Appendix B

Solutions to Problems in Chapter 2

B.1 Problem 2.1

Figure B.1a shows the two parallel wires with a distance of $d = 1$ cm. Each wire carries a DC current with a magnitude of $I_0 = 10$ mA.

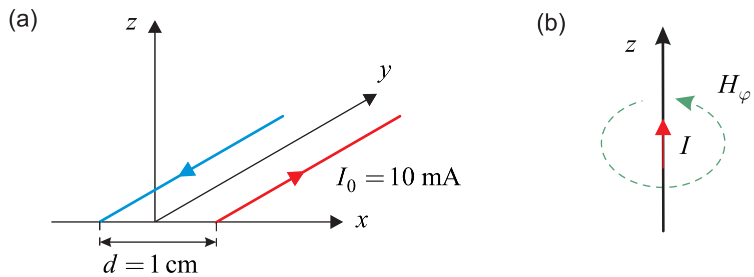


Figure B.1: Two parallel wires in y-direction; (b) single wire in z-direction

Magnetic field of a single wire

We start by looking at the magnetic field of a *single wire*. In contrast to our original problem the current shall flow in z-direction in order to apply the cylindrical coordinate system.

We use Ampere's law and write

$$\oint_{C(A)} \vec{H} \cdot \mathrm{d}\vec{s} = \iint_{A} \left(\vec{J} + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\to 0} \right) \cdot \mathrm{d}\vec{A} \tag{B.1}$$

where the time derivative is zero for DC current ($\partial/\partial t \to 0$).

From the right-hand rule in Section 2.1.2.2 we expect a magnetic field circulating around the single wire. Figure B.1b and B.2a illustrate the magnetic field lines. The magnetic field has only a φ-component. There are no R- and z-components. The magnitude of the magnetic field is *constant* on circles around the z-axis. Therefore, we choose an integration area A that corresponds to a circle centred around the z-axis (Figure B.2b).

Lets look at the left-hand side of Equation (B.1). The boundary curve $C(A)$ of the line integral equals the circumference of the circle with radius R. Along the boundary curve the

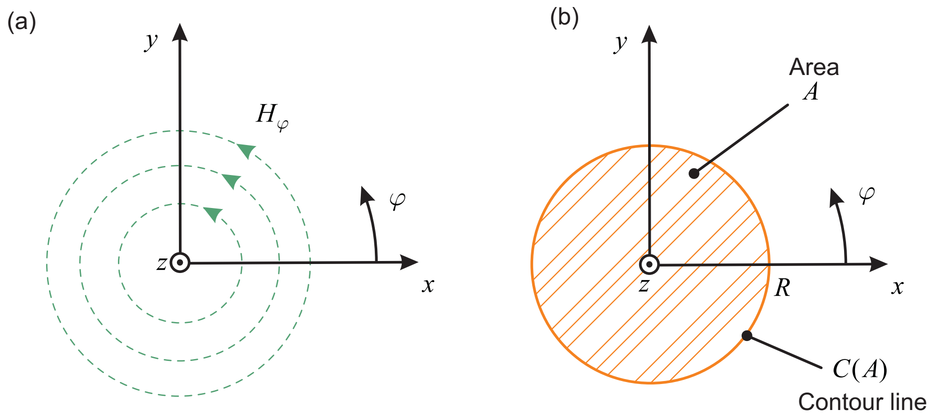


Figure B.2: (a) Distribution of magnetic field strength and (b) integration area

magnitude of the magnetic field strength is constant and the vector of the magnetic field strength points into the direction of the line segment. Hence, using cylindrical coordinates the left-hand side of Equation (B.1) becomes

$$\oint\limits_{C(A)} \vec{H} \cdot \mathrm{d}\vec{s} = \int\limits_0^{2\pi} H_\varphi \mathrm{d}s_\varphi = \int\limits_0^{2\pi} H_\varphi R \mathrm{d}\varphi = 2\pi R H_\varphi \tag{B.2}$$

The right-hand side of Equation (B.1) equals the current through the area A.

$$\iint\limits_A \vec{J} \cdot \mathrm{d}\vec{A} = I \tag{B.3}$$

Bringing together left- and right-hand side allows us to calculate the magnetic field strength of a single wire as

$$H_\varphi = \frac{I}{2\pi R} \quad \text{or} \quad \vec{H} = \frac{I}{2\pi R}\vec{e}_\varphi \tag{B.4}$$

a) Magnetic field of two parallel wires in a cross-sectional area (xz-plane)

Our original problem consists of two parallel wires running in y-direction and separated by a distance d. The total magnetic field strength is given by superposition as shown in Figure B.3.

b) Magnetic field of two parallel wires on the x-axis

On the x-axis the magnitude of magnetic field $H_1(x)$ of the (blue) wire (Figure B.1) is determined by Equation (B.4). The orientation (direction of the vector) is evaluated by inspecting the arrows in Figure B.3

$$\vec{H}_1(x) = \frac{I}{2\pi(x + d/2)}\vec{e}_z \tag{B.5}$$

Likewise, the magnetic field $H_2(x)$ of the (red) wire (Figure B.1) is given by

$$\vec{H}_2(x) = \frac{-I}{2\pi(x - d/2)}\vec{e}_z \tag{B.6}$$

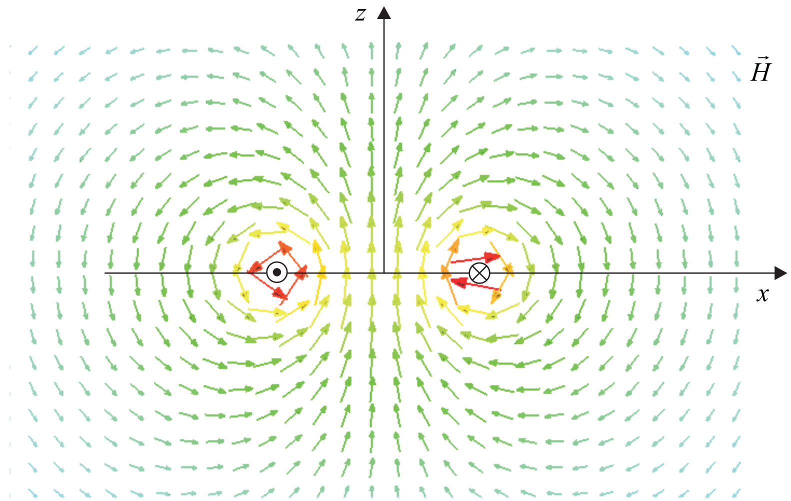


Figure B.3: Arrows of the magnetic field strength in a cross-section (xz-plane)

The total field strength of both wires then becomes

$$\begin{aligned}\vec{H}(x) = \vec{H}_1(x) + \vec{H}_2(x) &= \left[\frac{I}{2\pi(x+d/2)} + \frac{-I}{2\pi(x-d/2)}\right]\vec{e}_z \\ &= \frac{I}{2\pi}\left[\frac{x-d/2-(x+d/2)}{x^2-(d/2)^2}\right]\vec{e}_z = -\frac{I}{2\pi}\left[\frac{d}{x^2-d^2/4}\right]\vec{e}_z \end{aligned} \tag{B.7}$$

Figure B.4 shows a plot of the magnetic field strength $H_z(x)$ as a function of x.

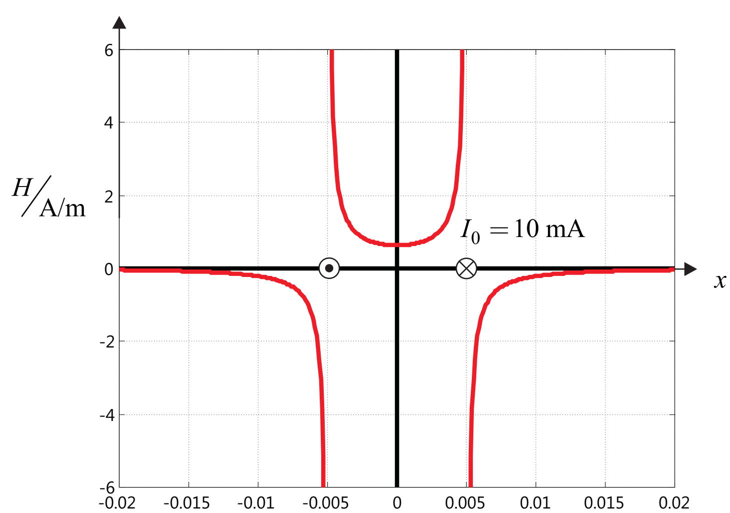


Figure B.4: Magnitude of the magnetic field strength $H_z(x)$ along the x-axis

B.2 Problem 2.2

Figure B.5 shows an air-filled coaxial line ($\varepsilon_\mathrm{r} = \mu_\mathrm{r} = 1$) with DC current $I = 1$ A flowing in inner and outer conductor. The electric conductivity σ of the conductors shall be finite.

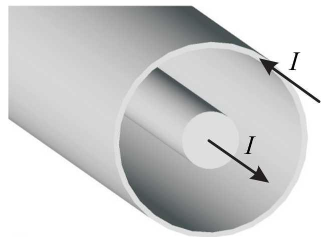


Figure B.5: Geometry of a coaxial line

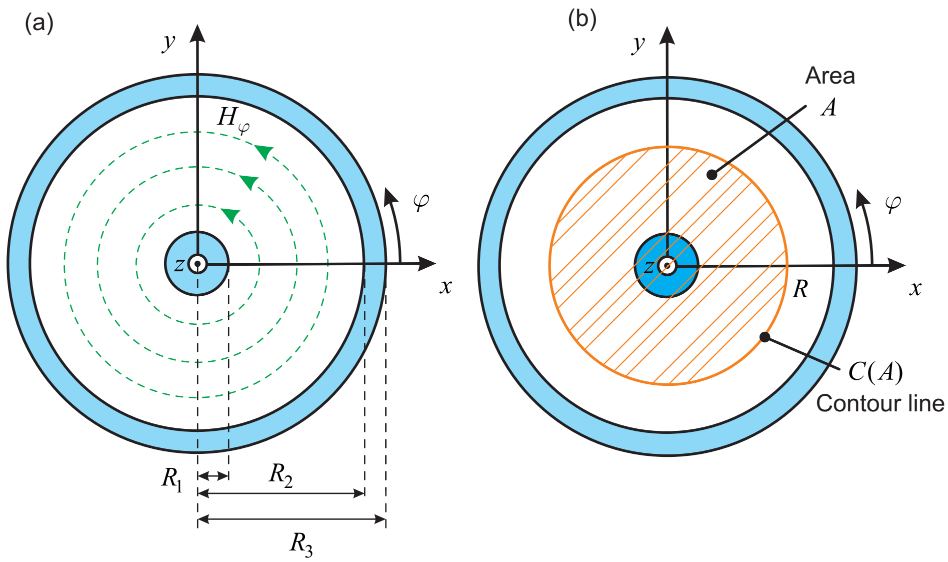


Figure B.6: (a) Distribution of magnetic field strength and (b) integration area

Magnetic field strength on the line

Let us start with Ampere's law.

$$\oint\limits_{C(A)} \vec{H} \cdot \mathrm{d}\vec{s} = \iint\limits_{A} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot \mathrm{d}\vec{A} \tag{B.8}$$

In order to apply cylindrical coordinates we assume that the line extends in z-direction (Figure B.6). As in Problem 2.1 we expect a circulating magnetic field in φ-direction and select a circular integration area A with radius R.

We evaluate Ampere's law in four different regions:

1. Inside the inner conductor ($R \leq R_1$)

2. Between inner and outer conductor (air-filled region) ($R_1 \leq R \leq R_2$)
3. Inside the outer conductor ($R_2 \leq R \leq R_3$)
4. Beyond the outer conductor ($R \geq R_3$)

In the non-time-varying case (DC) the currents of inner and outer conductor spread homogeneously across the cross-sectional areas. Hence, the corresponding current densities J_i and J_o are

$$J_\mathrm{i} = \frac{I}{\pi R_1^2} \tag{B.9}$$

$$J_\mathrm{o} = \frac{I}{\pi \left(R_3^2 - R_2^2\right)} \tag{B.10}$$

a1) Magnetic field strength in the inner conductor ($R \leq R_1$)

Ampere's law for steady currents is

$$\oint\limits_{C(A)} \vec{H} \cdot \mathrm{d}\vec{s} = \iint\limits_{A} \left(\vec{J} + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\to 0} \right) \cdot \mathrm{d}\vec{A} \tag{B.11}$$

Since $R \leq R_1$ only a fraction of the current I passes through the integration area A. With $\mathrm{d}\vec{A} = \vec{e}_z$ and $\vec{J} = J_\mathrm{i}\vec{e}_z = \text{const.}$ we write

$$H_\varphi 2\pi R = \pi R^2 J_\mathrm{i} \tag{B.12}$$

Therefore, the magnetic field is given as

$$H_\varphi = \frac{J_\mathrm{i} R}{2} = \frac{IR}{2\pi R_1^2} \quad \text{for} \quad R \leq R_1 \tag{B.13}$$

a2) Magnetic field between inner and outer conductor ($R_1 \leq R \leq R_2$)

Now, the integration area A includes the complete inner conductor and the full current I passes through the area.

$$H_\varphi 2\pi R = I \tag{B.14}$$

The magnetic field then becomes

$$H_\varphi = \frac{I}{2\pi R} = \frac{J_\mathrm{i} R_1^2}{2R} \quad \text{for} \quad R_1 \leq R \leq R_2 \tag{B.15}$$

a3) Magnetic field inside the outer conductor ($R_2 \leq R \leq R_3$)

Now the full current of the inner conductor and a fraction of the current in the outer conductor flow through the integration area A. The currents in inner and outer conductor flow in opposite directions.

$$
\begin{aligned}
H_\varphi 2\pi R &= I - \int_0^{2\pi}\int_{R_2}^{R} \vec{J}_\mathrm{o}\cdot \mathrm{d}\vec{A}_z \quad \text{where} \quad \mathrm{d}\vec{A}_z = R\,\mathrm{d}\varphi\,\mathrm{d}R\,\vec{e}_z \\
&= I - J_\mathrm{o}2\pi \int_{R_2}^{R} R\,\mathrm{d}R = I - J_\mathrm{o}2\pi\left[\frac{1}{2}R^2\right]_{R_2}^{R} \\
&= I - J_\mathrm{o}2\pi\left[\frac{1}{2}R^2 - \frac{1}{2}R_2^2\right] = I - \frac{I}{\pi R_3^2 - \pi R_2^2}\left[\pi R^2 - \pi R_2^2\right]
\end{aligned}
\tag{B.16}
$$

For the magnetic field we get

$$
H_\varphi = \frac{I}{2\pi R}\left(1 - \frac{R^2 - R_2^2}{R_3^2 - R_2^2}\right) = \frac{I}{2\pi R}\left(\frac{R_3^2 - R^2}{R_3^2 - R_2^2}\right) \quad \text{for} \quad R_2 \le R \le R_3 \tag{B.17}
$$

a4) Magnetic field beyond the outer conductor ($R \ge R_3$)

Now the integration area includes both conductors so the net current is zero.

$$
H_\varphi 2\pi R = 0 \tag{B.18}
$$

Consequently, there is no magnetic field outside the coaxial line.

$$
H_\varphi = 0 \quad \text{for} \quad R \ge R_3 \tag{B.19}
$$

Figure B.7 shows the magnitude of the magnetic field strength on the x-axis.

