Homework #3

6.1 Part a) X-axis is $\log(t_p)$

Y-axis is $\Phi_{logis}\left(\frac{\log(t_p)-\mu}{\sigma}\right)$

Part b) See Attached plot (Next page) the one with smaller slope is loglogistic(1,2)

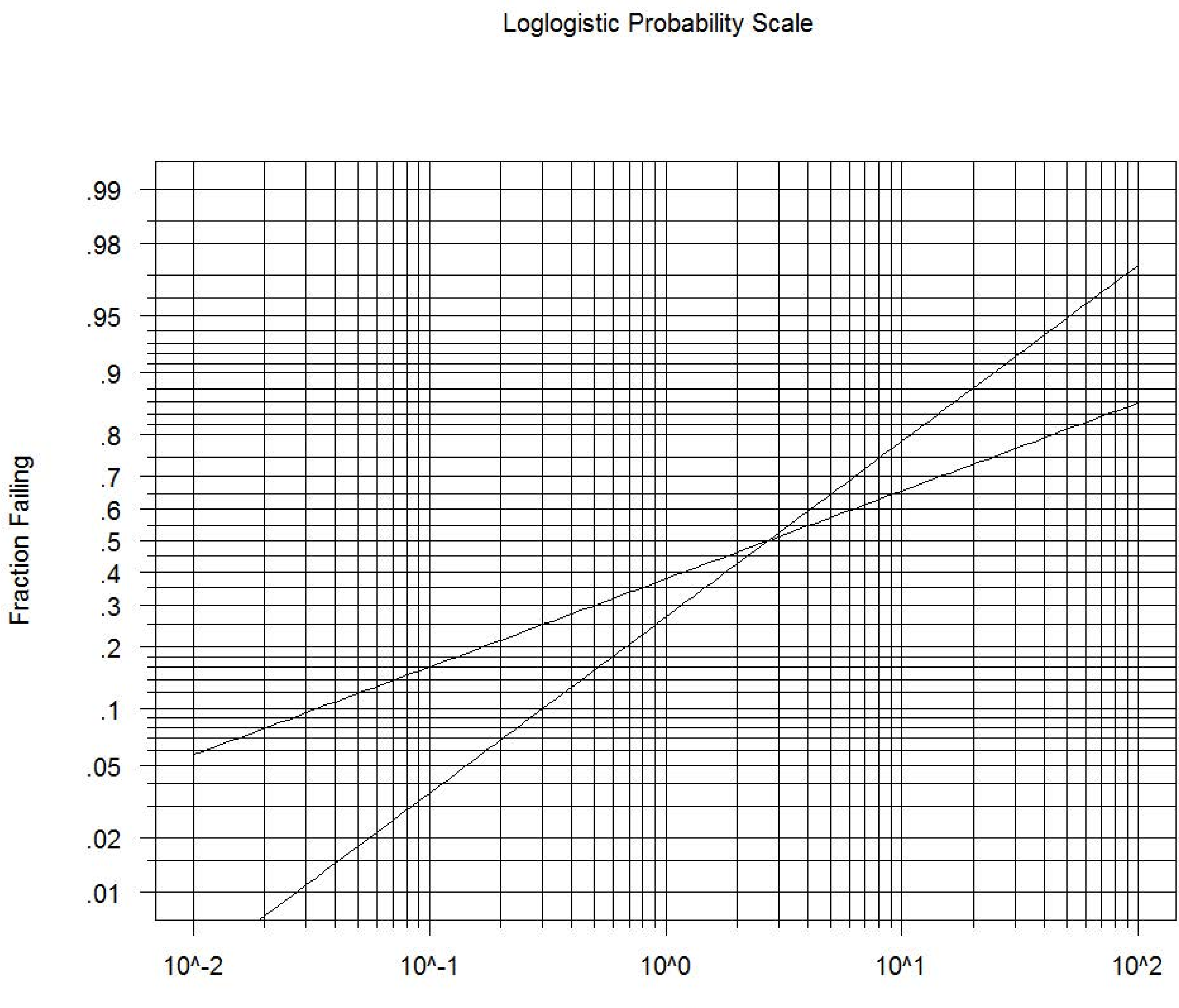
Part c)

$$F(x|\mu,\sigma) = \Phi_{logis}\left(\frac{\log(x)-\mu}{\sigma}\right) \Rightarrow F(\exp(\mu)|\mu,\sigma) =$$

$$\Phi_{logis}\left(\frac{\log(\exp(\mu))-\mu}{\sigma}\right) = \Phi_{logis}(0) = \boxed{.5}$$

