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Bayesian and non-Bayesian inference and lifetime data applications for additive modified Weibull extension distribution with bathtub-shaped failure rate function

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SUPPLEMENTARY MATERIAL

Bayesian and non-Bayesian inference and lifetime data applications for additive modified Weibull extension distribution with bathtub-shaped failure rate function

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M1. LOG-LIKELIHOOD FUNCTION AND DERIVATIVES

The log-likelihood function of the additive modified Weibull extension (AMWE) distribution is

$$\ell(\varphi) = n\log\gamma - n\gamma\log\theta + (\gamma-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}(x_i/\theta)^{\gamma} + \sum_{i=1}^{n}\log(c_i) - \sum_{i=1}^{n}(\lambda a_i + a_i^{\alpha}), \tag{1}$$

where $\varphi = (\gamma, \alpha, \theta, \lambda)'$ is the parameter vector.

M1.2. First derivatives of the log-likelihood function

The estimate $\widehat{\varphi} = \left(\widehat{\gamma}, \widehat{\alpha}, \widehat{\theta}, \widehat{\lambda}\right)'$ of $\varphi = (\gamma, \alpha, \theta, \lambda)'$ is determined by maximizing the log-likelihood function ($\ell(\varphi)$) for each of the AMWE parameters. Thus, we have the following score functions.

$$\frac{\partial\ell}{\partial\lambda} = \sum_{i=1}^{n}\frac{1}{c_i} - \sum_{i=1}^{n}a_i, \tag{2}$$

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=1}^{n}\frac{a_i^{\alpha-1}}{c_i}\left(1+\alpha\log(a_i)\right) - \sum_{i=1}^{n}a_i^{\alpha}\log(a_i), \tag{3}$$

$$\frac{\partial\ell}{\partial\theta} = -\frac{n\gamma}{\theta} - \frac{\gamma}{\theta}\sum_{i=1}^{n}(x_i/\theta)^{\gamma} + \frac{\gamma\alpha(\alpha-1)}{\theta}\sum_{i=1}^{n}\frac{a_i^{\alpha}b_i}{a_i^{2}c_i} + \frac{\gamma}{\theta}\sum_{i=1}^{n}\frac{b_i}{c_i} \tag{4}$$

$$\frac{\partial\ell}{\partial\gamma} = \frac{n}{\gamma} - n\log\theta + \sum_{i=1}^{n}\log x_i + \alpha(\alpha-1)\sum_{i=1}^{n}\frac{a_i^{\alpha}b_i}{a_i^{2}c_i}\log(x_i/\theta)$$
$$+ \sum_{i=1}^{n}(x_i/\theta)^{\gamma}\log(x_i/\theta) - \sum_{i=1}^{n}b_i c_i\log(x_i/\theta), \tag{5}$$

where $a_i = e^{(x_i/\theta)^{\gamma}} - 1$, $b_i = (x_i/\theta)^{\gamma}e^{(x_i/\theta)^{\gamma}}$, and $c_i = \lambda + \alpha a_i^{\alpha-1}$. Solving Eqns (2) - (5) analytically may be intractable. Thus, a numerical approach is adopted to obtain the maximum likelihood estimates (MLEs) of the parameters $\varphi = (\gamma, \alpha, \theta, \lambda)'$ with a good set of initial values using **optim** function from R-base **stats** package in R statistical software.

M1.2: Second derivative of the log-likelihood function

$$\frac{\partial^2 \ell(\varphi)}{\partial \lambda^2} = -c_i^{-2}, \tag{6}$$

$$\frac{\partial^2 \ell(\varphi)}{\partial \alpha \partial \lambda} = \frac{a_i^{\alpha-1}(1+\alpha\log(a_i))}{c_i^2}, \tag{7}$$

$$\frac{\partial^2 \ell(\varphi)}{\partial \lambda \partial \theta} = -\frac{\gamma}{\theta}(1-\alpha(\alpha-1)a_i^{\alpha-2}c_i^{-2})b_i, \tag{8}$$

$$\frac{\partial^2 \ell(\varphi)}{\partial \lambda \partial \gamma} = (1-\alpha(\alpha-1)a_i^{\alpha-2}c_i^{-2})b_i\log(x_i/\theta), \tag{9}$$

$$\frac{\partial^2 \ell(\varphi)}{\partial \alpha^2} = (2+\alpha\log(a_i))\,a_i^{\alpha-1}\frac{\log(a_i)}{c_i} - a_i^{2(\alpha-1)}\left(\frac{1+\log(a_i)}{c}\right)^2 + a_i^{\alpha}\log^2(a_i), \tag{10}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varphi)}{\partial \alpha \partial \gamma} &= -\alpha(\alpha-1)(1+\alpha\log(a_i))\frac{b_i}{a_i c_i^2}a_i^{2(\alpha-1)}\log(x_i/\theta) + (1+\alpha\log(a_i))\frac{b_i}{a_i}a_i^{\alpha}\log(x_i/\theta)\\ &\quad + \frac{b_i}{a_i c_i}a_i^{\alpha-1}[\alpha+(\alpha-1)(1+\alpha\log(a_i))]\log(x_i/\theta),\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varphi)}{\partial \alpha \partial \theta} &= \frac{\gamma(\alpha-1)}{\theta}(1+\alpha\log(a_i))\frac{b_i}{a_i c_i^2}a_i^{2(\alpha-1)} - \frac{\gamma}{\theta}(1+\alpha\log(a_i))\frac{b_i}{a_i}a_i^{\alpha}\\ &\quad -\frac{\gamma}{\theta}[1+(\alpha-1)(1+\alpha\log(a_i))]\frac{b_i}{a_i}a_i^{\alpha-1},\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varphi)}{\partial \theta^2} &= \frac{n\gamma}{\theta^2} + \frac{\gamma\alpha(\alpha-1)}{\theta^2}\left[\frac{1}{a_i} + \gamma\left(\frac{\alpha-1}{a_i} + (x/\theta)^{\gamma} + 1\right) - \alpha\gamma(\alpha-1)\frac{a_i^{\alpha-3}b_i}{c_i}\right]\frac{a_i^{\alpha}b_i}{a_i c_i}\\ &\quad + \frac{\gamma^2\lambda}{\theta^2}\left[\frac{1}{\gamma} + 1 + (x_i/\theta)^{\gamma}\right]b_i + \frac{\gamma^2\alpha}{\theta^2}\left[\frac{1}{\gamma} + 1 + (x_i/\theta)^{\gamma} + \frac{(\alpha-1)}{a_i}\right]\frac{a_i^{\alpha}b_i}{a_i}\\ &\quad + \frac{\gamma^2}{\theta^2}\left[\frac{1}{\gamma} + 1 + (x/\theta)^{\gamma} - \frac{b_i}{a_i+1}\right]\frac{b_i}{a_i+1},\end{aligned} \tag{13}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varphi)}{\partial \gamma^2} &= -\frac{n}{\gamma^2} + \alpha(\alpha-1)\left[\left(1+(x_i/\theta)^{\gamma} + (\alpha-2)\frac{b_i}{a_i}\right)c_i - \alpha(\alpha-1)a_i^{\alpha-2}b_i\right]a_i^{\alpha}b_i\left(\frac{\log(x_i/\theta)}{a_i c_i}\right)^2\\ &\quad -\lambda(1+(x_i/\theta)^{\gamma})b_i\log^2(x_i/\theta) - \alpha\left[1+(x_i/\theta)^{\gamma} + (\alpha-1)\frac{b_i}{a_i}\right]a_i^{\alpha-1}b_i\log^2(x_i/\theta)\\ &\quad + b_i\left(\frac{\log(x_i/\theta)}{a_i+1}\right)^2[(a_i+1)(1+(x_i/\theta)^{\gamma}) - b_i],\end{aligned} \tag{14}$$

$$\begin{aligned}\frac{\partial^2 \ell(\varphi)}{\partial \theta \partial \gamma} &= -\frac{n}{\theta} - \frac{\gamma\lambda}{\theta}\left[\frac{a_i^{\alpha}b_i}{a_i^2 c_i} + \gamma\left[\left(1+(x_i/\theta)^{\gamma} + (\alpha-2)\frac{b_i}{a_i}\right)c_i - \alpha(\alpha-1)a_i^{\alpha-2}b_i\right]\right]\frac{a_i^{\alpha}b_i}{a_i^2c_i^2}\log(x_i/\theta)\\ &\quad -\frac{\lambda}{\theta}[1+\gamma(1+(x_i/\theta)^{\gamma})\log(x_i/\theta)]\,b_i - \frac{\alpha}{\theta}\left[b_i a_i^{\alpha-1} + \gamma\left[1+(x_i/\theta)^{\gamma} + (\alpha-1)\frac{b_i}{a_i}\right]a_i^{\alpha-1}b_i\log(x_i/\theta)\right]\\ &\quad -\frac{1}{\theta}\left[1+\gamma\frac{\log(x_i/\theta)}{(a_i+1)}[(a_i+1)(1+(x_i/\theta)^{\gamma}) - b_i]\right]\frac{b_1}{a_i+1},\end{aligned} \tag{15}$$

where $a_i = e^{(x_i/\theta)^{\gamma}} - 1$, $b_i = (x_i/\theta)^{\gamma}e^{(x_i/\theta)^{\gamma}}$, and $c_i = \lambda + \alpha a_i^{\alpha-1}$.

M2: SOME RESULTS AND PLOTS FOR THE THREE REAL DATA APPLICATIONS

M2.1: Aarset data

TABLE 1 Reliability point estimates for the fitted AMWE parameters; Aarset data set

x	$\hat{R}_{MLE}(x)$	$\hat{R}_{BE}(x)$	x	$\hat{R}_{MLE}(x)$	$\hat{R}_{BE}(x)$
0.1	0.9873	0.9874	46.0	0.4601	0.4569
0.2	0.9803	0.9805	47.0	0.4540	0.4508
1.0	0.9459	0.9461	50.0	0.4363	0.4329
2.0	0.9166	0.9167	55.0	0.4084	0.4048
3.0	0.8927	0.8927	60.0	0.3823	0.3786
6.0	0.8354	0.8351	63.0	0.3675	0.3638
7.0	0.8191	0.8187	67.0	0.3487	0.3448
11.0	0.7620	0.7611	72.0	0.3265	0.3225
12.0	0.7492	0.7482	75.0	0.3138	0.3098
18.0	0.6809	0.6793	79.0	0.2970	0.2931
21.0	0.6508	0.6490	82.0	0.2716	0.2701
32.0	0.5563	0.5537	83.0	0.2446	0.2471
36.0	0.5265	0.5237	84.0	0.1880	0.1987
40.0	0.4987	0.4957	85.0	0.0945	0.1133
45.0	0.4663	0.4631	86.0	0.0153	0.0260

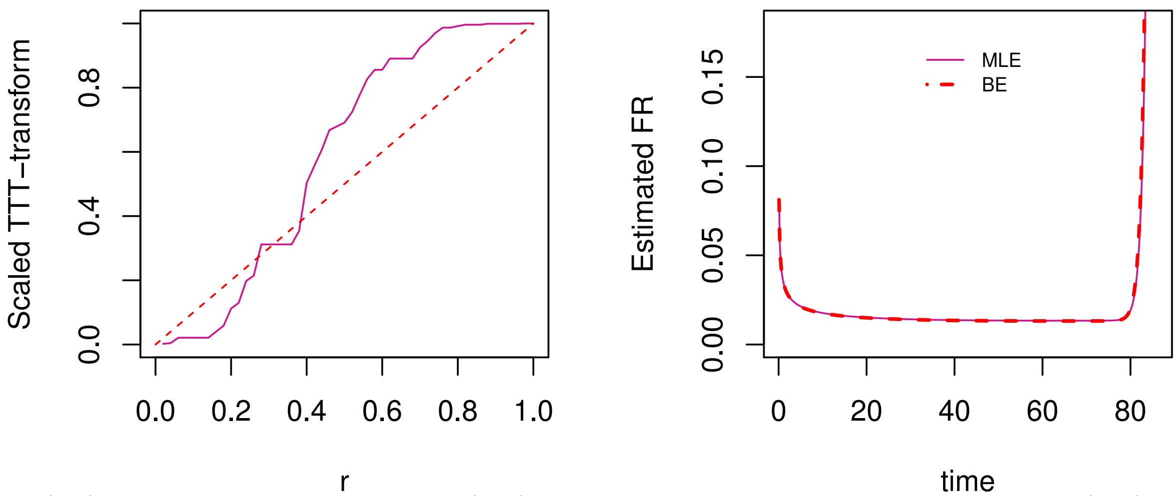

FIGURE 1 (Left) Empirical scaled TTT-transform plot and (right) estimated failure rate functions of the fitted AMWE for MLE (solid) and BE (dashed); Aarset data set

M2.2: Meeker-Escoba data

TABLE 2 Reliability point estimates of MLEs, BEs and their 95% interval estimates for the fitted AMWE parameters; Meeker-Escoba data set

x	$\hat{R}_{MLE}(x)$	$\hat{R}_{BE}(x)$	x	$\hat{R}_{MLE}(x)$	$\hat{R}_{BE}(x)$
2	0.9568	0.9440	147	0.4731	0.4579
10	0.8803	0.8581	173	0.4330	0.4202
13	0.8595	0.8358	181	0.4212	0.4091
23	0.8022	0.7762	212	0.3755	0.3653
23	0.8022	0.7762	245	0.3184	0.3083
28	0.7781	0.7517	247	0.3142	0.3041
30	0.7691	0.7426	261	0.2807	0.2700
65	0.6463	0.6217	266	0.2662	0.2554
80	0.6063	0.5833	275	0.2347	0.2243
88	0.5869	0.5649	293	0.1261	0.1199
106	0.5475	0.5277	300	0.0569	0.0491
143	0.4797	0.4640	300	0.0569	0.0491

