Ex 3.14

As hinted at in the question I will itterate through Alg. 3.3 for $m = 1, 2$ and verify that $\hat{\phi}_{2,j} = 0$ $\forall j = 1, \dots, p$.

This solves the problem as $\hat{\phi} = 0 \Rightarrow Z = 0 \Rightarrow$ Part (c) of the algorithm will no longer update $\hat{y}$ for further values of m.

For $m=1$:

1. $x_j^{(0)} = x_j$ and $\hat{y}^{(0)} = \bar{y}\mathbf{1}$

2. (a) $Z_1 = \sum_{j=1}^{p} \hat{\phi}_{1,j} x_j$, where $\hat{\phi}_{1,j} = \langle x_j^{(0)}, y \rangle = \langle x_j, y \rangle$

(b) $\hat{\theta}_1 = \dfrac{\langle Z_1, y \rangle}{\langle Z_1, Z_1 \rangle}$

(c) $\hat{y}^{(1)} = \bar{y}\mathbf{1} + \hat{\theta}_1 Z_1$

(d) $x_j^{(1)} = x_j - \dfrac{\langle Z_1, x_j \rangle}{\langle Z_1, Z_1 \rangle} \cdot Z_1$

For $m=2$:

$$Z_2 = \sum_{j=1}^{p} \hat{\phi}_{2j} x_j^{(2)}, \quad \text{where } \hat{\phi}_{2,j} = \langle x_j^{(1)}, y \rangle$$

W.t.s. $\hat{\phi}_{2,j} = 0 \quad \forall j = 1, \dots, p$

$$\Rightarrow \hat{\phi}_{2,j} = \langle x_j^{(1)}, y \rangle = \left\langle x_j - \frac{\langle Z_1, x_j \rangle}{\langle Z_1, Z_1 \rangle} \cdot Z_1, y \right\rangle \quad \text{(From (d))}$$

$$= \langle x_j, y \rangle - \left\langle \frac{\langle Z_1, x_j \rangle}{\langle Z_1, Z_1 \rangle} Z_1, y \right\rangle \quad \text{(From (a))}$$

$$= \hat{\phi}_{1,j} - \frac{\langle Z_1, x_j \rangle}{\langle Z_1, Z_1 \rangle} \langle Z_1, y \rangle \quad (*)$$

Now solve each component of (*) in turn:

① $\langle z_1, x_j \rangle = \langle \sum_{j=1}^{p} \hat{\phi}_{1j} x_j , x_j \rangle$ (from (a))

$= \hat{\phi}_{1j} \langle x_j , x_j \rangle$ as $\langle x_i, x_j \rangle = 0 \ \forall i \neq j$

as x_j's are orthogonal

$= \hat{\phi}_{1j} \| x_j \|^2$

② $\langle z_1, z_1 \rangle = \langle \sum_{j=1}^{p} \hat{\phi}_{1j} x_j , \sum_{j=1}^{p} \hat{\phi}_{1,j} x_j \rangle$

$= \langle \hat{\phi}_{11} x_1 , \sum_{j=1}^{p} \hat{\phi}_{1,j} x_j \rangle + \langle \hat{\phi}_{12} x_2 , \sum_{j=1}^{p} \hat{\phi}_{1,j} x_j \rangle + \ldots$

$= \sum_{j=1}^{p} \hat{\phi}_{1,j}^{2} \| x_j \|^2$ again since x_j's are $\perp$

③ $\langle z_1 , y \rangle = \langle \sum_{j=1}^{p} \hat{\phi}_{1j} , x_j , y \rangle$

$= \sum_{j=1}^{p} \hat{\phi}_{1j} \langle x_j , y \rangle = \sum_{j=1}^{p} \hat{\phi}_{1j}^{2}$

Subbing ①, ②, ③ back into (*)

$$\hat{\phi}_{2,j} = \hat{\phi}_{1j} - \frac{\hat{\phi}_{1,j} \| x_j \|^2}{\sum_{j=1}^{p} \hat{\phi}_{1j}^{2} \| x_j \|^2} \cdot \sum_{j=1}^{p} \hat{\phi}_{1,j}^{2}$$

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Now $\sigma_j^2 = \frac{1}{N} \sum_{i=1}^{N} (x_j^{(i)} - \bar{x}_j)^2$

$\Rightarrow 1 = \frac{1}{N} \sum_{i=1}^{N} (x_j^{(i)} - 0)^2$

$\Rightarrow N = \| x_j \|^2$

So,

$$\hat{\phi}_{2,j} = \hat{\phi}_{1,j} - \frac{\hat{\phi}_{1,j} \cancel{N}}{\cancel{N} \cancel{\sum_{j} \hat{\phi}_{1,j}^{2}}} \cdot \cancel{\sum_{j=1}^{p} \hat{\phi}_{1,j}^{2}}$$

$= \hat{\phi}_{1,j} - \hat{\phi}_{1,j} = 0 \quad \forall j = 1, \ldots, p$