Maximum Likelihood Estimation for the Sectional Model Involving Two and Three-Parameter Weibull Distributions with Type-I Right Censored Data

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Maximum Likelihood Estimation for the Sectional Model Involving Two and Three-Parameter Weibull Distributions with Type-I Right Censored Data

A. B. Oliveira Neto, *Member, IEEE*, E. G. Costa, *Senior Member, IEEE*, and L. N. T. Alves, *Member, IEEE*

Abstract*—Among the Weibull models are the sectional models, which enable modeling of failure rates of complex behaviors. In this context, the sectional model involving two and three-parameter Weibull distribution stands out, from now on denominate $\mathbf{WS_{2P3P}}$. The $\mathbf{WS_{2P3P}}$ is capable of modeling failure rates of increasing or decreasing, increasing and constant, constant and decreasing, and upside-down bathtub curve behavior. Several studies were carried out to characterize and improve the failure data modeling by $\mathbf{WS_{2P3P}}$. However, regarding the parameter estimates of $\mathbf{WS_{2P3P}}$, in the literature only estimates were made based on the Weibull probability paper (WPP). Thus, in this paper, we propose a mathematical development based on the maximum likelihood estimation (MLE) for estimating the of parameters of $\mathbf{WS_{2P3P}}$ with Type-I right censored data, which is one of the most used censoring in recent research. Additionally, we developed and implemented an algorithm based on the Newton-Raphson (NR) method capable of assisting the numerical estimate of the parameters of $\mathbf{WS_{2P3P}}$. We approached a real case study in reliability modeling of failure data from an engineering problem to validate the applicability of this work. Finally, with the improvements in the parameter estimates of $\mathbf{WS_{2P3P}}$, properly proven, this work contributes to improving the reliability modeling in different areas of knowledge.

***Index Terms*—Maximum likelihood estimation, Newton-Raphson, parameter estimates, reliability modeling, sectional model, three-parameter Weibull distribution, two-parameter Weibull distribution, Type-I right censored data, $\mathbf{WS_{2P3P}}$.**

NOMENCLATURE

Acronyms and Abbreviation

CDF	Cumulative distribution function.
ECDF	Empirical cumulative distribution function.
EHF	Empirical hazard function.
EPDF	Empirical probability density function.
ERF	Empirical reliability function.
HF	Hazard function (or failure rate function).
MLE	Maximum likelihood estimation.
NR	Newton-Raphson.
PDF	Probability density function.
RF	Reliability function.
WPP	Weibull probability paper.
$\mathrm{WS_{2P3P}}$	Sectional model involving two and three-parameter Weibull distributions.

Notation

$F(t)$, $R(t)$, $f(t)$, $h(t)$	[CDF, RF, PDF, HF] of the population.
j	Index to sub-population j, $j = 1, 2$ unless otherwise specified.
$F_j(t)$, $R_j(t)$, $f_j(t)$, $h_j(t)$	[CDF, RF, PDF, HF] of sub-population j.
D_1	$= -(t/\alpha_1)^{\beta_1}$
D_2	$= -[(t_i - \gamma)/\alpha_2]^{\beta_2}$
α_j, β_j	Scale, shape parameters for $F_j(t)$.
γ	Position parameter for $F_2(t)$.
t_s	Sectioning additive parameter.
r	Samples observed.
n	Total samples.
t_i	Data set, with $1 \le i \le r$.
x	$\ln(t)$.
y	$\ln\{-\ln[R(t)]\}$.
C_y	Fitting plot: $y(x)$ vs x.
L_j	Straight line, $y_{L_j} = \beta_j([x - \ln(\alpha_j)])$.
(x_I, y_I)	Intersection point between L_1 and L_2.
(x_s, y_s)	Theoretical sectioning point.
t_c	Value of a prefixed censoring.
$L(\theta)$	Maximum likelihood function.
θ	Set of parameters to be estimated.
i_s	Number of samples observed up to t_s.
l	Log-likelihood function.
A	$= {\alpha_1}^{\beta_1}(\beta_2/\beta_1/\alpha_2)^{\beta_2}$.
B	$= 1/(\beta_1 - \beta_2)$.
C	$= 1 - \beta_2/\beta_1$.

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θ^{k+1}	Matrix of iteratively estimated parameters.
θ^0	Matrix of initial values.
k	Number of iterations.
k_{max}	Maximum number of iterations.
δ, φ	Prefixed tolerances.
$F_X(\theta)$	Set of nonlinear equations obtained by MLE.
$F_X(\theta^k)$	Matrix obtained by applying θ_k to $F_X(\theta)$.
$J_F(\theta^k)$	Jacobian matrix of $F_X(\theta)$ for θ^k.
$J_{F_{UL}}$	Upper limit value of the Jacobian matrix.
$J_{F_{LL}}$	Low limit value of the Jacobian matrix.
T	Transposed matrix.
$\overrightarrow{\theta}$	Matrix of estimated parameters
$\overrightarrow{\alpha_1}, \overrightarrow{\beta_1}, \overrightarrow{\alpha_2}, \overrightarrow{\beta_2}, \vec{\gamma}, \overrightarrow{t_s}$	Estimated parameters.

I. Introduction

RELIABILITY modeling employing variations of Weibull distributions has been performed in several fields of science. Among the Weibull models are the sectional models, which enable modeling of failure rates of complex behaviors [1]. In this context, the sectional model involving two and three-parameter Weibull distributions stands out, from now on denominate $\mathrm{WS_{2P3P}}$. The $\mathrm{WS_{2P3P}}$ introduced by Jiang and Murthy [2] presents, as its main characteristic, the ability to model nonmonotone and monotonous failure rates, thus enabling the realization of reliability studies of lifetime data.

Regarding the parameter estimates of $\mathrm{WS_{2P3P}}$, Jiang and Murthy [2] proposed a graphical procedure based on Weibull Probability Paper (WPP). Murthy and Jiang [3] developed a study to characterize the shape parameters of $\mathrm{WS_{2P3P}}$ when they analyzed the probability density and failure rate functions. Murthy *et al.* [4] presented a study based on the least squares adjustment to choose the type of Weibull model in the reliability analysis and also estimated the parameters of $\mathrm{WS_{2P3P}}$ using WPP, including for censored data. Zhang and Dwight [5] developed a study for the optimal choice of Weibull models for failure data based on a graphical analysis and used WPP to parameter estimates of $\mathrm{WS_{2P3P}}$. However, estimates based only on WPP, especially when data of medium and large sizes are analyzed, can generate unsatisfactory results, requiring estimates by robust statistical methods such as method of maximum likelihood estimation (MLE) [4], [6].

In this context, MLE has been used recurrently in the parameter estimates of two-parameter Weibull and three-parameter Weibull distributions. For the two-parameters Weibull distribution MLE was used in the studies developed by: Cohen [7] and Harter and Moore [8], when they considered samples complete and censored, with following improvements implemented by Rockette *et al.* [9]; Sirvanci and Yang [10], when they considered Type-I censored data; Flygare *et al.* [11], when they examined data by interval; Odell *et al.* [12], when they developed an algorithm based on the Newton-Raphson (NR) method for censored data; Keats *et al.* [13], when they made a numerical estimate based on NR complete and censored data; Balakrishnan and Kateri [14], when they proposed an alternative graphical method of greater practicality considering complete and censored data; NG and Wang [15] and Cheng *et al.* [16], when they made numerical estimates based on NR for Type-I censored data; Li *et al.* [17], when they elaborated a classification model in modeling approach of accelerated degradation test data; Qiu *et al.* [18], when they modeled the time to failure (TTF) of data race software failures.

MLE was also used to estimate three-parameter Weibull distribution parameters in studies investigated by: Lemon [19], when he considered censored data; Archer [20], when he developed a computational technique based on NR considering complete and censored data; Gourdin *et al.* [21], when they developed an algorithm based on NR considering complete data and Type-I right censored data; Hirose [22], when he considered the generalized extreme value distribution and NR; Ismail [23], when he developed an algorithm based on NR for Type-I and II censoring; Yang *et al.* [24], when they elaborated an evolutionary algorithm, converting the likelihood function maximization process into an optimization problem.

In addition, parameter estimates using MLE in reliability studies, with or without censored data, involving more than one model with at least one Weibull distribution were presented for Weibull models of the type of: competitive risk [25]–[32]; mixed distribution [33], [34]; mixture [35]–[39]. These researches demonstrate which MLE has been used to improve parameter estimates and analysis in the reliability modeling of failure rates of complex behaviors involving Weibull distributions.

In general, for sectional models, MLE results in complex estimators or a set of equations that must necessarily be worked out numerically. Thus, the NR method is one of the most used when the parameter estimate is performed by MLE in models involving Weibull distributions, with complete or censored data, with specific algorithms elaboration or associated to another method, as presented in [12]–[16], [20]–[23], [29], [37]. Therefore, observing the literature, there are fundamentals for investigating the MLE and NR for parameter estimates of $\mathrm{WS_{2P3P}}$, including Type-I censored data.

Reliability studies with censored data are widely studied because life data are analyzed before all sample units fail [40]. Among the types of censoring, Type-I right censoring stands out, in which the observation comes to an end when a prefixed censoring time is reached [41]. Type-I censoring has received special attention recently [29], [42]–[46], as it is one of the most popular and practical life tests [43].

This research has as main objective to improve the parameter estimates of $\mathrm{WS_{2P3P}}$ and its reliability modeling. Therefore, we set specific objectives as following:

1) Establish guidelines for estimating parameters of $\mathrm{WS_{2P3P}}$ employing WPP with Type-I right censored data.
2) Propose a mathematical development based on the MLE for estimating the of parameters of $\mathrm{WS_{2P3P}}$ with Type-I right censored data.
3) Develop and implement an algorithm based on the NR method capable of assisting the numerical solution of the set of equations resulting from the MLE. The initial value for the numerical solution is obtained through the estimate by WPP.

4) Approach a real case study in reliability modeling of failure data from an engineering problem to validate the applicability of this work.

The remainder of the paper is organized as follows. In Section II we presented a description of the WS_{2P3P}. In Section III we presented the parameter estimates of WS_{2P3P} through WPP and MLE for Type-I right censored data. Section IV, an algorithm based on NR capable of assisting in the numerical resolution of the expressions resulting from the MLE is introduced. In Section V illustrates the application of the proposed research using a real case study in reliability modeling of failure data from an engineering problem. Section VI presents the main conclusions of this paper. In addition, in Appendices A, B, and C, summarized mathematical expressions were included, for purposes of consultation and application in the algorithm based on NR. Finally, we have the references section of this paper.