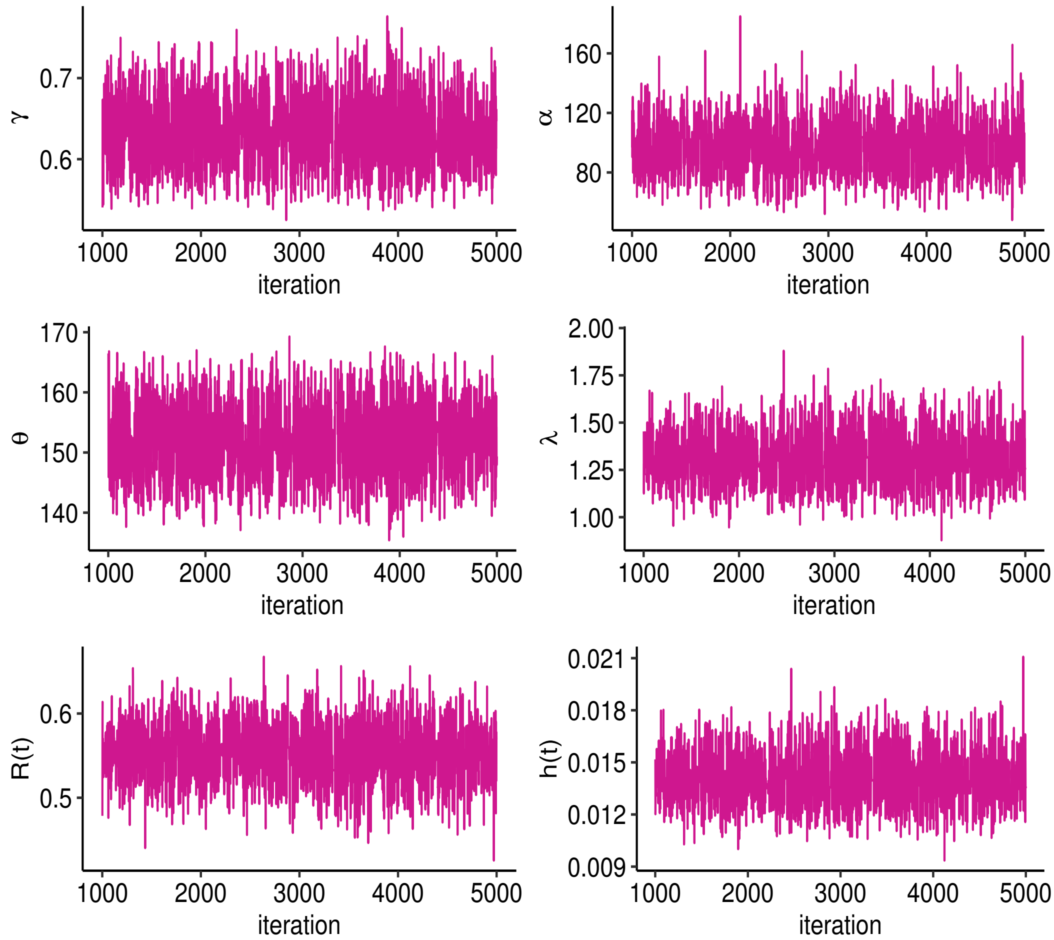

FIGURE 2 Trace plots of the fitted AMWE parameters produced by IM sampling; Aarset data set

M2.3: Rats survival times

TABLE 3 Reliability point estimates of MLEs, BEs and their 95% interval estimates for the fitted AMWE parameters; rats survival times

x	$\hat{R}_{MLE}(x)$	$\hat{R}_{BE}(x)$	x	$\hat{R}_{MLE}(x)$	$\hat{R}_{BE}(x)$
40	0.9696	0.9772	62	0.9022	0.9184
69	0.8711	0.8895	77	0.8298	0.8500
83	0.7951	0.8160	88	0.7639	0.7849
94	0.7240	0.7444	101	0.6743	0.6930
109	0.6139	0.6294	115	0.5662	0.5785
123	0.4993	0.5058	125	0.4818	0.4867
128	0.4550	0.4571	136	0.3787	0.3720
137	0.3685	0.3605	152	0.1881	0.1551
153	0.1739	0.1391	160	0.0726	0.0341
165	0.0162	0.0012			

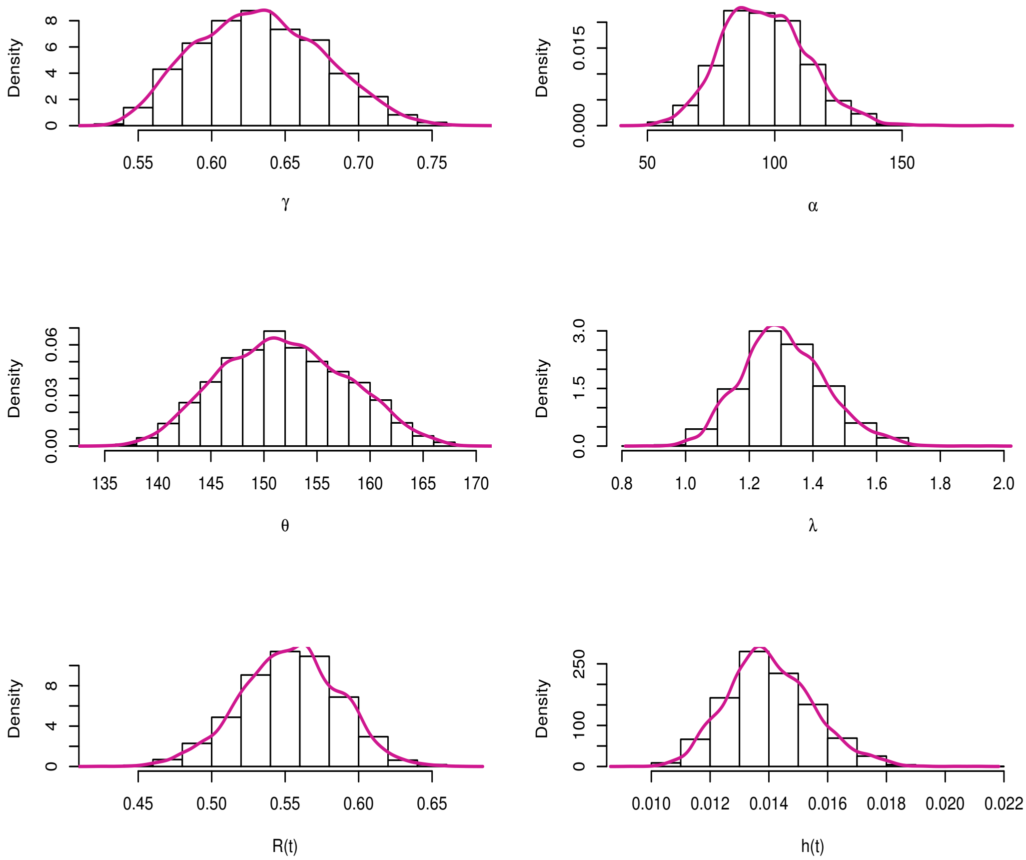

FIGURE 3 Histogram and kernel densities of γ, α, θ, and λ; Aarset data set

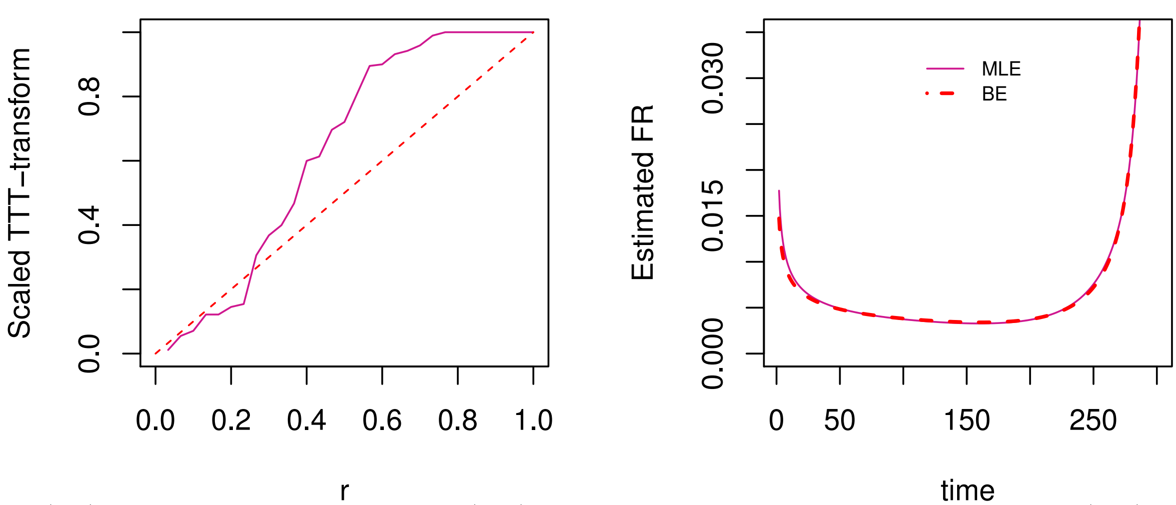

FIGURE 4 (Left) Empirical scaled TTT-transform plot and (right) estimated failure rate functions of the fitted AMWE for MLE (solid) and BE (dashed); Meeker-Escoba data set

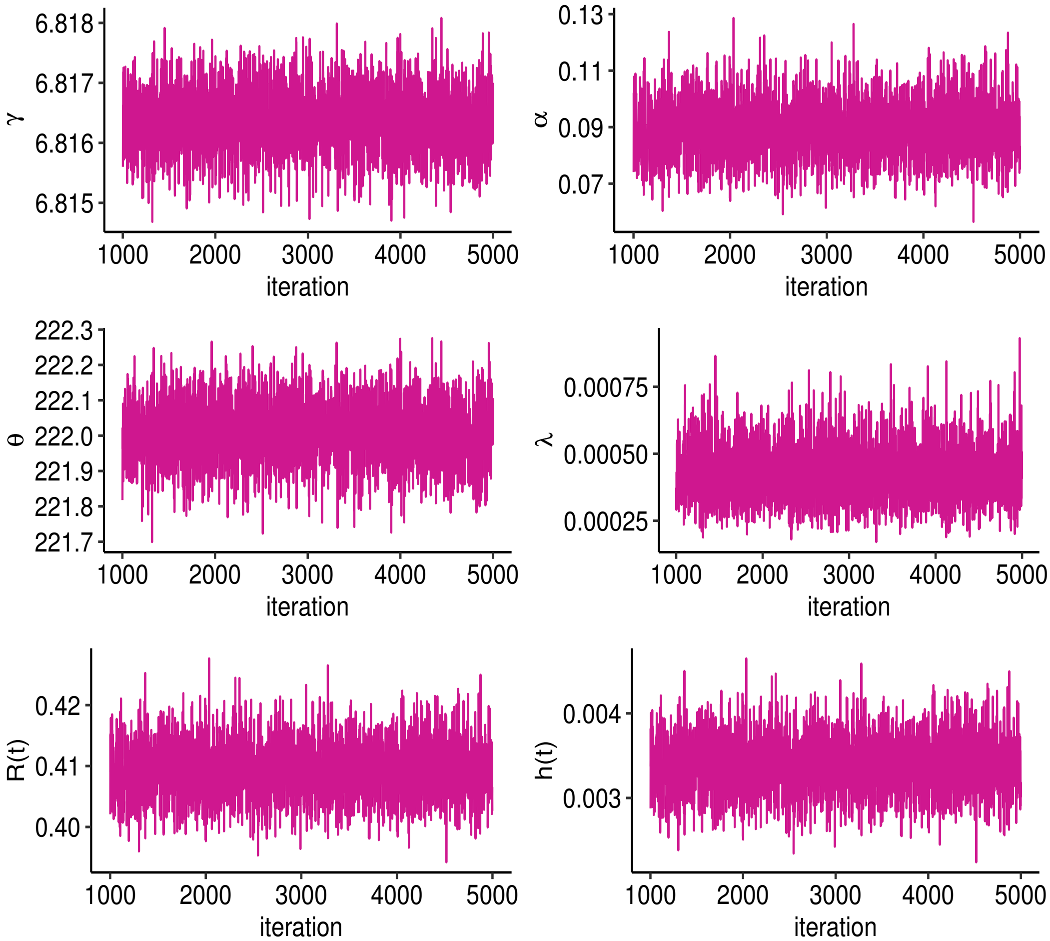

FIGURE 5 Trace plots of the fitted AExtEW parameters produced by IM sampling; Meeker-Escoba data set

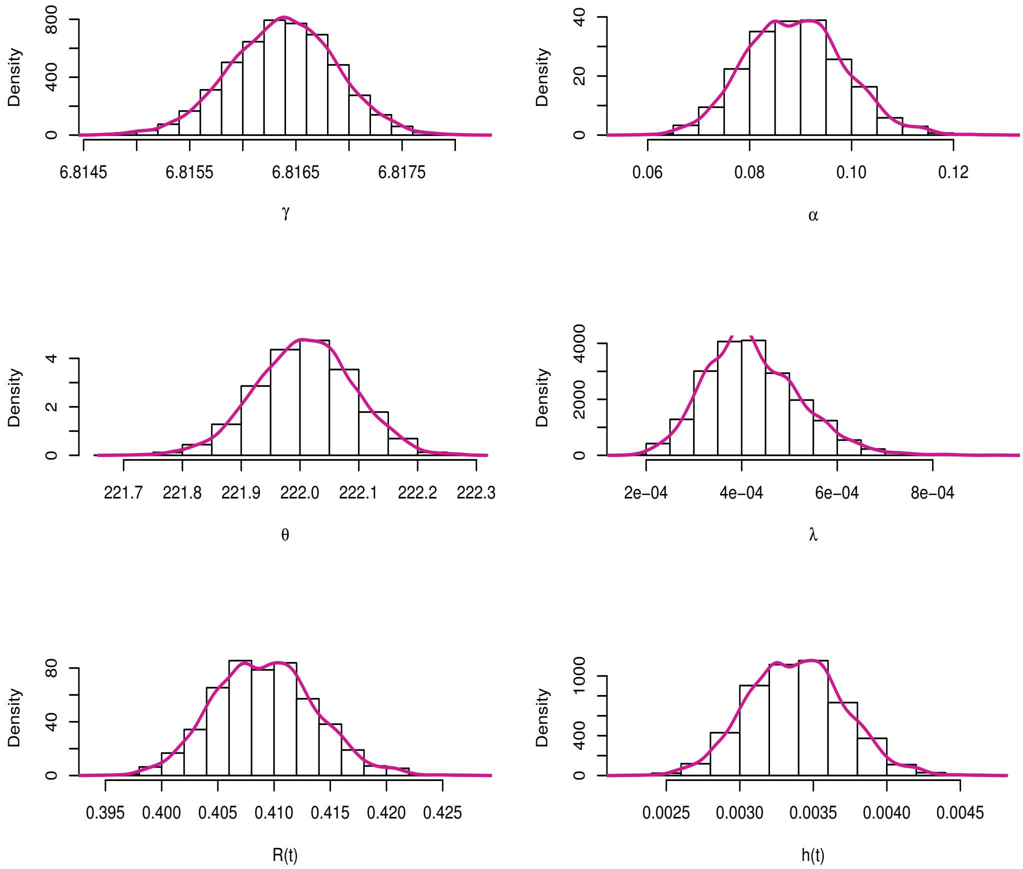

FIGURE 6 Histogram and kernel densities of γ, α, θ, and λ; Meeker-Escoba data set

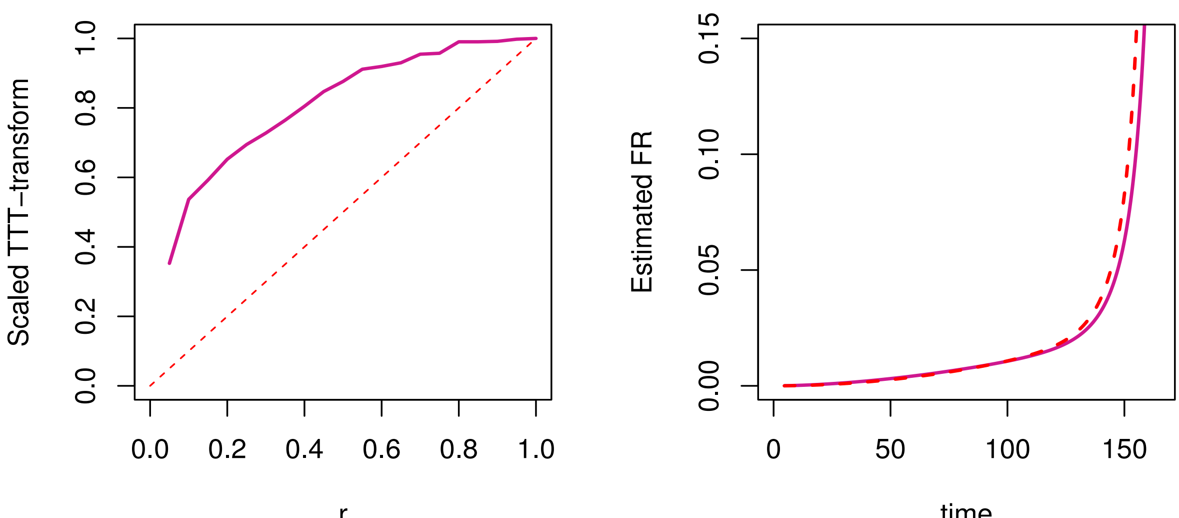

FIGURE 7 (Left) Empirical scaled TTT-transform plot and (right) estimated failure rate functions of the fitted AMWE for MLE (solid) and BE (dashed); rats survival times