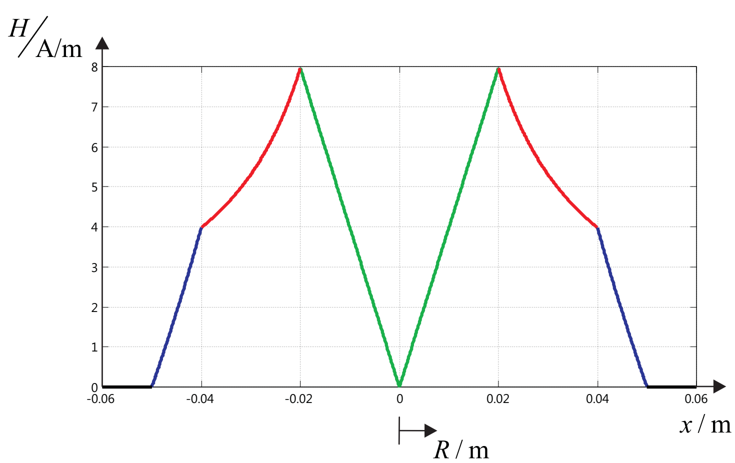


Figure B.7: Magnitude of the magnetic field strength along the x-axis (green: inside the inner conductor; red: in the air-filled region; blue: in the outer conductor; black: outside the coaxial line)

b) Inductance of a line segment (length $\Delta\ell$)

The magnetic field energy W_m is given by

$$W_\mathrm{m} = \iiint_V \frac{1}{2}\vec{B}\cdot\vec{H}\,\mathrm{d}v = \frac{1}{2}LI^2 \quad \text{where} \quad \vec{B} = \mu\vec{H} \tag{B.20}$$

The integral covers all areas where the magnetic field is non-zero. In Problem section 2.2a we considered three regions with non-zero field.

b1) Magnetic field energy within the inner conductor ($R \leq R_1$)

In cylindrical coordinates we get

$$\begin{aligned} W_\mathrm{m,i} &= \int_0^{R_1}\int_0^{2\pi}\int_0^{\Delta\ell} \frac{1}{2}\mu_0\left(\frac{IR}{2\pi R_1^2}\right)^2 \underbrace{\mathrm{d}z\,\mathrm{d}\varphi\,R\mathrm{d}R}_{\mathrm{d}v} \\ &= \frac{1}{2}\mu_0\frac{I^2}{(2\pi)^2R_1^4}\underbrace{\int_0^{\Delta\ell}\mathrm{d}z}_{\Delta\ell}\underbrace{\int_0^{2\pi}\mathrm{d}\varphi}_{2\pi}\underbrace{\int_0^{R_1}R^2R\mathrm{d}R}_{\frac{1}{4}R^4\big|_0^{R_1}=\frac{1}{4}R_1^4} \end{aligned} \tag{B.21}$$

The magnetic energy inside the inner conductor is

$$W_\mathrm{m,i} = \frac{1}{2}\mu_0(\Delta\ell)\frac{I^2}{8\pi} = \frac{1}{2}L_\mathrm{i}I^2 \tag{B.22}$$

The inductance per unit length becomes

$$L_\mathrm{i}' = \frac{L_\mathrm{i}}{\Delta\ell} = \frac{\mu_0}{8\pi} \tag{B.23}$$

b2) Magnetic field energy in the air-filled region between inner and outer conductor ($R_1 \leq R \leq R_2$)

In cylindrical coordinates we get

$$\begin{aligned} W_\mathrm{m,air} &= \int_{R_1}^{R_2}\int_0^{2\pi}\int_0^{\Delta\ell} \frac{1}{2}\mu_0\left(\frac{I}{2\pi R}\right)^2 \underbrace{\mathrm{d}z\,\mathrm{d}\varphi\,R\mathrm{d}R}_{\mathrm{d}v} \\ &= \frac{1}{2}\mu_0\frac{I^2}{(2\pi)^2}\underbrace{\int_0^{\Delta\ell}\mathrm{d}z}_{\Delta\ell}\underbrace{\int_0^{2\pi}\mathrm{d}\varphi}_{2\pi}\underbrace{\int_{R_1}^{R_2}\frac{1}{R^2}R\mathrm{d}R}_{\ln R\big|_{R_1}^{R_2}=\ln\left(\frac{R_2}{R_1}\right)} \end{aligned} \tag{B.24}$$

The magnetic energy then is

$$W_\mathrm{m,air} = \frac{1}{2}\mu_0\ell\frac{I^2}{2\pi}\ln\left(\frac{R_2}{R_1}\right) = \frac{1}{2}L_\mathrm{air}I^2 \tag{B.25}$$

The inductance per unit length becomes

$$L_\mathrm{air}' = \frac{L_\mathrm{air}}{\Delta\ell} = \frac{\mu_0}{2\pi}\ln\left(\frac{R_2}{R_1}\right) \tag{B.26}$$

b3) Magnetic field energy inside the outer conductor ($R_2 \leq R \leq R_3$)

In cylindrical coordinates we start with

$$W_{\mathrm{m,o}} = \int_{R_2}^{R_3} \int_{0}^{2\pi} \int_{0}^{\Delta\ell} \frac{1}{2}\mu_0 \left(\frac{I}{2\pi R} \left(\frac{R_3^2 - R^2}{R_3^2 - R_2^2} \right) \right)^2 \underbrace{\mathrm{d}z\, \mathrm{d}\varphi\, R \mathrm{d}R}_{\mathrm{d}v} \tag{B.27}$$

and calculate the magnetic energy as

$$W_{\mathrm{m,o}} = \frac{1}{2}\mu_0 \frac{I^2 2\pi(\Delta\ell)}{(2\pi)^2 \left(R_3^2 - R_2^2\right)} \underbrace{\int_{R_\mathrm{A}}^{R_0} \left[\frac{R_3^4}{R^2} - 2R_3^2 R + R^3 \right] R \mathrm{d}R}_{R_3^4 \ln R\big|_{R_2}^{R_3} - 2R_3^2 \frac{1}{2} R^2\big|_{R_2}^{R_3} + \frac{1}{4} R^4\big|_{R_2}^{R_3}} \tag{B.28}$$

The inductance per unit length is

$$L'_\mathrm{o} = \frac{L_\mathrm{o}}{\Delta\ell} = \frac{\mu_0}{2\pi} \left[\frac{R_3^4}{\left(R_3^2 - R_2^2\right)} \ln\left(\frac{R_3}{R_2}\right) + \frac{R_2^2 - 3R_3^2}{4\left(R_3^2 - R_2^2\right)} \right] \tag{B.29}$$

b4) Total inductance per unit length

In the DC-case the total inductance per unit length of the coaxial cable is given as the sum of the three previously determined inductances.

$$\begin{aligned} L'_\mathrm{DC} &= L'_\mathrm{i} + L'_\mathrm{air} + L'_\mathrm{o} \\ &= \frac{\mu_0}{2\pi} \left[\frac{1}{4} + \ln\left(\frac{R_2}{R_1}\right) + \frac{R_3^4}{\left(R_3^2 - R_2^2\right)} \ln\left(\frac{R_3}{R_2}\right) + \frac{R_2^2 - 3R_3^2}{4\left(R_3^2 - R_2^2\right)} \right] \end{aligned} \tag{B.30}$$

c) Inductance per unit length in the RF domain

The calculations above considered non-time-variant (DC) currents and fields, where the current density spreads homogeneously over the cross-sectional area of a conductor. In the case of RF fields the *skin effect* forces the current to flow in a superficial region of a conductor. Hence, the conductors are nearly free of fields. Therefore, the inductance per unit length in the RF-case is

$$L'_\mathrm{RF} = L'_\mathrm{air} < L'_\mathrm{DC} \tag{B.31}$$

B.3 Problem 2.3

a) Amplitude of the electric field strength

The characteristic impedance of a medium is given as

$$Z_\mathrm{F} = \frac{E_0}{H_0} \tag{B.32}$$

Hence, the amplitude of the electric field is

$$E_0 = Z_\mathrm{F} H_0 = 1200 \frac{\mathrm{V}}{\mathrm{m}} \tag{B.33}$$

b) Relative permittivity

Inside a dielectric medium the characteristic impedance is

$$Z_{\mathrm{F}} = \frac{Z_{\mathrm{F0}}}{\sqrt{\varepsilon_{\mathrm{r}}}} \tag{B.34}$$

So, we calculate the relative permittivity as

$$\varepsilon_{\mathrm{r}} = \left(\frac{Z_{\mathrm{F0}}}{Z_{\mathrm{F}}}\right)^2 = \left(\frac{377\,\Omega}{300\,\Omega}\right) = 1.5792 \tag{B.35}$$

c) Wave number

The wave number k is given by

$$k = \frac{2\pi}{\lambda} \tag{B.36}$$

By using the velocity of light

$$c = \lambda f = \frac{c_0}{\sqrt{\varepsilon_{\mathrm{r}}}} \tag{B.37}$$

we get

$$k = \frac{2\pi}{c_0} f \sqrt{\varepsilon_{\mathrm{r}}} = 1.313 \frac{1}{\mathrm{m}} \tag{B.38}$$

d) Velocity

Finally, the velocity of an electromagnetic wave in our dielectric medium is

$$c = \frac{c_0}{\sqrt{\varepsilon_{\mathrm{r}}}} = 2.387 \cdot 10^8 \frac{\mathrm{m}}{\mathrm{s}} \tag{B.39}$$

B.4 Problem 2.4

a) Magnetic field inside the coil

Figure B.8a shows a long circular coil (diameter D, length ℓ, number of turns n). A DC-current I flows through the wire. From the right-hand rule we expect a magnetic field in axial direction.

In order to calculate the magnetic field we once again start with Ampere's law in the non-time-variant case.

$$\oint\limits_{C(A)} \vec{H} \cdot \mathrm{d}\vec{s} = \iint\limits_{A} \left(\vec{J} + \underbrace{\frac{\partial \vec{D}}{\partial t}}_{\to 0} \right) \cdot \mathrm{d}\vec{A} \tag{B.40}$$

In order to evaluate the integrals we have to make some approximations that are valid for long coils. We assume that

- the magnetic field inside the coil is constant ($H_{\mathrm{i}} = \text{const.}$)
- the magnetic field outside the coil is negligible compared to the field inside the coil ($H_{\mathrm{o}} = 0$)

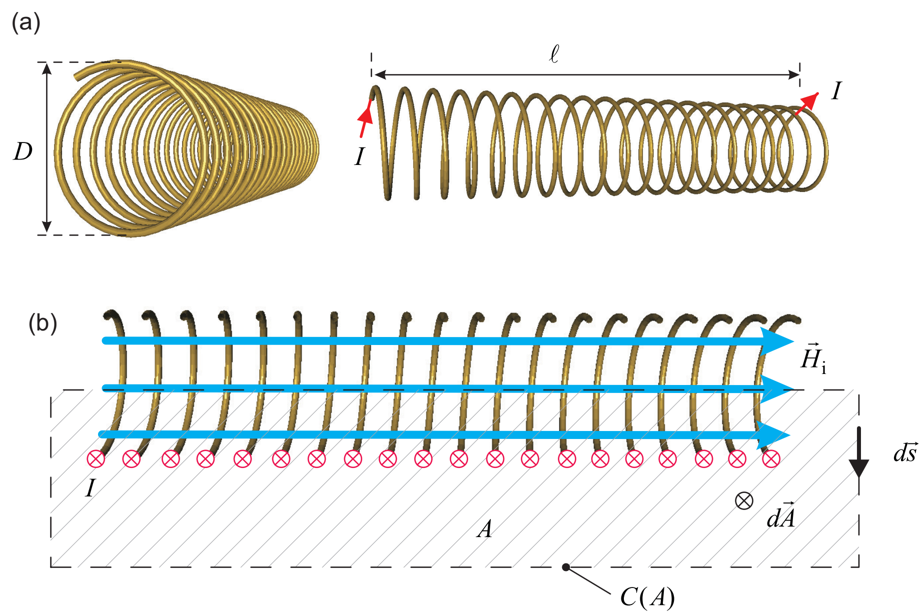


Figure B.8: Magnetic field in a long circular coil

We choose an integration path (boundary curve $C(A)$) along the axis of the coil (Figure B.8b). The path is closed in the exterior region of the coil. The left-hand side of Equation (B.40) gives us

$$\oint_{C(A)} \vec{H} \cdot \mathrm{d}\vec{s} = H_\mathrm{i}\ell \tag{B.41}$$

The current I runs n-times through the integration area A. Hence, the right-hand side of Equation (B.40) yields

$$\iint_A \vec{J} \cdot \mathrm{d}\vec{A} = nI \tag{B.42}$$

So, the constant magnetic field inside the coil is

$$H_\mathrm{i} = \frac{nI}{\ell} \tag{B.43}$$

b) Magnetic energy inside the coil

The magnetic field energy is given by

$$W_\mathrm{m} = \iiint_V \frac{1}{2}\vec{B} \cdot \vec{H}\mathrm{d}v \quad \text{where} \quad \vec{B} = \mu\vec{H} \tag{B.44}$$

