Ex 5.1

Wish to show:

$$\beta_1 \mathbb{I}(x < \xi_1) + x\beta_2 \mathbb{I}(x < \xi_1) + x^2\beta_3 \mathbb{I}(x < \xi_2) + x^3\beta_4 \mathbb{I}(x < \xi_2)$$
$$+ \beta_5 \mathbb{I}(\xi_1 \le x < \xi_2) + x\beta_6 \mathbb{I}(\xi_1 \le x < \xi_2) + x^2\beta_7 \mathbb{I}(\xi_1 < x < \xi_2) + x^3\beta_8 \mathbb{I}(\xi_1 < x < \xi_2)$$
$$+ \beta_9 \mathbb{I}(\xi_2 \le x) + x\beta_{10} \mathbb{I}(\xi_2 \le x) + x^2\beta_{11} \mathbb{I}(\xi_2 < x) + x^3\beta_{12} \mathbb{I}(\xi_2 \le x)$$

With constraints

$$\beta_1 + \xi_1\beta_2 + \xi_1^2\beta_3 + \xi_1^3\beta_4 = \beta_5 + \xi_1\beta_6 + \xi_1^2\beta_7 + \xi_1^3\beta_8 \qquad (1a)$$
$$\beta_2 + 2\xi_1\beta_3 + 3\xi_1^2\beta_4 = \beta_6 + 2\xi_1\beta_7 + 3\xi_1^2\beta_8 \qquad (1b)$$
$$2\beta_3 + 6\xi_1\beta_4 = 2\beta_7 + 6\xi_1\beta_8 \qquad (1c)$$
$$\beta_5 + \xi_2\beta_6 + \xi_2^2\beta_7 + \xi_2^3\beta_8 = \beta_9 + \xi_2\beta_{10} + \xi_2^2\beta_{11} + \xi_2^3\beta_{12} \qquad (2a)$$
$$\beta_6 + 2\xi_2\beta_7 + 3\xi_2^2\beta_8 = \beta_{10} + 2\xi_2\beta_{11} + 3\xi_2^2\beta_{12} \qquad (2b)$$
$$2\beta_7 + 6\xi_2\beta_8 = 2\beta_{11} + 6\xi_2\beta_{12} \qquad (2c)$$

is equivalent to the following (unconstrained) expression:

$$\alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 (x - \xi_1)_+^3 + \alpha_6 (x - \xi_2)_+^3 \qquad (*)$$

Now we can reduce (*) to 3 cases:

(a) $x < \xi_1$

$$\Rightarrow \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 = \beta_1 + x\beta_2 + x^2\beta_3 + x^3\beta_4$$

(b) $\xi_1 \le x < \xi_2$

$$\Rightarrow \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 (x - \xi_1)^3$$
$$= \alpha_1 - \xi_1^3\alpha_5 + x(\alpha_2 + 3\alpha_5\xi_1^2) + x^2(\alpha_3 - 3\alpha_5\xi_1) + x^3(\alpha_4 + \alpha_5)$$
$$= \beta_5 + x\beta_6 + x^2\beta_7 + x^3\beta_8$$

(c) $\xi_2 \le x$

$$\Rightarrow \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \alpha_5 (x - \xi_1)^3 + \alpha_6 (x - \xi_2)^3$$
$$= \alpha_1 - \alpha_5\xi_1^3 - \alpha_6\xi_2^3 + x(\alpha_2 + 3\alpha_5\xi_1^2 + 3\alpha_6\xi_2^2) + x^2(\alpha_3 - 3\alpha_5\xi_1 - 3\alpha_6\xi_2)$$
$$+ x^3(\alpha_4 + \alpha_5 + \alpha_6)$$
$$= \beta_9 + x\beta_{10} + x^2\beta_{11} + x^3\beta_{12}$$

Now we just need to show that if we set the coefficients of the powers to be equal in (a), (b) and (c) then the constraints (1a) - (2c) are all satisfied.

e.g. setting $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$, $\alpha_1 - \xi_1^3 \alpha_5 = \beta_5$, $\alpha_2 + 3\alpha_5 \xi_1^2 = \beta_6$, $\alpha_3 - 3\alpha_5 \xi_1 = \beta_7$ and $\alpha_4 + \alpha_5 = \beta_8$ we solve

(1) $2\beta_7 + 6\xi_1\beta_8 = 2(\alpha_3 - 3\alpha_5\xi_1) + 6\xi_1(\alpha_4 + \alpha_5)$

$= 2\alpha_3 - 6\alpha_5\xi_1 + 6\xi_1\alpha_4 + 6\alpha_5\xi_1 = 2\alpha_3 + 6\xi_1\alpha_4$

$= 2\beta_3 + 6\xi_1\beta_4$ (satisfying 1c)

(2) $\beta_6 + 2\xi_1\beta_7 + 3\xi_1^2\beta_8 = \alpha_2 + 3\alpha_5\xi_1^2 + 2(\alpha_3 - 3\alpha_5\xi_1)\xi_1$
$+ 3\xi_1^2(\alpha_4 + \alpha_5)$

$= \alpha_2 + 3\alpha_5\xi_1^2 + 2\alpha_3\xi_1 - 6\alpha_5\xi_1^2 + 3\xi_1^2\alpha_4 + 3\alpha_5\xi_1^2$

$= \alpha_2 + 2\alpha_3\xi_1 + 3\alpha_4\xi_1^2$

$= \beta_2 + 2\beta_3\xi_1 + 3\beta_4\xi_1^2$ (satisfying 1b)

(3) $\beta_5 + \xi_1\beta_6 + \xi_1^2\beta_7 + \xi_1^3\beta_8$

$= \alpha_1 - \cancel{\alpha_5\xi_1^3} + \alpha_2\xi_1 + \cancel{3\alpha_5\xi_1^3} + \alpha_3\xi_1^2 - \cancel{3\alpha_5\xi_1^3} + \alpha_4\xi_1^3 + \cancel{\alpha_5\xi_1^3}$

$= \alpha_1 + \alpha_2\xi_1 + \alpha_3\xi_1^2 + \alpha_4\xi_1^3$

$= \beta_1 + \xi_1\beta_2 + \xi_1^2\beta_3 + \xi_1^3\beta_4$ (satisfying 1a)

And in the exact same way it can be shown that setting $\alpha_1 - \alpha_5\xi_1^2 - \alpha_6\xi_2^3 = \beta_9$, $\alpha_2 + 3\alpha_5\xi_1^2 + 3\alpha_6\xi_2^2 = \beta_{10}$, $\alpha_3 - 3\alpha_5\xi_1 - 3\alpha_6\xi_2 = \beta_{11}$ and $\alpha_4 + \alpha_5 + \alpha_6 = \beta_{12}$ will satisfy constraints 2a, 2b and 2c.

Therefore we have found the α values such that (*) is equivalent to the original expression and satisfies the 6 constraints enforcing up to second derivative continuity at the knots.

Since (*) = eqn 5.3 this proves the truncated power basis functions represent a basis for a cubic spline with knots at ξ_1 and ξ_2.