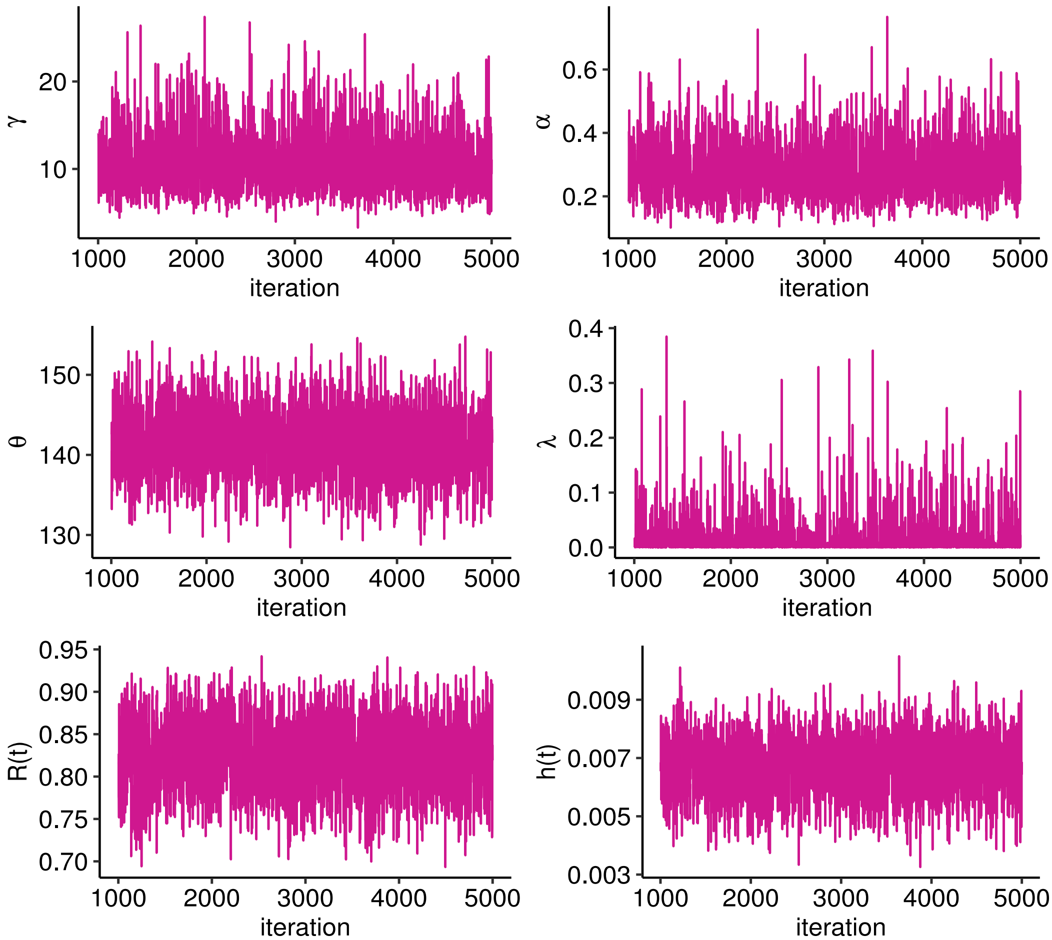

FIGURE 8 Trace plots of the fitted AExtEW parameters produced by IM sampling; rats survival times

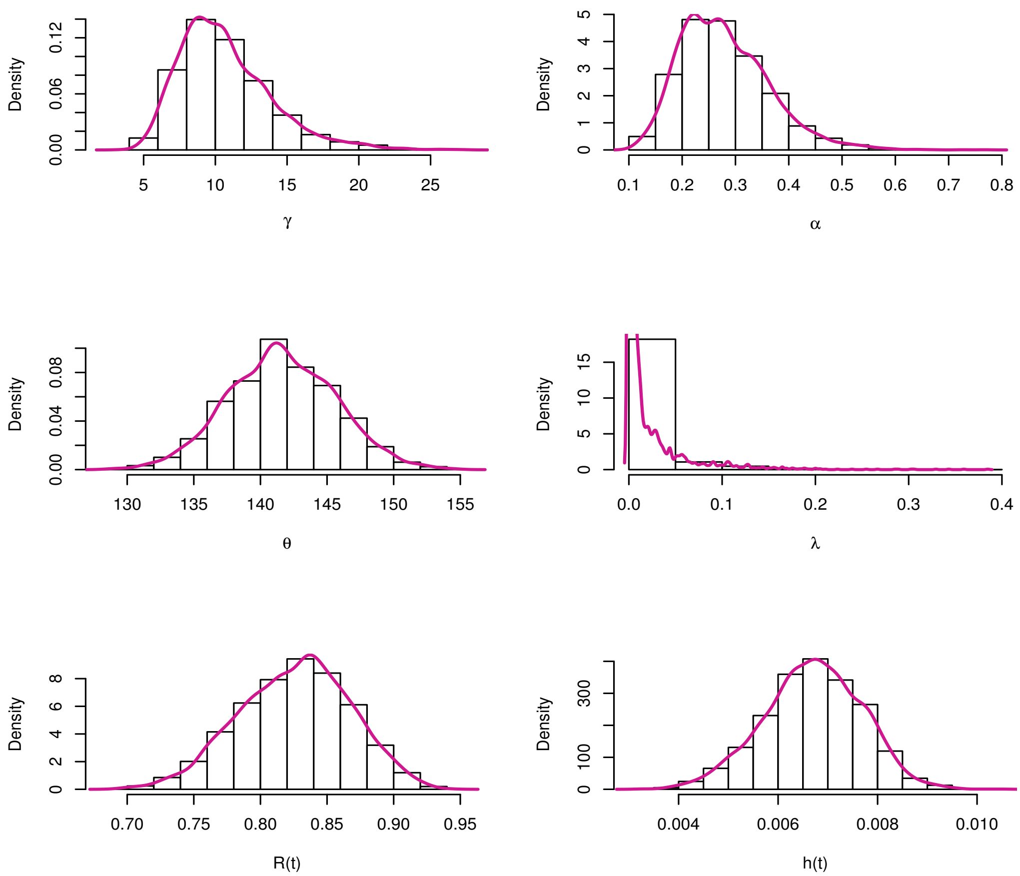

FIGURE 9 Histogram and kernel densities of γ, α, θ, and λ; Rats survival time

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RESEARCH ARTICLE

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Summary

To adequately study some complex lifetime data sets such as a bathtub-shaped failure rate data usually requires a flexible mixture of two-lifetime distributions with at least one of the parent models having non-monotone failure rate function. The resulted family of distributions are termed as the most credible class of distributions because of its simple characterization. In this paper, we introduce a new class of distributions, called additive modified Weibull extension distribution. The new model is flexible and can describe different failure rate data sets, particularly the bathtub-shaped failure rate data with a long-flat useful period or lifetime. We present some of the basic distributional properties of the model, including the mean residual life, mean time to failure, moments, mean deviations, Bonferroni curve and Lorenz curve, and order statistics. The parameter estimation of the model was approached by maximum likelihood and Bayesian methods. The asymptotic confidence intervals and Bayesian credible intervals of the parameters are obtained. Applications of the proposed distribution were demonstrated using three different lifetime data sets from reliability engineering and biological studies. The results show the dominant performance of the new model among the compared Weibull extended models for fitting lifetime data with bathtub-shaped failure rate function as proved by some goodness-of-fit tests.

KEYWORDS:
Modified Weibull extension distribution, bathtub-shaped failure rate, Bayesian inference, maximum likelihood method, lifetime data analysis

1 | INTRODUCTION

Over the past decades, many lifetime distributions have been introduced based on the Weibull distribution's modifications, extensions or generalizations to cope with its limitations for modeling non-monotone failure rate (FR) data. A majority of these extended distributions have either bathtub-shaped or up-side-down bathtub FR functions, making it possible for them to model monotone and non-monotone FR functions regardless of their parent distributions. The bathtub-shaped FR function is the most realistic which occurs in many real-life systems Dey, Sharma, and Mesfioui (2017), such as car failures, electricity generator failures, and mortality life, among others, and can be divided into three segments. At time zero (early life), the FR noticeably decreases due to early failures that might happen due to design faults, initial implementation issues, or birth defects in the case of human beings. The middle part of the FR function is nearly flat (constant)- regarded as a useful lifetime or stable period. The last part is the increasing or wear-out period due to reasons such as material degradation or fatigue. Following are some of the Weibull generalizations, modifications or extension proposed to cope with the above described FR function. Mudholkar and Srivastava Mudholkar and Srivastava

(1993) defined an exponentiated Weibull (EW) distribution for analyzing bathtub and up-side-down bathtub FR in addition to the increasing and decreasing FR inherited from Weibull distribution. Xie and Lai (1996) developed a mixture of two Weibull distribution called additive Weibull (AddW) distribution with four standard FR shapes (increasing, decreasing, bathtub and unimodal). The cumulative distribution functions (CDFs) of the EW and AddW models are, respectively, given by

$$F(x) = \left(1 - e^{-(\alpha x)^{\gamma}}\right)^{\theta}, \quad \alpha, \gamma, \theta > 0,\, x > 0,$$

$$F(x) = 1 - e^{-\gamma x^{\alpha} - \beta x^{\lambda}}, \quad x > 0,\, \gamma, \beta \geq 0,\, \alpha > 0,\, 1 > \lambda > 0.$$

Xie, Tang, and Goh (2002) introduced a three-parameter modified Weibull extension (MWE) distribution with CDF given by

$$F(x) = 1 - e^{\lambda\left(1 - e^{(x/\theta)^{\gamma}}\right)}, \quad x > 0,\, \lambda = \theta\alpha > 0,\, \gamma, \theta > 0.$$

The MWE distribution has increasing, decreasing and V-shaped bathtub FR function. The model reduces to Chen distribution developed by Chen (2000) for $\theta = 1$. Subsequently, Lai, Xie, and Murthy (2003) proposed modified Weibull (MW) distribution with CDF

$$F(x) = 1 - e^{-\alpha x^{\theta} e^{\gamma x}}, \quad x > 0,\, \theta, \alpha > 0,\, \gamma \geq 0.$$

Other Weibull extended distributions developed include the odd Weibull (OW) (Cooray 2006), (Jiang, Xie, & Tang 2008), beta Weibull (BW) (Lee, Famoye, & Olumolade 2007), beta modified Weibull (BMW) (Silva, Ortega, & Cordeiro 2010), new modified Weibull (NMW) (Almalki & Yuan 2013), exponentiated modified Weibull extension (EMWE) (Sarhan & Apaloo 2013), dditive modified Weibull (AMW) (He, Cui, & Du 2016), Weibull-Dagum (WD) (Tahir, Cordeiro, Mansoor, Zubair, & Alizadeh 2016), and Beta Sarhan–Zaindin modified Weibull (Saboor, Bakouch, & Nauman Khan 2016) distributions. In 2020, Thach and Briš (Thach & Bris 2020) modified the Almalki and Yaun's NMW model called the improved new modified Weibull (INMW) distribution. The INMW model is more flexible and appeared to perform better than the NWM for fitting bathtub FR data. More recently, Shakhatreh, Lemonte, and Cordeiro (2020) and Ahmad and Ghazal (2020) also defined a five-parameter generalized extended exponential Weibull (GExtEW) and exponentiated additive Weibull (EAddW) distributions. The GExtEW distribution is an extension of the exponential-Weibull (ExW) (Cordeiro, Ortega, & Lemonte 2014) - a mixture of exponential and Weibull distributions. While the EAddW model is an extension of the AddW distribution. Both the GExtEW and EAddW distributions presented bathtub-shaped FRs.

One major concern is that the bathtub for most of the existing Weibull extended distributions are either U or V-shapes as there is no interval for which the FR function, $h(t)$, is a constant (Glaser 1980; Lai, Xie, & Murthy 2001; Shakhatreh, Lemonte, & Moreno–Arenas 2019). Even some of these distributions have a constant region in the bathtub FR (for example, (Ahmad & Ghazal 2020; Sarhan & Apaloo 2013; Xie & Lai 1996)), they fail to provide a better fit for some bathtub FR data sets, such as the data sets considered by in Aarset (Aarset 1987), Meeker and Escoba (Meeker & Escoba 1998), and Tang et al. (Tang et al. 2015), among others.

According to Nelson Nelson (1990) and He et al. He et al. (2016), the number of a model's parameters can be viewed as the measure of complexity, and more flexible distributions usually require more parameters. Thus, to fit a particular data set, we want the model that can best maintain the balance between the goodness of fit and the model complexity. One approach to producing such a flexible model is by mixing two distributions with at least one distribution having non-monotone FR. The additive modified Weibull, lognormal modified Weibull, Weibull-Chen and additive Chen-Weibull distributions defined by He et al. (2016), Shakhatreh et al. (2019), Tarvirdizade and Ahmadpour (2021), and Thach and Briš (2021) are recently introduced in this regard, respectively. Shakhatreh et al. (2019) describe the class of distributions from this approach as the most realistic family of distribution due to its simple characterization.

Though the MWE distribution has increasing, decreasing and bathtub FR function. The shape of it bathtub is approximately V-shape and hence may not provide good fit to data sets with either U-shape bathtub FR or the bathtub-shaped FR with constant region. This has encouraged Sarhan and Apaloo (2013) to extend the model and defined the exponentiated modified Weibull extension (EMWE) distribution. The EMWE produce good bathtub shape of the FR than the MWE model. Although the latter is more flexible with better bathtub shape of the FR. However, neither the MWE nor the EMWE model give bathtub-shaped FR function with long-flat region. Thus, in this paper, we are motivated by the aforementioned rationale on mixing two distributions to introduce a new flexible lifetime model by combining the MWE distribution and the Bourguignon, Silva, and Cordeiro (2014) three-parameter Weibull Weibull (WW) distribution considering two independent components placed in a series system. The new distribution is referred to as additive modified Weibull extension (AMWE) distribution. The AMWE assumed that the two components have MWE and WW distribution, respectively. Our contributions are to (i) develop a flexible lifetime model that can properly analyze different lifetime data sets from a diverse class of problems; (ii) accommodate many types of FR functions, such as bathtub-shaped FR data with flat or long-flat useful lifetime often encountered in many real-life systems; (iii) generalized both MWE and WW distributions to widen their individual ability for analyzing lifetime data. The novelty of this study is that, we introduced a new bathtub distribution as a mixture model resulted from two Weibull extended models both with non-monotone FR functions.

