Proof of Sherman-Morrison formula

Sherman-Morrison states that for: A, an invertible $P \times P$ matrix; u, v, $P \times 1$ Column Vectors; λ, constant.

$$(A + \lambda u v^T)^{-1} = \left(A^{-1} - \frac{\lambda A^{-1} u v^T A^{-1}}{1 + \lambda v^T A^{-1} u}\right)$$

We will show that:

$$(*) \quad (A + \lambda u v^T)\left(A^{-1} - \frac{\lambda A^{-1} u v^T A^{-1}}{1 + \lambda v^T A^{-1} u}\right) = \mathbb{I}$$

Now letting K (constant) replace the denominator we can expand the LHS:

$$\Rightarrow (A + \lambda u v^T)(A^{-1} - K \lambda A^{-1} u v^T A^{-1})$$

$$= \mathbb{I} - K \lambda u v^T A^{-1} + \lambda u v^T A^{-1} + - K \lambda^2 u v^T A^{-1} u v^T A^{-1}$$

$$= \mathbb{I} + \underbrace{(\lambda - K\lambda - K\lambda^2 u v^T A^{-1})}_{\text{Find } K \text{ s.t. this expession is } = 0} u v^T A^{-1}$$

$$\lambda - \lambda K - \lambda^2 K u v^T A^{-1} = 0$$

$$K(1 + \lambda u v^T A^{-1}) = 1$$

$$\Rightarrow K(1 + \lambda v^T A^{-1} u) = 1$$

This holds if $K = (1 + \lambda v^T A^{-1} u)^{-1}$ which is exactly as K is defined, thus proving $(*)$.