The integral extends over the inner volume of the coil where the magnetic field is non-zero. We get:

$$
\begin{aligned}
W_{\mathrm{m}} &= \iiint\limits_V \frac{1}{2}\mu_0 H_{\mathrm{i}}^2 \mathrm{d}v = \frac{1}{2}\mu_0 H_{\mathrm{i}}^2 \pi \left(\frac{D}{2}\right)^2 \ell \\
&= \frac{1}{2}\mu_0 \frac{n^2 I^2}{\ell^2} \underbrace{\pi \left(\frac{D}{2}\right)^2 \ell}_{V} = \frac{1}{2}\mu_0 \frac{n^2 I^2}{\ell} \underbrace{\pi \left(\frac{D}{2}\right)^2}_{A}
\end{aligned}
\tag{B.45}
$$

c) Inductance of the coil

Inductance and magnetic field energy are linked by the following relation:

$$
W_{\mathrm{m}} = \iiint\limits_V \frac{1}{2}\vec{B}\cdot\vec{H}\mathrm{d}v = \frac{1}{2}LI^2 \quad \text{where} \quad \vec{B} = \mu\vec{H} \tag{B.46}
$$

In Problem section 2.4b we calculated the magnetic energy as

$$
W_{\mathrm{m}} = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 \frac{n^2 I^2}{\ell} \underbrace{\pi \left(\frac{D}{2}\right)^2}_{A} \tag{B.47}
$$

So, the inductance of the long cylindrical coil becomes

$$
L = \mu_0 \frac{n^2}{\ell} \underbrace{\pi \left(\frac{D}{2}\right)^2}_{A} = \mu_0 \frac{n^2}{\ell} A \tag{B.48}
$$

B.5 Problem 2.5

a) Calculation of the magnetic field strength

The vector of the electric field strength is given as

$$
\vec{E} = E_0 \mathrm{e}^{-jkx} \vec{e}_z \tag{B.49}
$$

Maxwell's second equation (Faraday's law) is

$$
\nabla \times \vec{E} = -j\omega\vec{B} = -j\omega\mu\vec{H} \quad \text{where} \quad \mu = \mu_0\mu_{\mathrm{r}} \tag{B.50}
$$

In Cartesian coordinates we express the curl-operator as

$$
\nabla \times \vec{E} = \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)\vec{e}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\vec{e}_y + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\vec{e}_z \tag{B.51}
$$

The electric field vector has a z-component and is a function of x only. So, the expression reduces to

$$
\nabla \times \vec{E} = -\frac{\partial E_z}{\partial x}\vec{e}_y = jkE_0 \mathrm{e}^{-jkx}\vec{e}_y = -j\omega\mu\vec{H} \tag{B.52}
$$

Hence, the magnetic field is

$$\vec{H} = -\frac{k}{\omega\mu} E_0 \mathrm{e}^{-jkx} \vec{e}_y \tag{B.53}$$

We rewrite our result using basic formulas for speed of light and characteristic impedance.

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0 \varepsilon_\mathrm{r} \mu_0 \mu_\mathrm{r}}} \quad \text{and} \quad Z_\mathrm{F} = \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu_0 \mu_\mathrm{r}}{\varepsilon_0 \varepsilon_\mathrm{r}}} \tag{B.54}$$

Finally, our result equals Equation (2.104).

$$\vec{H} = -\frac{1}{\mu} \sqrt{\varepsilon\mu} E_0 \mathrm{e}^{-jkx} \vec{e}_y = -\underbrace{\sqrt{\frac{\varepsilon}{\mu}}}_{1/Z_\mathrm{F}} E_0 \mathrm{e}^{-jkx} \vec{e}_y = -\underbrace{\frac{E_0}{Z_\mathrm{F}}}_{H_0} \mathrm{e}^{-jkx} \vec{e}_y \tag{B.55}$$

b) Power flow through an area of $A = 1\,\mathbf{m}^2$

The power density is determined by the Poynting vector $\vec{S}$ as

$$\vec{S} = \frac{1}{2} \vec{E} \times \vec{H} = \frac{1}{2} E_0 H_0 \vec{e}_x = \frac{1}{2Z_\mathrm{F}} E_0^2 \vec{e}_x \tag{B.56}$$

Integrating the power density over an area of A gives us

$$P = S \cdot A = \frac{1}{2Z_\mathrm{F}} E_0^2 A \tag{B.57}$$

(Last modified: 09.08.2012)

Appendix C

Solutions to Problems in Chapter 3

C.1 Problem 3.1

In Example 3.6 (book page 90) a complex load impedance $Z_{A2} = (37.5 + j150)\,\Omega$ was connected to a transmission line (characteristic impedance $Z_{02} = 75\,\Omega$; length of line $\ell_t/\lambda = 0.194$). By using the Smith diagram our result for the input impedance was $Z_{in} \approx (15 - j75)\,\Omega$. We introduced the Smith-chart as a *graphical* tool to avoid *complex-valued* calculations.

For comparison we use Equation (3.78) to calculate the input impedance *analytically*. We repeat Equation (3.78) here for convenience.

$$Z_{in} = \mathrm{A}\frac{1 + j\dfrac{Z_0}{Z_A}\tan(\beta\ell_t)}{1 + j\dfrac{Z_A}{Z_0}\tan(\beta\ell_t)} \tag{C.1}$$

The complex calculations may be performed *manually*. However, in engineering practice mathematical tools (e.g. MATLAB from MathWorks, Inc.) are available. Using MATLAB we get $Z_{in} = (14.6 - j75.2)\,\Omega$. The numerical and graphical results are in good agreement.

C.2 Problem 3.2

In our problem the resistances are given as $R_I = 15\,\Omega$ and $R_A = 10\,\Omega$. Hence, the reflection coefficients become

$$r_{in} = \frac{R_I - Z_0}{R_I + Z_0} = -0.538 \qquad \text{and} \qquad r_A = \frac{R_A - Z_0}{R_A + Z_0} = -0.667 \tag{C.2}$$

Figure 3.24 (book page 102) lists the relevant equations for our problem. The magnitudes of the forward-propagating and reflected pulses are

$$U_{1f} = U_0\frac{Z_0}{R_I + Z_0} = 769\,\mathrm{mV} \tag{C.3}$$

$$U_{1r} = r_A U_{1f} = -513\,\mathrm{mV} \tag{C.4}$$

$$U_{2f} = r_I r_A U_{1f} = 276\,\mathrm{mV} \tag{C.5}$$

$$U_{2r} = r_I r_A^2 U_{1f} = -184\,\mathrm{mV} \tag{C.6}$$

$$U_{3f} = r_I^2 r_A^2 U_{1f} = 99.1\,\mathrm{mV} \tag{C.7}$$

At the transmission line terminals forward-propagating and reflected pulses add up, so we observe the following time signal at the input terminals

$$u_{\mathrm{in}}(t) = U_{1\mathrm{f}} = 769\,\mathrm{mV} \quad \text{for} \quad 0 \le t \le t_{\mathrm{p}} \tag{C.8}$$

$$u_{\mathrm{in}}(t) = (1 + r_{\mathrm{I}})\,U_{1\mathrm{r}} = -237\,\mathrm{mV} \quad \text{for} \quad 2t_{\mathrm{D}} \le t \le 2t_{\mathrm{D}} + t_{\mathrm{p}} \tag{C.9}$$

$$u_{\mathrm{in}}(t) = (1 + r_{\mathrm{I}})\,U_{2\mathrm{r}} = -85\,\mathrm{mV} \quad \text{for} \quad 4t_{\mathrm{D}} \le t \le 4t_{\mathrm{D}} + t_{\mathrm{p}} \tag{C.10}$$

At the output (load) terminal we see

$$u_{\mathrm{A}}(t) = (1 + r_{\mathrm{A}})\,U_{1\mathrm{f}} = 256\,\mathrm{mV} \quad \text{for} \quad t_{\mathrm{D}} \le t \le t_{\mathrm{D}} + t_{\mathrm{p}} \tag{C.11}$$

$$u_{\mathrm{A}}(t) = (1 + r_{\mathrm{A}})\,U_{2\mathrm{f}} = 92\,\mathrm{mV} \quad \text{for} \quad 3t_{\mathrm{D}} \le t \le 3t_{\mathrm{D}} + t_{\mathrm{p}} \tag{C.12}$$

Figure C.1 shows a graphical representation of the time signals. (The numerical simulation was done with ADS circuit simulation software from Agilent, Inc.)

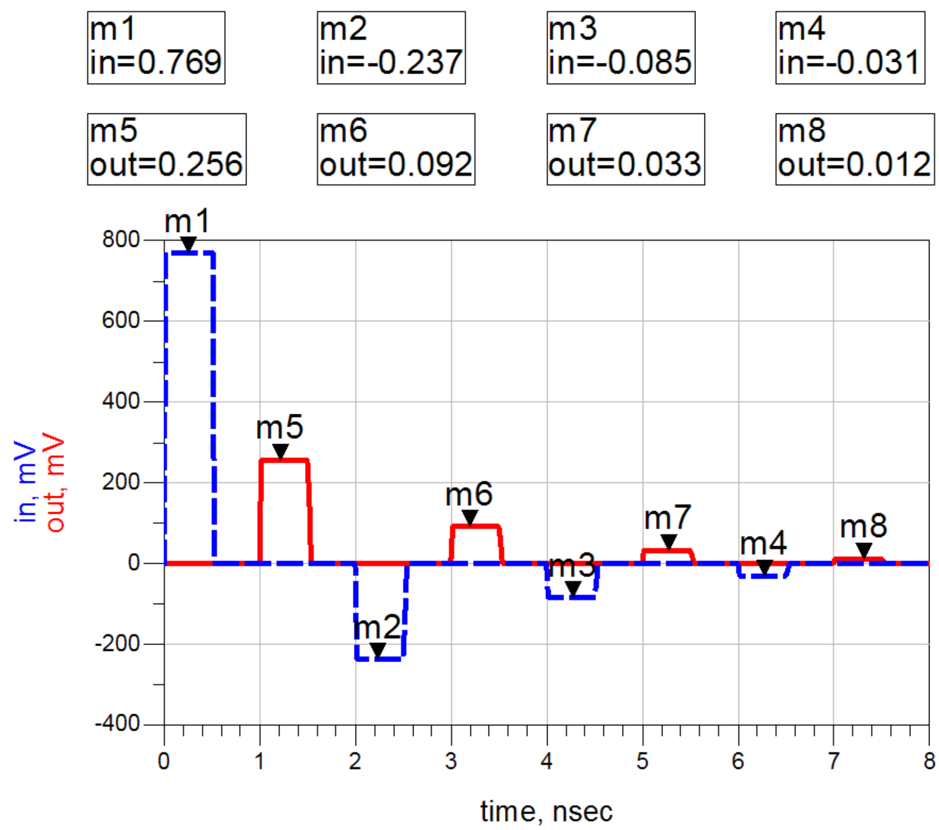


Figure C.1: Time plots of the voltages at the transmission line terminals (ADS simulation result)

C.3 Problem 3.3

A load impedance $Z_{\mathrm{A}} = (120 - j80)\,\Omega$ is matched to a $50\,\Omega$-source impedance by a matching network. The network starts (seen from the load) with a serial line and includes an open-ended stub line, both having line impedances of $Z_0 = 50\,\Omega$. The frequency of operation is $f = 1$ GHz.

The problem may be solved with freeware computerized Smith chart tools available on the Internet. Figure C.2 shows the result using a commercial tool (ADS from Agilent, Inc.). The serial line rotates the (normalized) load impedance $z_A = Z_A/Z_0$ around the center of the diagram (clockwise rotation) until is reaches the circle of unity normalized conductance. Following this circle we approach the matching point (center of the diagram).

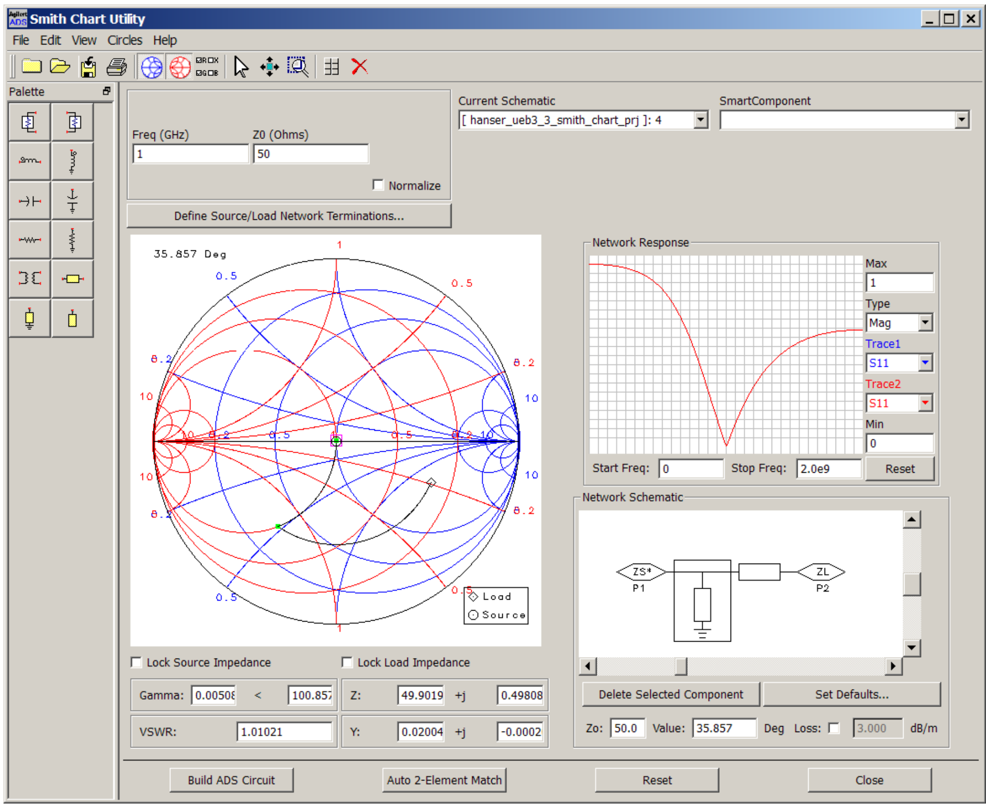


Figure C.2: Design of a matching network with ADS Smith-chart tool

The software displays the appropriate *electrical line lengths* in degree. We can calculate physical line lengths by using frequency and speed of propagation. For an air-filled line ($c = c_0$) our result would be

$$\frac{\ell_{\text{serial}}}{\lambda} = \frac{50.461^\circ}{360^\circ} \quad \rightarrow \quad \ell_{\text{serial}} = \frac{50.461^\circ}{360^\circ}\lambda = \frac{50.461^\circ}{360^\circ} \cdot \frac{c_0}{f} = 4.2\,\text{cm} \tag{C.13}$$

$$\frac{\ell_{\text{stub}}}{\lambda} = \frac{35.857^\circ}{360^\circ} \quad \rightarrow \quad \ell_{\text{stub}} = \frac{35.857^\circ}{360^\circ}\lambda = \frac{35.857^\circ}{360^\circ} \cdot \frac{c_0}{f} = 3.0\,\text{cm} \tag{C.14}$$

In order to evaluate our results we perform a circuit simulation with ADS. Figure C.3 shows the circuit for a s-parameter simulation in the frequency range from 0.1 to 2 GHz. The transmission lines are represented by physical line models (TLINP=Transmission LINe Physical). The transmission line parameters are

- characteristic impedance $Z = Z_0 = 50\,\Omega$,
- geometric length L and
- relative permittivity $K = \varepsilon_r = 1$.