II. WS_{2P3P} Description

The cumulative distribution function (CDF) of WS_{2P3P}, with a total of 6 parameters, has the following representation (1) [2].

$$F(t) = \begin{cases} F_1(\mathrm{t}) \\ F_2(\mathrm{t}) \end{cases} = \begin{cases} 1-\exp(D_1), & 0 \le t \le t_s, \\ 1-\exp(D_2), & t_s < t < \infty. \end{cases} \tag{1}$$

Where: $D_1 = -(t/\alpha_1)^{\beta_1}$; $D_2 = -[(t-\gamma)/\alpha_2]^{\beta_2}$; α_1 and α_2 are scale parameters; β_1 and β_2 are shape parameters; γ is the position parameter; t_s is the sectioning additive parameter. Moreover, it has the following conditions: α_1, β_1, α_2, β_2, and $\gamma > 0$.

In addition, in (2) and (3) are presented the reliability function (RF) and probability density function (PDF) of WS_{2P3P}.

$$R(t) = \begin{cases} R_1(\mathrm{t}) \\ R_2(\mathrm{t}) \end{cases} = \begin{cases} \exp(D_1), & 0 \le t \le t_s, \\ \exp(D_2), & t_s < t < \infty. \end{cases} \tag{2}$$

$$f(t) = \begin{cases} f_1(\mathrm{t}) \\ f_2(\mathrm{t}) \end{cases} = \begin{cases} -(\beta_1/t)\, D_1 \exp(D_1), & 0 \le t \le t_s, \\ -[\beta_2/(t-\gamma)]\, D_2 \exp(D_2), & t_s < t < \infty. \end{cases} \tag{3}$$

By imposing the continuity condition of $R(t)$ and $\mathrm{f}(t)$ on t_s we obtained two dependent parameters which are estimated according to (4) and (5) [2]:

$$t_s = \left[{\alpha_1}^{\beta_1}(\beta_2/\beta_1/\alpha_2)^{\beta_2}\right]^{1/(\beta_1-\beta_2)}, \tag{4}$$

$$\gamma = (1-\beta_2/\beta_1)t_s. \tag{5}$$

Thus $\beta_1 > \beta_2$ and consequently $\gamma < t_s$.

Additionally, the hazard function (HF) of WS_{2P3P} is expressed as (6).

$$h(t) = \begin{cases} h_1(\mathrm{t}) \\ h_2(\mathrm{t}) \end{cases} = \begin{cases} -(\beta_1/t)D_1, & 0 \le t \le t_s, \\ -[\beta_2/(t-\gamma)]D_2, & t_s < t < \infty. \end{cases} \tag{6}$$

The shapes of the HF of WS_{2P3P} with respect to the shape

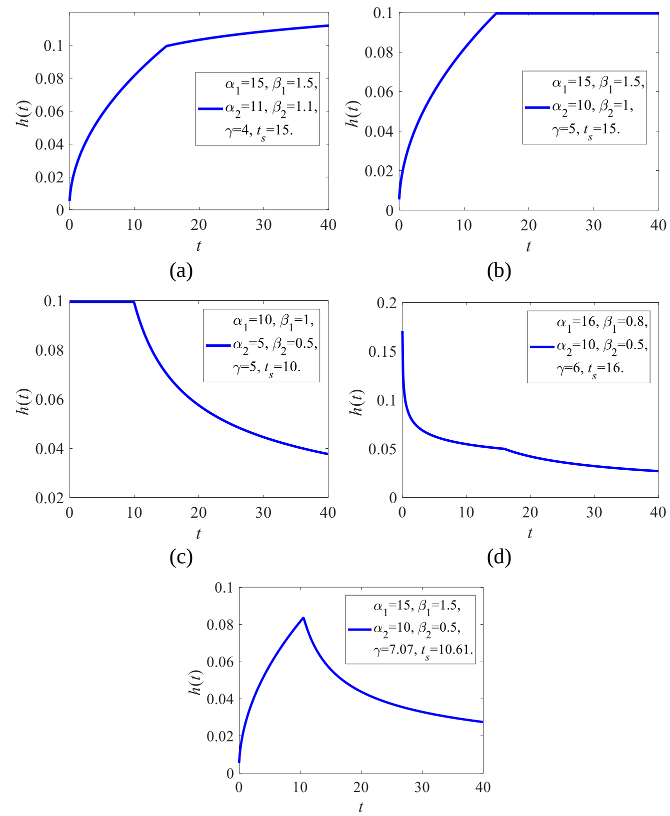

Fig. 1. Plots the hazard functions of WS_{2P3P} under different parameter settings.

parameters $\beta_1$and β_1 are summarized in [1][4], as follows.

1 – Increasing if $\beta_1 > 1$ and $\beta_2 > 1$, e.g., Fig. 1 (a).

2 – Increasing and constant if $\beta_1 > 1$ and $\beta_2 = 1$, e.g., Fig. 1 (b).

3 – Constant and decreasing if $\beta_1 = 1$ and $\beta_2 < 1$, e.g., Fig. 1 (c).

3 – Decreasing if $\beta_1 < 1$ and $\beta_2 < 1$, e.g., Fig. 1 (d).

5 – Upside down bathtub shaped (unimodal) if $\beta_1 > 1$ e $\beta_2 < 1$, e.g., Fig. 1 (e).

For the purpose of visually illustrating the behavior of $h(t)$, in Fig. 1 we presented hazard functions with some adjustments of specific parameters based on the shapes presented in [3].

III. Parameter Estimates of WS_{2P3P}

A. Estimates of WS_{2P3P} Employing WPP

The estimation of the parameters of WS_{2P3P} employing WPP for uncensored data was presented by Jiang and Murthy [2]. Therefore, the transformation $x = \ln(t)$ and $y = \ln\{-\ln[R(t)]\}$ was considered, which resulted in expression (7):

$$y = y(x) = \begin{cases} \beta_1[x - \ln\alpha_1], & -\infty \le x \le \ln(t_s), \\ \beta_2[\ln(e^x - \gamma) - \ln\alpha_2], & \ln(t_s) < x < \infty. \end{cases} \tag{7}$$

A graphic illustration of y versus x for the WS_{2P3P} employing WPP is shown in Fig. 2. Denoting by C_y the curve representing the theoretical expression (7) for the example in Fig. 2, when performing an asymptotic analysis around C_y, we obtained (8) and (9), respectively, for the lines L_1 and L_2 [2].

$$y_{L_1} = \beta_1([x - \ln(\alpha_1)]). \tag{8}$$

$$y_{L_2} = \beta_2([x - \ln(\alpha_2)]). \tag{9}$$

The intercept coordinate (x_I, y_I) between L_1 and L_2 was obtained as follows (10) and (11).

$$x_I = [\beta_1 \ln(\alpha_1) - \beta_2 \ln(\alpha_2)]/(\beta_1 - \beta_2). \tag{10}$$

$$y_I = [\beta_1\beta_2 \ln(\alpha_1/\alpha_2)]/(\beta_1 - \beta_2). \tag{11}$$

Using (4), (10), and (11), the theoretical sectioning point of the model can be estimated as:

$$x_s = \ln t_s = x_I - [\beta_2 \ln(\beta_1/\beta_2)]/(\beta_1 - \beta_2), \tag{12}$$

$$y_s = y(x)|_{x=x_s} = y_I - [\beta_1\beta_2 \ln(\beta_1/\beta_2)]/(\beta_1 - \beta_2). \tag{13}$$

Thus, considering Jiang and Murthy [2] and Nelson [40], for Type-I censored data, from a total of n samples, with r observed or failed samples up to t_c value of a prefixed censoring, the estimation using WPP is done according to steps 1-7 as follows.

Step 1: Reorder the data in ascending order, where t_i, with $1 \leq i \leq r$, is the data set.

Step 2: Calculate x_i and y_i, with $1 \leq i \leq r$, where $x_i = \ln(t_i)$ and $y_i = \ln\{-\ln[1 - i/(n+1)]\}$ [40].

Step 3: Plot y_i versus x_i.

Step 4: Fit the line L_1 to the left side of the graph in Fig. 2 and get β_1 for the slope of the line and α_1 for the intersection of L_1 with the x axis.

Step 5: Carry out the asymptotic fit of the line L_2 on the graph in Fig. 2 and obtain β_2 through the slope of the line and α_2 through the intersection of L_2 with the x axis.

Step 6: Calculate t_s and γ using (4) and (5).

Step 7: Use (12) and (13) and observe whether the following conditions are satisfied: $x_I > x_s$ and $y_I > y_s$. If not, repeat steps 5 and 6.

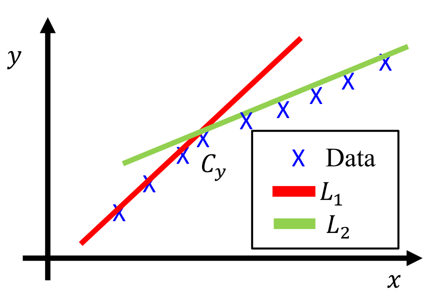

Fig. 2. Graphic illustration of y versus x in WPP, as well as L_1, L_2 and C_y for WS_{2P3P}.

In this paper, the estimate made by WPP will be used as the initial value in the numerical solution of the expressions resulting from the maximum likelihood function.

B. *Estimate Parameters of WS_{2P3P} Employing MLE*

MLE for Type-I right censored data, with r samples observed in relation to a total of n samples, can be represented as the expression (14) [47].

$$L(\theta) = \prod_{i=1}^{r} f(t_i) \prod_{i=r+1}^{n} R(t_i). \tag{14}$$

Where: $L(\theta)$ is the maximum likelihood function; θ represents the set of parameters to be estimated.

In the case of WS_{2P3P}, in addition to r and n, it is necessary to consider the number of samples i_s observed up to t_s. Thus, the $L(\theta)$ for Type-I right censored data for WS_{2P3P} is represented by expression (15):

$$L(\theta) = \prod_{i=1}^{i_s} f_1(t_i) \prod_{i=i_s+1}^{r} f_2(t_i) \prod_{i=r+1}^{n} R_2(t_i). \tag{15}$$

Applying (2) and (3) to (15), we obtained (16):

$$L(\theta) = \left(\frac{\beta_1}{{\alpha_1}^{\beta_1}}\right)^{i_s} \left(\frac{\beta_2}{{\alpha_2}^{\beta_2}}\right)^{(r-i_s)} \prod_{i=1}^{i_s} (t_i)^{(\beta_1-1)} R_1(t_i) \prod_{i=i_s+1}^{r} (t_i - \gamma)^{(\beta_1-1)} \prod_{i=i_s+1}^{n} R_2(t_i). \tag{16}$$

Applying the logarithm to the maximum likelihood function of (16), we obtained (17):

$$\begin{aligned} l = \ln L(\theta) = {} & i_s(\ln \beta_1 - \beta_1 \ln \alpha_1) \\ & + (r - i_s)(\ln \beta_2 - \beta_2 \ln \alpha_2) \\ & + \sum_{i=1}^{i_s} \{(\beta_1 - 1)\ln t_i + \ln[R_1(t_i)]\} \\ & + \sum_{i=1}^{i_s} (\beta_1 - 1)\ln(t_i - \gamma) \\ & + \sum_{i=1}^{i_s} \ln[R_2(t_i)]. \end{aligned} \tag{17}$$

Now an analysis of the first order partial derivatives of l is done with the objective of obtaining the maximization of $L(\theta)$.

