Ex 3.13

Given $\hat{y}^{PCR}_{(M)} = \bar{y}\mathbb{1} + \sum_{m=1}^{M} \hat{\theta}_m z_m \quad (1)$

Where z_m's are orthogonal, want to show expression (3.62).

$$(1) = \bar{y}\mathbb{1} + \sum_{m=1}^{M} \hat{\theta}_m X v_m \quad \text{(by definition)}$$

$$= \bar{y}\mathbb{1} + \sum_{m=1}^{M} X \hat{\theta}_m v_m \quad (\hat{\theta}_m\text{'s are constants})$$

$$= \bar{y}\mathbb{1} + X \underbrace{\sum_{m=1}^{M} \hat{\theta}_m v_m}_{= \hat{\beta}^{pcr}_{(M)}} \quad (X \text{ not dependant on } M)$$

Now W.t.s. when $M = p$, $\hat{\beta}^{pcr} = \hat{\beta}^{ls}$.

Recall:

$$\boxed{\text{Singular Value decomp. } X = UDV^T}$$

$$\hat{\beta}^{ls} = (X^T X)^{-1} X^T y$$

$$= (VDU^T UDV^T)^{-1} VDU^T y$$

$$= (VD^2V^T)^{-1} VDU^T y \qquad U^T U = \mathbb{I},\ D = \text{diagonal}$$

$$= (V^T)^{-1}(D^2)^{-1}(V)^{-1} V D U^T y$$

$$= V(D^2)^{-1} D U^T y \qquad V^T V = \mathbb{I} \Leftrightarrow V = (V^T)^{-1}$$

$$= V D^{-1} U^T y$$

And $\hat{\beta}^{pcr}_{(p)} = \sum_{m=1}^{p} \hat{\theta}_m v_m$ and since $M = p$ $Z = UD$

$$= [v_1, v_2, \ldots, v_p] \begin{bmatrix} \hat{\theta}_1 \\ \vdots \\ \hat{\theta}_p \end{bmatrix} = V\hat{\theta} \quad (*) \qquad \text{(Just rewriting notation)}$$

Now

$$\hat{\theta} = (Z^T Z)^{-1} Z^T y$$

$$= (DU^T UD)^{-1} DU^T y$$

$$= D^{-1} U^T y$$

And subbing into $(*)$: $\hat{\beta}^{pcr}_{(M=p)} = VD^{-1}U^T y = \hat{\beta}^{ls}$