Ex 3.3 (a)

Given $\hat{\theta} = a^T\hat{\beta}$ and $\tilde{\theta} = C^T y$ where $\mathbb{E}(C^T y) = a^T\beta$

W.t.s.:

$$Var(a^T\hat{\beta}) \leq Var(C^T y)$$

Since $\tilde{\theta}$ is some other unbiased linear estimator we can quantify its difference from the L.S. estimate $\hat{\theta}$

e.g.

$$C^T = a^T(X^TX)^{-1}X^T + D \qquad (*)$$

$$\Rightarrow \tilde{\theta} = [a^T(X^TX)^{-1}X^T + D]y$$

$$= a^T(X^TX)^{-1}X^Ty + Dy = \hat{\theta} + Dy$$

Now, $\mathbb{E}(\tilde{\theta}) = \mathbb{E}(\hat{\theta}) + \mathbb{E}(Dy)$

$$= a^T\beta + DX\beta$$

Which is unbiased $\iff DX\beta = 0 \iff DX = 0 \qquad (**)$

Now examine $Var(\tilde{\theta})$

$$\Rightarrow Var(C^T y)$$

$$= C^T Var(y) C$$

$$= C^T Var(X\beta + \varepsilon) C$$

$$= \sigma^2 C^T C$$

Expanding using $(*)$:

$$\Rightarrow \sigma^2[(a^T(X^TX)^{-1}X^T + D)(a^T(X^TX)^{-1}X^T + D)^T]$$

$$= \sigma^2(a^T(X^TX)^{-1}X^T + D)(X(X^TX)^{-1}a + D^T)$$

$$= \sigma^2[(a^T(X^TX)^{-1}X^TX(X^TX)^{-1}a) + DX(X^TX)^{-1}a$$

$$+ a^T(X^TX)^{-1}X^TD^T + DD^T]$$

Using $(**)$ and identity $(AB)^T = B^TA^T$ we get

$$= \sigma^2[a^T(X^TX)^{-1}a + DD^T]$$

$$= a^T\sigma^2(X^TX)^{-1}a + \sigma^2 DD^T$$

$$= a^T Var(\hat{\beta})a + \sigma^2 DD^T \qquad \text{(using eqn 3.8)}$$

$$= Var(a^T\hat{\beta}) + \sigma^2 DD^T$$

$$= Var(\hat{\theta}) + \sigma^2 DD^T$$

(Note: $D = (1 \times N)$ vector so $DD^T \geq 0$ and $\sigma^2 \geq 0$ by definition)

$$\Rightarrow Var(\hat{\theta}) \leq Var(\tilde{\theta})$$