Once t_s and γ are dependent parameter α_1, β_1, α_2, and β_2, to make the analysis of the first and second order partial derivatives of l mathematically presentable, the following convention is adopted: $t_s = [A]^B$ and $\gamma = Ct_s$, where:

$$A = {\alpha_1}^{\beta_1}(\beta_2/\beta_1/\alpha_2)^{\beta_2}, \tag{18}$$

$$B = 1/(\beta_1 - \beta_2), \tag{19}$$

$$C = 1 - \beta_2/\beta_1. \tag{20}$$

Therefore, the first order partial derivatives of A, B and C in relation to α_1, β_1, α_2, and β_2 are shown in Table I. Then, we calculated the first order partial derivatives of t_s and γ in relation to α_1, β_1, α_2, and β_2 as presented in (21) to (28).

TABLE I
PARTIAL DERIVATIVES OF A, B AND C IN RELATION TO α_1, β_1, α_2, AND β_2.

	A	B	C
$\partial/\partial\alpha_1$	$(\beta_1/\alpha_1)A$	0	0
$\partial/\partial\beta_1$	$(\ln\alpha_1-\beta_2/\beta_1)A$	$-B^2$	$\beta_2/{\beta_1}^2$
$\partial/\partial\alpha_2$	$-(\beta_2/\alpha_2)A$	0	0
$\partial/\partial\beta_2$	$[1+\ln(\beta_2/\beta_1/\alpha_2)]A$	B^2	$-1/\beta_1$

$$\partial t_s/\partial\alpha_1=(\beta_1/\alpha_1)At_s, \tag{21}$$

$$\partial t_s/\partial\beta_1=Bt_s(\ln\alpha_1-\beta_2/\beta_1-B\ln A), \tag{22}$$

$$\partial t_s/\partial\alpha_2=-(\beta_2/\alpha_2)At_s, \tag{23}$$

$$\partial t_s/\partial\beta_2=Bt_s[1+\ln(\beta_2/\beta_1/\alpha_2)+B\ln A], \tag{24}$$

$$\partial\gamma/\partial\alpha_1=(\beta_1/\alpha_1)B\gamma, \tag{25}$$

$$\partial\gamma/\partial\beta_1=B\gamma(\ln\alpha_1-\beta_2/\beta_1-B\ln A)+(\beta_2/{\beta_1}^2)t_s, \tag{26}$$

$$\partial\gamma/\partial\alpha_2=-(\beta_2/\alpha_2)B\gamma, \tag{27}$$

$$\partial\gamma/\partial\beta_2=B\gamma[1+\ln(\beta_2/\beta_1/\alpha_2)+B\ln A]-t_s/\beta_1, \tag{28}$$

Now, the partial derivative of l in relation to the independent parameters of $\mathrm{WS_{2P3P}}$ was obtained, as presented in (29) to (32):

$$\frac{\partial\ln(L)}{\partial\alpha_1}=-i_s\frac{\beta_1}{\alpha_1}+\sum_{i=1}^{i_s}\frac{\partial D_1}{\partial\alpha_1}-\sum_{i=i_s+1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\alpha_1}+\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\alpha_1}, \tag{29}$$

$$\frac{\partial\ln(L)}{\partial\beta_1}=i_s\left(\frac{1}{\beta_1}-\ln\alpha_1\right)+\sum_{i=1}^{i_s}\left(\ln t_i+\frac{\partial D_1}{\partial\beta_1}\right)-\sum_{i=i_s+1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\beta_1}+\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\beta_1}, \tag{30}$$

$$\frac{\partial\ln(L)}{\partial\alpha_2}=-\frac{\beta_2}{\alpha_2}(r-i_s)-\sum_{i=i_s+1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\alpha_2}+\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\alpha_2}, \tag{31}$$

$$\frac{\partial\ln(L)}{\partial\beta_2}=(r-i_s)\left(\frac{1}{\beta_2}-\ln\alpha_2\right)+\sum_{i=i_s+1}^{r}\left[\ln(t_i-\gamma)-\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\beta_2}\right]+\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\beta_2}. \tag{32}$$

Thus, the first order partial derivatives of D_1 and D_2 in relation to α_1, β_1, α_2, and β_2 are presented (33) to (39).

$$\frac{\partial D_1}{\partial\alpha_1}=-\left(\frac{\beta_1}{\alpha_1}\right)D_1, \tag{33}$$

$$\frac{\partial D_1}{\partial\beta_1}=\ln\left(\frac{t_i}{\alpha_1}\right)D_1, \tag{34}$$

$$\frac{\partial D_1}{\partial\alpha_2}=\frac{\partial D_1}{\partial\beta_2}=0, \tag{35}$$

$$\frac{\partial D_2}{\partial\alpha_1}=-\left(\frac{\beta_2}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\alpha_1}D_2, \tag{36}$$

$$\frac{\partial D_2}{\partial\beta_1}=-\left(\frac{\beta_2}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\beta_1}D_2, \tag{37}$$

$$\frac{\partial D_2}{\partial\alpha_2}=-\beta_2\left[\left(\frac{1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\alpha_2}+\frac{1}{\alpha_2}\right]D_2, \tag{38}$$

$$\frac{\partial D_2}{\partial\beta_2}=\left[\ln\left(\frac{t_i-\gamma}{\alpha_2}\right)-\left(\frac{\beta_2}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\beta_2}\right]D_2. \tag{39}$$

Then, by equating the partial derivatives of l to zero, the estimator is found to parameters of $\mathrm{WS_{2P3P}}$ as shown in (40) to (43):

$$\alpha_1=\frac{i_s\beta_1}{\sum_{i=1}^{i_s}\frac{\partial D_1}{\partial\alpha_1}-\sum_{i=i_s+1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\alpha_1}+\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\alpha_1}},\ i_s>0, \tag{40}$$

$$\beta_1=\frac{i_s}{i_s\ln\alpha_1-\sum_{i=1}^{i_s}\left(\ln t_i+\frac{\partial D_1}{\partial\beta_1}\right)+\sum_{i=i_s+1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\beta_1}-\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\beta_1}},\ i_s>0, \tag{41}$$

$$\alpha_2=\frac{\beta_2(r-i_s)}{-\sum_{i=i_s+1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\alpha_2}+\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\alpha_2}},\ i_s\geq 0, \tag{42}$$

$$\beta_2=\frac{(r-i_s)}{(r-i_s)\ln\alpha_2-\sum_{i=i_s+1}^{r}\left[\ln(t_i-\gamma)-\left(\frac{\beta_2-1}{t_i-\gamma}\right)\frac{\partial\gamma}{\partial\beta_2}\right]-\sum_{i=i_s+1}^{n}\frac{\partial D_2}{\partial\beta_2}},\ i_s\geq 0. \tag{43}$$

As can be seen, expressions (40) and (41) are estimators for the condition of i_s>0, that is, no sample was observed before t_s. Thus, for situations in which there i_s=0, which can commonly occur in iterative numerical processes, α_1 and β_1 for i_s=0 are given by (44) to (45):

$$\alpha_1=\frac{\beta_1 B\gamma}{\left[\frac{\sum_{i=1}^{n}\frac{\partial D_2}{\partial\alpha_1}}{\sum_{i=1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)}\right]^{\frac{1}{r}}}, \tag{44}$$

$$\beta_1=\left\{\beta_2 t_s\Big/\left\{\left[\frac{\sum_{i=1}^{n}\frac{\partial D_2}{\partial\beta_1}}{\sum_{i=1}^{r}\left(\frac{\beta_2-1}{t_i-\gamma}\right)}\right]^{\frac{1}{r}}-B\gamma\left(\ln\alpha_1-\frac{\beta_2}{\beta_1}-B\ln A\right)\right\}\right\}^{\frac{1}{2}}. \tag{45}$$

Once the estimators for the parameters of $\mathrm{WS_{2P3P}}$, were through WPP, iterative numerical methods can be used to estimate roots that maximize $L(\theta)$. It is worth noting that, for eventual convergences, it is necessary that the iterative numerical methods used undergo adaptations as a function of the mathematical conditions of $\mathrm{WS_{2P3P}}$, as will be discussed in the implementation of the numerical solution in the next section.

IV. NUMERICAL SOLUTION FOR PARAMETERS OF $\mathrm{WS_{2P3P}}$

The maximization of the maximum likelihood function of (15) is obtained by solving the set of nonlinear equations of (29) to (32), denoted by $F_X(\theta)$. To solve this set of equations the literature points to the use of numerical methods, including the Newton-Raphson (NR) method. The iterative expression of NR, considering $F_X(\theta)$ and the independent parameters α_1, β_1, α_2, β_2 to be estimated, is presented in (46) [48].

$$\theta^{k+1} = \theta^k - {J_F}^{-1}(\theta^k)F_X(\theta^k),\ k = 0, 1, \dots, k_{max}\,. \tag{46}$$

Where: θ^{k+1} is the matrix of iteratively estimated parameters; θ^0 is a matrix of initial values, obtained in this paper by graphical estimation using WPP; k is the number of iterations to be performed until reaching a k_{max} or the prefixed tolerances δ and φ; $F_X(\theta^k)$ the matrix obtained by applying θ_k to $F_X(\theta)$, according to expression (47); $J_F(\theta^k)$ is the Jacobian matrix of $F_X(\theta)$ for θ^k, as shown in expression (48).

$$(\theta^k) = \begin{bmatrix} \frac{\partial \ln(L)}{\partial \alpha_1} \\ \frac{\partial \ln(L)}{\partial \beta_1} \\ \frac{\partial \ln(L)}{\partial \alpha_2} \\ \frac{\partial \ln(L)}{\partial \beta_2} \end{bmatrix}, \tag{47}$$

$$J_F(\theta^k) = \begin{bmatrix} \frac{\partial^2 \ln(L)}{\partial {\alpha_1}^2} & \frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \beta_1} & \frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \alpha_2} & \frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \beta_2} \\ \frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \alpha_1} & \frac{\partial \ln(L)}{\partial {\beta_1}^2} & \frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \alpha_2} & \frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \beta_2} \\ \frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \alpha_1} & \frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \beta_1} & \frac{\partial^2 \ln(L)}{\partial {\alpha_2}^2} & \frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \beta_2} \\ \frac{\partial^2 \ln(L)}{\partial \beta_2 \partial \alpha_1} & \frac{\partial^2 \ln(L)}{\partial \beta_2 \partial \beta_1} & \frac{\partial^2 \ln(L)}{\partial \beta_2 \partial \alpha_2} & \frac{\partial^2 \ln(L)}{\partial {\beta_2}^2} \end{bmatrix}. \tag{48}$$

