LOGM 634 - Homework Set #1 (Solutions)

Due 11 January 2017

From the Ebeling text - Exercise 2.2

A component has the following linear hazard rate, where t is in years:

$$\lambda(t) = 0.4t \qquad t \geq 0$$

a. Find $R(t)$ and determine the probability of a component failing within the first month of its operation.

To find $R(t)$, recall the relationship

$$R(t) = \exp\left[-\int_0^t \lambda(t)\, dt\right]$$

Using this relationship we find $R(t)$ as

$$R(t) = \exp\left[-\int_0^t 0.4t\, dt\right]$$

$$= \exp\left[-0.2t^2\right]$$

Using this equation we find the reliability after one month as

$$\begin{aligned}
R(t = 1/12 \text{ yr}) &= \exp\left[-0.2 \times \frac{1}{12^2}\right] \\
&= \exp\left[-0.2 \times \frac{1}{144}\right] \\
&= \exp\left[-0.2 \times \frac{1}{12^2}\right] \\
&= \exp\left[-0.001388889\right] \\
&= 0.9986121
\end{aligned}$$

b. What is the design life is a reliability of 0.95 is desired?

The design life $t_{0.95}$ is the value of t at which $R(t) = 0.95$. This value can be found as follows:

$$R(t_{0.95}) = 0.95$$

$$\exp\left[-0.2t_{0.95}^2\right] = 0.95$$

$$t_{0.95} = \sqrt{\frac{-\ln[0.95]}{0.2}}$$

$$t_{0.95} = 0.5064252 \text{ years}$$

From the Ebeling text - Exercise 2.4

The failure distribution is defined by

$$f(t) = \frac{3t^2}{10^9} \qquad 0 \le t \le 1000 \text{ hr}$$

a. What is the probability of failure within a $100-$ hr warranty period?

The probability of failure over an interval is determined by the CDF $F(t)$

$F(100)$ can be found for this exercise as follows

$$\begin{aligned} F(t) &= \int_0^{100} f(t)\, dt \\ &= \int_0^{100} \frac{3t^2}{10^9}\, dt \\ &= \left.\frac{t^3}{10^9}\right|_0^{100} \\ &= \frac{10^6}{10^9} - \frac{0}{10^9} \\ &= 0.001 \end{aligned}$$

b. Compute the MTTF

The mean (or expected value) for any distribution is defined as

$$\text{MTTF } = E[T] = \int_0^{\infty} t f(t)\, dt$$

Using the provided expression for $f(t)$ we can find the MTTF as

$$
\begin{aligned}
\text{MTTF} &= \int_0^{1000} t f(t)\, dt \\
&= \int_0^{1000} \frac{3t^3}{10^9}\, dt \\
&= \left.\frac{3t^4}{4 \times 10^9}\right|_0^{1000} \\
&= \frac{3 \times 10^{12}}{4 \times 10^9} - \frac{0}{4 \times 10^9} \\
&= \frac{3}{4} \times \frac{10^{12}}{10^9} \\
&= 0.75 \times 1000 \\
&= 750 \text{ hours}
\end{aligned}
$$

c. Find the design life for a reliability of 0.99

The design life $t_{0.99}$ is the value of t at which $R(t) = 0.99$. This value can be found as follows:

$$
\begin{aligned}
R(t_{0.99}) &= 0.99 \\
1 - \frac{t_{0.99}^3}{10^9} &= 0.99 \\
t_{0.99} &= \left[0.01 \times 10^9\right]^{1/3} \\
t_{0.99} &= 215.4435 \text{ hours}
\end{aligned}
$$

From the Ebeling text - Exercise 2.11

A new fuel injection system is experiencing high failure rates. This reliability function has been found to be

$$R(t) = (t+1)^{-3/2} \qquad t \geq 0$$

where t is measured in years. The reliability over its intended life of 2 yr is 0.19, which is unacceptable. Will a burn-in period of 6 months significantly improve upon this reliability? If so, by how much?

This question us to find the conditional reliability $R(2|0.5)$. That is, the reliability of a population of items after two years, considering only the items that have already survived a 6-month burn-in test.