Splida Code:

```
> probpaper("Loglogistic", x.range=c(0.01,40), grid=TRUE, Y.range=c(.011, .981))
> curve(x-1, add=T)  # This is the loglogistic (1,1)
> curve((x-1)/2, add=T)  # This is the loglogistic (1,2)
```


Loglogistic Probability Scale
Fraction Failing
.99
.98
.95
.9
.8
.7
.6
.5
.4
.3
.2
.1
.05
.02
.01
10^-2
10^-1
10^0
10^1
10^2
Time

```
#6.5
#Splida Code

# Part a)
lzbearing.ld <- frame.to.ld(file=SplidaDataName("lzbearing.txt"),
  response.column = 1,data.title = "Lieblein and Zelen Ball Bearing Failure Data",
  time.units = "Megacycles")

# Part b)
plot(lzbearing.ld,distribution="Weibull",grid=T)

# Part c)
plot(lzbearing.ld,distribution="Lognormal",grid=T)

# Part d)
# Both of these models seem adequate.
# The lognormal looks to be preferable over the Weibull
```

```
#I copied the data over to a new file "PhotoDetector_mod.txt"
#I deleted the comments
#Changed "Censored" to "Right"

PhotoDD.ld <- frame.to.ld(file=SplidaDataName("PhotoDetector_mod.txt"),
  data.title = "Silicon Photodiode Detectors",
  response.column = c(1,2), censor.column = 3, case.weight.column = 4,
  time.units = "Thousands of Hours")

plot(PhotoDD.ld,distribution="Weibull",grid=T)
plot(PhotoDD.ld,distribution="Exponential",grid=T)
plot(PhotoDD.ld,distribution="Lognormal",grid=T)

# From the probability plots I would choose the Weibull model
# The lognormal looks adequate
# The exponential doesn't look that great (although we can't rule it out)
```

7.1

$f_T(t) = \frac{1}{\theta} e^{-t/\theta}$ $t_1, t_2, \cdots, t_n$ iid $Exp(\theta)$

Part a) $L(\theta|T) = \prod_{i=1}^{n} \frac{1}{\theta} e^{-t_i/\theta} = \frac{1}{\theta^n} e^{-\Sigma t_i/\theta}$

$\ell(\theta|T) = -n\log(\theta) - \sum_{i=1}^{n} t_i/\theta$ $\frac{d\ell}{d\theta} = -\frac{n}{\theta} + \frac{\sum_{i=1}^{n} t_i}{\theta^2}$

Setting this equal to 0 $\Rightarrow$ $\frac{\sum_{i=1}^{n} t_i}{\theta^2} = \frac{n}{\theta}$ $\Rightarrow$ $\hat{\theta} = \frac{\sum_{i=1}^{n} t_i}{n} = \overline{t}$

$\frac{d^2\ell}{d\theta^2} = \frac{n}{\theta^2} - \frac{2\sum_{i=1}^{n} t_i}{\theta^3}\Bigg|_{\hat{\theta}} = \frac{n^3}{(\Sigma t_i)^2} - \frac{2n^3 \Sigma t_i}{(\Sigma t_i)^3} = -\frac{n^3 \Sigma t_i}{(\Sigma t_i)^3} < 0$ So $\hat{\theta}$ is the MLE.

Part b) $R(\theta) = \frac{L(\theta|T)}{L(\hat{\theta}|T)} = \frac{\theta^{-n} e^{-\Sigma t_i/\theta}}{\left(\frac{\Sigma t_i}{n}\right)^{-n} e^{-\Sigma t_i/(\Sigma t_i/n)}} = e^n e^{-n\bar{t}/\theta} \left(\frac{\bar{t}}{\theta}\right)^n$

Part c) When evaluating the true value of θ $-2\log(R(\theta)) \dot{\sim} \chi^2_1$