The AMWE distribution is flexible and its FR function has several shapes, including the bathtub-shaped with a long-flat useful region. In addition to the new proposed distribution, Bayesian estimation have been used together with the usual maximum likelihood method to estimate the unknown parameters of the distribution. This is important as Bayesian paradigm allows for more flexible parameter estimation and often yields a better result than the frequentist approach Ibrahim, Chen, and Sinha (2001). The applicability of the new AMWE distribution is demonstrated by fitting three failure-times data sets with two having bathtub-shaped FR Aarset (1987); Meeker and Escoba (1998) and Lawless (2003) with increasing FR function. The results of the applications illustrated that the new model provides a better fit to all the three data sets than other well-known Weibull extended distributions with bathtub FR function as evidenced by some goodness-of-fit tests, including the Akaike information criteria (AIC), Bayesian information criteria (BIC), and Kolmogorov Smirnov (KS) test.

After section 1, in which we justify the study problem and how to define a flexible distribution with bathtub-shaped FR function, we describe the proposed distribution in Section 2., including the model interpretation as well as some shapes of PDF and FR functions. Section 3 presents some properties of the distribution. Estimation of the distribution's parameters using the method of maximum likelihood and Bayesian approach are given in Sections 4. We illustrate the real data applications of the distribution in Section 5. Finally, we conclude the paper in Section 6.

2 | THE NEW DISTRIBUTION

The CDF of the additive modified Weibull extension with four parameters $\varphi = (\gamma, \alpha, \theta, \lambda)'$ is defined by

$$F(x) = 1 - e^{\lambda(1-e^{(x/\theta)^\gamma}) - (e^{(x/\theta)^\gamma}-1)^\alpha}, x > 0, \gamma, \alpha, \theta > 0, \lambda \geq 0. \quad (1)$$

Eqn (1) can be expressed as $F(x) = 1 - S_{MWE}(x)S_{WW}(x)$, where $S_{MWE}(x)$ and $S_{WW}(x)$ are the reliability functions of the MWE and WW distributions, respectively. Following is an alternative technique to define the AMWE distribution.

Suppose B is a random variable with probability density function (PDF) $\kappa(b) = (\lambda + \alpha b^{\alpha-1})e^{-\lambda b - \alpha b^\alpha}$, $b > 0$ and $\lambda, \alpha > 0$. Let $D(x)$ be valid CDF of an absolutely continuous random variable X. A family of extended CDF of X can take the form

$$F(x) = \int_0^{C(x)} (\lambda + \alpha b^{\alpha-1})e^{-\lambda b - \alpha b^\alpha} db = 1 - e^{\lambda C(x) - C(x)^\alpha}, \quad (2)$$

where $C(x) = D(x)/(1 - D(x))$ - the odd function of the CDF $D(x)$. Hence, the CDF in (1) is a special case of (2) taking $D(.)$ as the Weibull CDF, $D(x) = 1 - e^{-(x/\theta)^\gamma}$, $\theta, \gamma > 0$. Where the baseline PDF of the family $\kappa(b)$ is a two-parameter exponential-Weibull by Cordeiro et al. Cordeiro et al. (2014). The associated PDF, reliability/survival (RE) and failure rate (FR) functions are, respectively, given by

$$f(x) = \gamma\theta^{-\gamma}x^{\gamma-1}e^{(x/\theta)^\gamma}(\lambda + \alpha(e^{(x/\theta)^\gamma} - 1)^{\alpha-1})e^{\lambda(1-e^{(x/\theta)^\gamma}) - (e^{(x/\theta)^\gamma}-1)^\alpha}, x > 0, \quad (3)$$

$$R(x) = e^{-H(x)}, x > 0. \quad (4)$$

and

$$h(x) = \gamma\theta^{-\gamma}x^{\gamma-1}e^{(x/\theta)^\gamma}(\lambda + \alpha(e^{(x/\theta)^\gamma} - 1)^{\alpha-1}), x > 0, \quad (5)$$

where $H(x) = \lambda(e^{(x/\theta)^\gamma} - 1) + (e^{(x/\theta)^\gamma} - 1)^\alpha$ is the cumulative hazard function of the model. Notice that $h(x) = h_{MWE}(x) + h_{WW}(x)$, for $x > 0$, where $h_{MWE}(x) = \gamma\lambda\theta^{-\gamma}x^{\gamma-1}e^{(x/\theta)^\gamma}$ is the FR function of the MWE distribution and $h_{WW}(x) = \gamma\alpha\theta^{-\gamma}x^{\gamma-1}e^{(x/\theta)^\gamma}(e^{(x/\theta)^\gamma} - 1)^{\alpha-1}$ is the FR function of WW distribution by Xie et al. (2002) and Bourguignon et al. (2014), respectively. Glaser Glaser (1980) discuss the sufficient conditions to characterize a given distribution with non-monotone FR function. The FR function of the AMWE model, which depends on the two shape parameters $\gamma > 0$ and $\alpha > 0$, presents bathtub-shaped when $h'(x^*) = 0$ if and only if $x = x^*$ is the root of the equation:

$$\left[(\gamma - 1)x^{*-1} + \gamma\theta^{-\gamma}x^{*\gamma-1}\right] h(x^*) + (\alpha - 1)x^{*\gamma-1}(1 - e^{-(x^*/\theta)^\gamma})^{-1} = 0$$

It can be verified that the FR function is decreasing if $h'(x) < 0$ for $x < x^*$ and is increasing if $h'(x) > 0$ for $x > x^*$. **FIGURE** 1 presents the PDFs of the AMWE distribution. We observe that the density function has decreasing, unimodal, and modified bathtub (decreasing-increasing-decreasing) shapes among others, depending on the values of the parameters. The associated plots of the FR functions are displayed in **FIGURE** 2 - 3 at various values of parameters. The new model provides several shapes of the FR function for describing complicated lifetime data. The Bathtub-shaped FR function along with the long flat region (useful lifetime) of the AMWE distribution is displayed in **FIGURE** 3. It is very useful in reliability to have a bathtub-shaped FR with a long or long-flat useful lifetime, see, for instance, (Almalki & Yuan 2013; Shakhatreh et al. 2019). Interestingly, the FR function of AMWE possessed this property, and thus, AMWE can be very useful in this context.

The physical interpretation of the AMWE model can be applied to two components in a series system, with first component lifetime having MWE distribution and the second component following the WW distribution. Hence, the system's lifetime is taken as the minimum lifetime of the two components.

3 | DISTRIBUTION PROPERTIES

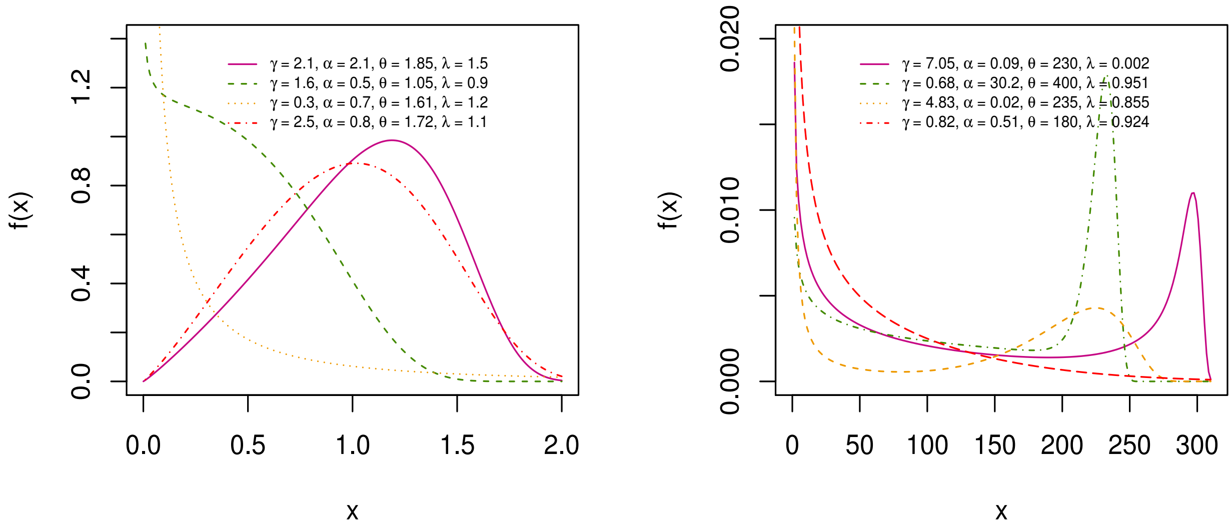

FIGURE 1 Probability density functions of the AMWE

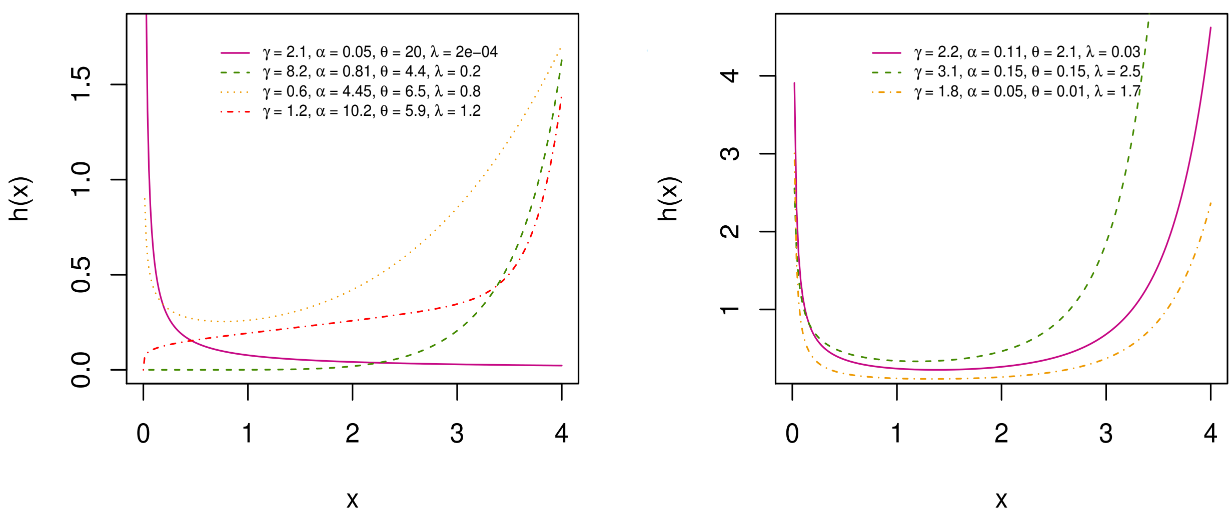

FIGURE 2 Failure rate functions of the AMWE; increasing, decreasing and bathtub, (left-panel) - bathtub-shaped with and without long useful lifetime (right-panel)for different values of parameters

3.1 | Quantiles, median and mode

The quantile x_q of the AMWE model is a real solution of the following non-linear equation

$$\lambda e^{(x/\theta)^{\gamma}} + (e^{(x/\theta)^{\gamma}} - 1)^{\alpha} = \lambda - \log(1-q)\,. \tag{6}$$

Eqn (6) does not have a closed-form solution in x_q, and thus, we adopt a numerical solution technique to determine the quantile. Following is a simple algorithm for simulating data from the AMWE$(\gamma, \alpha, \theta, \lambda)$ distribution.

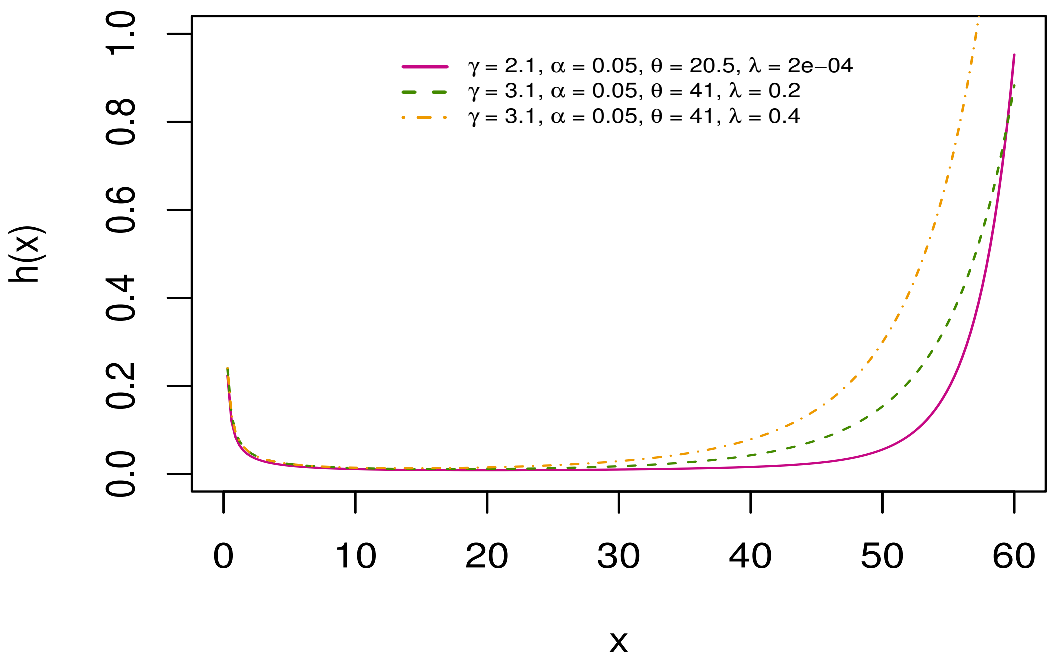

FIGURE 3 Bathtub-shaped failure rate with a long flat region (useful lifetime) of the AMWE distribution with different values of parameters

1. Generate $u_i \sim U(0,1), i = 1, 2, \dots, n$, and then
2. use (1) to solve for x_i in

$$\lambda e^{(x_i/\theta)^\gamma} + (e^{(x_i/\theta)^\gamma} - 1)^\alpha = \lambda - \log(1 - u_i).$$

To investigate the consistency of the generated from Eqn (6), we display the histograms of the simulated samples in **FIGURE** 4 . From **FIGURE** 4 , we observe that the generated samples are consistent. Eqn (6) gives the median of the AMWE random variable X at $q = 1/2$. The value of X for which AMWE density function $f(x) = h(x)S(x)$ attains its maximum is the mode of AMWE distribution. Thus, the solution of the equation

$$h'(x)S(x) + h(x)S'(x) = 0 \tag{7}$$

provides the mode of the distribution. Where $h'(x)$ and $S'(x)$ are the derivatives of the FR and RE functions, given respectively, as follows.

$$h'(x) = \left[(\gamma - 1)x^{-1} + \gamma\theta^{-\gamma}x^{\gamma-1}\right] h(x) + (\alpha - 1)x^{\gamma-1}(1 - e^{-(x/\theta)^\gamma})^{-1},$$
$$S'(x) = -\gamma\theta^{-\gamma}x^{\gamma-1}e^{(x/\theta)^\gamma}(\lambda + \alpha(e^{(x/\theta)^\gamma} - 1)^{\alpha-1})S(x).$$

Eqn 7 have no convenient closed form, and therefore, required some numerical approaches.