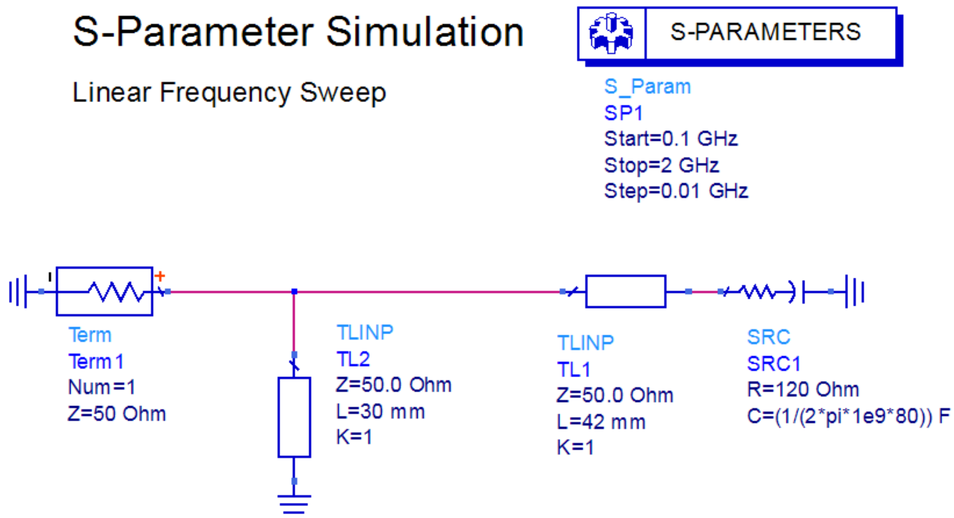


Figure C.3: Matching circuit of Problem 3.3

The negative reactance of $X_A = -j80\,\Omega$ at $f = 1$ GHz is represented by a capacitance.

$$jX_A = -j80\,\Omega = \frac{1}{j\omega C} \quad \rightarrow \quad C = \frac{1}{\omega 80\,\Omega} \tag{C.15}$$

The resulting s-parameter s_{11} is shown in Figure C.4. At $f = 1$ GHz the circuit is matched to the source impedance of $Z_0 = 50\,\Omega$.

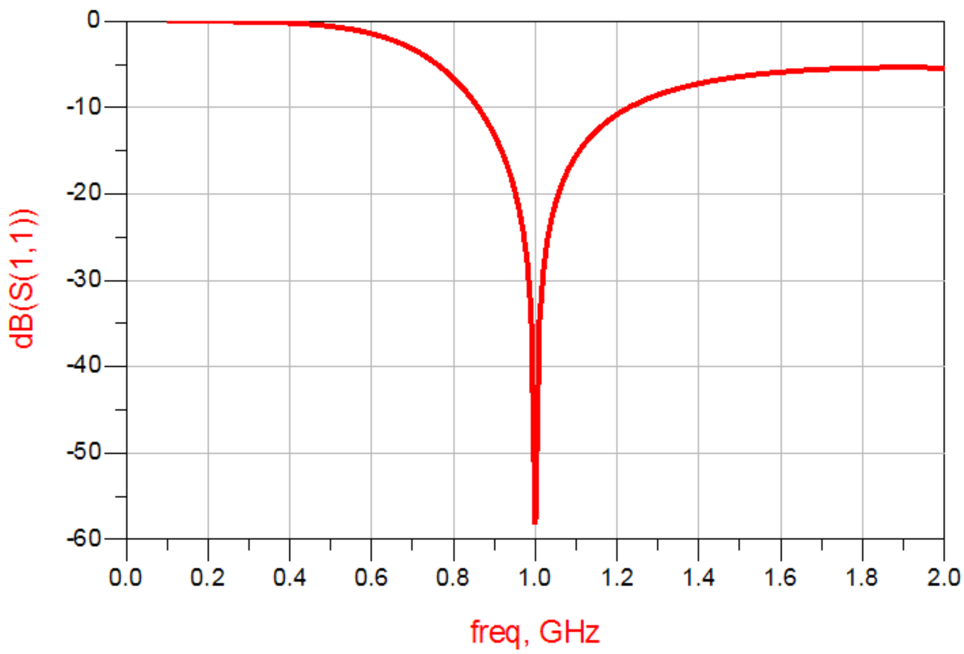


Figure C.4: Reflection coefficient s_{11} showing matching at $f = 1$ GHz

C.4 Problem 3.4

a) Speed of propagation

According to Equation (2.102) the speed of light in dielectric material is ($\mu_\mathrm{r} = 1$)

$$c = \frac{c_0}{\sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}}} = 2.5 \cdot 10^8\,\frac{\mathrm{m}}{\mathrm{s}} = 83.3\%\,c_0 \tag{C.16}$$

b) Characteristic line impedance

With

$$C' = \frac{C}{\ell} \quad \text{and} \quad \mu_\mathrm{r} = 1 \tag{C.17}$$

and Equation (3.68) we get

$$Z_0 = \frac{\sqrt{\varepsilon_\mathrm{r}\mu_\mathrm{r}}}{c_0 C'} = 32\,\Omega \tag{C.18}$$

c) Inductance per unit length

In order to calculate the inductance per unit length we start with Equation (3.65)

$$Z_0 = \sqrt{\frac{L'}{C'}} \tag{C.19}$$

By rearranging Equation (C.19) we get

$$L' = Z_0^2 C' = 128\frac{\mathrm{nH}}{\mathrm{m}} \tag{C.20}$$

d) Propagation constant

The propagation constant γ is

$$\gamma = \alpha + j\beta = j\omega\sqrt{L'C'} = j6.28\frac{1}{\mathrm{m}} \tag{C.21}$$

Alternatively, we can directly calculate the phase constant β

$$\beta = \frac{2\pi}{\lambda} \quad \text{with} \quad \lambda = \frac{c}{f} = \frac{c_0}{\sqrt{\varepsilon_\mathrm{r}}f} = 1\,\mathrm{m} \tag{C.22}$$

Consequently, the propagation constant is

$$\gamma = j\beta = j\frac{2\pi}{1\,\mathrm{m}} = j6.28\frac{1}{\mathrm{m}} \tag{C.23}$$

e) Input impedance

In our example the wavelength λ is

$$\lambda = \frac{c}{f} = \frac{c_0}{\sqrt{\varepsilon_\mathrm{r}}f} = 1\,\mathrm{m} \tag{C.24}$$

The geometric length is $\ell = 0.25\,\mathrm{m} = \lambda/4$. As shown in Section 3.1.8.3 an an open-ended quarter-wave line has a zero input impedance (short-circuit).

$$Z_\mathrm{in} = 0\,\Omega \tag{C.25}$$

C.5 Problem 3.5

a) Line length

The voltage $u_\mathrm{A}(t)$ shows that the delay time of the transmission line is $t_\mathrm{D} = 200$ ns. By using the basic cinematic equation $v = s/t$ we calculate the line length ℓ as

$$\ell = c_0 t_0 = 60\,\mathrm{m} \tag{C.26}$$

where c_0 is the speed of light in vacuum.

b) Calculation of circuit elements $R_\mathbf{AS}$, $R_\mathbf{AP}$ and C

At $t = 200$ ns the voltage pulse reaches the end of the line. The voltage across the capacitance is steady, hence, the capacitance behalves like a short-circuit. At that moment the load impedance Z_A0 is given by a parallel circuit of R_AS and R_AP.

$$Z_\mathrm{A0} = R_\mathrm{AP} \parallel R_\mathrm{AS} \tag{C.27}$$

The corresponding reflection coefficient is given by

$$r_\mathrm{A0} = \frac{Z_\mathrm{A0} - Z_0}{Z_\mathrm{A0} + Z_0} \tag{C.28}$$

According to Figure 3.22 the voltage at the load terminal is

$$u(t_0) = U_\mathrm{1f}\,(1 + r_\mathrm{A0}) \tag{C.29}$$

From Figure 3.27 we read a value of 0.8 V at t_0.

$$u(t_0) = U_\mathrm{1f}\,(1 + r_\mathrm{A0}) = 0.8\,\mathrm{V} \tag{C.30}$$

The voltage divider rule at the input terminals ($R_\mathrm{I} = Z_0$) yields a value of 1 V for the forward propagating voltage ($U_\mathrm{f1} = 1$ V). So, reflection coefficient and load impedance become

$$r_\mathrm{A0} = -0.2 \qquad \Rightarrow \qquad Z_\mathrm{A0} = Z_0 \frac{1 + r_\mathrm{A0}}{1 - r_\mathrm{A0}} = 16.67\,\Omega = R_\mathrm{AP} \parallel R_\mathrm{AS} \tag{C.31}$$

Next, we look at the voltage $u_\mathrm{A}(t)$ for $t \to \infty$. The capacitance is fully charged, there is no (DC) current through the capacitance. The capacitance behaves like an open circuit. Hence, the load impedance now becomes

$$Z_{\mathrm{A}\infty} = R_\mathrm{AP} \tag{C.32}$$

For $t \to \infty$ we see steady (DC) conditions. The loss-less transmission line represents a simple through-connection. From the voltage divider rule we get

$$\frac{u_\mathrm{A}(t \to \infty)}{2\,\mathrm{V}} = \frac{Z_{\mathrm{A}\infty}}{R_\mathrm{I} + Z_{\mathrm{A}\infty}} \tag{C.33}$$

Now, we can calculate the first circuit element as

$$Z_{\mathrm{A}\infty} = R_\mathrm{AP} = 80\,\Omega \tag{C.34}$$

Next, we can derive the second unknown resistance

$$\frac{1}{R_{\mathrm{AP}}} + \frac{1}{R_{\mathrm{AS}}} = \frac{1}{R_{\mathrm{AP}} \parallel R_{\mathrm{AS}}} \quad \rightarrow \quad R_{\mathrm{AS}} = 21\,\Omega \tag{C.35}$$

In order to determine the capacitance we take a look at the time plot of $u_{\mathrm{A}}(t)$. For $t \geq t_0$ the output voltage is given by the following relation.

$$u_{\mathrm{A}}(t) = 0.8\,\mathrm{V} + (1.524\,\mathrm{V} - 0.8\,\mathrm{V}) \left(1 - \mathrm{e}^{-\dfrac{t - t_0}{\tau}}\right) \quad \text{for} \quad t \geq t_0 \tag{C.36}$$

From $u_{\mathrm{A}}(t_{\mathrm{x}}) = 1.182$ V and Equation (C.36) we determine the time constant τ.

$$\tau = 100\,\mathrm{ns} \tag{C.37}$$

The exponential time constant τ depends on the capacitance and the effective resistance R.

$$\tau = RC \tag{C.38}$$

From basic circuit theory we derive the resistance R as

$$R = R_{\mathrm{AS}} + R_{\mathrm{AP}} \parallel Z_0 = 40\,\Omega \tag{C.39}$$

Finally, the capacitance C is

$$C = \frac{\tau}{R} = 2.5\,\mathrm{nF} \tag{C.40}$$

(Last modified: 13.08.2012)

Appendix D

Solutions to Problems in Chapter 4

D.1 Problem 4.1

Figure D.1 shows a network of three resistors R_1, R_2 and R_3. Equations 4.80 and 4.81 provide element values to terminate both, common and differential mode. In this problem we will derive these formulas.

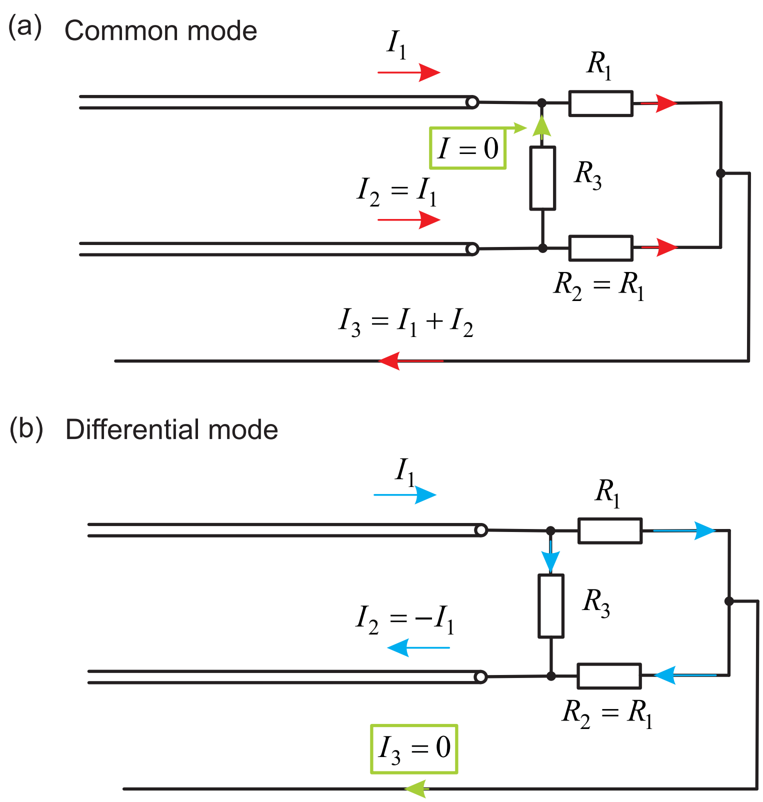


Figure D.1: Current definition for common mode and differential mode

Common Mode

The currents I_1 and I_2 on the signal lines are equal (Figure D.1a). The common ground acts as a return conductor. The differential voltage between the signal lines is zero. Hence, there is no current through resistor R_3.