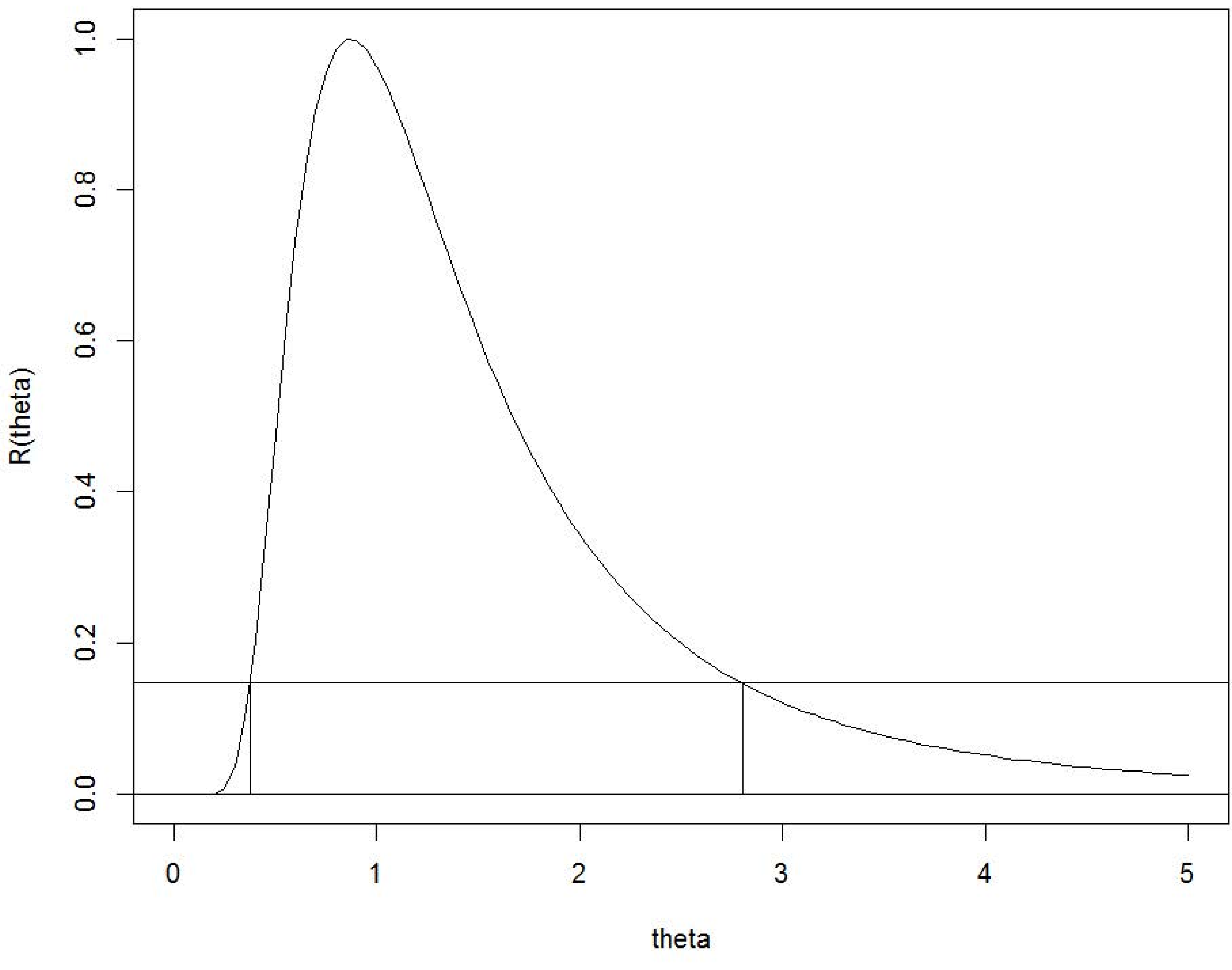
$\Rightarrow$ if $R(\theta) > \exp\left\{\frac{-\chi^2_{1,\alpha}}{2}\right\}$ then those values of θ are unlikely to have generated the data we observed. Hence if $R(\theta) < \exp\left\{\frac{-\chi^2_{1-\alpha,1}}{2}\right\}$ then θ will be in our $(1-\alpha)\%$ approximate CI.

Part d) $\hat{\theta} = .87$ for the CI θ must satisfy $e^{-3.8415/2} < e^4 e^{-4(.87)/\theta} \left(\frac{.87}{\theta}\right)^4$

the values where these two are equal are $(.375, 2.802)$

this is the approx 95% CI.

See Next page for the plot.



