Ex. 3.2

$$f(X) = \sum_{j=0}^{3} \beta_j X^j$$

approach ①:

for test point $x_0 = (1, x_0, x_0^2, x_0^3)^T$

Pointwise 95% CI: $x_0^T \hat{\beta} \pm Z^{(1-\alpha)} \sqrt{Var(x_0^T \hat{\beta})}$

Where $\hat{\beta} = (X^T X)^{-1} X^T y$ $\quad Z^{(1-0.025)} = 1.96$

and

$$Var(x_0^T \hat{\beta}) = \mathbb{E}(x_0^T \hat{\beta} \hat{\beta}^T x_0) - \mathbb{E}(x_0^T \hat{\beta}) \cdot \mathbb{E}(\hat{\beta}^T x_0)$$
$$= x_0^T Var(\hat{\beta}) x_0$$
$$= x_0^T (X^T X)^{-1} x_0 \cdot \sigma^2$$

overall CI @ x_0: $x_0^T (X^T X)^{-1} X^T y \pm 1.96 \sigma \sqrt{x_0^T (X^T X)^{-1} x_0}$

approach ②:

Global approach - $C_\beta = \{ \beta \mid (\hat{\beta} - \beta)^T X^T X (\hat{\beta} - \beta) \leq \hat{\sigma}^2 \chi^{2\,(1-\alpha)}_{p+1} \}$

$\chi^{2\,(1-0.05)}_{4} = 9.5$

The challenge here is that there is a potentially infinite number of β vectors of length 4 that find an upper/lower bound of the above eqn by setting it to an equality. Thus we may generate multiple confidence intervals from the above set.

Then for a given β vector we may calculate the confidence interval at x_0: $f(x_0) = x_0^T \beta$

or all together: $\{ x_0^T \beta : \beta \in C_\beta \}$