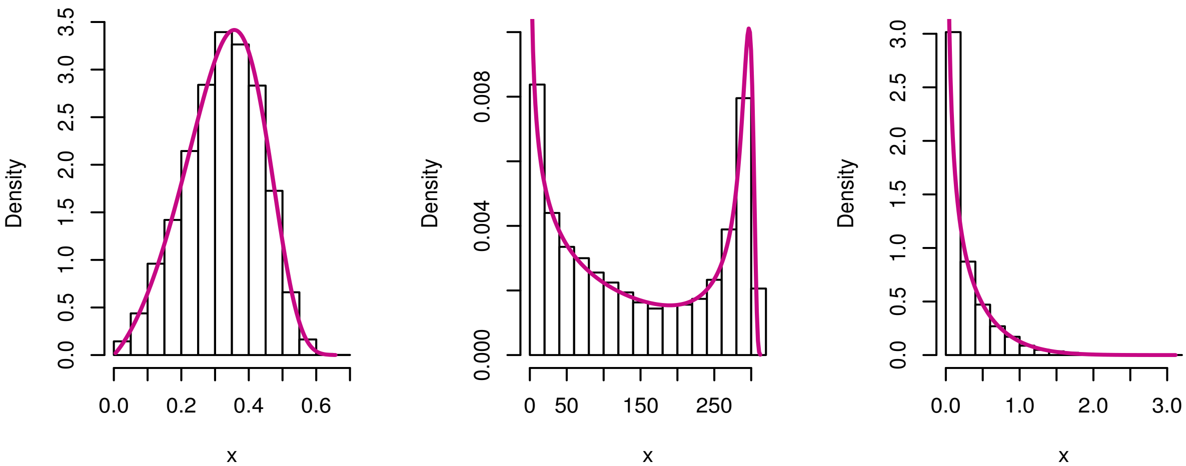

FIGURE 4 Histogram of three simulated samples each with $n = 10000$: $\gamma = 2.0, \alpha = 1.5, \theta = 0.5, \lambda = 0.5$ (left-panel), $\gamma = 7.0, \alpha = 0.1, \theta = 230, \lambda = 0.002$ (center-panel), and $\gamma = 0.8, \alpha = 0.5, \theta = 1.5, \lambda = 2.0$ (right-panel).

3.2 | Mean residual life

The mean residual life (MRL) represents the period from a specific time t to the failure time of an individual or system. That is, it measures the expected remaining lifetime of an individual or system with age t at present. For the AMWE model, the MRL is defined as

$$\begin{aligned}\mu_X(t) = E(X - t|X > t) &= \frac{1}{S(t)}\int_t^{+\infty} S(x)\,dx = \frac{1}{S(t)}\int_0^{+\infty} S(x+t)\,dx \\ &= \frac{e^{\lambda}}{S(t)}\sum_{i,j=0}^{+\infty}\sum_{p=0}^{+\infty}\upsilon_{i,j}\upsilon_p\int_0^{+\infty}(x+t)^{j\gamma}e^{-p\theta^{-\gamma}(x+t)^{\gamma}}dx \\ &= \frac{e^{\lambda e^{(t/\theta)^{\gamma}}+\left(e^{(t/\theta)^{\gamma}}-1\right)^{\alpha}}}{\gamma}\sum_{i,j=0}^{+\infty}\sum_{p=0}^{+\infty}\upsilon_{i,j}\upsilon_p\frac{\theta^{j\gamma+1}}{p^{(j\gamma+1)/\gamma}}\Gamma\left(\frac{j\gamma+1}{\gamma}\right),\end{aligned} \tag{8}$$

where $\upsilon_{i,j} = (-\lambda)^i(i)^j/i!j!\theta^{j\gamma}$, $\upsilon_p = \sum_{k,\ell=0}^{+\infty}(-1)^p\Gamma(k\alpha+\ell+1)\upsilon_{k,\ell}/p!\Gamma(k\alpha+\ell-p+1)$ and $\upsilon_{k,\ell} = (-1)^k\Gamma(k\alpha+\ell)/k!\ell!\Gamma(k\alpha)$, and $\Gamma(.)$ is the gamma function.

3.3 | Mean time to failure

We define the AMWE model mean time to failure (MTTF) as

$$\begin{aligned}\text{MTTF} &= \int_0^{+\infty} S(x)\,dx = \int_0^{+\infty} e^{\lambda\left(1-e^{(t/\theta)^{\gamma}}\right)-\left(e^{(t/\theta)^{\gamma}}-1\right)^{\alpha}}dx \\ &= e^{\lambda}\sum_{i,j=0}^{+\infty}\sum_{p=0}^{+\infty}\upsilon_{i,j}\upsilon_p\int_0^{+\infty}x^{j\gamma}e^{-p(x/\theta)^{\gamma}}dx \\ &= \frac{e^{\lambda}}{\gamma}\sum_{i,j=0}^{+\infty}\sum_{p=0}^{+\infty}\upsilon_{i,j}\upsilon_p\frac{\theta^{j\gamma+1}}{p^{(j\gamma+1)/\gamma}}\Gamma\left(\frac{j\gamma+1}{\gamma}\right),\end{aligned} \tag{9}$$

where $\upsilon_{i,j}$, $\upsilon_{k,\ell}$ and υ_p are given in Eqn (8), and $\Gamma(.)$ is the gamma function.

3.4 | Moments

For X $\sim$AMWE$(\gamma,\alpha,\theta,\lambda)$, the r^{th} non-central moment can be derived as follows by using integration by parts and Taylor expression of e^x with Binomial expression of $(1+x)^{b-1}$, for non-integer number b and $|x|<1$.

$$\begin{aligned}\mu_r &= \int_0^{+\infty} x^r dF(x) = \int_0^{+\infty} r x^{r-1} e^{-\lambda\left(1-e^{(t/\theta)^\gamma}\right)-\left(e^{(t/\theta)^\gamma}-1\right)^\alpha} dx \\ &= re^{\lambda}\sum_{i,j=0}^{+\infty}\sum_{p=0}^{+\infty}\upsilon_{i,j}\upsilon_p\int_0^{+\infty} x^{j\gamma+r-1}e^{-p(x/\theta)^\gamma}dx \\ &= \frac{re^{\lambda}}{\gamma}\sum_{i,j=0}^{+\infty}\sum_{p=0}^{+\infty}\upsilon_{i,j}\upsilon_p\frac{\theta^{j\gamma+r}}{p^{(j\gamma+r)/\gamma}}\Gamma\left(\frac{j\gamma+r}{\gamma}\right), \quad \text{for} \quad r=1,2,\ldots,\end{aligned} \tag{10}$$

where $\upsilon_{i,j}, \upsilon_{k,\ell}$ and υ_p are given in Eqn (8), and $\Gamma(.)$ is the gamma function. Lemma 1 provides the lower incomplete moments obtain from (10). The lemma is used in the subsequent section to compute the mean deviations, Bonferroni curve and Lorenz curve.

Lemma 1. Let X $\sim$AMWE$(\gamma,\alpha,\theta,\lambda)$, following a similar approach to (10), the r^{th} incomplete moments is given as

$$\zeta_r(z) = \frac{re^{\lambda}}{\gamma}\sum_{i,j=0}^{+\infty}\sum_{p=0}^{+\infty}\upsilon_{i,j}\upsilon_p\frac{\theta^{j\gamma+r}}{p^{(j\gamma+r)/\gamma}}\Gamma\left(\frac{j\gamma+r}{\gamma},\omega(z)\right), \quad for \quad r=1,2,\ldots, \tag{11}$$

where $\omega(z)=p(z/\theta)^\gamma$ and $\Gamma(.)$ is the incomplete gamma function.

3.5 | Mean deviations, Bonferroni curve and Lorenz curve

The mean deviation about the mean $\xi_1(X)$ and mean deviation about the median $\xi_2(X)$ of the AMWE random variable are defined by $\xi_1(X) = \int_0^\infty |x-\mu_1| f(x)\,dx = 2\mu F(\mu) - 2\zeta_1(z)(\mu)$ and $\xi_2(X) = \int_0^\infty |x-M| f(x)\,dx = \mu - 2\zeta_1(z)(M)$, respectively, where F(x) is the CDF of X, $\mu = E(x)$ is the mean of X from (10) at $r=1$, M - is the median of X, and ζ_1 defined in lemma 1 for $r=1$. The Bonferroni curve and Lorenz curve are two vital curves in insurance, econometrics and geography, among others. They represent the distribution of wealth within a population. We defined the Bonferroni and Lorenz curves by $B(d)=\zeta_1(q)/d\mu$ and $L(d)=\zeta_1(q)/\mu$, where μ is the mean of X, ζ_1 defined in lemma 1 for $r=1$, and $q(d)$ is the solution of the nonlinear quantile equation given in (6) at a given probability d.

3.6 | Order statistics

Let the random sample $X_1, X_2, \ldots, X_n$ follows the AMWE distribution with the p^{th} order statistic $X_{p:n}$, then the PDF of $X_{p:n}$ is given as

$$g_{p:n}(x) = \frac{1}{B(p,n-p+1)}F(x)^{p-1}\left[1-F(x)\right]^{n-k}f(t),$$

where $B(.,.)$ is the beta function. Considering Eqns (1) and (4), we have

$$F(x)^{p-1} = \left(1-e^{-H(x)}\right)^{p-1} = \sum_{s=0}^{p-1}\frac{(-1)^s\Gamma(p)}{s!\Gamma(p-s)}e^{-sH(x)},$$

and

$$\left[1-F(x)\right]^{n-k} = e^{-(n-p)H(x)}.$$

Thus, the PDF of $X_{p:n}$ for the AMWE is derived as

$$\begin{aligned}g_{p:n}(x) &= \frac{1}{B(p,n-p+1)}\sum_{s=0}^{p-1}\frac{(-1)^s\Gamma(p)}{s!\Gamma(p-s)}h(x)e^{-(n+s+1-p)H(x)} \\ &= \frac{\gamma\theta^{-\gamma}\Gamma(n)}{\Gamma(n-p+1)}\sum_{s=0}^{p-1}\frac{(-1)^s e^{(x/\theta)^\gamma}}{s!\Gamma(p-s)}x^{\gamma-1}(\lambda+\alpha(e^{(x/\theta)^\gamma}-1)^{\alpha-1})e^{-(n+s+1-p)H(x)},\end{aligned}$$

where $H(x)=\lambda(e^{(x/\theta)^\gamma}-1)+(e^{(x/\theta)^\gamma}-1)^\alpha$.

4 | PARAMETER ESTIMATION

In this section, we discuss the maximum likelihood and the Bayesian methods for estimating the AMWE parameters, the reliability/survival function $R(x)$ and the FR function $h(x)$.

4.1 | Maximum likelihood for non-censored observations

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the AMWE model with the vector of parameters $\varphi = (\gamma, \alpha, \theta, \lambda)'$. Then the log-likelihood function of φ from the PDF (3) is

$$\ell(\varphi) = n\log\gamma - n\gamma\log\theta + (\gamma-1)\sum_{i=1}^{n}\log x_i + \sum_{i=1}^{n}(x_i/\theta)^{\gamma} + \sum_{i=1}^{n}\log(c_i) - \sum_{i=1}^{n}(\lambda a_i + a_i^{\alpha}), \tag{12}$$

where $a_i = e^{(x_i/\theta)^{\gamma}} - 1$, and $c_i = \lambda + \alpha a_i^{\alpha-1}$. The estimate $\widehat{\varphi} = \left(\widehat{\gamma}, \widehat{\alpha}, \widehat{\theta}, \widehat{\lambda}\right)'$ of $(\gamma, \alpha, \theta, \lambda)'$ is determined by maximizing the log-likelihood function for each of the AMWE parameters. Thus, we have the following score functions.

$$\frac{\partial \ell}{\partial \lambda} = \sum_{i=1}^{n}\frac{1}{c_i} - \sum_{i=1}^{n} a_i, \tag{13}$$

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{n}\frac{a_i^{\alpha-1}}{c_i}\left(1 + \alpha\log(a_i)\right) - \sum_{i=1}^{n} a_i^{\alpha}\log(a_i), \tag{14}$$

$$\frac{\partial \ell}{\partial \theta} = -\frac{n\gamma}{\theta} - \frac{\gamma}{\theta}\sum_{i=1}^{n}(x_i/\theta)^{\gamma} + \frac{\gamma\alpha(\alpha-1)}{\theta}\sum_{i=1}^{n}\frac{a_i^{\alpha} b_i}{a_i^2 c_i} + \frac{\gamma}{\theta}\sum_{i=1}^{n}\frac{b_i}{c_i} \tag{15}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} - n\log\theta + \sum_{i=1}^{n}\log x_i + \alpha(\alpha-1)\sum_{i=1}^{n}\frac{a_i^{\alpha} b_i}{a_i^2 c_i}\log(x_i/\theta)$$
$$+ \sum_{i=1}^{n}(x_i/\theta)^{\gamma}\log(x_i/\theta) - \sum_{i=1}^{n} b_i c_i\log(x_i/\theta), \tag{16}$$

where $a_i = e^{(x_i/\theta)^{\gamma}} - 1$, $b_i = (x_i/\theta)^{\gamma} e^{(x_i/\theta)^{\gamma}}$, and $c_i = \lambda + \alpha a_i^{\alpha-1}$. Solving Eqns (13) - (16) analytically may be intractable. Thus, a numerical approach is adopted to obtain the maximum likelihood estimates (MLEs) of the parameters $\varphi = (\gamma, \alpha, \theta, \lambda)'$ with a good set of initial values. For the optimization, **goodness.fit** function from **AdequacyModel** (Marinho, Silva, Bourguignon, Cordeiro, & Nadarajah 2019) package in **R** (RStudio Team 2020) is used first; when **goodness.fit** fails to converge, we used **optim** with “BFGS” method.