R(theta)
theta
0.0
0.2
0.4
0.6
0.8
1.0
0
1
2
3
4
5

7.7

Part a) See Next page for plot (Yes, it seems to be adequate)

Part b) $TTT = 1,986$ hours

$$\hat{\theta} = \frac{TTT}{r}, \; r = 10 \Rightarrow \hat{\theta} = 198.6$$

Part c)

$$\hat{se}_{\hat{\theta}} = \sqrt{\left(-\frac{d^2 \ell(\theta)}{d\theta^2}\right)^{-1}}\Bigg|_{\hat{\theta}} = \frac{\hat{\theta}}{\sqrt{r}} = \frac{198.6}{\sqrt{10}} \approx 62.8$$

Part d)

$$CI_{Z_{\hat{\theta}}} = \hat{\theta} \pm z_{1-\alpha/2}\, \hat{se}_{\hat{\theta}} = [75.51, 321.69]$$

$$CI_{Z_{\log\hat{\theta}}} = \left[\hat{\theta}\exp\{-z_{1-\alpha/2}\hat{se}_{\hat{\theta}}/\hat{\theta}\}, \hat{\theta}\exp\{z_{1-\alpha/2}\hat{se}_{\hat{\theta}}/\hat{\theta}\}\right] = [106.86, 369.11]$$

$$CI_{Exact} = \left[\frac{2TTT}{\chi^2_{1-\alpha/2;2r}}, \frac{2TTT}{\chi^2_{\alpha/2;2r}}\right] = [116.24, 414.15]$$

I would be very comfortable using the exact CI, the one based on $Z_{\log\hat{\theta}}$ less so, but more than $Z_{\hat{\theta}}$.

Part e) Yes, the belief that the Data are $\exp(\theta)$ is a form of extrapolation from the past. [Also if we want to use this information at a Normal Voltage we will have to extrapolate it back to normal use conditions]

But for Estimating θ give data exponential then No.

Part f) $t_{.1}$ is a function (monotone) of θ i.e., $t_{.1} = -\theta \log(.9)$

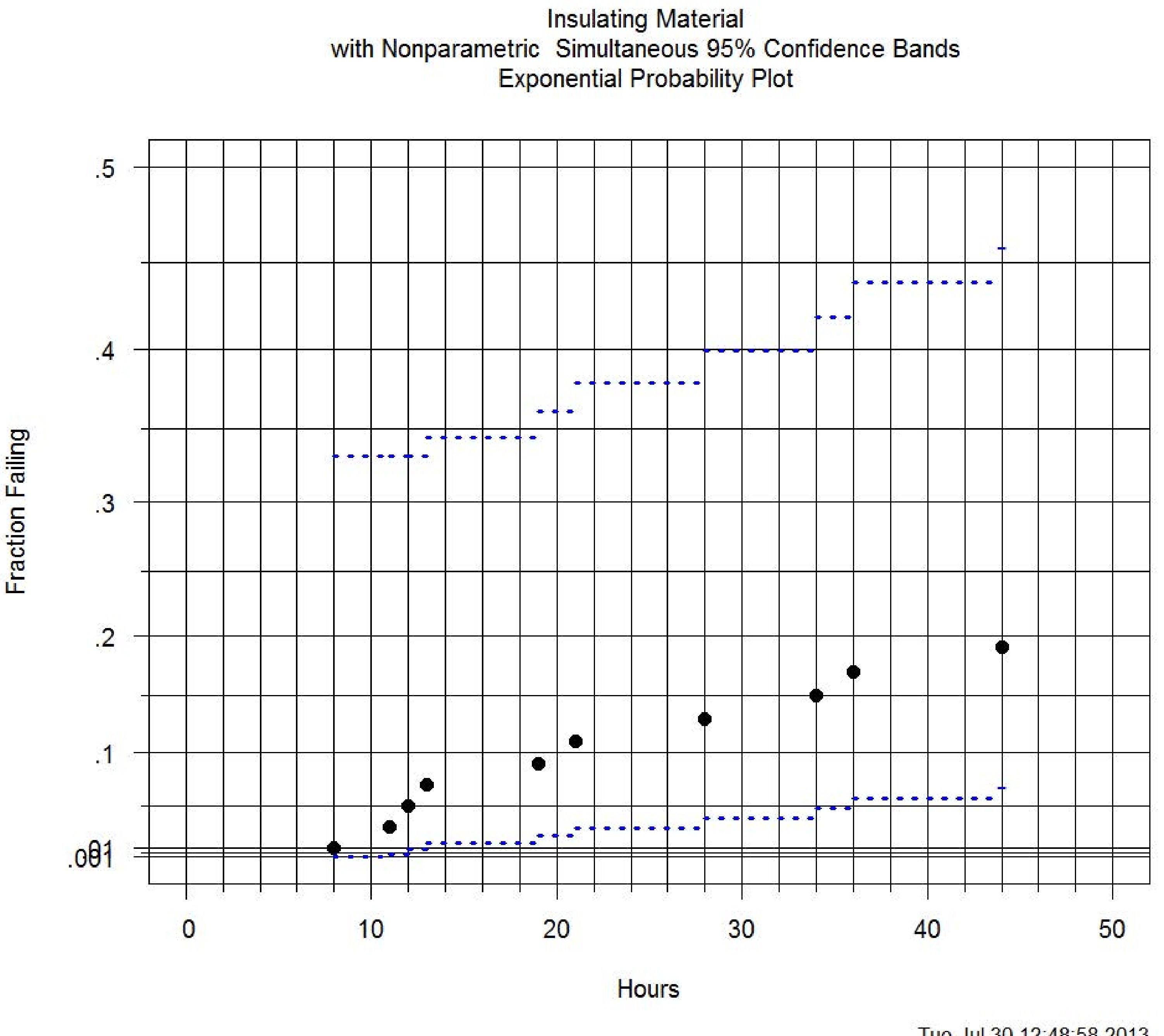
$\Rightarrow$ an Exact CI for $t_{.1}$ w/ type II censored data is $\left[\frac{2TTT(-\log(.9))}{\chi^2_{1-\alpha/2;2r}}, \frac{2TTT(-\log(.9))}{\chi^2_{\alpha/2;2r}}\right] = [12.25, 43.63]$