The elements of F_X are obtained by using (29) to (32). The elements of the Jacobian matrix J_F were calculated through the second order derivatives of $\ln[L(\theta)]$ in relation to the parameters α_1, β_1, α_2, β_2, as shown in (49) to (64).

$$\frac{\partial^2 \ln(L)}{\partial {\alpha_1}^2} = i_s \frac{\beta_1}{{\alpha_1}^2} + \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial {\alpha_1}^2} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial {\alpha_1}^2} + \left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial \gamma}{\partial \alpha_1}\right)^2\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial {\alpha_1}^2} \tag{49}$$

$$\frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \beta_1} = -\frac{i_s}{\alpha_1} + \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial \alpha_1 \partial \beta_1} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \beta_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \beta_1}\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \alpha_1 \partial \beta_1} \tag{50}$$

$$\frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \alpha_2} = \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial \alpha_1 \partial \alpha_2} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \alpha_2}\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \alpha_1 \partial \alpha_2} \tag{51}$$

$$\frac{\partial^2 \ln(L)}{\partial \alpha_1 \partial \beta_2} = \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial \alpha_1 \partial \beta_2} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left\{\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \beta_2} + \frac{\partial \gamma}{\partial \alpha_1}\left[\left(\frac{1}{\beta_2-1}\right) + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right]\right\} + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \alpha_1 \partial \beta_2} \tag{52}$$

$$\frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \alpha_1} = -\frac{i_s}{\alpha_1} + \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial \beta_1 \partial \alpha_1} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial \beta_1 \partial \alpha_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \beta_1}\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \beta_1 \partial \alpha_1} \tag{53}$$

$$\frac{\partial^2 \ln(L)}{\partial {\beta_1}^2} = -\frac{i_s}{{\beta_1}^2} + \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial {\beta_1}^2} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial {\beta_1}^2} + \left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial \gamma}{\partial \beta_1}\right)^2\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial {\beta_1}^2} \tag{54}$$

$$\frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \alpha_2} = \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial \beta_1 \partial \alpha_2} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial \beta_1 \partial \alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_1}\frac{\partial \gamma}{\partial \alpha_2}\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \beta_1 \partial \alpha_2} \tag{55}$$

$$\frac{\partial^2 \ln(L)}{\partial \beta_1 \partial \beta_2} = \sum_{i=1}^{i_s} \frac{\partial^2 D_1}{\partial \beta_1 \partial \beta_2} - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left\{\frac{\partial^2 \gamma}{\partial \beta_1 \partial \beta_2} + \frac{\partial \gamma}{\partial \beta_1}\left[\left(\frac{1}{\beta_2-1}\right) + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right]\right\} + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \beta_1 \partial \beta_2} \tag{56}$$

$$\frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \alpha_1} = -\sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \alpha_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \alpha_2}\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \alpha_2 \partial \alpha_1} \tag{57}$$

$$\frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \beta_1} = -\sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \beta_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_1}\frac{\partial \gamma}{\partial \alpha_2}\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \alpha_2 \partial \beta_1} \tag{58}$$

$$\frac{\partial^2 \ln(L)}{\partial {\alpha_2}^2} = \frac{\beta_2}{{\alpha_2}^2}(r - i_s) - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left[\frac{\partial^2 \gamma}{\partial {\alpha_2}^2} + \left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial \gamma}{\partial \alpha_2}\right)^2\right] + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial {\alpha_2}^2} \tag{59}$$

$$\frac{\partial^2 \ln(L)}{\partial \alpha_2 \partial \beta_2} = -\left(\frac{r-i_s}{\alpha_2}\right) - \sum_{i=i_s+1}^{r} \left(\frac{\beta_2-1}{t_i-\gamma}\right) \left\{\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \beta_2} + \frac{\partial \gamma}{\partial \alpha_2}\left[\left(\frac{1}{\beta_2-1}\right) + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right]\right\} + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \alpha_2 \partial \beta_2} \tag{60}$$

$$\frac{\partial^2 \ln(L)}{\partial \beta_2 \partial \alpha_1} = -\sum_{i=i_s+1}^{r} \left(\frac{1}{t_i-\gamma}\right) \left\{\frac{\partial \gamma}{\partial \alpha_1} + (\beta_2 - 1)\left[\frac{\partial^2 \gamma}{\partial \beta_2 \partial \alpha_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \beta_2}\right]\right\} + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \beta_2 \partial \alpha_1} \tag{61}$$

$$\frac{\partial^2 \ln(L)}{\partial \beta_2 \partial \beta_1} = -\sum_{i=i_s+1}^{r} \left(\frac{1}{t_i-\gamma}\right) \left\{\frac{\partial \gamma}{\partial \beta_1} + (\beta_2 - 1)\left[\frac{\partial^2 \gamma}{\partial \beta_2 \partial \beta_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_1}\frac{\partial \gamma}{\partial \beta_2}\right]\right\} + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \beta_2 \partial \beta_1} \tag{62}$$

$$\frac{\partial^2 \ln(L)}{\partial \beta_2 \partial \alpha_2} = -\left(\frac{r-i_s}{\alpha_2}\right) - \sum_{i=i_s+1}^{r} \left(\frac{1}{t_i-\gamma}\right) \left\{\left[\frac{\partial^2 \gamma}{\partial \beta_2 \partial \alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\frac{\partial \gamma}{\partial \beta_2}\right]\right\} + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial \beta_2 \partial \alpha_2} \tag{63}$$

$$\frac{\partial^2 \ln(L)}{\partial {\beta_2}^2} = -\left(\frac{i_s-r}{{\beta_2}^2}\right) - \sum_{i=i_s+1}^{r} \left(\frac{1}{t_i-\gamma}\right) \left\{\frac{\partial \gamma}{\partial \beta_2} + (\beta_2 - 1)\left[\frac{\partial^2 \gamma}{\partial {\beta_2}^2} + \frac{\partial \gamma}{\partial \beta_2}\left(\left(\frac{1}{\beta_2-1}\right) + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right)\right]\right\} + \sum_{i=i_s+1}^{n} \frac{\partial^2 D_2}{\partial {\beta_2}^2} \tag{64}$$

The second order derivatives of D_1, D_2 and γ in relation to the parameters α_1, α_2, β_1, and β_2 are included in Appendices A, B, and C, respectively.

For the WS_{2P3P} under study, the estimate based solely in (47) may not result in acceptable estimates. Since the existential mathematical conditions of the model must be respected for each iteration, namely: α_1, β_1, α_2, β_2, and $\gamma > 0$; $\beta_1 > \beta_2$; and $\gamma < t_s$.

In addition to the mathematical conditions, iterative numerical implementations in software can present problems of singularity of the Jacobian matrix [49] which can configure a forced singularity as a function of the dimensional difference of values of the Jacobian matrix. Therefore, in this paper, upper and lower limits are considered for values of the Jacobian matrix, named, respectively, as $J_{F_{UL}}$ and $J_{F_{LL}}$. These limiters are mainly associated with the numerical representation capability of each software used in the numerical solution, as well as the order of magnitude of the type of data under analysis. Additionally, the numerical solution must always be tested by conditions defined by the prefixed tolerances: δ related to the value $|\theta^{k+1} - \theta^k|$; and φ related to the value of $|F_X(\theta^k)|$.

In this context, to include the existential mathematical conditions and overcome practical computational problems, an algorithm based on NR was developed to solve the expressions obtained by MLE of WS_{2P3P} with Type-I right censoring, as can be seen in Fig. 3. In this paper, the algorithm based on NR presented in Fig. 3 was implemented in the ®MATLAB software.

Step 1: Obtain $\theta^0 = [\alpha_1{}^0\ \beta_1{}^0\ \alpha_2{}^0\ \beta_2{}^0]^T$ through WPP.
Step 2: Set J_F_LI and J_F_LS limits to $J_F(\theta^k)$.
Step 3: While $k \le k_{max}$, $|F_X(\theta^k)| > \varphi$ and error$> \delta$, do:

3.1 Calculate: $\theta^{k+1} = \theta^k - J_F{}^{-1}(\theta^k)F_X(\theta^k)$.
3.2 Calculate: error= $|\theta^{k+1} - \theta^k|$.
3.3 Test: if $k > 1$, do:

3.3.1 Test: if $J_{F_{ij}}(\theta^k) < J_{F_{LL}}$ or $J_{F_{ij}}(\theta^k) > J_{F_{UL}}$, for i, j=1, ... , 4, do:

3.3.1.1 Update: $J_{F_{ij}}(\theta^k) = J_{F_{ij}}(\theta^{k-1})$.

3.4 Test: if $\theta^{k+1}{}_{ij} < 0$, for $i = 1, \ldots, 4$ and j =1, do:

3.4.1 Update: $\theta^k{}_{ij} = \theta^0{}_{ij}$.

3.5 Test: if $\beta_1{}^{k+1} < \beta_2{}^{k+1}$, do:

3.5.1 Assign: $\beta_1{}^{k+1} = c$; $\beta_1{}^{k+1} = \beta_2{}^{k+1}$; and $\beta_2{}^{k+1} = c$.

Step 4: For when the solution is converged, do:

4.1 Display θ^{k+1}.

Fig. 3. Algorithm based on NR for numerical solution of equations obtained from MLE.

V. Real Case Study

In this section, a case study based on real failure data from an engineering problem is presented to illustrate the reliability modeling using the WS_{2P3P} with parameters initially estimated through WPP and, later, through MLE with the assist of the algorithm based on NR.

A. Database

The data was obtained from around 180 vehicles recording a total of around one million miles over just over a year of use around the world [50]. The purpose of failure modeling is to obtain the closest reliability representation of the actual failure behavior of vehicles. Vehicles are reasonably complex mechanical/electrical equipment as they have thousands of components that are prone to failure.

In this work, the database used was composed of data from real failures in a component appointed "accelerators" present in vehicles. The data presented in Table II are from [50] and organized according to [4] which used [2] and used WPP to parameter estimates of WS_{2P3P}. The data provides the distances traveled (in thousands of kilometers) before failure, or the item being suspended prior to failure, for a pre-production general purpose freight transport vehicle. Therefore, the independent variable is the distance covered, with 1000 km being the unit of measure [4].

TABLE II
Distance Traveled Before Failure or Censoring (Denoted by +) [4].

