Ex 5.7 (a)

Wish to show:

$$\int_a^b g''(x)h''(x)\,dx = 0$$

Where:

$g(x)$ = Natural Cubic Spline interpolant to the Pairs $\{x_i, \xi_i\}_1^N$

$h(x) = \tilde{g}(x) - g(x)$

$\tilde{g}(x)$ = Any other differentiable function

and $a < x_1 < \dots < x_N < b$

Using integration by Parts. e.g.

$$\int u\,dv = uv - \int v\,du$$

Now,

$$\int_a^b g''(x)h''(x) = g''(x)h'(x)\Big|_a^b - \int_a^b h'(x)g'''(x)$$

$$= -\int_a^b h'(x)g'''(x)$$

($g''(a) = g''(b) = 0$ as natural cubic splines are linear at the boundaries)

$$= -g'''(x)h(x)\Big|_a^b + \int_a^b h(x)g''''(x)$$

(integration by Parts again)

$$= -g'''(x)h(x)\Big|_a^b$$

(as $g''''(x) = 0$)

$$= -g'''(x)h(x)\Big|_{x_1}^{x_2} - \dots - g'''(x)h(x)\Big|_{x_{N-1}}^{x_N}$$

(as $g(x)$ outside (x_1, x_N) is linear)

$$= -\sum_{j=1}^{N-1} g'''(x)h(x)\Big|_{x_j}^{x_{j+1}}$$

$$= -\sum_{j=1}^{N-1}\left[g'''(x_{j+1}^-)h(x_{j+1}) - g'''(x_j^+)h(x_j)\right]$$

The 3rd derivative of a cubic Polynomial is constant between x_j and x_{j+1}, therefore $g'''(x_{j+1}^-) = g'''(x_j^+)$

$$= -\sum_{j=1}^{N-1} g'''(x_j^+)\left[h(x_{j+1}) - h(x_j)\right] = 0$$

(as required)

Ex 5.7 (b)

$$\int_a^b \tilde{g}''(t)^2 \, dt$$

$$= \int_a^b [h''(t) + g''(t)]^2 \, dt \qquad \text{(by definition)}$$

$$= \int_a^b h''(t)^2 dt + 2\underbrace{\int_a^b g''(t) h''(t) dt}_{\substack{= \\ 0 \\ \text{(by Part (a))}}} + \int_a^b g''(t)^2 dt$$

$$= \int_a^b h''(t)^2 dt + \int_a^b g''(t)^2 dt \geq \int_a^b g''(t)^2 dt$$

as required. Additionally, equality only occours when:

$$\int_a^b h''(t)^2 \, dt = 0$$

Which implies $h(t)$ is linear. $(*)$

Now since both $\tilde{g}$ and g interpolate the N pairs we have $h(x_i) = \tilde{g}(x_i) - g(x_i) = 0 \quad \forall i \in (1, \ldots, N)$

We are also told $N \geq 2$, so combining this with the above fact and $(*)$ we may conclude that $h(t) = 0$.

$\Rightarrow$ Equality only occours if $h(t) = 0$

Ex 5.7 (c)

Consider $g_0(x)$ to be the minimizer from all functions in the Sobolev space except for a cubic ~~knot~~ spline with knots at each of the x_i.

We may now construct a natural cubic spline with knots at each of the x_i, $g_s(x)$, such that:

$$\sum_{i=1}^{N} (y_i - g_0(x_i))^2 \geq \sum_{i=1}^{N} (y_i - g_s(x_i))^2$$

and by part (b):

$$\lambda \int_a^b g_0''(t)^2 \, dt \geq \lambda \int_a^b g_s''(t)^2 \, dt$$

$\Rightarrow$ $g_s(x)$ is the minimiser of the penalised least squares problem.