Additionally, we can determine the MLEs of $R(t)$ and $h(x)$ by applying the invariant property of the MLE (Casella & Berger 2001). Therefore, the MLEs of $R(t)$ and $h(x)$ are, respectively, given by

$$\hat{R}(x) = e^{\hat{\lambda}(1-e^{(x/\hat{\theta})^{\hat{\gamma}}}) - (e^{(x/\hat{\theta})^{\hat{\gamma}}}-1)^{\hat{\alpha}}}; \quad \hat{h}(x) = \hat{\gamma}\hat{\theta}^{-\hat{\gamma}} x^{\hat{\gamma}-1} e^{(x/\hat{\theta})^{\hat{\gamma}}}\left(\hat{\lambda} + \hat{\alpha}(e^{(x/\hat{\theta})^{\hat{\gamma}}} - 1)^{\hat{\alpha}-1}\right).$$

4.1.1 | Asymptotic confidence intervals

The exact distributions of the MLEs can be derived but quite difficult to have it closed forms. Hence, we derive an approximate confidence interval of the parameters based on the asymptotic normality distributions for the $\gamma, \alpha, \theta,$ and λ as $n \to \infty$, that is, $\sqrt{n}(\varphi - \hat{\varphi}) \sim N_4\left(0, I^{-1}\right)$ - a four-dimensional normal distribution with zero mean and variance-covariance matrix I^{-1}, where

$$I(\varphi) = -\begin{bmatrix} \frac{\partial^2\ell(\varphi)}{\partial\gamma^2} & \frac{\partial^2\ell(\varphi)}{\partial\gamma\partial\alpha} & \frac{\partial^2\ell(\varphi)}{\partial\gamma\partial\theta} & \frac{\partial^2\ell(\varphi)}{\partial\gamma\partial\lambda} \\ . & \frac{\partial^2\ell(\varphi)}{\partial\alpha^2} & \frac{\partial^2\ell(\varphi)}{\partial\alpha\partial\theta} & \frac{\partial^2\ell(\varphi)}{\partial\alpha\partial\lambda} \\ . & . & \frac{\partial^2\ell(\varphi)}{\partial\theta^2} & \frac{\partial^2\ell(\varphi)}{\partial\theta\partial\lambda} \\ . & . & . & \frac{\partial^2\ell(\varphi)}{\partial\lambda^2} \end{bmatrix}.$$

The elements of $I(\varphi)$ are given (see supp. material Section M1.1). Then we can have the approximate variance-covariance matrix evaluated at $\hat{\varphi} = (\hat{\gamma}, \hat{\alpha}, \hat{\theta}, \hat{\lambda})'$, the MLE of $(\gamma, \alpha, \theta, \lambda)'$ as

$$I^{-1}(\hat{\varphi}) = \begin{bmatrix} \mathrm{var}(\hat{\gamma}) & \mathrm{cov}(\hat{\gamma}, \hat{\alpha}) & \mathrm{cov}(\hat{\gamma}, \hat{\theta}) & \mathrm{cov}(\hat{\gamma}, \hat{\lambda}) \\ . & \mathrm{var}(\hat{\alpha}) & \mathrm{cov}(\hat{\alpha}, \hat{\theta}) & \mathrm{cov}(\hat{\alpha}, \hat{\lambda}) \\ . & . & \mathrm{var}(\hat{\theta}) & \mathrm{cov}(\hat{\theta}, \hat{\lambda}) \\ . & . & . & \mathrm{var}(\hat{\lambda}) \end{bmatrix}.$$

Hence, the $100(1-\delta)\%$ asymptotic confidence intervals (ACIs) for each φ_k is given by

$$ACI_k = \left[\hat{\varphi}_k - Z_{\frac{\delta}{2}}\sqrt{\hat{I}_{kk}}, \hat{\varphi}_k + Z_{\frac{\delta}{2}}\sqrt{\hat{I}_{kk}}\right],$$

where $\hat{I}_{kk}$ is the (k,k) diagonal elements of $I_n(\hat{\varphi})^{-1}$ for $k = 1, 2, 3, 4$ and $Z_{\frac{\delta}{2}}$ is the upper δ^{th} percentile of the standard normal distribution.

4.2 | Maximum likelihood for right-censored observations

Similarly, we defined the log-likelihood function of the AMWE model for right-censored data, as follows. Suppose (y_i, δ_i), $i = 1, 2, \ldots, n$ is a censored random sample, where y_i is a failure or survival time for $\delta_i = 1$ and censored time for $\delta_i = 0$. We write the log-likelihood function of the AMWE model as

$$\begin{aligned}\ell_C(\varphi) &= \sum_{i=1}^{n} \delta_i \log f(y_i) + \sum_{i=1}^{n} (1-\delta_i) \log R(y_i) \\ &= \nu \log\left(\gamma\theta^{-\gamma}\right) + (\gamma - 1)\sum_{i=1}^{n} \delta_i \log y_i + \sum_{i=1}^{n} \delta_i (y_i/\theta)^{\gamma} \\ &+ \sum_{i=1}^{n} \delta_i \log c_i - \lambda \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_i^{\alpha},\end{aligned} \tag{17}$$

where a_i and c_i are given in (12), $\nu = \sum_{i=1}^{n} \delta_i$, and $f(.)$ and $R(.)$ are the PDF (3) and survival function (4). The estimate $\widehat{\varphi} = \left(\widehat{\gamma}, \widehat{\alpha}, \widehat{\theta}, \widehat{\lambda}\right)'$ of $(\gamma, \alpha, \theta, \lambda)'$ can be obtained using the log-likelihood function (17) in a similar manner with the above non-censored case.

4.3 | Bayesian estimation

Bayesian inference is an alternative to frequentist inference and have been applied in various fields for parameter estimation, most especially when dealing with a complex situation. In this article, we also proposed the Bayesian paradigm for estimating the parameters of AMWE model (see, (Ibrahim et al. 2001; Robert 2007) for more details). We discuss and construct the Bayesian model by multiplying the likelihood function with a specified prior distribution $\phi(\varphi)$ for $\varphi = (\gamma, \alpha, \theta, \lambda)'$, to obtain the posterior distribution of φ, denoted by $\phi(\varphi|D)$, where the prior $\phi(\varphi)$ denote the distribution of φ before the data $D : x_1, x_2, \ldots, x_n$ are observed. Using Bayes theorem, the posterior distribution of $\varphi|D$ is defined as

$$\phi(\varphi|D) = \frac{L(D|\varphi)\phi(\varphi)}{\int_{\varphi} L(D|\varphi)\phi(\varphi)d\varphi} \propto L(D|\varphi)\phi(\varphi),$$

where $\int_{\varphi} L(D|\varphi)\phi(\varphi)d\varphi$ is the normalizing constant of the posterior distribution of φ, also called the marginal distribution of D. The likelihood function of AMWE is given by

$$L(D|\varphi) = \left(\gamma\theta^{-\gamma}\right)^{n} \left[\prod_{i=1}^{n} A_i\right] \sum_{i=1}^{n} x_i^{\gamma-1} e^{\sum_{i=1}^{n}\left\{\lambda(1-e^{(x_i/\theta)^{\gamma}}) - (e^{(x_i/\theta)^{\gamma}}-1)^{\alpha} + (x_i/\theta)^{\gamma}\right\}},$$

where $A_i = (\lambda + \alpha(e^{(x_i/\theta)^{\gamma}} - 1)^{\alpha-1})$. Suppose that the parameters γ, α, θ, and λ are independent random variables and following Kundu and Howlader (2010),Soliman, Abd-ellah, Abou-elheggag, and Ahmed (2012), and Thach and Briš (2021), these parameters are assumed to have gamma priors. Let $\phi(\varphi_k)$ denote the gamma distribution with parameters (α_k, γ_k) - also called hyper-parameters, with density given by

$$\phi(\varphi_k) = \frac{\alpha_k^{\gamma_k}}{\Gamma(\gamma_k)} \varphi_k^{\gamma_k - 1} e^{-\alpha_k \varphi_k}, \quad \alpha_k > 0, \gamma_k > 0, \; k = 1, 2, 3, 4.$$

Thus, $\gamma \sim \phi(\gamma|\alpha_1, \gamma_1)$, $\alpha \sim \phi(\alpha|\alpha_2, \gamma_2)$, $\theta \sim \phi(\theta|\alpha_3, \gamma_3)$ and $\lambda \sim \phi(\lambda|\alpha_4, \gamma_4)$, and then we have a typical joint prior $\phi(\varphi) = \phi(\gamma|\alpha_1, \gamma_1)\phi(\alpha|\alpha_2, \gamma_2)\phi(\theta|\alpha_3, \gamma_3)\phi(\lambda|\alpha_4, \gamma_4)$. For this study, we assigned the hyper-parameters values which yield means approximate to the MLEs of the estimands. The joint posterior distribution of $\varphi|D$ is given by

$$\begin{aligned}\phi(\varphi|D) &\propto \gamma^{n+\gamma_1-1}\alpha^{\gamma_2}\theta^{-n\gamma+\gamma_3-1}\lambda^{\gamma_4-1}\left[\prod_{i=1}^{n} A_i\right] \\ &\times e^{-\alpha_1\gamma - \alpha_2\alpha - \alpha_3\theta - \alpha_4\lambda + (\gamma-1)\sum_{i=1}^{n}\log x_i} e^{\sum_{i=1}^{n}\left\{\lambda(1-e^{(x_i/\theta)^{\gamma}}) - (e^{(x_i/\theta)^{\gamma}}-1)^{\alpha} + (x_i/\theta)^{\gamma}\right\}}.\end{aligned} \tag{18}$$

The marginal posterior density of γ, α, θ, and λ can be obtained from (18) as

$$\phi(\gamma|D) \propto \gamma^{n+\gamma_1-1}\theta^{-n\gamma}\left[\prod_{i=1}^{n} A_i\right] e^{-\alpha_1\gamma + (\gamma-1)\sum_{i=1}^{n}\log x_i} e^{\sum_{i=1}^{n}\left\{\lambda\left(1-e^{(x_i/\theta)^{\gamma}}\right) - \left(e^{(x_i/\theta)^{\gamma}}-1\right)^{\alpha} + (x_i/\theta)^{\gamma}\right\}}, \tag{19}$$

$$\phi(\alpha|D) \propto \alpha^{\gamma_2} e^{-\alpha_2\alpha}\left[\prod_{i=1}^{n} A_i\right] e^{-\sum_{i=1}^{n}\left(e^{(x_i/\theta)^{\gamma}}-1\right)^{\alpha}}, \tag{20}$$

$$\phi(\theta|D) \propto \theta^{-n\gamma+\gamma_3-1} e^{-\alpha_3\theta}\left[\prod_{i=1}^{n} A_i\right] e^{\sum_{i=1}^{n}\left\{\lambda\left(1-e^{(x_i/\theta)^{\gamma}}\right) - \left(e^{(x_i/\theta)^{\gamma}}-1\right)^{\alpha} + (x_i/\theta)^{\gamma}\right\}}, \tag{21}$$

$$\phi(\lambda|D) \propto \lambda^{\gamma_4-1} e^{-\alpha_4\lambda}\left[\prod_{i=1}^{n} A_i\right] e^{\sum_{i=1}^{n}\lambda\left(1-e^{(x_i/\theta)^{\gamma}}\right)}. \tag{22}$$

The closed-form of the marginal posterior densities (19) - (22) are not available, and hence, one needs to use Markov chain Monte Carlo (MCMC) methods. Here, we use the Metropolis-Hastings algorithm and the Gibbs sampling method to generate samples from the posterior distributions. The algorithm is a general MCMC technique first introduced by Metropolis, Rosenbluth, Rosenbluth, and Teller (1953) and later modified by Hastings (1970). A step-by-step details of the MCMC approach can be found in Gelfand and Smith (1990). We adopt the normal distribution as a proposal density for the Metropolis-Hasting algorithm and below is the description of the algorithm.

Using the samples drawn, the Bayes estimates (BE) of AMWE parameters γ, α, θ, and λ, reliability function $R(t)$ and FR function $h(t)$ under the

Algorithm 1

Given marginal posterior distributions $\phi(\gamma|\alpha,\theta,\lambda,D)$, $\phi(\alpha|\gamma,\theta,\lambda,D)$, $\phi(\theta|\gamma,\alpha,\lambda,D)$, $\phi(\lambda|\gamma,\alpha,\theta,D)$, and sample size N:

Step 1: Select a starting value of the chain $\gamma^{(0)}, \alpha^{(0)}, \theta^{(0)}, \lambda^{(0)}$.

Step 2: Set $m = 1$.

Step 3: Using the Metropolis–Hastings, generate $\gamma^{(m)}$ from $\phi(\gamma|\alpha^{(m-1)}, \theta^{(m-1)}, \lambda^{(m-1)}, D)$.

Step 4: Using the Metropolis–Hastings, generate $\alpha^{(m)}$ from $\phi(\alpha|\gamma^{(m)}, \theta^{(m-1)}, \lambda^{(m-1)}, D)$.

Step 5: Using the Metropolis–Hastings, generate $\theta^{(m)}$ from $\phi(\theta|\gamma^{(m)}, \alpha^{(m)}, \lambda^{(m-1)}, D)$.

Step 6: Using the Metropolis–Hastings, generate $\lambda^{(m)}$ from $\phi(\lambda|\gamma^{(m)}, \alpha^{(m)}, \theta^{(m)}, D)$.

Step 7: Set $m = m + 1$.

Step 8: Repeat step 2 to 7 until $m = N$ to obtain the samples of γ, α, θ, and λ with size N, respectively.

square loss function (SELF) are given by

$$\hat{\gamma} \approx \frac{1}{N-s}\sum_{m=s+1}^{N}\gamma^{(m)}; \qquad \hat{\alpha} \approx \frac{1}{N-s}\sum_{m=s+1}^{N}\alpha^{(m)}$$

$$\hat{\theta} \approx \frac{1}{N-s}\sum_{m=s+1}^{N}\theta^{(m)}; \qquad \hat{\lambda} \approx \frac{1}{N-s}\sum_{m=s+1}^{N}\lambda^{(m)}$$

$$\hat{R}(t) \approx \frac{1}{N-s}\sum_{m=s+1}^{N}R(t;\varphi_m); \qquad \hat{h}(t) \approx \frac{1}{N-s}\sum_{m=s+1}^{N}h(t;\varphi_m),$$

where s is the warm-up/burn-in period, which represents the number of iterations before the stationary samples are determined. The samples are sorted in ascending order as $\gamma_1 \le \gamma_2 \le \cdots \le \gamma_{N-s}$ to allow the computation of the $(1-\delta)100\%$ Bayes credible interval for γ, which is $[\gamma_{\frac{\delta}{2}(N-s)}, \gamma_{(1-\frac{\delta}{2})(N-s)}]$. Similarly, the Bayes credible intervals for α, θ and λ could be obtained. We also compute the approximate $(1-\delta)100\%$ highest posterior density (HPD) interval of φ, estimated as the interval that contains the highest posterior density. All Bayesian computations are carried out in R software using *LaplacesDemon* (Statisticat 2020) package.

5 | REAL DATA APPLICATIONS

In this section, we demonstrate the real applications of AMWE distribution with three different lifetime data sets and present a comparison with some Weibull extended bathtub distributions. The comparisons are done based on $-\ell(\hat{\varphi})$, AIC, BIC, and KS statistic. Bayes (BE) and its associated 95% HPD intervals of the parameters as well as the MLE and BE estimates of the reliability function are presented. Other important posterior summaries of the parameters are also provided.

5.1 | Example I: Lifetime of fifty devices

The data was first used by (Aarset 1987), and represent the failure times of $n = 50$ devices (see, **TABLE** 1). This data has been considered by many authors as the benchmark to fit and compare lifetime distributions with bathtub-shaped FR (see, for example Mudholkar and Srivastava (1993),Almalki and Yuan (2013),Shakhatreh et al. (2019),Ahmad and Ghazal (2020)). The scaled TTT-transform plot (see supp. material Section M2.1) presents the bathtub-shaped of the data.

The MLEs and measure of fit values of the AMWE together with MW, EW, AddW, EMWE, EAddW, and GExtEW models for fitting the data are provided in **TABLE** 2 . Among the fitted models, we find that the four parameters AMWE possessed the largest log-likelihood value, $\ell(\hat{\varphi})$ and produce the smallest AIC, BIC, and KS values which depicts that it best fitted the data compare to MW, EW, AddW, EMWE, EAddW, and GExtEW. Further, the AMWE distribution has fewer parameters than EAddW and GExtEW and the same number of parameters with AddW and EMWE distributions. **FIGURE** 5 presents the estimated PDFs and FR functions of the AMWE, MW, EW, AddW, EMWE, EAddW, and GExtEW models for fitting the Aarset data. The AMWE is shown to be close to the non-parametric estimate (represented by the "Step function") of the FR function, which was obtained using the “pehaz” function in the "muhaz" package (Gentleman 2019).

TABLE 1 The lifetime of 50 electronic devices.