0.478	0.959	1.847 +	3.904	6.711 +
0.484 +	1.071 +	2.400	4.443 +	6.835 +
0.583	1.318 +	2.550 +	4.829	6.947 +
0.626 +	1.377	2.568 +	5.328	7.878 +
0.753	1.472 +	2.639	5.562	7.884 +
0.753	1.534	2.944	5.900 +	10.263 +
0.801	1.579 +	2.981	6.122	11.019
0.834	1.610 +	3.392	6.226 +	12.986
0.850 +	1.729 +	3.393	6.331	13.103 +
0.944	1.792 +	3.791 +	6.531	23.245 +

The data presented in Table II are considered to be of medium size. For analysis purposes in this research, after ordering the data, it was observed that que n=50 and r=25.

B. Parameter Estimates

The parameter estimates of WS_{2P3P} using WPP resulted in α_1=1190, β_1 =4.68, α_2=8720, β_2 =0.699, γ =511 and t_s =601, as also presented by Murthy *et al.* [4]. In Fig. 4 the plotting of data by WPP can be observed using the procedures of Sec. II-A.

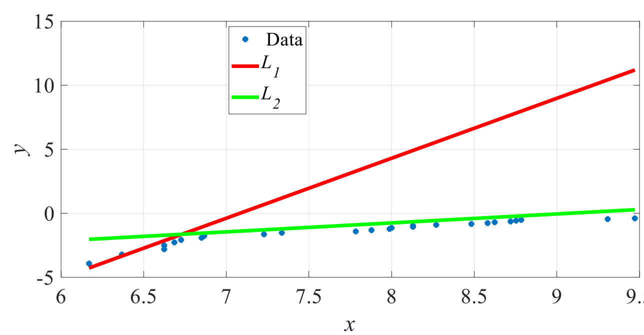

Fig. 4. Graphic representation of y versus x in WPP with lines L_1 and L_2 for WS_{2P3P}.

Now, the values estimated by WPP are used as initial values for the numerical solution based on the MLE and the algorithm based on the NR method. For this purpose, in Table III we presented the prefixed tolerances δ and φ, the limits of the Jacobian matrix $J_{F_{UL}}$ and $J_{F_{LL}}$ and the maximum number of iterations k_{max} are presented. In Table III also we presented are the estimates of the independent parameters $\overrightarrow{\alpha_1}$, $\overrightarrow{\beta_1}$, $\overrightarrow{\alpha_2}$, $\overrightarrow{\beta_2}$ which satisfy the system of equations (29) to (30), as well as the dependent parameters $\vec{\gamma}$ and $\overrightarrow{t_s}$.

TABLE III
PARAMETER ESTIMATES OF WS_{2P3P} CONSIDERING MLE AND AN ALGORITHM BASED ON NR.

Algorithm Settings					
δ	φ	$J_{F\,UL}$	$J_{F\,LL}$	k_{max}	
10^{-8}	10^{-10}	10^{15}	10^{-15}	5000	
Independent Parameters				Dependent Parameters	
$\overrightarrow{\alpha_1}$	$\overrightarrow{\beta_1}$	$\overrightarrow{\alpha_2}$	$\overrightarrow{\beta_2}$	$\vec{\gamma}$	$\overrightarrow{t_s}$
735	10.77	23842	0.53	499.42	525.27

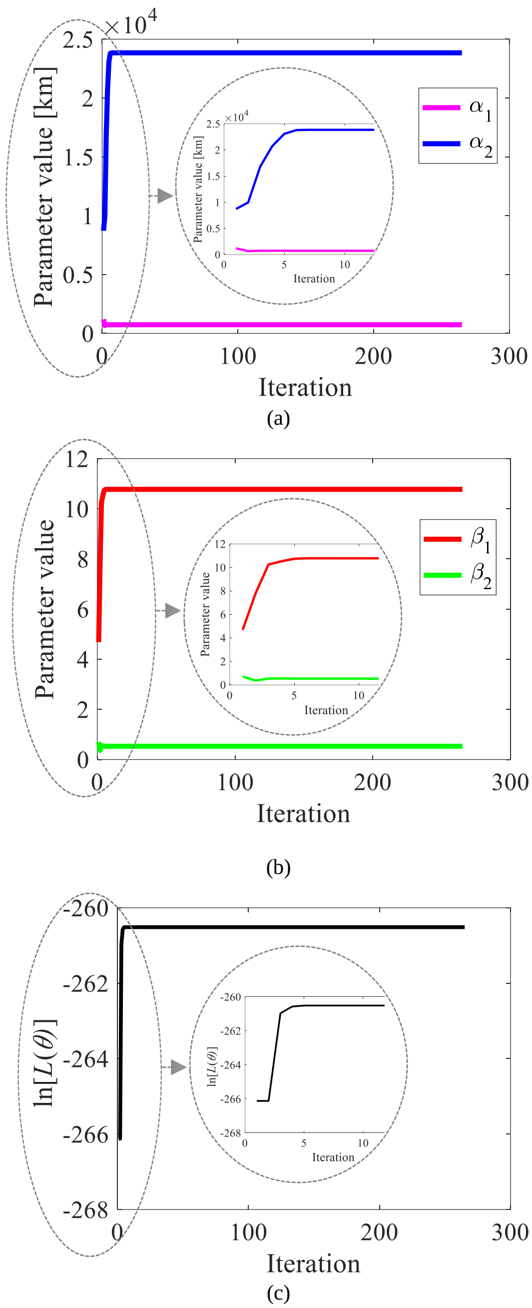

Fig. 5. Iteration versus the behavior of: (a) α_1 and α_2; (b) β_1 and β_2; and (c) $\ln[L(\theta)]$.

Complementarily, the behavior of the parameters as a function of the number of iterations, as well as the values of $\ln[L(\theta)]$ can be seen in Fig. 5. It is noteworthy that for the case under study convergence was achieved in iteration number 265 and resulted in $\ln[L(\vec{\theta})]$=-260.51. It can be observed that the convergence behavior in the parameter estimates and in the $\ln[L(\theta)]$ value was quickly reached, needing more interactions just to reach the prefixed tolerances.

For purposes of comparison and visualization of the graphic adjustment, in Fig. 6 the data and lines L_1 and L_2 are plotted considering the parameters estimated by MLE with the aid of the algorithm based on NR, as shown in Table III. Notably, it is possible to observe a proper fit.

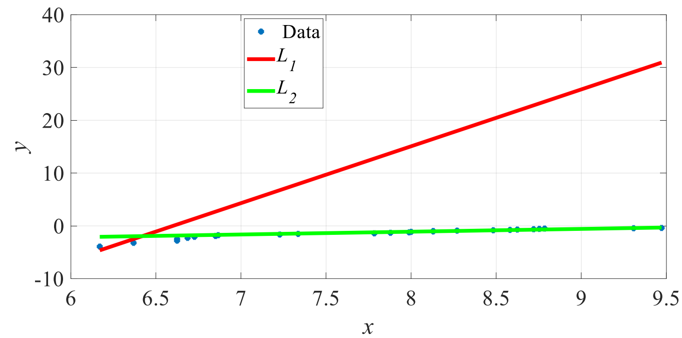

Fig. 6. Graphic representation of y versus x in WPP, with lines L_1 and L_2, for WS_{2P3P} considering the parameters estimated by MLE and NR.

C. *Reliability Representations*

The reliability representations $F(t)$, $R(t)$, $f(t)$ and $h(t)$ considering the empirical data, the parameters estimated only by WPP and estimated by MLE, with the assist of the algorithm based on NR, as shown in Figs. 7 and 8.

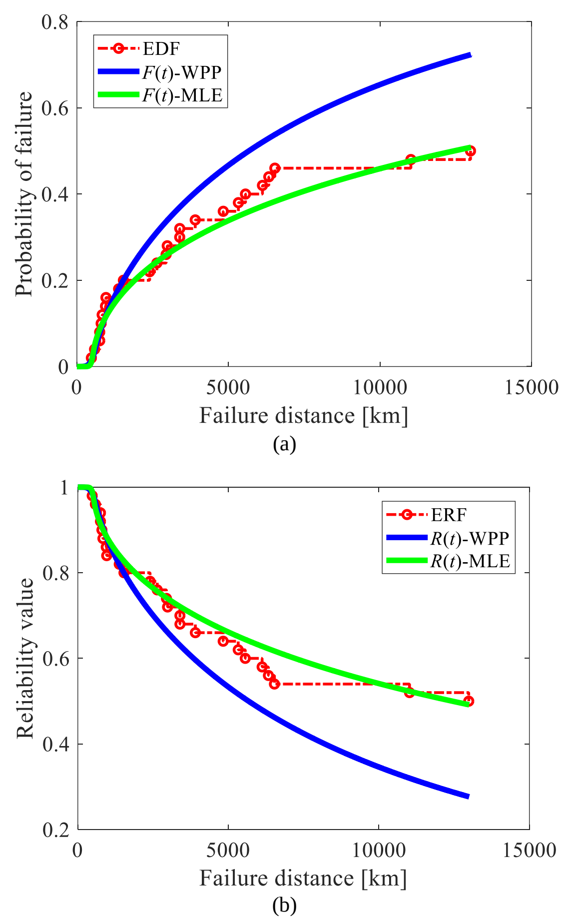

Fig. 7. (a) ECDF and assumed CDF; (b) ERF and assumed RF.

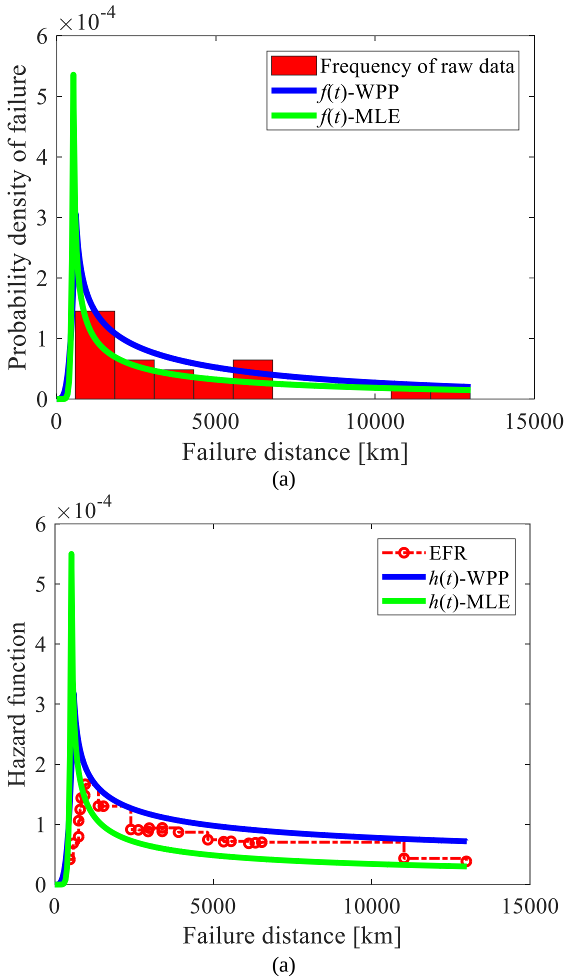

Fig. 8. (a) EPDF and assumed PDF; (b) EHF and assumed HF.

