Ex 5.6

Since we know the period T we can simply apply the following transformation to every element of X:

$$x_{new} = x_{old} - nT$$

Where n is the maximum integer value such that $x_{new} > 0$

After this transformation we may apply any of the usual basis expansions on the domain $(0, T)$.

We might additionally like to add one further constraint that the function is continuous at the left and right bounds. e.g.

$$f(0) = f(T)$$

So for a periodic 'global' cubic polynomial where we fit a single function and constrain it to be continuous at the boundaries we have:

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3$$

Such that $f(0) = f(T)$

$$\Rightarrow \beta_0 = \beta_0 + \beta_1 T + \beta_2 T^2 + \beta_3 T^3$$

$$\Rightarrow \beta_1 = -(\beta_2 T + \beta_3 T^2)$$

So

$$f(x) = \beta_0 - (\beta_2 T + \beta_3 T^2)x + \beta_2 x^2 + \beta_3 x^3$$

$$= \beta_0 + \beta_2(x^2 - Tx) + \beta_3(x^3 - T^2 x)$$

Giving the following basis:

$$h_1(x) = 1 \quad , \quad h_2(x) = x^2 - Tx \quad , \quad h_3(x) = x^3 - T^2 x$$

Alternatively, we could fit a periodic cubic polynomial spline with knots at ξ_1 and ξ_2.

Starting from the basis in eqn 5.3 (derived in exercise 5.1), we have:

$$f(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \beta_4 (x-\xi_1)_+^3 + \beta_5 (x-\xi_2)_+^3$$

we add the constraint:

$$f(0) = f(T)$$

$$\Rightarrow \beta_0 = \beta_0 + \beta_1 T + \beta_2 T^2 + \beta_3 T^3 + \beta_4 (T-\xi_1)_+^3 + \beta_5 (T-\xi_2)^3$$

$$\Rightarrow \beta_1 = -\left(\beta_2 T + \beta_3 T^2 + \beta_4 \frac{(T-\xi_1)^3}{T} + \beta_5 \frac{(T-\xi_2)^3}{T}\right)$$

so,

$$f(x) = \beta_0 - \left(\beta_2 T + \beta_3 T^2 + \beta_4 \frac{(T-\xi_1)^3}{T} + \beta_5 \frac{(T-\xi_2)^3}{T}\right) + \beta_2 x^2$$

$$+ \beta_3 x^3 + \beta_4 (x-\xi_1)_+^3 + \beta_5 (x-\xi_2)_+^3$$

$$= \beta_0 + \beta_2 (x^2 - Tx) + \beta_3 (x^3 - T^2 x) + \beta_4 \left[(x-\xi_1)_+^3 - \frac{(T-\xi_1)^3}{T} x\right]$$

$$+ \beta_5 \left[(x-\xi_2)_+^3 - \frac{(T-\xi_2)^3}{T} x\right]$$

Giving the following basis

$$h_1(x) = 1, \quad h_2(x) = x^2 - Tx, \quad h_3(x) = x^3 - T^2 x$$

$$h_4(x) = (x-\xi_1)_+^3 - \frac{(T-\xi_1)^3}{T} x$$

$$h_5(x) = (x-\xi_2)_+^3 - \frac{(T-\xi_2)^3}{T} x$$

And we also notice that this can easily be extended to any number of knots K.