0.1	0.2	1.0	1.0	1.0	1..0	1.0	2.0	3.0	6.0
7.0	11	12	18	18	18	18	18	21	32
36	40	45	46	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

We used Algorithm 1 to simulate 5000 samples from each of the marginal posterior distributions considering the first 1000 as warm-up. The last 4000 posterior samples were managed to determine the Bayes estimate (BE) of γ, α, θ, and λ, and their corresponding 95% HPD intervals (see, **TABLE** 3). From the table, we can see the MLEs and BEs together with their 95% asymptotic confidence intervals (ACIs) and HPD intervals, respectively. The intervals have well contained the estimates, with HPD having narrower intervals. Measuring the closeness between the empirical distribution and the fitted AMWE distribution for both the approaches using the Kolmogorov-Smirnov (KS) statistic indicates that the maximum likelihood slightly better fit with KS = 0.094 (p-value = 0.7725) than the Bayesian method (KS = 0.097 (p-value = 0.7362)).

The trace plots and marginal posterior densities of the Gibbs samples of γ, α, θ, λ, $\mathrm{R}(t = 32)$, and $\mathrm{h}(t = 32)$ of the AMWE distribution are given (see supp. material Section M2.1). The trace plots in reveal that the Gibbs sampler is mixing well. The densities provide good BEs under the SELF as they are distributed approximately symmetrically around the central values.

The MLEs and BEs of the AMWE reliability function at each failure time x is computed (see supp. material Section M2.1). We can see that both two estimation methods agree with each other at lower values of x, and BE provide better estimates at the upper values of x, for instance, at $x = 86.0$, the BE survival probability ($\hat{\mathrm{R}}_{\mathrm{BE}}(x)$) is 0.0260 compare to $\hat{\mathrm{R}}_{\mathrm{MLE}}(x)$ with 0.0153. Comparison between the MLEs and BEs of the AMWE FR function is shown (see supp. material Section M2.1). The two curves are in contact with each other and reveal a longer useful lifetime than the other fitted models and hence described the real behaviour of the data. We further give the corresponding P-P plot of the AMWE model and other models in **FIGURE** 6 to show more information about the appropriateness of the fitted models for the Aarset data graphically. From these figures, we can conclude that the AMWE distribution is a better option for the data used.

TABLE 2 MLEs, $\ell(\hat{\varphi})$, AIC, BIC, and KS statistic for the fitted models; Aarset data set

Model	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\ell(\hat{\varphi})$	AIC	BIC	KS (p-value)
3-parameters									
MW	0.023	0.063			0.360	-227.155	460.320	466.047	0.134 (0.328)
EW	4.671	0.011			0.145	-229.136	464.272	470.008	0.206 (0.0291)
4-parameters									
AMWE	0.630	96.711	151.931	1.290		-205.053	418.106	425.754	0.094 (0.7725)
AddW	1.622×10^{-8}	4.134	0.090	0.461		-221.714	451.427	459.075	0.126 (0.4073)
EMWE	3.165	7.020×10^{-5}	49.206		0.144	-213.816	435.632	443.280	0.140 (0.2832)
5-parameters									
EaddW	5.368×10^{-15}	7.541	4.902	0.056	345.18	-215.855	441.711	451.271	0.119 (0.4783)
GExEW	0.088	34.029	0.508	0.003	0.128	-221.587	453.173	462.733	0.151 (0.2035)

TABLE 3 MLEs, BEs and their 95% interval estimates for the fitted AMWE parameters; Aarset data set

Parameter	MLE	95% ACI	BE	95% HPD
γ	0.6302	[0.4478, 0.8126]	0.6336	[0.5548, 0.7144]
α	96.711	[41.121, 152.30]	95.548	[63.897, 128.03]
θ	151.93	[126.43, 177.43]	151.82	[140.68, 162.62]
λ	1.2904	[0.8549, 1.7260]	1.3080	[1.0709, 1.5703]

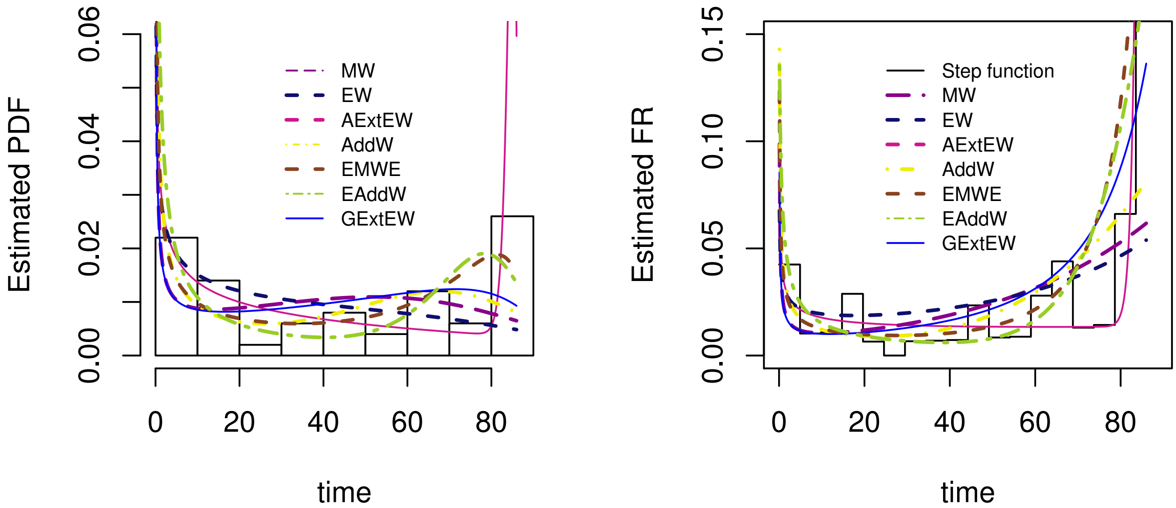

FIGURE 5 (Left) Estimated density functions and (right) estimated failure rate functions of the fitted models; Aarset data set

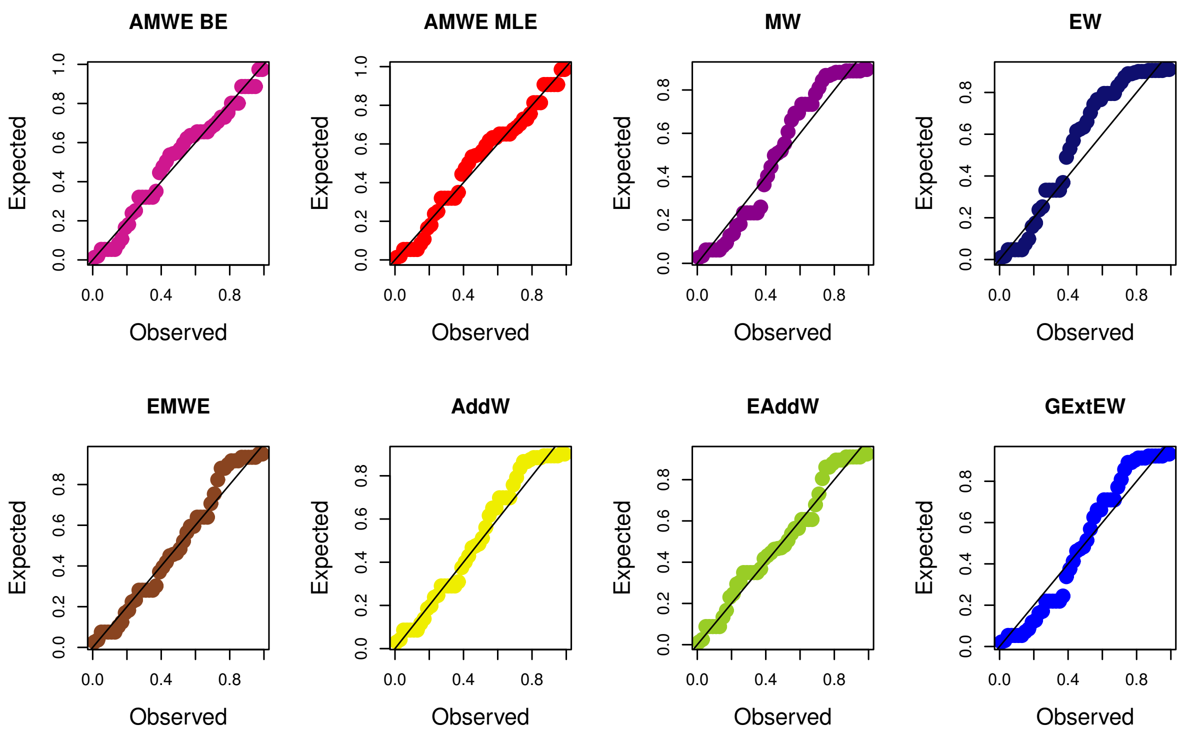

FIGURE 6 PP plots of the fitted AMWE distribution and other models; Aarset data set

5.2 | Example II: Lifetime of thirty devices

The second data used is obtained from Meeker and Escoba (1998), representing the failure and running times for $n = 30$ samples of devices from a large system field tracking study. It is a well-known lifetime dataset with the bathtub FR function used by many authors (for instance, Sarhan and Apaloo (2013),Almalki and Yuan (2013),Ahmad and Ghazal (2020)). The data is provided in **TABLE** 4, and its bathtub-shaped FR is shown by the convex followed by a concave scaled TTT-transform plot (see supp. material Section M2.2).

TABLE 4 Failure times of 30 devices.

275	13	147	23	181	30	65	10	300	173
106	300	300	212	300	300	300	2.0	261	293
88	247	28	143	300	23	300	80	245	266

The MLEs and measure of fit values of the fitted model for fitting the data are presented in **TABLE** 5 . Among the fitted distributions, we observe that the four parameters AMWE has the largest $\ell(\hat{\varphi})$ value with minimum values of AIC, BIC, and KS which indicates that it provides the best fit for the data compare to MW, EW, AddW, EMWE, EAddW, and GExtEW. Further, the AMWE distribution has fewer parameters than EAddW and GExtEW and the same number of parameters with AddW and EMWE distributions. **FIGURE** 7 presents the estimated PDFs and FR functions of the AMWE, MW, EW, AddW, EMWE, EAddW, and GExtEW models for fitting the Aarset data. The AMWE is shown to be close to the non-parametric estimate (represented by the "Step function") of the FR function, obtained using the “pehaz” function in the "muhaz" package **?**.

Similarly, as in the first illustration, we used Algorithm 1 to simulate 5000 samples from each of the marginal posterior distributions considering the first 1000 as burn-in. The last 4000 posterior samples were managed to determine the Bayes estimate (BE) of γ, α, θ, and λ, and their corresponding 95% HPD intervals (see, **TABLE** 6). From the table, we can see the MLEs and BEs together with their 95% asymptotic confidence intervals (ACIs) and HPD intervals, respectively. The intervals for the parameters have well contained the estimates, with HPD having narrower intervals. Closeness between the empirical distribution and the fitted AMWE distribution for the two estimation techniques is measured using the Kolmogorov-Smirnov (KS) statistic. The results indicated that the maximum likelihood produced better fit for the Meeker-Escoba data with KS = 0.149 (p-value = 0.5159) than the Bayesian method (KS = 0.2176 (p-value = 0.1168)).

Trace plots and marginal posterior densities of the Gibbs samples of γ, α, θ, λ, $\mathrm{R}(t = 181)$, and $h(t = 181)$ of the AMWE distribution are presented (see supp. material Section M2.2). The trace plots show that the Gibbs sampler is mixing well while densities indicate good BEs under the SELF as they are distributed approximately symmetrically around the central values.

We present the MLEs and BEs of the AMWE reliability function at each failure time x is obtained (see supp. material Section M2.2). We can see that both two estimation methods slightly agree with each other over all the values of x, with MLE having relatively better estimates. For instance, at $x = 106.0$, the MLE survival probability ($\hat{R}_{MLE}(x)$) is 0.5475 compare to $\hat{R}_{BE}(x)$ with 0.5277. The comparison between the MLEs and BEs of the AMWE FR function is shown graphically (see supp. material Section M2.2). The two curves are in contact with each other and depict a better bathtub-shaped than the other fitted models and hence described the real behaviour of the data. Supplement to the aforementioned results, we give the corresponding P-P plot of the AMWE model and other competing models in **FIGURE** 8 to demonstrate more information about the appropriateness of the fitted models for the data set graphically. From these figures, one can infer that the AMWE distribution is a good option for the Meeker and Escoba data.

TABLE 5 MLEs, $\ell(\hat{\varphi})$,AIC, BIC, and KS of the fitted models; Meeker-Escoba data set

Model	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\ell(\hat{\varphi})$	**AIC**	**BIC**	**KS** (p-value)
3-parameters									
MW	0.0182	0.0071			0.4529	-178.065	362.131	366.334	0.178 (0.2989)
EW	5.5320	0.0030			0.1620	-177.915	361.829	366.033	0.216 (0.1219)
4-parameters									
AMWE	6.896	0.095	229.2	0.002		-163.076	334.151	339.756	0.149 (0.5159)
AddW	4.560×10^{-8}	3.026	0.033	0.485		-177.771	363.543	369.147	0.178 (0.2996)
EMWE	4.469	5.407×10^{-6}	197.48		0.129	-166.388	340.775	346.380	0.181 (0.2787)
5-parameters									
EaddW	3.915×10^{-8}	3.093	3.473	0.064	137.46	-175.422	360.844	367.850	0.154 (0.4737)
GExEW	0.028	1.463	2.262	0.003	114.40	-183.031	376.062	383.0682	0.206 (0.1556)

TABLE 6 MLEs, BEs and their 95% interval estimates for the fitted AMWE parameters; Meeker-Escoba data set

Parameter	MLE	95% CI	BE	95% HPD
γ	6.896	[6.8741, 6.9180]	6.816	[6.8154, 6.8173]
α	0.095	[0.0587, 0.1320]	0.089	[0.0706, 0.1070]
θ	229.2	[224.87, 233.55]	222.0	[221.85, 222.17]
λ	0.0017	[0.0011, 0.0023]	0.0004	[0.0002, 0.0006]

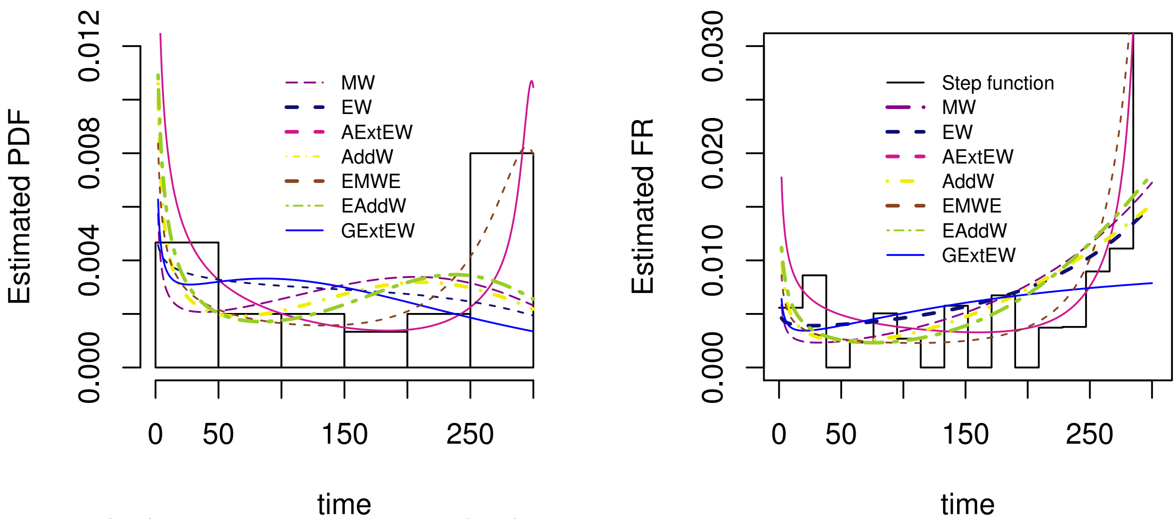

FIGURE 7 (Left) Estimated density functions and (right) estimated failure rate functions of the fitted models; Meeker-Escoba data set

5.3 | Example III: Survival times of twenty male rats

The third data is obtained from (Lawless 2003) (chapter 4, p168), and it represents the survival times in weeks for $n = 20$ male rats which are exposed to a high level of radiation. We present the data in **TABLE** 7, and its increasing FR is shown by the concave scaled TTT-transform plot (see supp. material Section M2.3).

