Ex 6.1

① For Nadaraya-Watson Kernel Smooth with fixed metric bandwidth λ and a Gaussian Kernel, We will Show that $\hat{f}(x_0)$ is differentiable. Where

$$\hat{f}(x_0) = \frac{\sum_{i=1}^{N} \phi(x_i; x_0, \lambda)\, y_i}{\sum_{i=1}^{N} \phi(x_i; x_0, \lambda)}$$

With

$$\phi(x_i; x_0, \lambda) = \frac{1}{\lambda\sqrt{2\pi}} \exp\left(-\frac{1}{2\lambda}(x_i - x_0)^2\right)$$

Now we simply need to observe that since ϕ is differentiable and by the Product rule, Clearly this Kernel is differentiable e.g. $\frac{d\hat{f}(x_0)}{dx_0}$ exists.

② The Epanechnikov quadratic kernel does not have the Same Property as there is a discontinuity at the boundary.

$$K_\lambda(x_0, x) = D\left(\frac{|x_0 - x|}{\lambda}\right)$$

$$\text{where } D(t) = \begin{cases} \frac{3}{4}(1-t^2) & \text{if } |t| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

We will show that, in this case, $\frac{d}{dx_0}\hat{f}(x_0) \neq 0$ as $|t| \to 1$ from inside the boundary while clearly we do have zero when approaching from the other side of the boundary.

Now, $\frac{d}{dx_0}\left(\hat{f}(x_0)\right) \Rightarrow$ reduces to evaluating the kernel again, $K_\lambda(x_0, x)$.
(Product rule)

Evaluating, $\frac{d}{dx_0} K_\lambda(x_0, x) = \overset{(a)}{\frac{dK}{dt}} \cdot \overset{(b)}{\frac{dt}{dx_0}}$

(a) $\frac{dK}{dt} = -\frac{3}{2}t = -\frac{3}{2}\frac{|x - x_0|}{\lambda}$

(b) Now, since we are evaluating from inside the boundary as $|t| \to 1$ we have:

$$\frac{dt}{dx_0} = \frac{d}{dx_0}\frac{|x - x_0|}{\lambda} = \frac{d}{dx_0}\frac{(x \pm \varepsilon) - x_0}{\lambda} \quad \left(\begin{array}{c}\text{where } \pm\varepsilon \text{ is } -\text{'ve} \\ \text{on RHS and } +\text{'ve} \\ \text{on LHS and} \\ \varepsilon \to 0\end{array}\right)$$

$$= -1/\lambda$$

$\Rightarrow \frac{d}{dx_0} K_\lambda(x_0, x) = +\frac{3}{2}\frac{|x - x_0|}{\lambda^2} = \frac{3/2\, t}{\lambda} \neq 0$ as $|t| \to 1$

Proving a discontinuity exists at the boundary as required.