$\Rightarrow$ an Exact CI for $h(50|\theta)$ w/ type II censoring is $\left[\frac{\chi^2_{\alpha/2;2r}}{2TTT}, \frac{\chi^2_{1-\alpha/2;2r}}{2TTT}\right] = [.00024, .00086]$ since $1/\theta$ is a monotone function of θ

$\Rightarrow$ Since $F(50|\theta)$ is a monotone function of θ an Exact CI for $F(50|\theta)$ w/ type II censoring is $\left[1-\exp\left\{-\frac{50\chi^2_{\alpha/2;2r}}{2TTT}\right\}, 1-\exp\left\{-\frac{50\chi^2_{1-\alpha/2;2r}}{2TTT}\right\}\right]$

$= [.114, .350]$

for estimating $t_{.1}$, $h(50|\theta)$, $F(50|\theta)$ there is no Extrapolation if the Exponential is the <u>known</u> dist'n & we have type II censoring.



Insulating Material
with Nonparametric Simultaneous 95% Confidence Bands
Exponential Probability Plot
Fraction Failing
.5
.4
.3
.2
.1
.001
.01
0
10
20
30
40
50
Hours
Tue Jul 30 12:48:58 2013

7.8

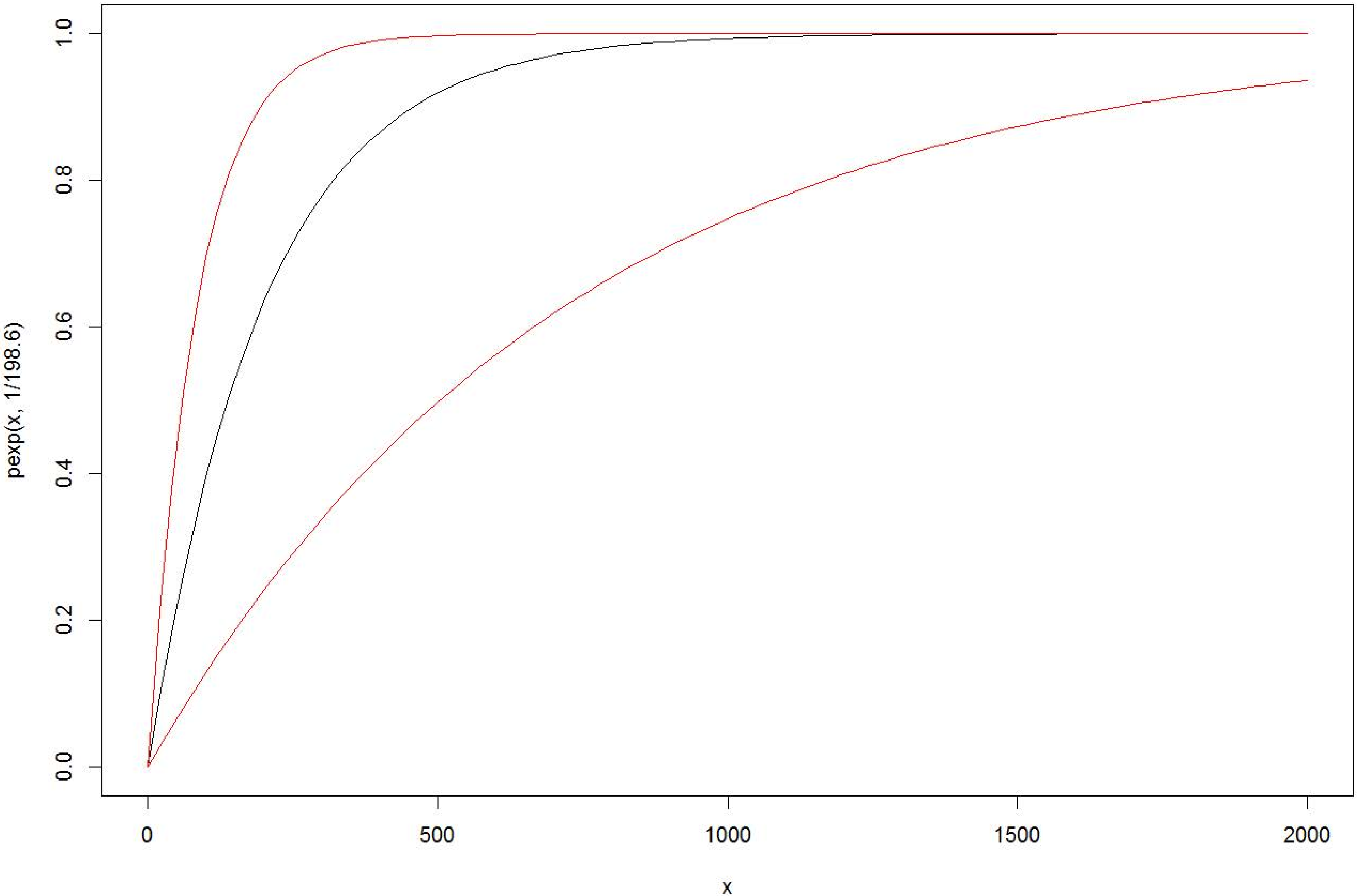
Basically, I am not sure if I am implementing the simultaneous intervals correctly but here is my stab at it.

Using the information from part f) of # 7.7 we have a $(1-\alpha)\%$ pointwise interval is $\left[1-\exp\left\{-\frac{t\,\chi^2_{\alpha/2;2r}}{2TTT}\right\},\ 1-\exp\left\{-\frac{t\,\chi^2_{1-\alpha/2;2r}}{2TTT}\right\}\right]$

for $F(t|\theta)$

to implement the simultaneous intervals Let's pick α^* Such that $\Phi_{norm}(3.06) = \alpha^* \approx .0011067$ [3.06 comes from table 3.5 on page 61] (Last row) this will give roughly simultaneous intervals for $F(t,\theta)$ for $F(t,\theta) \in [.1,.9]$.

the Plot of this is on the next page.


pexp(x, 1/198.6)
0.0
0.2
0.4
0.6
0.8
1.0
0
500
1000
1500
2000
x

$\boxed{8.1}$

part a) Use the Splida command

mleprobplot(lzbearing.ld, distribution="Lognormal", grid=T)

part b) same as ↑ with distribution="Weibull"

part c) they both seem like good models. I would prefer the Lognormal to the Weibull because it has a more linear trend in the probability plots.

