Ex 5.4

Note: To keep notation clear in this question I will represent the number of knot with $\bar{K}$.

Now enforcing linearity at the boundries to obtain these constraints. At the LHS boundry ($k=1$) where $x \leq \xi_1$:

$$f(x) = \sum_{j=0}^{3} \beta_j x^j = \alpha_0 + \alpha_1 x$$

$$\Rightarrow \quad \beta_2 = 0 \quad \text{and} \quad \beta_3 = 0$$

and at the RHS boundary ($k = \bar{K}$) where $\xi_{\bar{K}} < x$:

$$f(x) = \beta_0 + \beta_1 x + \sum_k \theta_k (x - \xi_k)^3 = \gamma_0 + \gamma_1 x$$

equating the coeffients of the 2nd and 3rd powers we obtain:

$$-3 \sum_k \theta_k \xi_k x^2 = 0 \quad \Rightarrow \quad \sum_{k=1}^{\bar{K}} \theta_k \xi_k = 0 \quad ①$$

$$\sum_k \theta_k x^3 = 0 \quad \Rightarrow \quad \sum_{k=1}^{\bar{K}} \theta_k = 0 \quad ②$$

Therefore the truncated power series representation becomes:

$$f(x) = \beta_0 + \beta_1 x + \underbrace{\sum_{k=1}^{\bar{K}} \theta_k (x - \xi_k)_+^3}_{(*)} \quad \text{with constraints ① and ②.}$$

We can rewrite (*) as:

$$\theta_{\bar{K}} (x - \xi_{\bar{K}})_+^3 + \sum_{k=1}^{\bar{K}-1} \theta_k (x - \xi_k)_+^3$$

$$= \sum_{k=1}^{\bar{K}-1} \theta_k \left[(x - \xi_k)_+^3 - (x - \xi_{\bar{K}})_+^3 \right]$$

$$\left(\text{Since } ② \text{ implies } \theta_{\bar{K}} = -\sum_{k=1}^{\bar{K}-1} \theta_k\right)$$

$$= \theta_{\bar{K}-1}\left[(x-\xi_{\bar{K}-1})_+^3 - (x-\xi_{\bar{K}})_+^3\right] + \sum_{k=1}^{\bar{K}-2} \theta_k \left[(x-\xi_k)_+^3 - (x-\xi_{\bar{K}})_+^3\right]$$

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① implies that:

$$\theta_{\bar{K}}\xi_{\bar{K}} + \theta_{\bar{K}-1}\xi_{\bar{K}-1} + \sum_{k=1}^{\bar{K}-2} \theta_k \xi_k = 0$$

$$\Rightarrow -\sum_{k=1}^{\bar{K}-1} \theta_k \xi_{\bar{K}} + \theta_{\bar{K}-1}\xi_{\bar{K}-1} + \sum_{k=1}^{\bar{K}-2} \theta_k \xi_k = 0 \quad \left(\text{using } ② \text{ again}\right)$$

$$\Rightarrow \sum_{k=1}^{\bar{K}-2} \theta_k(\xi_k - \xi_{\bar{K}}) - \theta_{\bar{K}-1}\xi_{\bar{K}} + \theta_{\bar{K}-1}\xi_{\bar{K}-1} = 0$$

$$\Rightarrow \theta_{\bar{K}-1} = \frac{1}{\xi_{\bar{K}} - \xi_{\bar{K}-1}} \sum_{k=1}^{\bar{K}-2} \theta_k (\xi_k - \xi_{\bar{K}})$$

$$= \sum_{k=1}^{\bar{K}-2} \theta_k \left\{ \frac{(\xi_k - \xi_{\bar{K}})\left[(x-\xi_{\bar{K}-1})_+^3 - (x-\xi_{\bar{K}})_+^3\right]}{\xi_{\bar{K}} - \xi_{\bar{K}-1}} + \left[(x-\xi_k)_+^3 - (x-\xi_{\bar{K}})_+^3\right] \right\}$$

$$= \sum_{k=1}^{\bar{K}-2} \theta_k \left\{ (\xi_k - \xi_{\bar{K}}) d_{\bar{K}-1}(x) + (\xi_{\bar{K}} - \xi_k) d_k(x) \right\}$$

$$= \sum_{k=1}^{\bar{K}-2} \theta_k (\xi_{\bar{K}} - \xi_k)\left[d_k(x) - d_{\bar{K}-1}(x)\right]$$

Finally we can sub this back in for (*) to obtain:

$$f(x) = \beta_0 + \beta_1 x + \sum_{k=1}^{\bar{K}-2} \beta_{k+1}\left[d_k(x) - d_{\bar{K}-1}(x)\right]$$

where $\beta_{k+1} = \theta_k(\xi_{\bar{K}} - \xi_k)$

Which gives us the basis described in (5.4) and (5.5).