At the end of the line a common mode signal *sees* R_1 and R_2 in parallel. In order to avoid reflections at the end of the line the parallel circuit must equal the common mode characteristic impedance.

$$Z_{0,\mathrm{cm}} = R_1 \parallel R_2 \tag{D.1}$$

If we choose $R_1 = R_2$ the parallel circuit becomes

$$R_1 \parallel R_2 = \frac{R_1}{2} = \frac{R_2}{2} = Z_{0,\mathrm{cm}} \tag{D.2}$$

and the resistors are

$$R_1 = R_2 = 2Z_{0,\mathrm{cm}} \tag{D.3}$$

Differential Mode

In differential mode the currents I_1 and I_2 on the signal lines are of equal magnitude but opposite sign (Figure D.1b). There is no net current in the ground conductor ($I_3 = 0$). At the end of the line a differential mode signal sees R_3 in parallel with the sum of R_1 and R_2.

$$Z_{0,\mathrm{diff}} = R_3 \parallel (R_1 + R_2) \tag{D.4}$$

With Equation D.3 we get immediately

$$R_3 = \frac{4Z_{0,\mathrm{cm}}}{4Z_{0,\mathrm{cm}} - Z_{0,\mathrm{diff}}} \tag{D.5}$$

Example

Let us put some numbers in here. Figure 4.29 (book page 153) shows differential and common mode characteristic impedances of a coupled microstrip line. The characteristic impedance of the single ended microstrip line is $Z_0 = 50\,\Omega$ at $f = 5$ GHz. The transmission line is characterized by: substrate height $h = 635\,\mu$m, relative permittivity $\varepsilon_\mathrm{r} = 9.8$, trace width $w = 600\,\mu$m, thickness of metallization $t = 10\,\mu$m. The spacing s determines the differential and common mode characteristic impedances in Figure 4.29.

Let us consider the case $s/h = 1$. From Figure 4.29 we read

$$Z_{0,\mathrm{cm}} \approx 28.5\,\Omega \qquad \text{and} \qquad Z_{0,\mathrm{diff}} \approx 85.0\,\Omega \tag{D.6}$$

Hence, the termination network consists of the following three resistors.

$$R_1 = R_2 = 57.0\,\Omega \qquad \text{and} \qquad R_3 = 334.1\,\Omega \tag{D.7}$$

In order to evaluate our result we perform a circuit simulation with a coupled microstrip line. Figure D.2 shows the schematics for common mode and differential mode excitations. Additionally, we use the transmission line tool *LineCalc* (Agilent Inc.) to calculate more accurate characteristic impedances than those approximate values read from Figure 4.29. The schematic with the modified values of R_1, R_2 and R_3 are also shown in Figure D.2.

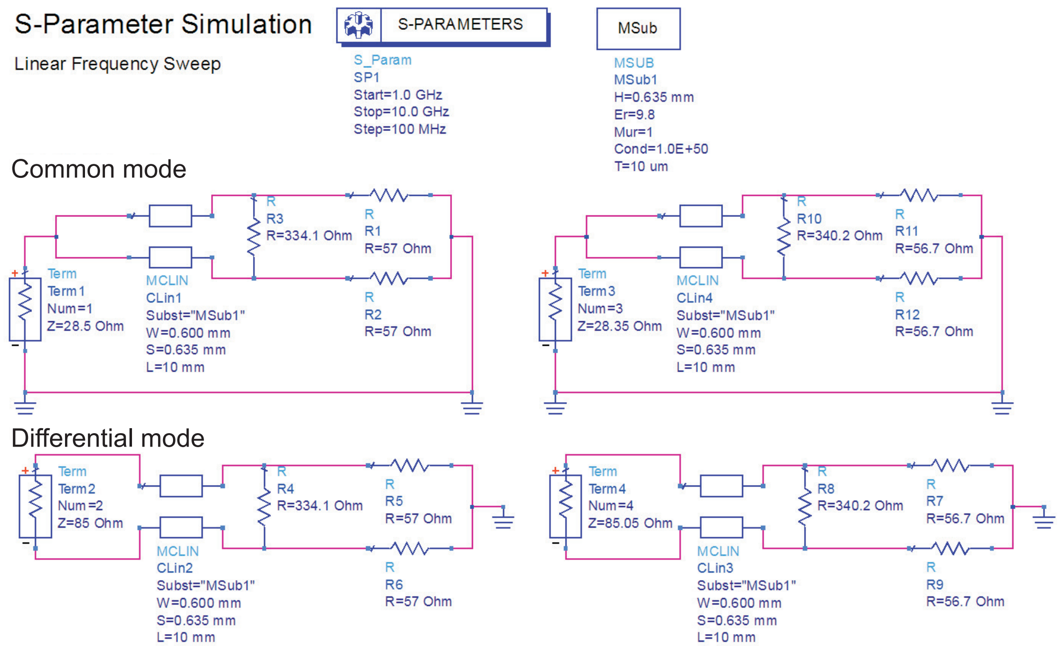


Figure D.2: ADS circuit simulation of line termination: schematics with approximate values for R_1, R_2 and R_3 (left) and schematics with more accurate values (right)

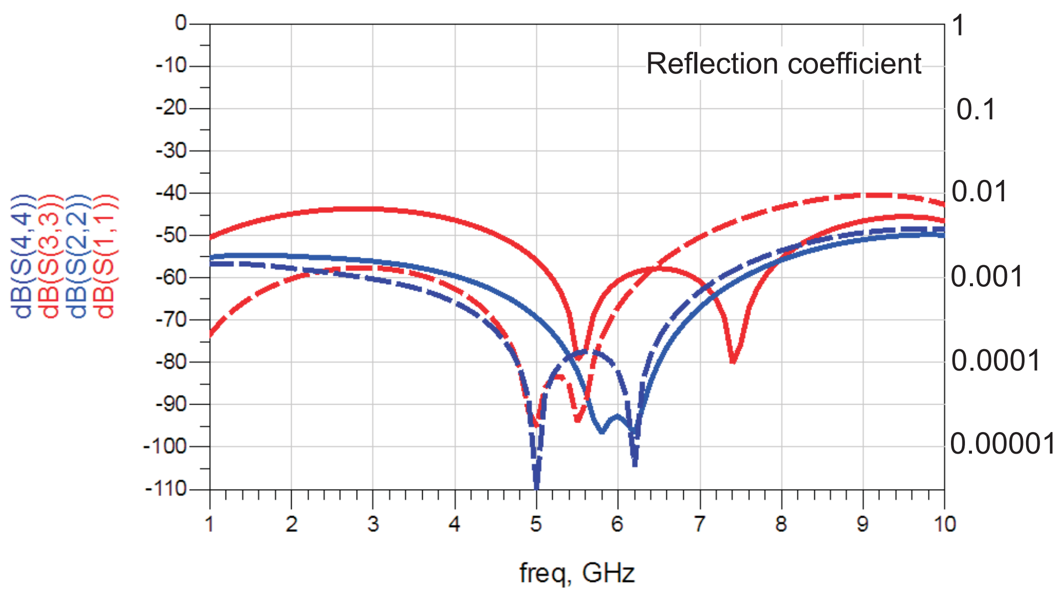


Figure D.3: Simulated s-parameter results (solid lines: approximate values; dashed lines: more accurate values)

Figure D.3 shows the reflection coefficients $r_i = s_{ii}$ for all four circuits. The magnitudes of the reflection coefficients are given in dB and in linear scale. Solid lines represent the simulations with the approximate values. Dashed lines represent simulations with more accurate values. The curves indicate appropriate matching (low reflection) of common and differential mode signals.

Due to the dispersive nature of microstrip lines the characteristic impedances vary with frequency. Hence, small reflections occur.

D.2 Problem 4.2

The cut-off frequencies of higher order modes are given by Equation 4.64 (book page 139).

$$f_{c,mn} = \frac{c_0}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \tag{D.8}$$

Table D.1 lists wave types and cut-off frequencies.

Table D.1: Cut-off frequencies $f_{c,mn}$ in a R260 waveguide ($a = 8.636$ mm and $b = 4.318$ mm)

Wavemode	H_{10}	H_{20}	H_{01}	H_{11}, E_{11}	H_{21}, E_{21}	H_{20}
$f_{c,mn}$	17.37 GHz	34.74 GHz	34.74 GHz	38.84 GHz	49.13 GHz,	52.11 GHz

The characteristic impedance of the fundamental wave type (H_{10} or TE_{10}) is given by Equation 4.62 (book page 138).

$$Z_0^{H_{10}} = \frac{\pi^2 b}{8a} \cdot \frac{Z_{F0}}{\sqrt{1-(f_c/f)^2}} = 312\,\Omega \tag{D.9}$$

D.3 Problem 4.3

Figure D.4 shows two coaxial lines ($Z_{01} = 75\,\Omega$ and $Z_{02} = 125\,\Omega$) matched by a quarterwave transformer at $f = 10$ GHz.

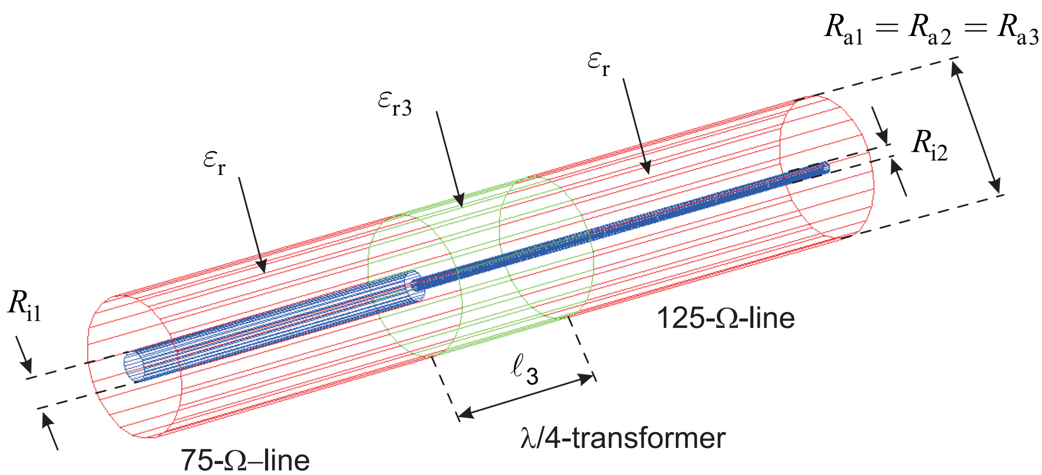


Figure D.4: Quarterwave transformer for matching two lines at $f = 10$ GHz

a) Relative permittivity

The relative permittivity can be derived from the reduced speed of propagation c.

$$c = \frac{c_0}{\sqrt{\varepsilon_r}} = 81\%\, c_0 \qquad \Rightarrow \qquad \varepsilon_r = \left(\frac{1}{0.81}\right)^2 = 1.524 \tag{D.10}$$

b) Inner radii of coaxial lines

The characteristic impedance of a coaxial line is given by Equation 4.12 (book page 115).

$$Z_0 = \frac{60\,\Omega}{\sqrt{\varepsilon_r}} \ln\left(\frac{R_o}{R_i}\right) \tag{D.11}$$

From the known outer radii we calculate the inner radii.

$$R_{i1} = R_{o1} \exp\left(-\frac{Z_0}{60\,\Omega\sqrt{\varepsilon_r}}\right) = 0.427\,\text{mm} \qquad \text{and} \qquad R_{i2} = 0.1528\,\text{mm} \tag{D.12}$$

c) Characteristic impedance of quarterwave transformer

For a quarterwave transformer ($\ell = \lambda/4$) the load impedance Z_A and input impedance Z_{in} are linked by the following equation

$$Z_{in} Z_A = Z_0^2 \tag{D.13}$$

In our problem we have $Z_A = Z_{01}$ and $Z_{in} = Z_{02}$. Therefore, the characteristic impedance Z_{03} of the quarterwave transformer is

$$Z_{03} = \sqrt{Z_{01} Z_{02}} = 96.82\,\Omega \tag{D.14}$$

d) Relative permittivity ε_{r3}

Inner and outer radii of the quarterwave transformer are given ($R_{o3} = R_{o2}$, $R_{i3} = R_{i2}$). In order to obtain the desired characteristic impedance Z_{03} we select an appropriate relative permittivity ε_{r3}.

$$Z_{03} = \frac{60\,\Omega}{\sqrt{\varepsilon_{r3}}} \ln\left(\frac{R_{o3}}{R_{i3}}\right) = \frac{60\,\Omega}{\sqrt{\varepsilon_{r3}}} \ln\left(\frac{R_{o2}}{R_{i2}}\right) \quad \Rightarrow \quad \varepsilon_{r3} = \left(\frac{60\,\Omega}{Z_{03}} \ln\left(\frac{R_{o2}}{R_{i2}}\right)\right)^2 = 2.541 \tag{D.15}$$

e) Length of quarterwave transformer

The wavelength λ in the quarterwave transformer is

$$\lambda = \frac{c}{f} = \frac{c_0}{\sqrt{\varepsilon_{r3}} f} \tag{D.16}$$

Hence, the length of the line is

$$\ell_3 = \frac{\lambda}{4} = 4.705\,\text{mm} \tag{D.17}$$

For comparison we use a circuit simulation (ADS) (see Figure D.5). Figure D.6 shows the simulation result. The reflection coefficient is minimum at $f = 10$ GHz.