$\boxed{8.5}$ part a) $F(t|\eta,\beta=2)=1-e^{-(t/\eta)^2}$ $\quad f(t)=\frac{2}{\eta}\left(\frac{t}{\eta}\right)e^{-(t/\eta)^2}$

$$L(\eta)=\prod_{i=1}^{9} f(t_i)\prod_{i=10}^{100} 1-F(t_i)=\left(\frac{2^9}{\eta^{18}}\left(\prod_{i=1}^{9} t_i\right)e^{-\frac{1}{\eta^2}\sum_{i=1}^{9}t_i^2}\right)\left(e^{-\frac{1}{\eta^2}\sum_{i=10}^{100}t_i^2}\right)$$

$$\propto \frac{\prod_{i=1}^{9} t_i}{\eta^{18}} e^{-\frac{1}{\eta^2}\sum_{i=1}^{100}t_i^2} \Rightarrow \ell(\eta)=C-18\log(\eta)-\frac{1}{\eta^2}\sum_{i=1}^{100}t_i^2$$

$$\frac{d\ell}{d\eta}=-\frac{18}{\eta}+\frac{2\sum_{i=1}^{100}t_i^2}{\eta^3} \Rightarrow \eta^2=\frac{\sum_{i=1}^{100}t_i^2}{9} \Rightarrow \hat{\eta}=\sqrt{\sum_{i=1}^{100}t_i^2}/3$$

$$\frac{d^2\ell}{d\eta^2}=+\frac{18}{\eta^2}-\frac{6\sum t_i^2}{\eta^4}\Bigg|_{\hat{\eta}}=\frac{2\cdot 9^2}{(\sum t_i^2)}-\frac{6\cdot 9^2}{(\sum t_i^2)}=\frac{9^2}{\sum(t_i^2)}(-4)<0 \quad \text{So } \hat{\eta}_{MLE} \text{ is the}$$

$$\sqrt{\sum_{i=1}^{100}t_i^2}=\sqrt{314815} \text{ (in thousands of cycles)} \Rightarrow \hat{\eta}=323.94$$

The practical interpretation of $\hat{\eta}$ is the square root of sum of the square individual item's time on test divided by the number of failures.

8.5 Part b) $\hat{se}_{\hat{\eta}} = \sqrt{\left(-\frac{(-4)81}{\sum_{i=1}^{\hat{}} t_i^2}\right)^{-1}} = \frac{\sqrt{\sum t_i^2}}{18} \approx 53.99$

part c) I could not find where they give formulas for a conservative CI so I will just use one for $Z_{\log(\hat{\eta})} \Rightarrow$

$$[\underset{\sim}{\eta}, \tilde{\eta}] = \left[\hat{\eta}\exp\{-z_{1-\alpha/2}\,\hat{se}_{\hat{\eta}}/\hat{\eta}\}, \hat{\eta}\exp\{z_{1-\alpha/2}\,\hat{se}_{\hat{\eta}}/\hat{\eta}\}\right]$$

