2.4 Page 18

Show $f(x) = E(Y | X = x)$ minimises $EPE(f) = E(Y - f(X))^2$

① Claim: $E_{X,Y}(g(X, Y)) = E_X(E_{Y|X}(g(X,Y) | X))$

Proof: (for discrete case, same for continuous)
unroll RHS:

$$\Rightarrow E_X\left(\sum_Y g(X, y_i) \mathbb{P}(Y = y_i | X)\right)$$

$$= \sum_x \left(\sum_Y g(x_i, y_i) \mathbb{P}(Y = y_i | x_i)\right) P(x_i)$$

bayes rule

$$= \sum_{xy} g(x_i, y_i) \mathbb{P}(x_i, y_i)$$

$$= E_{x,y}(g(x_i, y_i))$$

This takes us from $2.9 \to 2.11$

② Now we wish to find f to minimise $E_x E_{Y|x}([Y - f]^2 | X)$

Again consider discrete case:
To minimise $\sum_x \underbrace{\left(\sum_Y (Y - f(X))^2 \mathbb{P}(Y = y_i | x_i)\right)}$ $P(x_i)$

is equivalent to minimising

So $f(x) = \text{argmin}_c\, E_{Y|x}([Y - c]^2 | X = x)$

$$\frac{\partial}{\partial f} \int_Y ([y - c]^2)\, \mathbb{P}(y|x)\, dy = 0$$

$$= \int_Y -2\,[y - c]\, \mathbb{P}(y|x)\, dy$$

$$\Rightarrow \int_Y y\, \mathbb{P}(y|x)\, dy = c \int_Y \mathbb{P}(y|x)\, dy$$

$$\Rightarrow \mathbb{E}(Y | X = x) = c = f(x)$$