In Fig. 7 it can be seen that both the estimates by WPP and by MLE result in visually adequate modeling of the initial behaviors of empirical cumulative density function (ECDF) and empirical reliability function (ERF). When analyzing the period from sectioning to the end of the observation, it is possible to visually verify that the modeling obtained by MLE is closer to the empirical values. These behaviors are also obtained by observing the modeling of the empirical probability density function (EPDF) and empirical hazard function (EHF) presented in Fig. 8. In addition, the characteristic of upside-down bathtub curve of EHF is properly modeled by the $h(t)$ of $\mathrm{WS_{2P3P}}$.

Therefore, when analyzing the modeling of $\mathrm{WS_{2P3P}}$ with the parameters estimated by MLE, it was found, in addition to the maximization of the likelihood function, a better graphical modeling of the reliability representations $F(t)$, $R(t)$, $f(t)$ and $h(t)$.

VI. Conclusion

In the literature, the parameter estimates of $\mathrm{WS_{2P3P}}$ has been done only by WPP configuring a non-robust estimation. The literature also points out: the capability the modelling reliability problems with failure rates of complex behavior (increasing, increasing and constant, decreasing, decreasing and constant, and upside-down bathtub curve) by $\mathrm{WS_{2P3P}}$; the use of MLE and the NR method for estimating parameters of complex models, involving more than one model, with at least one Weibull distribution; and a tendency to use Type-I right censoring as one of the most practical and popular life tests.

In this motivational context, in this paper was presented a research that provides robustness and improvement in the parameter estimates of $\mathrm{WS_{2P3P}}$, as well as in the reliability modeling with Type-I right censored data, arising from modeling problems in different areas of knowledge in which the EFR presents one sectioning in its behavior and $\beta_1 > \beta_2$.

Additionally, we presented guidelines for initial estimate of the parameters of $\mathrm{WS_{2P3P}}$ with Type-I right censored data through WPP. Subsequently, we presented a mathematical development based on MLE for parameter estimates of $\mathrm{WS_{2P3P}}$ with Type-I right censored data. To assist in the numerical solution of the sets of equations resulting from the MLE, an algorithm based on the NR method was developed, implemented and effectively applied.

Furthermore, through a real case study containing failure data of a vehicle component in an engineering problem, it was possible to evaluate the applicability of $\mathrm{WS_{2P3P}}$. This also made it possible to validate the parameter estimates by MLE with the assist of the algorithm based on NR.

In future works, we intend to improve the estimates of the sectional model involving two distributions of two-parameter Weibull since it is able to model failure rates that present one sectioning and $\beta_1 < \beta_2$, complementing the insufficiencies of $\mathrm{WS_{2P3P}}$. Moreover, comparative studies between the improved sectional models may be approached, including analysis of confidence intervals.

Appendix A

This appendix presents the second order partial derivatives of D_1 in relation to the independent parameters of $\mathrm{WS_{2P3P}}$, as can be seen in (A.1) to (A.8).

$$\frac{\partial^2 D_1}{\partial {\alpha_1}^2} = \frac{\beta_1}{{\alpha_1}^2}(1+\beta_1)D_1. \tag{A.1}$$

$$\frac{\partial^2 D_1}{\partial \alpha_1 \partial \beta_1} = -\frac{1}{\alpha_1}\left[1+\beta_1 \ln\left(\frac{t_i}{\alpha_1}\right)\right] D_1. \tag{A.2}$$

$$\frac{\partial^2 D_1}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 D_1}{\partial \alpha_1 \partial \beta_2} = 0. \tag{A.3}$$

$$\frac{\partial^2 D_1}{\partial \beta_1 \partial \alpha_1} = D_1\left\{\left[\ln\left(\frac{t_i}{\alpha_1}\right)\right]^2 - \frac{1}{\alpha_1}\right\}. \tag{A.4}$$

$$\frac{\partial^2 D_1}{\partial {\beta_1}^2} = D_1\left[\ln\left(\frac{t_i}{\alpha_1}\right)\right]^2. \tag{A.5}$$

$$\frac{\partial^2 D_1}{\partial \beta_1 \partial \alpha_2} = \frac{\partial^2 D_1}{\partial \beta_1 \partial \beta_2} = 0. \tag{A.6}$$

$$\frac{\partial^2 D_1}{\partial \alpha_2 \partial \alpha_1} = \frac{\partial^2 D_1}{\partial \alpha_2 \partial \beta_1} = \frac{\partial^2 D_1}{\partial {\alpha_2}^2} = \frac{\partial^2 D_1}{\partial \alpha_2 \partial \beta_2} = 0. \tag{A.7}$$

$$\frac{\partial^2 D_1}{\partial \beta_2 \partial \alpha_1} = \frac{\partial^2 D_1}{\partial \beta_2 \partial \beta_1} = \frac{\partial^2 D_1}{\partial \beta_2 \partial \alpha_2} = \frac{\partial^2 D_1}{\partial {\beta_2}^2} = 0. \tag{A.8}$$

APPENDIX B

This appendix presents the second order partial derivatives of D_2 in relation to the independent parameters of $\mathrm{WS}_{\mathrm{2P3P}}$, as can be seen in (B.1) to (B.8).

$$\frac{\partial^2 D_2}{\partial \alpha_1{}^2} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial D_2}{\partial \alpha_1} + D_2\left[\frac{\partial^2 \gamma}{\partial \alpha_1{}^2} + \left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial \gamma}{\partial \alpha_1}\right)^2\right]\right\}. \tag{B.1}$$

$$\frac{\partial^2 D_2}{\partial \alpha_1 \partial \beta_1} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial D_2}{\partial \beta_1} + D_2\left[\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \beta_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \beta_1}\right]\right\}. \tag{B.2}$$

$$\frac{\partial^2 D_2}{\partial \alpha_1 \partial \alpha_2} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial D_2}{\partial \alpha_2} + D_2\left[\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \alpha_2}\right]\right\}. \tag{B.3}$$

$$\frac{\partial^2 D_2}{\partial \alpha_1 \partial \beta_2} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial D_2}{\partial \beta_2} + D_2\left[\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \beta_2} + \frac{\partial \gamma}{\partial \alpha_1}\left(\frac{1}{\beta_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right)\right]\right\}. \tag{B.4}$$

$$\frac{\partial^2 D_2}{\partial \beta_1 \partial \alpha_1} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \beta_1}\frac{\partial D_2}{\partial \alpha_1} + D_2\left[\frac{\partial^2 \gamma}{\partial \beta_1 \partial \alpha_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \beta_1}\right]\right\}. \tag{B.5}$$

$$\frac{\partial^2 D_2}{\partial \beta_1{}^2} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \beta_1}\frac{\partial D_2}{\partial \beta_1} + D_2\left[\frac{\partial^2 \gamma}{\partial \beta_1{}^2} + \left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial \gamma}{\partial \beta_1}\right)^2\right]\right\}. \tag{B.6}$$

$$\frac{\partial^2 D_2}{\partial \beta_1 \partial \alpha_2} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \beta_1}\frac{\partial D_2}{\partial \alpha_2} + D_2\left[\frac{\partial^2 \gamma}{\partial \beta_1 \partial \alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_1}\frac{\partial \gamma}{\partial \alpha_2}\right]\right\}. \tag{B.7}$$

$$\frac{\partial^2 D_2}{\partial \beta_1 \partial \beta_2} = -\left(\frac{\beta_2}{t_i-\gamma}\right)\left\{\frac{\partial \gamma}{\partial \beta_1}\frac{\partial D_2}{\partial \beta_2} + D_2\left[\frac{\partial^2 \gamma}{\partial \beta_1 \partial \beta_2} + \frac{\partial \gamma}{\partial \beta_1}\left(\frac{1}{\beta_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right)\right]\right\}. \tag{B.8}$$

$$\frac{\partial^2 D_2}{\partial \alpha_2 \partial \alpha_1} = -\beta_2\left\{\frac{\partial D_2}{\partial \alpha_1}\left[\frac{1}{\alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\right] + \left(\frac{D_2}{t_i-\gamma}\right)\left[\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \alpha_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \alpha_2}\right]\right\}. \tag{B.9}$$

$$\frac{\partial^2 D_2}{\partial \alpha_2 \partial \beta_1} = -\beta_2\left\{\frac{\partial D_2}{\partial \beta_1}\left[\frac{1}{\alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\right] + \left(\frac{D_2}{t_i-\gamma}\right)\left[\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \beta_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_1}\frac{\partial \gamma}{\partial \alpha_2}\right]\right\}. \tag{B.10}$$

$$\frac{\partial^2 D_2}{\partial \alpha_2{}^2} = -\beta_2\left\{\frac{\partial D_2}{\partial \alpha_2}\left[\frac{1}{\alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\right] + D_2\left[\left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial^2 \gamma}{\partial \alpha_2{}^2} + \left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial \gamma}{\partial \alpha_2}\right)^2\right) - \frac{1}{\alpha_2{}^2}\right]\right\}. \tag{B.11}$$

$$\frac{\partial^2 D_2}{\partial \alpha_2 \partial \beta_2} = -\beta_2\left\{\frac{\partial D_2}{\partial \beta_2}\left[\frac{1}{\alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\right] + \left(\frac{D_2}{t_i-\gamma}\right)\left[\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \beta_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\frac{\partial \gamma}{\partial \beta_2}\right]\right\} - D_2\left[\frac{1}{\alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\right]. \tag{B.12}$$

$$\frac{\partial^2 D_2}{\partial \beta_2 \partial \alpha_1} = \frac{\partial D_2}{\partial \alpha_1}\left[\ln\left(\frac{t_i-\gamma}{\alpha_2}\right) - \left(\frac{\beta_2}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right] - \left(\frac{D_2}{t_i-\gamma}\right)\left[\frac{\partial \gamma}{\partial \alpha_1} + \beta_2\left(\frac{\partial^2 \gamma}{\partial \beta_2 \partial \alpha_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_1}\frac{\partial \gamma}{\partial \beta_2}\right)\right]. \tag{B.13}$$

$$\frac{\partial^2 D_2}{\partial \beta_2 \partial \beta_1} = \frac{\partial D_2}{\partial \beta_1}\left[\ln\left(\frac{t_i-\gamma}{\alpha_2}\right) - \left(\frac{\beta_2}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right] - \left(\frac{D_2}{t_i-\gamma}\right)\left[\frac{\partial \gamma}{\partial \beta_1} + \beta_2\left(\frac{\partial^2 \gamma}{\partial \beta_2 \partial \beta_1} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_1}\frac{\partial \gamma}{\partial \beta_2}\right)\right]. \tag{B.14}$$

