Ex. 3.8

We will examine the QR decomposition of X and notice what remains when we put it in terms of Q_2.

$X = QR$, Now if $\hat{Z}_i = \frac{Z_i}{\|Z_i\|}$ (as described by alg 3.1)

We have: $Q = [\hat{Z}_0, \hat{Z}_1, \ldots, \hat{Z}_p]$ (dim: $N \times (P+1)$)

and by def of D and Γ:

$$R = \left[\begin{array}{c|c} \multicolumn{2}{c}{\langle \hat{Z}_0, 1\rangle, \langle \hat{Z}_0, x_1\rangle, \ldots, \langle \hat{Z}_0, x_p\rangle} \\ \hline 0 & \\ \vdots & R_2 \\ 0 & \end{array}\right]$$

We will call the first row R_1, a $1 \times (P+1)$ vector. Then we are left with $\bar{O}_p$ a $P \times 1$ vector of O's and R_2, the $P \times P$ remainder of R.

Solving R_1 using:

Note: This is the sum of the N elements of the vector x_i

$\langle \hat{Z}_0, 1\rangle = N/\sqrt{N}$ and $\langle \hat{Z}_0, x_i\rangle = \frac{\sum x_i}{N}$

$$\Rightarrow R_1 = 1/\sqrt{N}\,[N, \sum x_1, \sum x_2, \ldots, \sum x_p]$$

Thus we can decompose QR:

$$X = QR = \hat{Z}_0 R_1 + Q_2 [\bar{O}_p, R_2]$$

$$\underset{(N\times 1)(1\times(P+1))}{\hat{Z}_0 R_1} = \begin{bmatrix} 1 & \bar{x}_1 & & \bar{x}_p \\ 1 & \bar{x}_1 & \cdots & \bar{x}_p \\ \vdots & \vdots & & \vdots \\ 1 & \bar{x}_1 & & \bar{x}_p \end{bmatrix} \quad (N \times P+1)$$

$$\underset{(N \times P+1)}{X - \hat{Z}_0 R_1} = \underset{(N \times P+1)}{[\bar{O}_N, Q_2 R_2]}$$

$$\Rightarrow [\bar{O}_N, \tilde{X}] = [\bar{O}_N, Q_2 R_2]$$

So $\tilde{X} = Q_2 R_2 \Rightarrow Q_2$ spans the column space of $\tilde{X}$

And by definition of SVD for $\tilde{X} = UDV^T$, we know that U also spans the column space of $\tilde{X}$.

Under what circumstances is $Q_2 = U$?

Suppose $\tilde{X}$ is orthogonal, then:

$\tilde{X} = Z_2 D_2^{-1} D_2 \Gamma_2 = Q_2 R_2$

Here we find:

$Z_2 = [X_1 - \bar{X}_1, \dots, X_P - \bar{X}_P]$

$D_2 = \mathrm{diag}(\|X_1 - \bar{X}_1\|, \dots, \|X_P - \bar{X}_P\|)$

$\Gamma_2 = \mathbb{I}_P$

$\Rightarrow Q_2 = [\hat{x}_1, \dots, \hat{x}_P]$ where $\hat{x}_i = \dfrac{x_i - \bar{x}_i}{\|x_i - \bar{x}_i\|}$

Now we simply notice that $Q_2 D_2 \Gamma_2$ is a valid SVD of $\tilde{X}$ where $Q_2 = U$, $D_2 = D$ and $\Gamma_2 = V^T$.

And since we know that the SVD is unique up to a sign flip then we ~~a~~ can conclude that if $\tilde{X}$ is orthogonal then $Q_2 = U$ up to a sign flip.