This quanitiy is found as follows:

$$\begin{aligned} R(2|0.5) &= \frac{R(2.5)}{R(0.5)} \\ &= \frac{(2.5+1)^{-3/2}}{(0.5+1)^{-3/2}} \\ &= \frac{0.1527207}{0.5443311} \\ &= 0.2805659 \end{aligned}$$

This values is larger, implying that a 6-month burn-in would increase the reliability of the fielded population of items

From the Ebeling text - Exercise 3.1

A component experiences chance (CFR) failures with an MTTF of 1100 hr. Find the following:

a. The reliability for a 200-hr mission

A constant failure rate (CFR) implies an exponential distribution. The reliability function for the exponential distribution is expressed as

$$R(t) = \exp[-\lambda t]$$

We are told that MTTF $= 1100$, which implies that $\lambda = 1/\text{MTTF} = 0.00091$. With λ known, we find that

$$R(200) = \exp[-0.00091 \times 200] = 0.8336013$$

b. The design life for a 0.90 reliability

The design life $t_{0.90}$ is the value of t at which $R(t) = 0.90$. This value can be found as follows:

$$R(t_{0.90}) = 0.90$$

$$\exp[-0.00091 \cdot t_{0.90}] = 0.90$$

$$t_{0.90} = \left[\frac{-\ln[0.90]}{0.00091}\right]$$

$$t_{0.90} = 115.7808 \text{ hours}$$

c. The median time to failure

The median time to failure is the same as the design life $t_{0.50}$ which is the value of t at which $R(t) = 0.50$. This value can be found as follows:

$$R(t_{0.50}) = 0.50$$

$$\exp[-0.00091 \cdot t_{0.50}] = 0.50$$

$$t_{0.50} = \left[\frac{-\ln[0.50]}{0.00091}\right]$$

$$t_{0.50} = 761.7002 \text{ hours}$$

From the Ebeling text - Exercise 3.2

A CFR system with $\lambda = 0.0004$ **has been operating for** 1000 **hr. What is the probability that it will fail in the next** 100 **hr? The next** 1000 **hr?**

This exercise is asking us to find the conditional reliabilities $R(100|1000)$ and $R(1000|1000)$. These values can be found as follows.

$$
\begin{aligned}
R(100|1000) &= \frac{R(1100)}{R(1000)} \\
&= \frac{\exp[-0.0004 \cdot 1100]}{\exp[-0.0004 \cdot 1000]} \\
&= \exp[-0.0004 \cdot (1100 - 1000)] \\
&= \exp[-0.004 \cdot 100]
\end{aligned}
$$

$$
\begin{aligned}
R(1000|1000) &= \frac{R(2000)}{R(1000)} \\
&= \frac{\exp[-0.0004 \cdot 2000]}{\exp[-0.0004 \cdot 1000]} \\
&= \exp[-0.0004 \cdot (2000 - 1000)] \\
&= \exp[-0.004 \cdot 1000]
\end{aligned}
$$

This exercise demonstrates the memoryless property of the exponential distribution. In both cases the resulting conditional reliability is exactly the same as if the system had just started operating.

Summary Exercise (30 Points)

A relay circuit has an average failure rate of 4 failures every 3 years. The circuit's failure times follow an exponential distribution.

a. What is the probability that the circuit will survive for one year without failure?

The relay circuit has a constant failure rate of $\lambda = 4/3$. Therefore, the probability that a circuit will survive the first year is expressed as

$$\begin{aligned} R(1) &= \exp[-4/3 \cdot 1] \\ &= 0.2635971 \end{aligned}$$

b. What is the probability that there will be more than two failures in the first year?

This question pertains to the relationship between the exponential and Poisson distributions.

Suppose that we are concerned about the reliability of a circuit in a subsystem. We only have two of these circuits on hand and - due to manufacturing issues - we cannot get any more for 1 year. We're concerned that both of these circuits will fail before we can get more into supply.

We've been told that the failure times for these circuits follow an exponential distribution with a rate parameter $\lambda = 4/3$. Therefore, over a one-year period we would expect to see $\lambda \cdot t = 4/3 \cdot 1 = 1.333$ failures in a typical year. but what is the probability that this is NOT a typical year and three or more failures occur?

Mathematically speaking, subsystem failure means that the sum of three exponentially-distributed failure times is less than 1-year. The Poisson distribution expresses the probability that n exponential distributed events occur by some time t. The pdf for the Poisson distribution is expressed as:

$$f(n) = \frac{e^{-\lambda t} \lambda t^n}{n!}$$

To determine the probability that $n = 3$ events occur in 1 year, we first determine the expected number of events in a year. This value is the rate at which the events occur, which is $\lambda = 4/3$ multipied by the time of interest. Therefore, $\lambda t = 4/3 \times 1 = 4/3$.

Now that we know this expected number of events, we can find probability that $n = 3$ events occur using the Poisson pdf. This value is

$$f(n = 3) = \frac{e^{-4/3} 4/3^3}{3!} = 0.1041371$$

Finally, the probability that $n > 2$ is equal to $1 - P(n \leq 2)$ which is expressed as

$$1 - F(n = 2) = 1 - \frac{e^{-4/3} 4/3^0}{0!} + \frac{e^{-4/3} 4/3^1}{1!} + \frac{e^{-4/3} 4/3^2}{2!} = 0.1506314$$