$$\frac{\partial^2 D_2}{\partial \beta_2 \partial \alpha_2} = \frac{\partial D_2}{\partial \alpha_2}\left[\ln\left(\frac{t_i-\gamma}{\alpha_2}\right) - \left(\frac{\beta_2}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right] - D_2\left\{\frac{1}{\alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\left[\frac{\partial \gamma}{\partial \alpha_2} + \beta_2\left(\frac{\partial^2 \gamma}{\partial \beta_2 \partial \alpha_2} + \left(\frac{1}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \alpha_2}\frac{\partial \gamma}{\partial \beta_2}\right)\right]\right\} \tag{B.15}$$

$$\frac{\partial^2 D_2}{\partial \beta_2{}^2} = \frac{\partial D_2}{\partial \beta_2}\left[\ln\left(\frac{t_i-\gamma}{\alpha_2}\right) - \left(\frac{\beta_2}{t_i-\gamma}\right)\frac{\partial \gamma}{\partial \beta_2}\right] - \left(\frac{D_2}{t_i-\gamma}\right)\left[2\frac{\partial \gamma}{\partial \beta_2} + \beta_2\left(\frac{\partial^2 \gamma}{\partial \beta_2{}^2} + \left(\frac{1}{t_i-\gamma}\right)\left(\frac{\partial \gamma}{\partial \beta_2}\right)^2\right)\right]. \tag{B.16}$$

APPENDIX C

This appendix presents the second order partial derivatives of γ in relation to the independent parameters of $\mathrm{WS}_{\mathrm{2P3P}}$, as can be seen in (C.1) to (C.16).

$$\frac{\partial^2 \gamma}{\partial \alpha_1{}^2} = \frac{\beta_1}{\alpha_1}B\left(\frac{\partial \gamma}{\partial \alpha_1} - \frac{\gamma}{\alpha_1}\right). \tag{C.1}$$

$$\frac{\partial^2 \gamma}{\alpha_1 \partial \beta_1} = \frac{B}{\alpha_1}\left[\beta_1\left(\frac{\partial \gamma}{\partial \beta_1} - B\gamma\right) + \gamma\right]. \tag{C.2}$$

$$\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \alpha_2} = \frac{\beta_1}{\alpha_1}B\frac{\partial \gamma}{\partial \alpha_2}. \tag{C.3}$$

$$\frac{\partial^2 \gamma}{\partial \alpha_1 \partial \beta_2} = \frac{\beta_1}{\alpha_1}B\left(\frac{\partial \gamma}{\partial \beta_2} + B\gamma\right). \tag{C.4}$$

$$\frac{\partial^2 \gamma}{\partial \beta_1 \partial \alpha_1} = B\left[\gamma\left(\frac{1}{\alpha_1} - \frac{B}{A}\frac{\partial A}{\partial \alpha_1}\right) + \frac{\partial \gamma}{\partial \alpha_1}\left(\ln \alpha_1 - \frac{\beta_2}{\beta_1} - B\ln A\right)\right] + \frac{\beta_2}{\beta_1{}^2}\frac{\partial t_s}{\partial \alpha_1}. \tag{C.5}$$

$$\frac{\partial^2 \gamma}{\partial \beta_1{}^2} = B\left\{\gamma\left[\frac{\beta_2}{\beta_1{}^2} - B\left(\frac{1}{A}\frac{\partial A}{\partial \beta_1} - B\ln A\right)\right] + \left(\frac{\partial \gamma}{\partial \beta_1} - B\gamma\right)\left(\ln \alpha_1 - \frac{\beta_2}{\beta_1} - B\ln A\right)\right\} + \frac{\beta_2}{\beta_1{}^2}\left(\frac{\partial t_s}{\partial \beta_1} - \frac{2}{\beta_1}t_s\right). \tag{C.6}$$

$$\frac{\partial^2 \gamma}{\partial \beta_1 \partial \alpha_2} = B\left[-\gamma\frac{B}{A}\frac{\partial A}{\partial \alpha_2} + \frac{\partial \gamma}{\partial \alpha_2}\left(\ln \alpha_1 - \frac{\beta_2}{\beta_1} - B\ln A\right)\right] + \frac{\beta_2}{\beta_1{}^2}\frac{\partial t_s}{\partial \alpha_2}. \tag{C.7}$$

$$\frac{\partial^2 \gamma}{\partial \beta_1 \partial \beta_2} = B\left\{-\gamma\left[\frac{1}{\beta_1} + B\left(\frac{1}{A}\frac{\partial A}{\partial \beta_2} + B\ln A\right)\right] + \left(\frac{\partial \gamma}{\partial \beta_2} + B\gamma\right)\left(\ln \alpha_1 - \frac{\beta_2}{\beta_1} - B\ln A\right)\right\} + \frac{1}{\beta_1{}^2}\left(\beta_2\frac{\partial t_s}{\partial \beta_2} + t_s\right). \tag{C.8}$$

$$\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \alpha_1} = -\frac{\beta_2}{\alpha_2}B\left(\frac{\partial \gamma}{\partial \alpha_1}\right). \tag{C.9}$$

$$\frac{\partial^2 \gamma}{\partial \alpha_2 \partial \beta_1} = -\frac{\beta_2}{\alpha_2}B\left(\frac{\partial \gamma}{\partial \beta_1} - B\gamma\right). \tag{C.10}$$

$$\frac{\partial^2\gamma}{\partial{\alpha_2}^2} = -\frac{\beta_2}{\alpha_2} B\left(\frac{\partial\gamma}{\partial\alpha_2} - \frac{\gamma}{\alpha_2}\right). \tag{C.11}$$

$$\frac{\partial^2\gamma}{\partial\alpha_2\partial\beta_2} = -\frac{B}{\alpha_2}\left[\beta_1\left(\frac{\partial\gamma}{\partial\beta_2} + B\gamma\right) + \gamma\right]. \tag{C.12}$$

$$\frac{\partial^2\gamma}{\partial\beta_2\partial\alpha_1} = B\left[\gamma\frac{B}{A}\frac{\partial A}{\partial\alpha_1} + \frac{\partial\gamma}{\partial\alpha_1}\left(1 + \ln\left(\frac{\beta_2}{\beta_1\alpha_2}\right) + B\ln A\right)\right] - \frac{1}{\beta_1}\frac{\partial t_s}{\partial\alpha_1}. \tag{C.13}$$

$$\frac{\partial^2\gamma}{\partial\beta_2\partial\beta_1} = B\left\{\gamma\left[-\frac{1}{\beta_1} + B\left(\frac{1}{A}\frac{\partial A}{\partial\beta_1} - B\ln A\right)\right] + \left(\frac{\partial\gamma}{\partial\beta_1} - B\gamma\right)\left(1 + \ln\left(\frac{\beta_2}{\beta_1\alpha_2}\right) + B\ln A\right)\right\} - \frac{1}{\beta_1}\left(\frac{\partial t_s}{\partial\beta_1} - \frac{t_s}{\beta_1}\right). \tag{C.14}$$

$$\frac{\partial^2\gamma}{\partial\beta_2\partial\alpha_2} = B\left[\gamma\left(-\frac{1}{\alpha_2} + \frac{B}{A}\frac{\partial A}{\partial\alpha_2}\right) + \frac{\partial\gamma}{\partial\alpha_2}\left(1 + \ln\left(\frac{\beta_2}{\beta_1\alpha_2}\right) + B\ln A\right)\right] - \frac{1}{\beta_1}\frac{\partial t_s}{\partial\alpha_2}. \tag{C.15}$$

$$\frac{\partial^2\gamma}{\partial{\beta_2}^2} = B\left\{-\gamma\left[\frac{1}{\beta_2} + B\left(\frac{1}{A}\frac{\partial A}{\partial\beta_2} + B\ln A\right)\right] + \left(\frac{\partial\gamma}{\partial\beta_2} + B\gamma\right)\left(1 + \ln\left(\frac{\beta_2}{\beta_1\alpha_2}\right) + B\ln A\right)\right\} - \frac{1}{\beta_1}\frac{\partial t_s}{\partial\beta_2}. \tag{C.16}$$

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References

[1] D. N. P. Murthy, M. Xie, and R. Jiang, "Type III (d) Weibull models," *Weibull Models*, New Jersey, EUA: John Wiley and Sons, 2004, pp. 209-211.

[2] R. Jiang and D. N. P. Murthy, "Reliability modeling involving two Weibull distributions," *Reliab. Eng. Syst. Saf.*, vol. 47, no. 3, pp. 187-198, 1995, 10.1016/0951-8320(94)00045-P.

[3] D. N. P. Murthy and R. Jiang, "Parametric study of sectional models involving two Weibull distributions," *Reliab. Eng. Syst. Saf.*, vol. 56, no. 2, pp. 151-159, May, 1997, 10.1016/S0951-8320(96)00114-7.

[4] D. N. P. Murthy, M. Bulmer, and J. A. Eccleston, "Weibull model selection for reliability modelling," *Reliab. Eng. Syst. Saf.*, vol. 86, no. 3, pp. 257-267, Dec., 2004, 10.1016/j.ress.2004.01.014.

[5] T. Zhang and R. Dwight, "Choosing an optimal model for failure data analysis by graphical approach," *Reliab. Eng. Syst. Saf.*, vol. 115, pp. 111-123, Jul., 2013, 10.1016/j.ress.2013.02.004.

[6] L. F. Zhang, M. Xie, and L. C. Tang, "A study of two estimation approaches for parameters of Weibull distribution based on WPP," *Reliab. Eng. Syst. Saf.*, vol. 92, no. 3, pp. 360-368, Mar., 2007, 10.1016/j.ress.2006.04.008.

[7] A. C. Cohen, "Maximum likelihood estimation in the Weibull distribution based on complete and on censored Samples," *Technometrics*, vol. 7, no. 4, pp. 579-588, Nov., 1965, 10.2307/1266397.

[8] H. L. Harter and A. H. Moore, "Maximum-likelihood estimation of the parameters of Gamma and Weibull populations from complete and from censored samples," *Technometrics*, vol. 7, no. 4, pp. 639-643, Nov., 1965, 10.2307/1266401.

[9] H. Rockette, C. Antle, and L. A. Klimko, "Maximum likelihood estimation with the Weibull model," *J. Am. Stat. Assoc.*, vol. 69, no. 345, pp. 246-249, Mar., 1974, 10.2307/2285536.

[10] M. Sirvanci and G. Yang, "Estimation of the Weibull Parameters under Type I Censoring," *J. Am. Stat. Assoc.*, vol. 79, no. 385, pp. 183-187, Mar, 1984, 10.1080/01621459.1984.10477082.