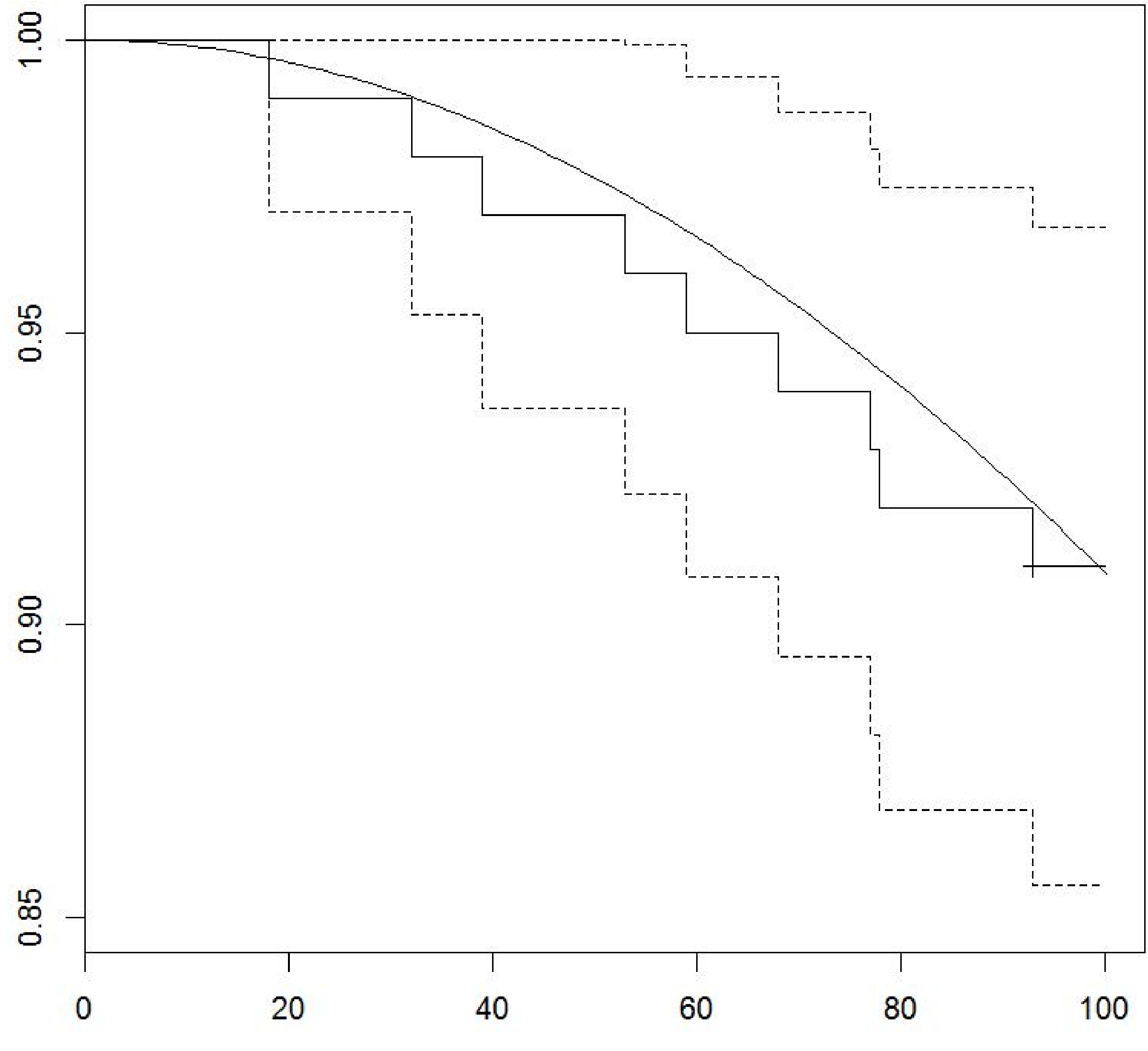
$$= [233.67, 449.09]$$

part d) See the next page for the plot. (I am looking @ $S(t)$ vs $F(t)$)
this Weibull seems to have an adequate fit

part e) $\hat{t}_{.1} = \hat{\eta}\sqrt{(-\log(.9))} \approx$ 105.15 (in thousands of cycles)

part f) Just multiplying the CI from part c) by $\sqrt{(-\log(.9))}$ we get

$= [75.85, 145.77]$


1.00
0.95
0.90
0.85
0
20
40
60
80
100

```
# 8.11

#Part a)
Fan.ld <- frame.to.ld(file=SplidaDataName("Fan.txt"),skip=1,
              response.column = 1, censor.column = 2,
          case.weight.column = 3,data.title = "Fan Failure Data",
           time.units = "Hours")
mleprobplot(Fan.ld, distribution = "Exponential",grid=T)

#Part b)
mleprobplot(Fan.ld, distribution = "Weibull",grid=T)

#Part c)
# Since the CI from the Splida output in part b) is based on the
# Relative likelihood and it includes 1 for beta/sigma
# Therefore we would fail to reject the null beta=1

# Using the Exp distn would suggest the replacement policy could
# be constant over time.
```

8.11 part d) Assuming an Exponential Distⁿ

$$\hat{\theta} = 28703 \quad \Rightarrow \quad \hat{t}_{.1} = \hat{\theta}(-\log(.9)) \approx 3024$$

$$\widehat{se}_{\hat{\theta}} = 8285.94 \quad \Rightarrow \quad \widehat{se}_{\hat{t}_{.1}} = \widehat{se}_{\hat{\theta}}(-\log(.9)) \approx 873.01$$

95% CI for $t_{.1}$ is $\left[\hat{t}_{.1} \pm 1.96\,\widehat{se}_{\hat{t}_{.1}}\right] = [1312.9, 4735.1]$

part e)

95% CI for $t_{.1}$ is $\left[\hat{t}_{.1}\exp\{-z_{1-\alpha/2}\,\widehat{se}_{\hat{t}_{.1}}/\hat{t}_{.1}\},\ \hat{t}_{.1}\exp\{z_{1-\alpha/2}\,\widehat{se}_{\hat{t}_{.1}}/\hat{t}_{.1}\}\right]$

$$= [1717.3, 5325]$$