TABLE 7 Survival times of 20 male rats (in weeks) exposed to high radiation.

152	152	115	109	137	88	94	77	160	165
125	40	128	123	136	101	62	153	83	69

TABLE 8 presents the MLEs of the AMWE together with MW, EW, EMWE, EAddW, and GExtEW models for fitting the data set. The measure of fit values for the fitted models are also provided in **TABLE** 8. Between the fitted distributions, we notice that the four parameters AMWE has the largest log-likelihood value, $\ell(\hat{\varphi})$ with minimum values of AIC, BIC, and KS, respectively, which shows that the new model offers the best fit for the data compare to MW, EW, EMWE, EAddW, and GExtEW. However, the competing models most especially, the MW, EW and EMWE have presents a reasonable fit for the data which may be due to the data increasing FR that is as well possessed by the models. **FIGURE** 9 describes the estimated PDFs and FR functions of the AMWE, MW, EW, EMWE, EAddW, and GExtEW models for fitting the rats survival times data. The AMWE is shown to be close to the non-parametric estimate.

Equally, as in other demonstrations, we applied Algorithm 1 to simulate 5000 samples from each of the marginal posterior distributions considering the first 1000 as warm-up. The last 4000 posterior samples were handled to compute the Bayes estimate (BE) of γ, α, θ and λ, and the

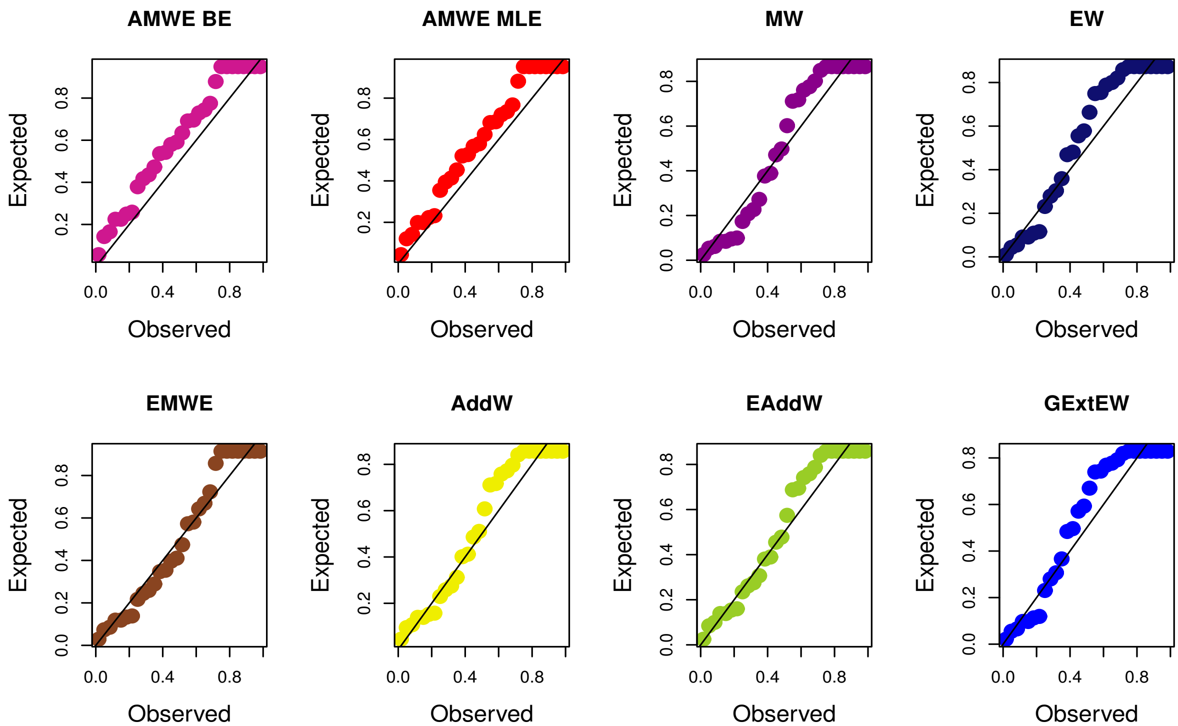

FIGURE 8 PP plots of the fitted AMWE distribution and other models; meeker and Escoba data set

corresponding 95% HPD intervals (see, **TABLE** 9). From the table, we can see the MLEs and BEs together with their 95% asymptotic confidence intervals (ACIs) and HPD intervals, respectively. In this case, the BE reveal a slightly better fit with KS = 0.110 (p-value = 0.9676) than the maximum likelihood with KS = 0.118 (p-value = 0.9435). The intervals for the parameters have well contained the estimates, with HPD giving thinner intervals.

We provides the trace plots and marginal posterior densities of the Gibbs samples of $\gamma, \alpha, \theta,\ \lambda, \mathrm{R}(t = 77)$, and $\mathrm{h}(t = 77)$ of the AMWE distribution (See supp. material Section M2.3). The trace plots in Figure 5 shows that the Gibbs sampler is mixing well. The densities provide good BEs under the SELF.

We also obtain the MLEs and BEs of the AMWE probability of survival at each survival time x (See supp. material Section M2.3). We notice that the BEs maintain a better estimate compare to the MLEs. For example, at x = 94.0, the BE survival probability ($\hat{R}_{BE}(x)$) is 0.7444 compare to $\hat{R}_{MLE}(x)$ with 0.7240. Comparison between the MLEs and BEs of the AMWE FR function is equally shown graphically (see supp. material Section M2.3). The two curves are in contact with each other and describe the actual behaviour of the male rats survival times. In addition, we provide the P-P plot of the AMWE model and other competing distributions in **FIGURE** 10 to demonstrate more details about the suitability of the fitted models for fitting the rats survival times set graphically. From these figures, one can observe that the AMWE and EMWE distribution are two better option for the data.

Other related posterior summaries of the parameters including the standard deviations (SDs), Monte Carlo errors (MC errors), Median and 95% credible intervals (Cred. I.) are given in **TABLE** 10.

6 | CONCLUSION

This paper defined and studied a new generalized Weibull distribution called the modified Weibull-Weibull (MWW) distribution. It is a three-parameter flexible distribution with the ability to accommodates monotone and non-monotone failure rates lifetime data. We obtain explicit expressions for the quantile function, moment generating function, moments, incomplete moments, mean deviation about the mean, mean deviation about the median, Lorenz curve, Rényi entropy, and Mathai-Houbold entropy. We conduct a Monte Carlo simulation study to obtain some numerical results for the mean, variance, skewness, kurtosis, mean deviation about the mean and mean deviation about the median. We also characterize the MWW model based on two truncated moments and in terms of the hazard function. Estimation of the distribution parameters is performed using the method of maximum likelihood, and the estimation method is assessed by Monte Carlo simulation experiments which yield

TABLE 8 MLEs, $\ell(\hat{\varphi})$, AIC, BIC, and KS of the fitted models; rats survival times

Model	$\hat{\gamma}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\beta}$	$\ell(\hat{\varphi})$	AIC	BIC	KS (p-value)
3-parameters									
MW	0.0318	0.0071			0.1673	-98.9846	203.9692	206.9564	0.126 (0.9105)
EW	2.0352	0.0108			2.5191	-100.056	206.1124	209.0996	0.131 (0.8824)
4-parameters									
AMWE	10.652	0.2576	142.0	0.0040		-97.242	202.4836	206.4665	0.118 (0.9435)
EMWE	0.4598	9.09×10^{-9}	0.2221		0.3495	-98.4733	204.9466	208.9295	0.118 (0.9420)
5-parameters									
EaddW	0.0040	1.0097	0.0159	1.0792	12.630	-101.4476	212.8953	217.8739	0.147 (0.7774)
GExEW	0.0301	1.2473	0.6382	0.0181	21.4262	-100.997	211.994	216.9727	0.137 (0.8468)

TABLE 9 MLEs, BEs and their 95% interval estimates for the fitted AMWE parameters; rats survival times

Parameter	MLE	95% CI	BE	95% HPD
γ	10.652	[0.0000, 23.529]	10.698	[5.0284, 16.972]
α	0.2576	[0.0000, 0.5784]	0.2788	[0.1325, 0.4400]
θ	141.95	[129.19, 154.71]	141.62	[134.14, 149.84]
λ	0.0040	[0.0000, 0.0504]	0.0154	[1.4×10^{-7}, 0.078]

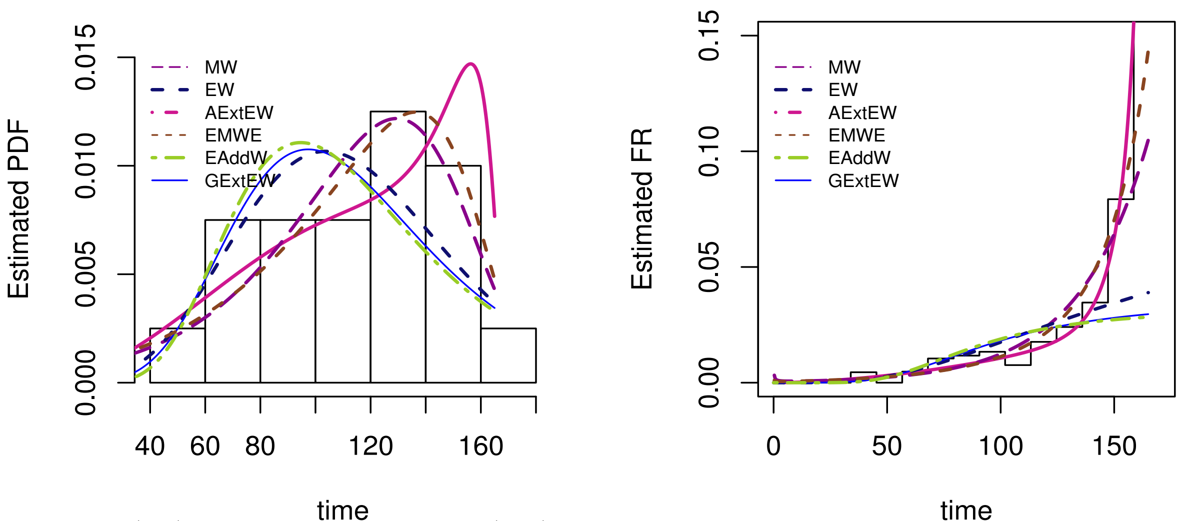

FIGURE 9 (Left) Estimated density functions and (right) estimated failure rate functions of the fitted models; rats survival times

consistent estimates in the samples considered. Two failure time data having non-monotone failure rate functions are analyzed to demonstrate the potentiality of the distribution. The two applications have explored the flexibility of the MWW model over the WW model and other Weibull extended distributions.

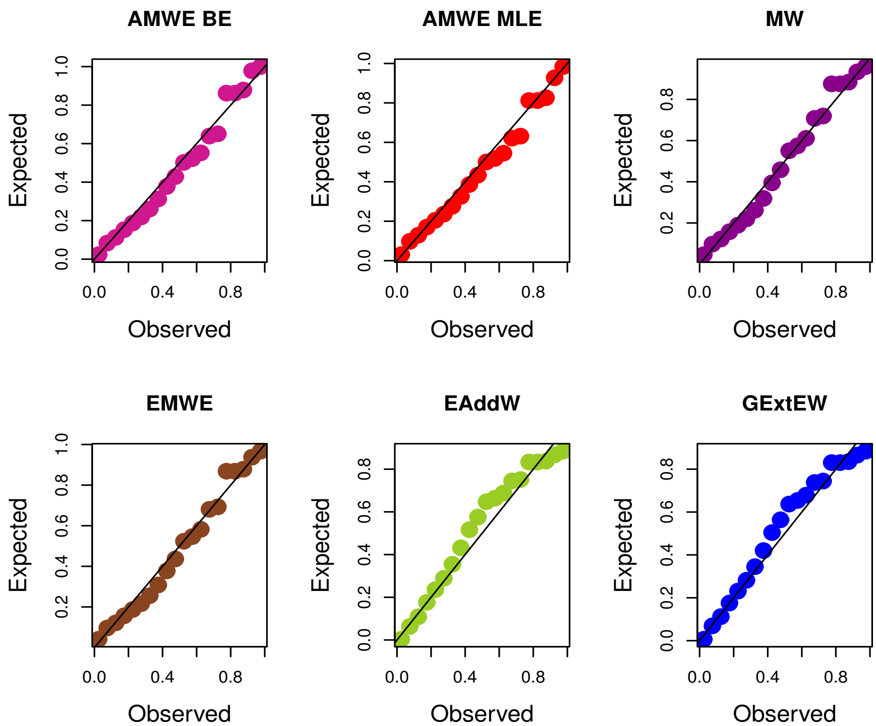

FIGURE 10 PP plots of the fitted AMWE distribution and other models; Rats survival times

TABLE 10 Other posterior summary statistics of the fitted AMWE parameters for the three data sets

Data	Parameter	BE	SD	MC error	Median	95% Cred. I.
Aarset	γ	0.6336	0.0426	0.0012	0.6321	[0.5585, 0.7189]
	α	95.548	16.645	0.4679	94.365	[66.369, 131.59]
	θ	151.83	5.8704	0.1723	151.66	[141.11, 163.07]
	λ	1.3080	0.1312	0.0035	1.2997	[1.0791, 1.5898]
Meeker-Escoba	γ	6.8164	0.0005	9.39×10^{-6}	6.8164	[6.8154, 6.8173]
	α	0.0888	0.0096	0.0002	0.0885	[0.0715, 0.1085]
	θ	222.01	0.0827	0.0016	222.01	[221.84, 222.17]
	λ	0.0004	9.83×10^{-5}	1.83×10^{-6}	0.0004	[0.0003, 0.0006]
Rats survival times	γ	10.698	3.2490	0.0742	10.167	[5.9179,18.717]
	α	0.2788	0.0832	0.0019	0.2666	[0.1504, 0.4687]
	θ	141.62	4.0419	0.0894	141.55	[133.83, 149.62]
	λ	0.0154	0.0351	0.0007	0.0027	[2×10^{-5}, 0.1122]

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Author contributions

Financial disclosure

Conflict of interest

The authors declare no potential conflict of interests.

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