\beta}^{ridge} = \sum_{j=1}^{p} u_j \frac{d_j^2}{d_j^2+\lambda} u_j^T y$$