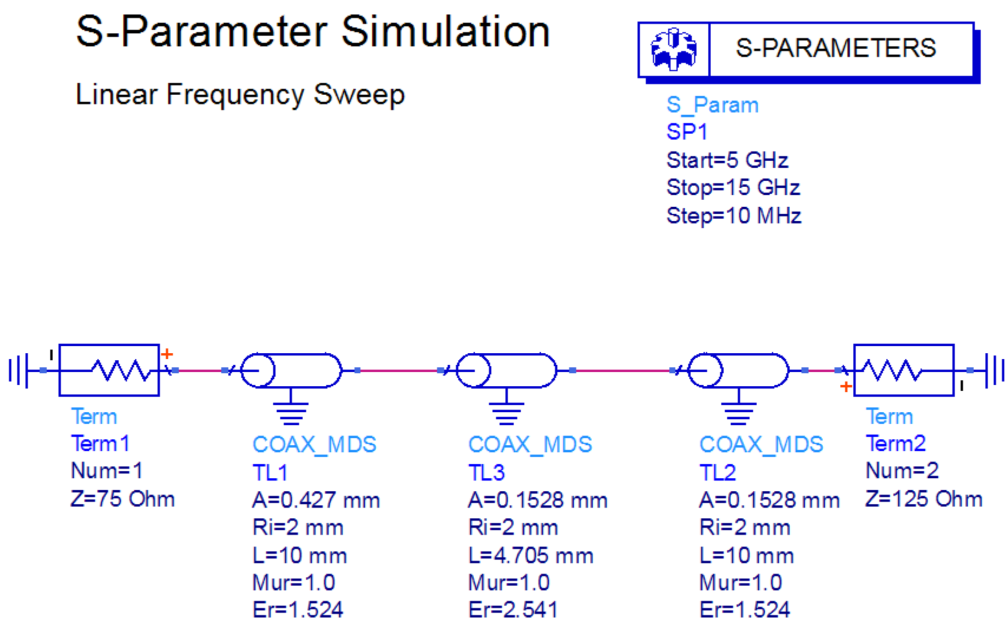


Figure D.5: Schematic for s-parameter simulation with ADS

Figure D.6: Simulation result shows matching at $f = 10$ GHz

D.4 Problem 4.4

The resonance frequencies $f_{\mathrm{R},mnp}$ of a rectangular cavity are given by Equation 4.69 (book page 142).

$$f_{\mathrm{R},mnp} = \frac{c_0}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{c}\right)^2} \tag{D.18}$$

Table D.2 lists the resonance frequencies with increasing frequencies for ($a = 5$ cm, $b = 7$ cm, $c = 9$ cm). Side a is the longest side. The frequency with the lowest frequency is given by $m = 0$ and $n = p = 1$. The electric field has *one* maximum ($n = p = 1$) in each horizontal direction. The electric field in vertical direction is constant (*no* maximum; $m = 0$).

Table D.2: Resonance frequencies

m	n	p	f_R/GHz
0	1	1	2.715
1	0	1	3.432
1	1	0	3.687
0	1	2	3.963
1	1	1	4.046

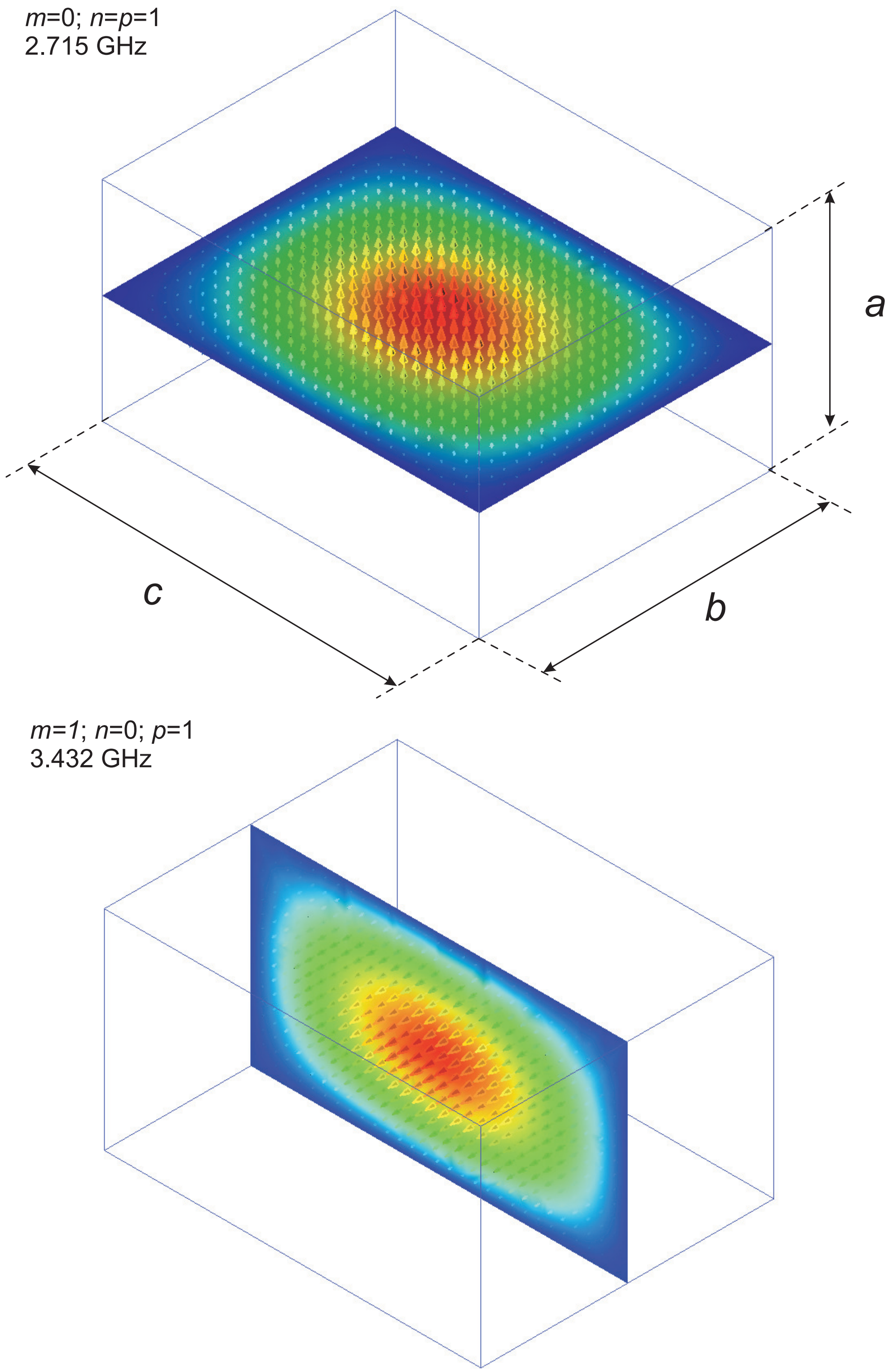


Figure D.7: Electric field distribution of the first two modes

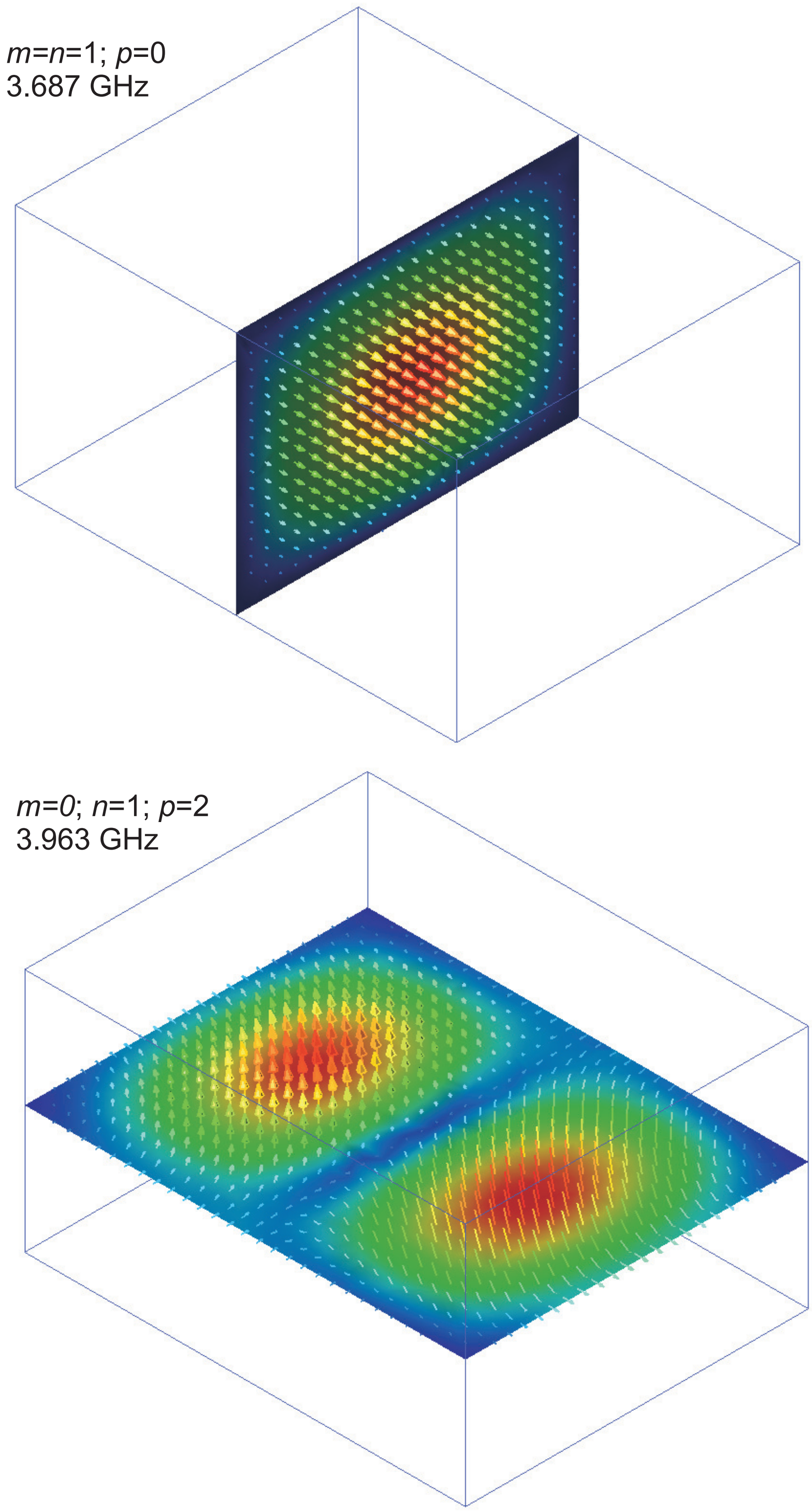


Figure D.8: Electric field distribution of the third and fourth mode

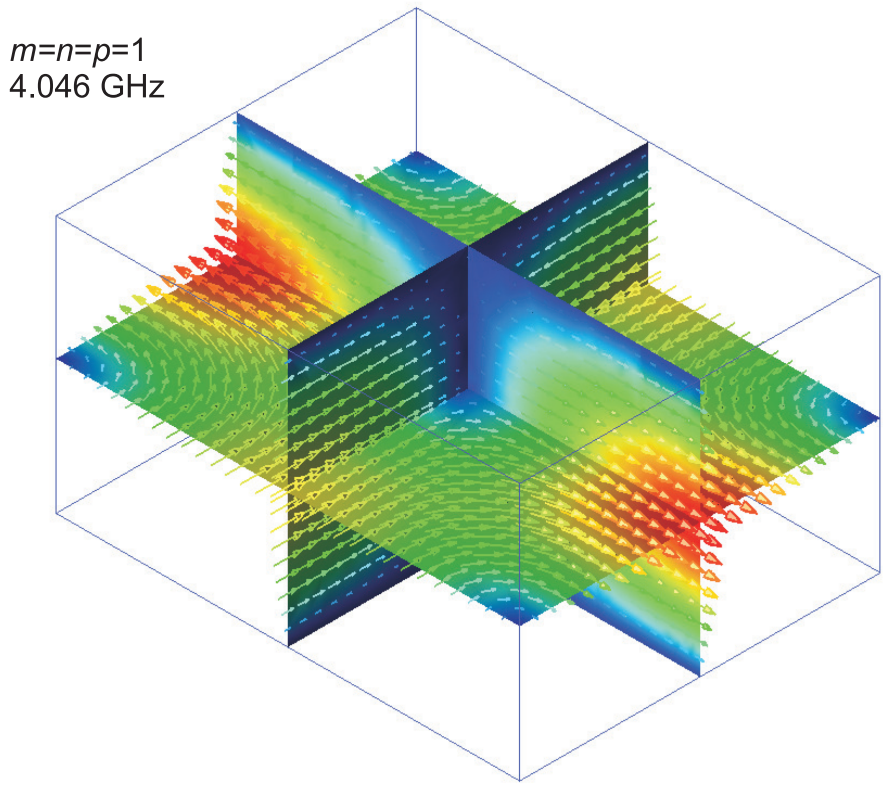


Figure D.9: Electric field distribution of the fifth mode

(Last modified: 27.12.2012)

Appendix E

Solutions to Problems in Chapter 5

E.1 Problem 5.1

Properties of the two-port network

- The network is mismatched at both ports ($s_{11} \neq 0$ and $s_{22} \neq 0$).
- The network is reciprocal ($s_{12} = s_{21}$).
- The network is symmetrical ($s_{12} = s_{21}$ and $s_{11} = s_{22}$)
- The network is lossless, since its matrix is a *unitary* matrix. ($\mathbf{S}^{\mathrm{T}}\mathbf{S}^{*} = \mathbf{I}$).