[11] M. E. Flygare, J. A. Austin, and R. M. Buckwalter, "Maximum likelihood estimation for the 2-parameter Weibull distribution based on interval-data," *IEEE Trans. Reliab.*, vol. 34, no. 1, pp. 57-59, Apr., 1985, 10.1109/TR.1985.5221930.

[12] P. M. Odell, K. M. Anderson, and R. B. D'Agostino, "Maximum likelihood estimation for interval-censored data using a Weibull-based Accelerated failure time model," *Biometrics*, vol. 48, no. 3, pp. 951-959, Sep., 1992, 10.2307/2532360.

[13] J. B. Keats, F. R. Lawrence, and F. K. Wang, "Weibull maximum likelihood parameter estimates with censored data," *J. Qual. Technol.*, vol. 29, no. 1, pp. 105-110, 1997, 10.1080/00224065.1997.11979730.

[14] N. Balakrishnan and M. Kateri, "On the maximum likelihood estimation of parameters of Weibull distribution based on complete and censored data," *Stat. Probab. Lett.*, vol. 78, no. 17, pp. 2971-2975, Dec., 2008, 10.1016/j.spl.2008.05.019.

[15] H. K. T. NG and Z. Wang, "Statistical estimation for the parameters of Weibull distribution based on progressively Type-I interval censored sample," *J. Stat. Comput. Simul.* , vol. 79, no. 2, pp. 145-159, Dec., 2008, 10.1080/00949650701648822.

[16] C. Cheng, J. Chen, and Z. Li, "A new algorithm for maximum likelihood estimation with progressive Type-I interval censored data," *Commun. Stat. Simul. Comput.* , vol. 39, no. 4, pp. 750-766, Apr., 2010, 10.1080/03610911003637406.

[17] L. Li, H. K. NG, A. H. Algarni, A. M. Almarashi, and Z. A. Abo-Eleneen, "A model-ranking approach for estimation based on accelerated degradation test data," *IEEE Trans. Reliab.*, vol. 69, no. 2, pp. 484-496, Jun., 2020, 10.1109/TR.2020.2976786.

[18] K. Qiu, Z. Zheng, K. S. Trivedi, and B. Yin, "Stress testing with influencing factors to accelerate data race software failures," *IEEE Trans. Reliab.*, vol. 69, no. 1, pp. 3-21, Mar., 2020, 10.1109/TR.2019.2895052.

[19] G. H. Lemon, "Maximum likelihood estimation for the three parameter Weibull distribution based on censored samples," *Technometrics*, vol. 17, no. 2, pp. 247-254, May, 1975, 10.2307/1268358.

[20] N. A. Archer, "A computational technique for maximum likelihood estimation with Weibull models," *IEEE Trans. Reliab.*, vol. 29, no. 1, pp. 57-62, Apr, 1980, 10.1109/TR.1980.5220713.

[21] E. Gourdin, P. Hansen, and B. Jaumard, "Finding maximum likelihood estimators for the three-parameter weibull distribution," *J. Glob. Optm.*, vol. 5, pp. 373-397, Dec., 1994, 10.1007/BF01096687.

[22] H. Hirose, "Maximum likelihood estimation in the 3-parameter Weibull distribution: a look through the generalized extreme-value distribution," *IEEE Trans. Dielectr. Electr. Insul.*, vol. 3, no.1, pp. 43-55, Feb., 1996, 10.1109/94.485513.

[23] A. A. Ismail, "Estimating the parameters of Weibull distribution and the acceleration factor from hybrid partially accelerated life test," *Appl. Math. Model.*, vol. 36, no.7, pp. 2920-2925, Jul., 2012, 10.1016/j.apm.2011.09.083.

[24] F.Yang, H. Ren, and Z. Hu, "Maximum likelihood estimation for three-parameter Weibull distribution using evolutionary strategy," *Math. Probl. Eng.*, vol. 3, no.1, pp. 43-55, May., 2019, 10.1155/2019/6281781.

[25] J. I. McCool, "Estimation of Weibull parameters with competing-mode censoring," *IEEE Trans. Reliab.*, vol. R-25, no. 1, pp. 25-31, Apr., 1976, 10.1109/TR.1976.5214946.

[26] T. Shioka and Y. Nonaka, "Maximum likelihood estimation of Weibull parameters for two independent competing risk," *IEEE Trans. Reliab.*, vol. 40, no. 1, pp. 71-74, Apr., 1991, 10.1109/24.75338.

[27] F. Pascual, "Accelerated life test planning with independent Weibull competing risks with known shape parameter," *IEEE Trans. Reliab.*, vol. 56, no. 1, pp. 85-93, Mar., 2007, 10.1109/TR.2006.890885.

[28] S. J. Wu and S. R. Huang, "Planning progressive Type-I interval censoring life tests with competing risks," *IEEE Trans. Reliab.*, vol. 63, no. 2, pp. 511-522, Jun., 2014, 10.1109/TR.2014.2313708.

[29] A. Zhang, Y. Shi, and M. Wu, "Statistical inference for competing risks model in step-stress partially accelerated life tests with progressively Type-I hybrid censored Weibull life data," *J. Comput. Appl. Math.*, vol. 297, no. 1, pp. 65-74, May, 2016, 10.1016/j.cam.2015.11.002.

[30] L. Wang, "Inference for Weibull competing risks data under generalized progressive hybrid censoring," *IEEE Trans. Reliab.*, vol. 67, no. 3, pp. 998-1007, Sept., 2018, 10.1109/TR.2018.2828436.

[31] D. Samanta, A. Gupta, and D. Kundu, "Analysis of Weibull step-stress model in presence of competing risk," *IEEE Trans. Reliab.*, vol. 68, no. 2, pp. 420-438, Jun., 2019, 10.1109/TR.2019.2896319.

[32] T. H. Fan, Y. F. Wang, and S. K. Ju, "A competing risks model with multiply censored reliability data under multivariate Weibull distributions," *IEEE Trans. Reliab.*, vol. 68, no. 2, pp. 462-475, Jun., 2019, 10.1109/TR.2019.2907518.

[33] S. W. Cheng and J. C. Fu, "Estimation of mixed Weibull parameters in life testing," *IEEE Trans. Reliab.*, vol. R-31, no. 4, pp. 377-381, Oct., 1982, 10.1109/TR.1982.5221382.

[34] S. Jiang and D. Kececioglu, "Maximum likelihood estimates, from censored data, for mixed-Weibull distributions," *IEEE Trans. Reliab.*, vol. 41, no. 2, pp. 248-255, Jun., 1992, 10.1109/24.257791.

[35] R. C. Jansen, "Maximum likelihood in a generalized linear finite mixture model by using the EM algorithm," *Biometrics*, vol. 49, no. 1, pp. 227-231, Mar., 1993, 10.2307/2532616.

[36] A. M. Rezali and A. A. A. Wakeel, "Mixture Weibull distributions for fitting failure times data," *Appl. Math. Comput.* , vol. 219, no. 24, pp. 11358-11364, Aug., 2013, 10.1016/j.amc.2013.05.062.

[37] E. E. Elmahdy and A. Aboutahoun, "A new approach for parameter estimation of finite Weibull mixture distributions for reliability modeling," *Appl. Math. Model.*, vol. 37, no. 4, pp. 1800-1810, Feb., 2013, 10.1016/j.apm.2012.04.023.

[38] E. M. Assis, G. A. C. Lima, A. Prestes, F. Marinho, and L. A. N. Costa, "*q*-Weibull applied to brazilian hydropower equipment," *IEEE Trans. Reliab.*, vol. 68, no. 1, pp. 122-132, Mar., 2019, 10.1109/TR.2018.2864550.

[39] H. Rami, A. D. E. Maliani, and M. E. Hassouni, "A finite mixture of Weibull-based statistical model for texture retrieval in the complex wavelet domain," *IEEE Access*, vol. 7, no. 1, pp. 130144-130155, Sept., 2019, 10.1109/ACCESS.2019.2939466.

[40] W. Nelson, "Probability plots of singly censored data," *Applied Life Data Analysis*, EUA: John Wiley and Sons, 1982, pp. 120-121.

[41] N. Balakrishnan and E. Cramer, "Progressive censoring: data and models," *The Art of Progressive Censoring: Applications to Reliability and Quality*, Basel, Switzerland: Birkhäuser, 2014, pp. 10-11.

[42] X. Jia, S. Nadarajah, and B. Guo, "Exact inference on Weibull parameters with multiply Type-I censored data," *IEEE Trans. Reliab.*, vol. 67, no. 2, pp. 432-445, Jun., 2018, 10.1109/TR.2018.2799967.

[43] Q. Zhao, X. Jia, and B. Guo, "Parameter estimation for the two-parameter exponentiated Weibull distribution based on multiply Type-I censored data," *IEEE Access*, vol. 7, pp. 45485-45493, Apr. 2019, 10.1109/ACCESS.2019.2909088.

[44] D. Han, "On the existence of the optimal step-stress accelerated life tests under progressive Type-I censoring," *IEEE Trans. Reliab.*, vol. 69, no. 3, pp. 903-915, Sept., 2020, 10.1109/TR.2019.2915773.

[45] L. Gao, Z. S. Ye, W. Chen, P. Qian, and J. Pan, "Accelerated life test planning for minimizing misclassification risks," *IEEE Trans. Reliab.*, vol. 70, no. 2, pp. 459-471, Jun., 2021, 10.1109/TR.2020.2969033.

[46] D. Han and T. Bai, "Parameter estimation using EM algorithm for lifetimes from step-stress and constant-stress accelerated life tests with interval monitoring," *IEEE Trans. Reliab.*, vol. 70, no. 1, pp. 49-64, Mar., 2021, 10.1109/TR.2019.2957710.

[47] J. P. Klein and M. L. Moeschberger, "Censoring and truncation," *Survival Analysis: Techniques for Censored and Truncated Data*, New York, EUA: Springer Science, 1997, pp. 69.

[48] E. T. Lee and J. W. Wang, "Appendix A: Newton-Raphson method," *Statistical Methods for Survival Data Analysis* , 3rd ed., New Jersey, EUA: John Wiley and Sons, 2003, pp. 428-432.

[49] A. Sintov, A. Borum, and T. Bretl, "Motion planning of fully actuated closed kinematic chains with revolute joints: a comparative analysis," *IEEE Robot. Autom. Lett.*, vol. 3, no. 4, pp. 2886-2893, Oct., 2018, 10.1109/LRA.2018.2846806.

[50] A. D. S Carter, "Design for reliability" and "Solutions to exercises," *Mechanical Reliability*, 2nd ed., London, UK: Macmillan Education, 1986, pp. 303-307, pp. 455-457.

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