Ex 6.2

We will begin by proving some identities by regressing X against itself rather than y in the local setting.

Replacing each y_i with $[1, x_i, x_i^2, \dots, x_i^K]$, eqns 6.8 and 6.9 become:

$$b(x_0)^T (B^T W(x_0) B)^{-1} B^T W(x_0) B = \sum_{i=1}^{N} l_i(x_0) [1, x_i, x_i^2, \dots]$$

$$\Rightarrow b(x_0)^T = \sum_{i=1}^{N} l_i(x_0) [1, x_i, x_i^2, \dots]$$

$$\Rightarrow [1, x_0, x_0^2, \dots, x_0^K] = \left[\sum_{i=1}^{N} l_i(x_0), \sum_{i=1}^{N} l_i(x_0) x_i, \dots\right] \quad (*)$$

Now, defining $b_j(x_0) = \sum_{i=1}^{N} (x_i - x_0)^j l_i(x_0)$

① $b_0(x_0) = \sum_{i=1}^{N} l_i(x_0) = 1$ (by equating first elements in $(*)$)

② $b_1(x_0) = \sum_{i=1}^{N} (x_i - x_0) l_i(x_0) = 0$ (equating second elements in $(*)$)

③ W.t.s. $b_j(x_0) = 0 \quad \forall j \geq 1$

$$b_j(x_0) = \sum_{i=1}^{N} (x_i - x_0)^j l_i(x_0)$$

$$= \sum_{i=1}^{N} \sum_{k=0}^{j} \binom{j}{k} (-x_0)^{j-k} x_i^k l_i(x_0) \quad \text{(binomial theorem)}$$

$$= \sum_{k=0}^{j} \left\{ \binom{j}{k} (-x_0)^{j-k} \sum_{i=1}^{N} x_i^k l_i(x_0) \right\}$$

$$= \sum_{k=0}^{j} \binom{j}{k} (-x_0)^{j-k} x_0^k \quad \left\{\text{equating LHS and RHS of } (*) \text{ for any } k\right\}$$

$$= (x_0 - x_0)^j \quad \text{(binomial theorem again)}$$

$$= 0$$

Now apply ①, ② and ③ to the expected value of $\hat{f}$ (eqn 6.10)

$$\Rightarrow \mathbb{E}(\hat{f}(x_0)) = f(x_0) \overbrace{\sum_{i=1}^{N} l_i(x_0)}^{=1\ (\text{by ①})} + f'(x_0) \overbrace{\sum_{i=1}^{N} (x_i - x_0) l_i(x_0)}^{=0\ (\text{by ②})}$$

$$+ \frac{f''(x_0)}{2} \overbrace{\sum_{i=1}^{N} (x_i - x_0)^2 l_i(x_0)}^{=0\ (\text{by ③})} + R$$

$$= f(x_0) + R$$

$$\Rightarrow \text{bias} = \mathbb{E}(\hat{f}(x_0)) - f(x_0) = R$$

We also observe that for polynomial of degree K all other taylor expansion terms in R up to degree K are also equal to zero. Therefore, for a polynomial of degree K the bias will consist of the taylor expansion terms of degree $K+1$ and greater.