The unitary matrix condition yields

$$\mathbf{S}^{\mathrm{T}}\mathbf{S}^{*} = \begin{pmatrix} \frac{5}{13} & j\frac{12}{13} \\ j\frac{12}{13} & \frac{5}{13} \end{pmatrix} \begin{pmatrix} \frac{5}{13} & -j\frac{12}{13} \\ -j\frac{12}{13} & \frac{5}{13} \end{pmatrix} \tag{E.1}$$

$$= \begin{pmatrix} \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 & 0 \\ 0 & \left(\frac{5}{13}\right)^2 + \left(\frac{12}{13}\right)^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I} \tag{E.2}$$

Reflection loss and insertion loss

The reflection losses at port 1 and port 2 are

$$RL_1 = 20\lg\left|\frac{1}{s_{11}}\right| = 8.3\,\mathrm{dB} \qquad \text{and} \qquad RL_2 = 20\lg\left|\frac{1}{s_{22}}\right| = 8.3\,\mathrm{dB} \tag{E.3}$$

The insertion loss is given by

$$IL_{12} = 20\lg\left|\frac{1}{s_{12}}\right| = 0.695\,\mathrm{dB} \qquad \text{and} \qquad IL_{21} = 20\lg\left|\frac{1}{s_{21}}\right| = 0.695\,\mathrm{dB} \tag{E.4}$$

Renormalization of scattering matrix from 50 Ω to 100 Ω

The set of equations required for the renormalization is given in Section 5.5.3 ($\mathbf{S} = \mathbf{S}_{50\,\Omega}$ and $\mathbf{S}_{\mathrm{new}} = \mathbf{S}_{100\,\Omega}$).

$$\mathbf{S}_{\text{new}} = \frac{1}{\det \mathbf{S}} \begin{pmatrix} (s_{11} - r)(1 - rs_{22}) + rs_{12}s_{21} & s_{12}\left(1 - r^2\right) \\ s_{21}\left(1 - r^2\right) & (s_{22} - r)(1 - rs_{11}) + rs_{12}s_{21} \end{pmatrix} \tag{E.5}$$

where

$$\det \mathbf{S} = (1 - rs_{11})(1 - rs_{22}) - r^2 s_{12}s_{21} \tag{E.6}$$

and

$$r = \frac{Z_{0,\text{new}} - Z_0}{Z_{0,\text{new}} + Z_0} \tag{E.7}$$

Using the given values we get

$$r = \frac{1}{3} \qquad \text{and} \qquad \det \mathbf{S} = \frac{100}{117} \tag{E.8}$$

Hence, the renormalized matrix is

$$\mathbf{S}_{\text{new}} = \frac{1}{25}\begin{pmatrix} -7 & j24 \\ j24 & -7 \end{pmatrix} = \begin{pmatrix} -0.28 & j0.96 \\ j0.96 & -0.28 \end{pmatrix} \tag{E.9}$$

Validation by circuit simulation (ADS)

We will validate our result by using a circuit simulator (ADS from Agilent, Inc.). First, we write our initial scattering parameter in a s*n*p-file (see Section 5.7).

s*n*p-files (n = number of ports) contain frequency dependent scattering parameters from simulation or measurement in ASCII format. Here, we use the file to define scattering parameters.

```
! s2p file for Problem 5.1

#   GHz   S   MA   R   50

! freq  magS11  angS11  magS21  angS21  magS12  angS12  magS22  angS22
! freq  5/13    0deg    12/13   90deg   12/13   90deg   5/13    0deg

  1.0    0.3846153846    0    0.923076923   90    0.923076923   90    0.3846153846    0
  2.0    0.3846153846    0    0.923076923   90    0.923076923   90    0.3846153846    0
```

Figure E.1: s2p-file for s-parameter definition

Figure E.1 shows the s2p-file. Lines with an exclamation mark (!) are comment line. The line with the sharp symbol (#) defines the format of the subsequent data lines: frequency in GHz (GHz), s-parameters (S) are given with magnitude and phase (MA). The port reference impedance (R) is $50\,\Omega$ (50).

In our problem the s-parameters do not vary with frequency. In order to calculate and plot the parameters with ADS an arbitrary frequency range from 1 GHz to 2 GHz is chosen. The s-parameter data is give in decimal nomination.

We use the previously defined s2p-file in a circuit simulation. Figure E.2 shows the schematic. The two-port network is connected to $50\,\Omega$ as well as $100\,\Omega$ terminals.

Figure E.3 shows the simulation results. The upper two plots show the s-parameters normalized to $50\,\Omega$. The absolute values (magnitude) correspond to the given values of $5/13 \approx 0.385$

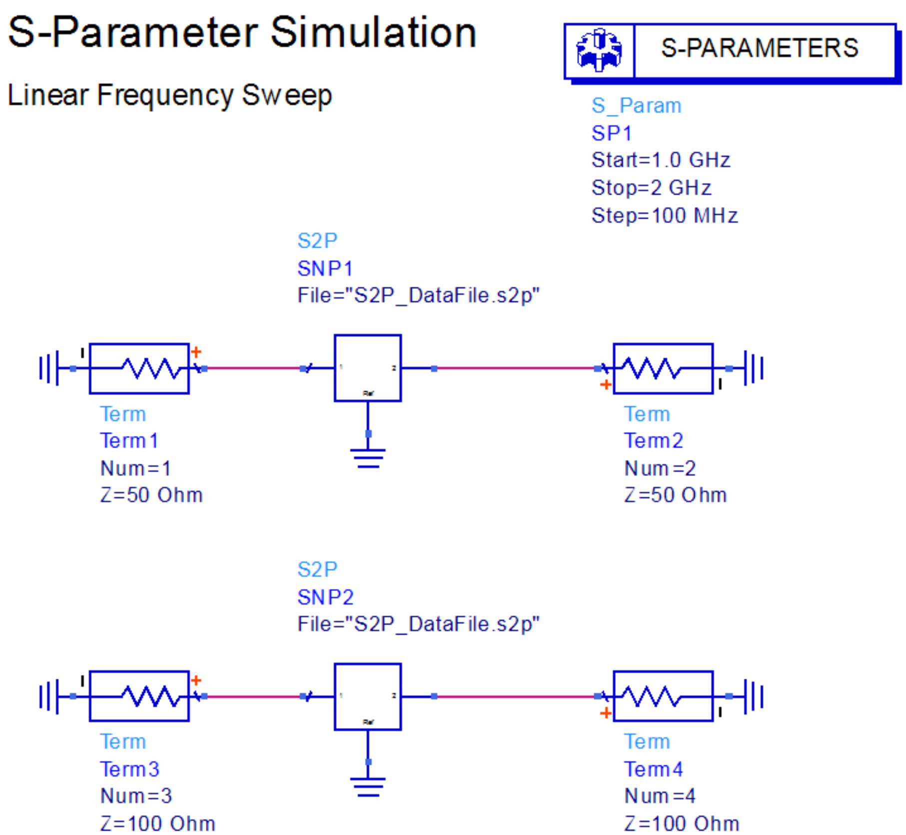


Figure E.2: Schematic for renormalization of scattering parameters

and $12/13 \approx 0.923$. The phases are zero degree (positive real value) for the reflection coefficients and 90 degree (positive imaginary value) for the transmission coefficients.

The lower two plots show the s-parameters for a reference value of $100\,\Omega$. The absolute values are 0.28 and 0.96. The faces are 180 degree (negative real values) for the reflection coefficients and 90 degree (positive imaginary values) for the transmission coefficients. The results agree with our calculated values in Equation E.9.

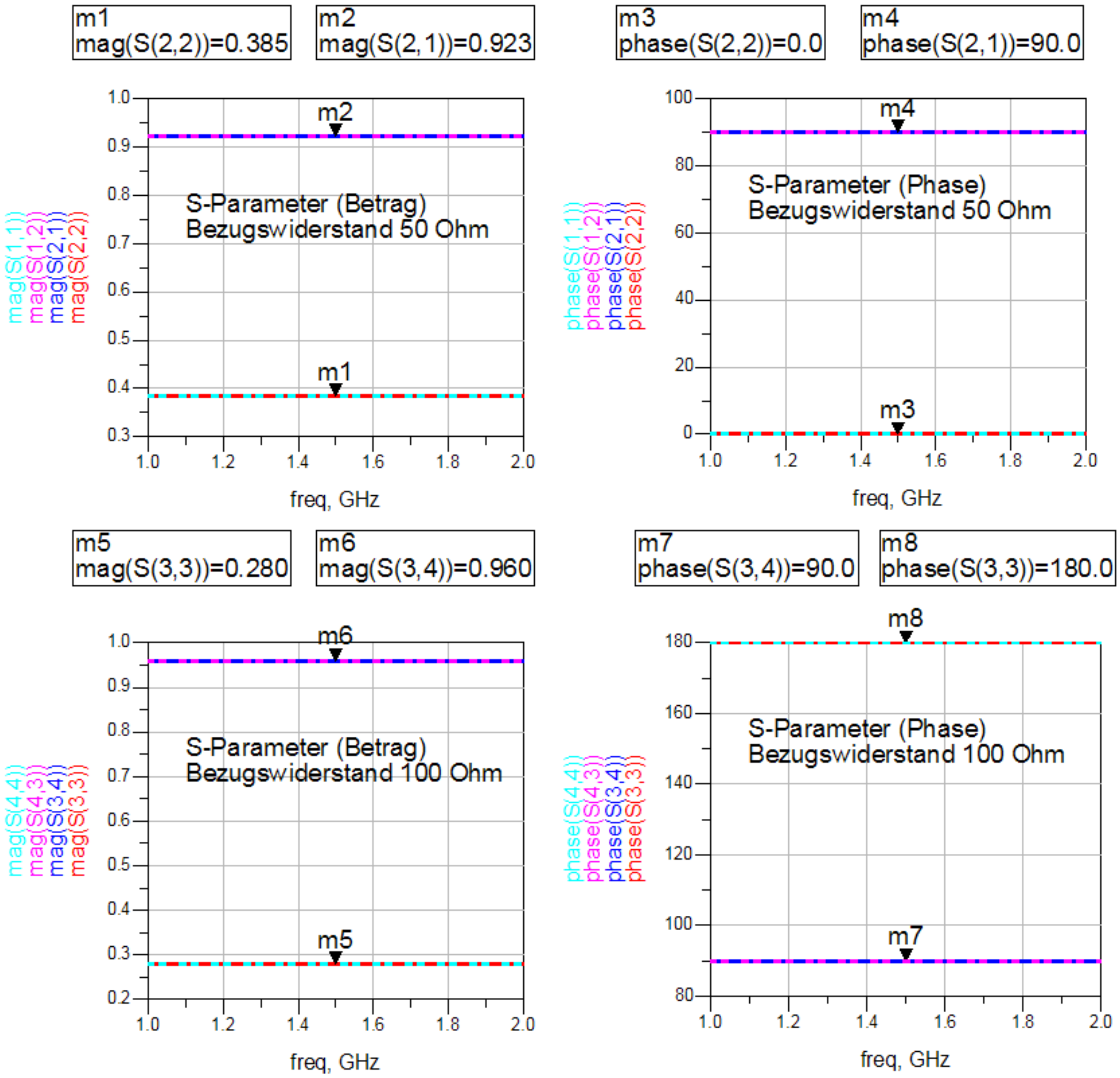


Figure E.3: Simulation results: s-parameters for $Z_0 = 50\,\Omega$ (upper plots) and s-parameters for $Z_{0,\text{new}} = 100\,\Omega$ (lower plots)

E.2 Problem 5.2

Antenna input impedance

The reflection coefficient r_{A} of an antenna is determined by its input impedance Z_{A} and the port reference impedance Z_0.

$$r_{\mathrm{A}} = \frac{Z_{\mathrm{A}} - Z_0}{Z_{\mathrm{A}} + Z_0} \tag{E.10}$$

Solving Equation E.10 for Z_{A} yields

$$Z_{\mathrm{A}} = Z_0 \frac{1 + r_{\mathrm{A}}}{1 - r_{\mathrm{A}}} \tag{E.11}$$

We rewrite the given reflection coefficient r_{A}

$$r_{\mathrm{A}} = 0.4\mathrm{e}^{-j20^\circ} = 0.4\left(\cos(20^\circ) - j\sin(20^\circ)\right) = \mathrm{Re}\left\{r_{\mathrm{A}}\right\} + j\,\mathrm{Im}\left\{r_{\mathrm{A}}\right\} = 0.367 - j0.137 \tag{E.12}$$

Substituting our result in Equation E.11 yields

$$\begin{aligned} Z_{\mathrm{A}} &= Z_0 \frac{1 + r_{\mathrm{A}}}{1 - r_{\mathrm{A}}} = Z_0 \frac{1 + \mathrm{Re}\left\{r_{\mathrm{A}}\right\} + j\,\mathrm{Im}\left\{r_{\mathrm{A}}\right\}}{1 - \mathrm{Re}\left\{r_{\mathrm{A}}\right\} - j\,\mathrm{Im}\left\{r_{\mathrm{A}}\right\}} && \text{(E.13)} \\ &= Z_0 \frac{\left(1 + \mathrm{Re}\left\{r_{\mathrm{A}}\right\} + j\,\mathrm{Im}\left\{r_{\mathrm{A}}\right\}\right)\left(1 - \mathrm{Re}\left\{r_{\mathrm{A}}\right\} + j\,\mathrm{Im}\left\{r_{\mathrm{A}}\right\}\right)}{\left(1 - \mathrm{Re}\left\{r_{\mathrm{A}}\right\}\right)^2 + \left(\mathrm{Im}\left\{r_{\mathrm{A}}\right\}\right)^2} && \text{(E.14)} \\ &= Z_0 \frac{1 - \left(\mathrm{Re}\left\{r_{\mathrm{A}}\right\}\right)^2 - \left(\mathrm{Im}\left\{r_{\mathrm{A}}\right\}\right)^2 + j2\,\mathrm{Im}\left\{r_{\mathrm{A}}\right\}}{\left(1 - \mathrm{Re}\left\{r_{\mathrm{A}}\right\}\right)^2 + \left(\mathrm{Im}\left\{r_{\mathrm{A}}\right\}\right)^2} = (102.9 - j33.51)\,\Omega && \text{(E.15)} \end{aligned}$$

Reflection coefficient for port reference impedance of $Z_{\mathbf{0,new}} = 75\,\Omega$

The reflection coefficient with respect to a port reference impedance of $Z_{0,\mathrm{new}} = 75\,\Omega$ is

$$\begin{aligned} r_{\mathrm{A,new}} &= \frac{Z_{\mathrm{A}} - Z_{0,\mathrm{new}}}{Z_{\mathrm{A}} + Z_{0,\mathrm{new}}} = \frac{\mathrm{Re}\left\{Z_{\mathrm{A}}\right\} + j\,\mathrm{Im}\left\{Z_{\mathrm{A}}\right\} - Z_{0,\mathrm{new}}}{\mathrm{Re}\left\{Z_{\mathrm{A}}\right\} + j\,\mathrm{Im}\left\{Z_{\mathrm{A}}\right\} + Z_{0,\mathrm{new}}} && \text{(E.16)} \\ &= \frac{\left(\mathrm{Re}\left\{Z_{\mathrm{A}}\right\} - Z_{0,\mathrm{new}} + j\,\mathrm{Im}\left\{Z_{\mathrm{A}}\right\}\right)\left(\mathrm{Re}\left\{Z_{\mathrm{A}}\right\} + Z_{0,\mathrm{new}} - j\,\mathrm{Im}\left\{Z_{\mathrm{A}}\right\}\right)}{\left(\mathrm{Re}\left\{Z_{\mathrm{A}}\right\} + Z_{0,\mathrm{new}}\right)^2 + \left(\mathrm{Im}\left\{Z_{\mathrm{A}}\right\}\right)^2} && \text{(E.17)} \\ &= 0.1857 - j0.1534 && \text{(E.18)} \end{aligned}$$

