Ex. 3.9

First we will examine $\hat{y}$ when we ~~added~~ add a single additional variable.

We know that for X_i we may find $Z_0, \ldots, Z_{q-1}$ via algorithm 3.1 e.g.

for j in $1, 2, \ldots, q$ $\quad \hat{\gamma}_{\ell,j} = \dfrac{\langle Z_\ell, X_j \rangle}{\langle Z_\ell, Z_\ell \rangle}$, $\ell = 0, 1, \ldots, j-1$

and $Z_j = X_j - \sum_{k=0}^{j-1} \hat{\gamma}_{k,j} Z_k$

$$\text{Then } Q = \underbrace{\begin{bmatrix} \vdots & \vdots & & \\ Z_0 & Z_1 & \cdots & Z_{q+1} \\ \vdots & \vdots & & \end{bmatrix}}_{N \times (q+1)} \underbrace{\begin{bmatrix} \frac{1}{\|Z_0\|} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\|Z_{q+1}\|} \end{bmatrix}}_{(q+1) \times (q+1)}$$

Then $\hat{y} = QQ^T y$ and $RSS = \sum_{i=1}^{N} (\hat{y}_i - y_i)^2$

Now suppose we add a single new variable in X_2, then we obtain an aditional vector Z_{q+1}.

Notation: We will denote the N elements of a vector $Z_j = (Z_{j,1}, Z_{j,2}, \ldots, Z_{j,N})$

and $\hat{Z}_{j,i} = \dfrac{Z_{j,i}}{\|Z_j\|}$

Now:
$$Q = \begin{bmatrix} \hat{Z}_{0,1} & \cdots & \hat{Z}_{0,N} \\ \vdots & & \vdots \\ \hat{Z}_{q+1,1} & \cdots & \hat{Z}_{q+1,N} \end{bmatrix}$$

and
$$QQ^T = \underbrace{\begin{bmatrix} \sum_{i=0}^{q+1} \hat{Z}_{i,1} \hat{Z}_{i,1} & \cdots & \sum_{i=0}^{q+1} \hat{Z}_{i,1} \hat{Z}_{i,N} \\ \vdots & & \vdots \\ \sum_{i=0}^{q+1} \hat{Z}_{i,N} \hat{Z}_{i,1} & & \sum_{i=0}^{q+1} \hat{Z}_{i,N} \hat{Z}_{i,N} \end{bmatrix}}_{N \times N}$$

Now we notice that this is

$$QQ^T = \underbrace{QQ^T_{original}}_{(QQ^T \text{ for first } q \text{ variables})} + \underbrace{QQ^T_{New}}_{\begin{bmatrix} \hat{Z}_{q+1,1}\hat{Z}_{q+1,1} & \cdots & \hat{Z}_{q+1,1}\hat{Z}_{q+1,N} \\ \vdots & & \vdots \\ \hat{Z}_{q+1,N}\hat{Z}_{q+1,1} & \cdots & \hat{Z}_{q+1,N}\hat{Z}_{q+1,N} \end{bmatrix}}$$

Now we may efficiently calculate the RSS for each of $P-q$ new variables. For new variable K we obtain RSS_K via:

$$\hat{Y}_K = QQ^T Y = \overbrace{QQ^T_{original} Y}^{\text{static}} + QQ^T_{New} Y$$

$$\Rightarrow RSS_K = \sum_{i=1}^{N} (\hat{Y}_{Ki} - Y_i)^2$$