Hence, magnitude and phase of the reflection coefficient are

$$\begin{aligned} \left|r_{\mathrm{A,new}}\right| &= \sqrt{\left(\mathrm{Re}\left\{r_{\mathrm{A,new}}\right\}\right)^2 + \left(\mathrm{Im}\left\{r_{\mathrm{A,new}}\right\}\right)^2} = 0.241 && \text{(E.19)} \\ \angle r_{\mathrm{A,new}} &= \arctan\left(\frac{\mathrm{Im}\left\{r_{\mathrm{A,new}}\right\}}{\mathrm{Re}\left\{r_{\mathrm{A,new}}\right\}}\right) = -39.56^\circ && \text{(E.20)} \end{aligned}$$

Reflected and accepted power

The reflected power is

$$P_b = \left|s_{11}\right|^2 P_a = 0.0581\,\mathrm{W} \tag{E.21}$$

The accepted power is

$$P_{\mathrm{acc}} = \left(1 - \left|s_{11}\right|^2\right) P_a = 0.9419\,\mathrm{W} \tag{E.22}$$

Validation by circuit simulation

We apply a circuit simulator (ADS from Agilent, Inc.) in order to validate our results. First, we define a s1p-file for the reflection coefficient with a reference impedance of $Z_0 = 50\,\Omega$ (see Figure E.4). Once again, we chose an arbitrary frequency range from 1 to 2 GHz for the s-parameter representation.

```
! s1p File for Problem 5.2
!
#   GHz   S   MA   R   50


! freq1  magS11  angS11
! freq2  magS11  angS11

  1.0     0.4     -20
  2.0     0.4     -20
```

Figure E.4: s1p-file with reflection coefficient (port reference impedance $Z_0 = 50\,\Omega$)

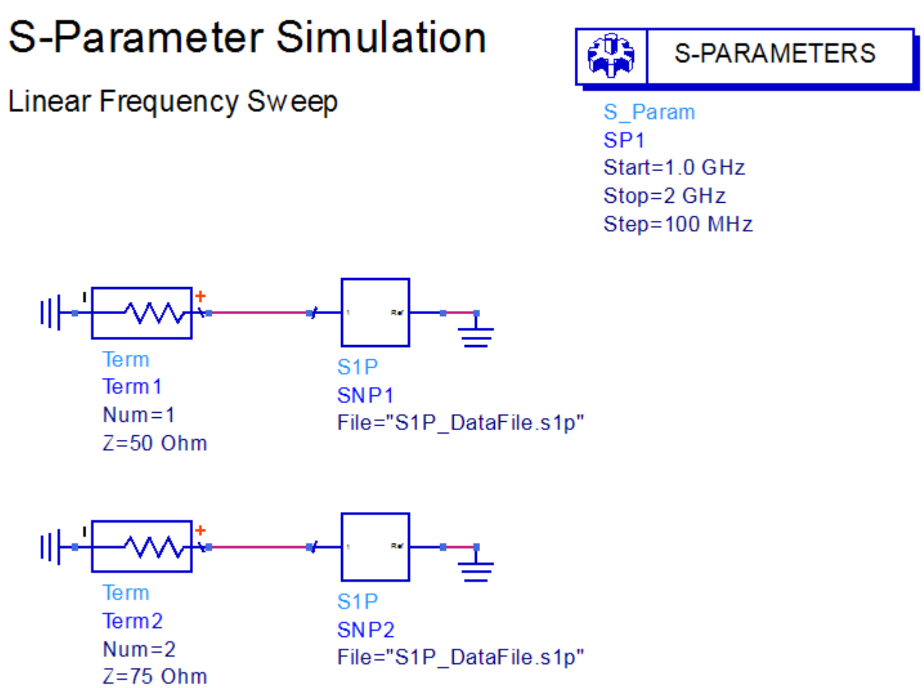


Figure E.5: ADS schematic

The schematic of the circuit is shown in Figure E.5. The s-parameter and impedance results are displayed in Figure E.6. There is good agreement between circuit simulation and calculation.

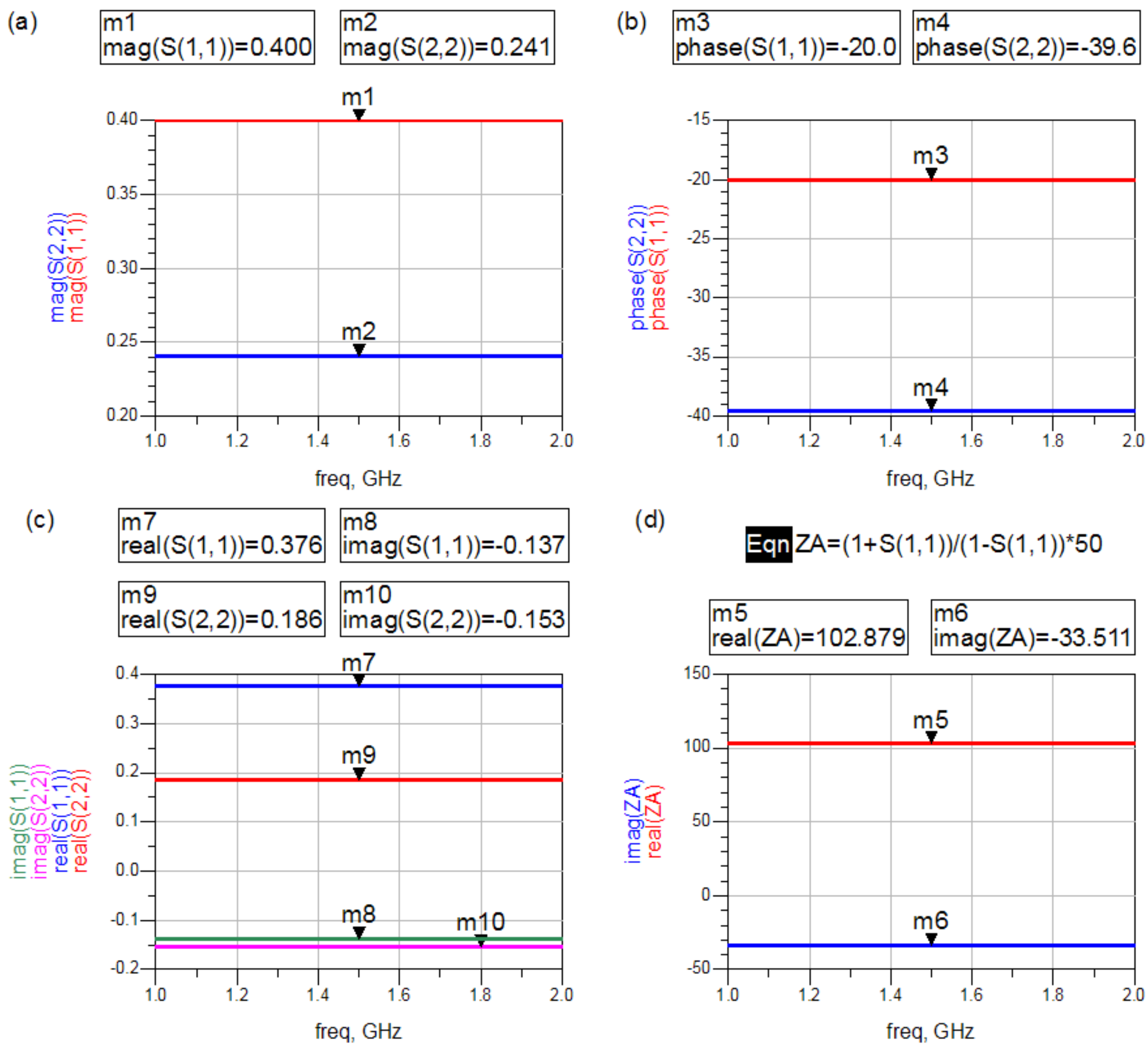


Figure E.6: (a) Magnitude and (b) phase of reflection coefficients for port reference impedances of 50 Ω (red) and 75 Ω (blue). (c) Real and imaginary part of reflection coefficients. (d) Real and imaginary part of input impedance.

E.3 Problem 5.3

Scattering matrix of a two port network with series impedance

Figure E.7 shows a circuit with a two-port network consisting of a series impedance Z. Example 5.3 (see book page 171) gives us the procedure to calculate the scattering matrix. First, we need the input impedance.

$$Z_{\mathrm{in1}} = Z + Z_0 \tag{E.23}$$

The reflection coefficient then becomes

$$s_{11} = \frac{Z_{\mathrm{in1}} - Z_0}{Z_{\mathrm{in1}} + Z_0} = \frac{Z_0}{2Z_0 + Z} = s_{22} \tag{E.24}$$

Since the network is symmetrical we get $s_{22} = s_{11}$.

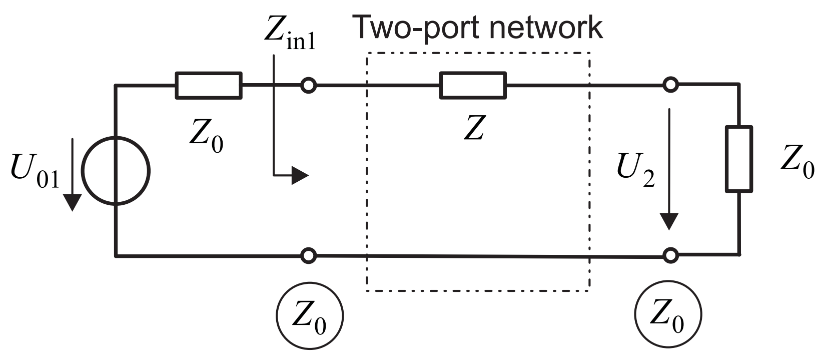


Figure E.7: Two-port network with a series impedance Z

The transmission coefficient s_{21} is given as

$$s_{21} = \frac{2U_2}{U_{01}}\sqrt{\frac{Z_0}{Z_0}} = 2\frac{Z_0}{Z_0 + Z + Z_0} = \frac{2Z_0}{2Z_0 + Z} = s_{12} \tag{E.25}$$

The voltage ratio U_2/U_{01} is determined by the voltage divider rule. Due to reciprocity we get $s_{12} = s_{21}$. Hence, the scattering matrix $\mathbf{S}$ is

$$\mathbf{S} = \frac{1}{2Z_0 + Z}\begin{pmatrix} Z & 2Z_0 \\ 2Z_0 & Z \end{pmatrix} \tag{E.26}$$

Special cases

We evaluate our result by looking at two special cases. First, we consider a short circuit ($Z = 0$), i.e. the two-port network becomes a direct through-connection.

The scattering matrix then is

$$\mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{E.27}$$

As expected there is no reflection ($s_{ii} = 0$) and full transmission ($|s_{ij}| = 1$). Next, we consider an open connection ($Z \to \infty$). The scattering matrix now reads

$$\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{E.28}$$

As expected there is full reflection ($|s_{ii}| = 1$) and no transmission ($s_{ij} = 0$).

Scattering matrix of T-network

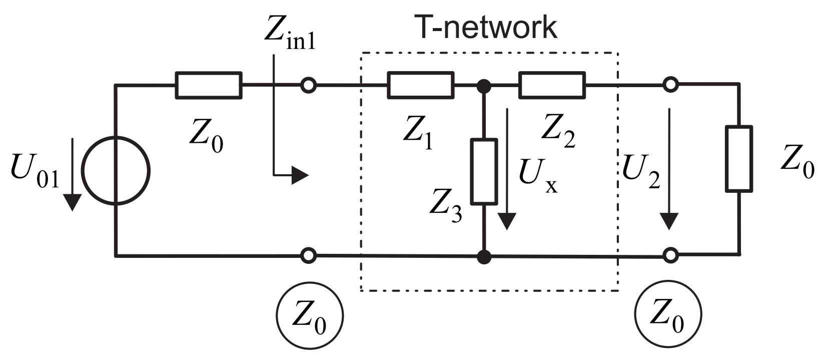


Figure E.8: Two-port circuit with a T-network

Figure E.8 shows the circuit. First, we need the input impedance.

$$Z_{\mathrm{in1}} = Z_1 + Z_3 \parallel (Z_2 + Z_0) = Z_1 + \frac{Z_3\,(Z_2 + Z_0)}{Z_3 + Z_2 + Z_0} \tag{E.29}$$

After a short calculation the input reflection coefficient is

$$s_{11} = \frac{Z_{\mathrm{in1}} - Z_0}{Z_{\mathrm{in1}} + Z_0} \tag{E.30}$$

$$= \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3 + Z_1 Z_0 - Z_2 Z_0 - Z_0^2}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3 + Z_1 Z_0 + Z_2 Z_0 + 2 Z_3 Z_0 + Z_0^2} \tag{E.31}$$

By swapping the indices 1 and 2 we get the output reflection coefficient.

$$s_{22} = \frac{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3 - Z_1 Z_0 + Z_2 Z_0 - Z_0^2}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3 + Z_1 Z_0 + Z_2 Z_0 + 2 Z_3 Z_0 + Z_0^2} \tag{E.32}$$

The calculation of the transmission factors involves the voltage ratio U_2/U_{01}. With an auxiliary quantity U_{x} we first determine the following voltage ratios.

$$\frac{U_2}{U_{\mathrm{x}}} = \frac{Z_0}{Z_2 + Z_0} \qquad \text{and} \qquad \frac{U_{\mathrm{x}}}{U_{01}} = \frac{Z_3 \parallel (Z_2 + Z_0)}{Z_0 + Z_1 + Z_3 \parallel (Z_2 + Z_0)} \tag{E.33}$$

Finally, we get

$$s_{21} = \frac{2U_2}{U_{01}}\sqrt{\frac{Z_0}{Z_0}} = 2\frac{U_2}{U_{\mathrm{x}}} \cdot \frac{U_{\mathrm{x}}}{U_{01}} = 2\frac{Z_0}{Z_2 + Z_0} \cdot \frac{Z_3 \parallel (Z_2 + Z_0)}{Z_0 + Z_1 + Z_3 \parallel (Z_2 + Z_0)} \tag{E.34}$$

$$= \frac{2 Z_3 Z_0}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3 + Z_1 Z_0 + Z_2 Z_0 + 2 Z_3 Z_0 + Z_0^2} = s_{12} \tag{E.35}$$

Due to reciprocity we find $s_{12} = s_{21}$.

Special case $Z_3 = 0$

In order to validate our formula we consider the case $Z_3 = 0$. In this case we expect zero transmission $s_{ij} = 0$ and indeed Equation E.35 is zero.

Equation E.31 becomes

$$s_{11} = \frac{Z_{\text{in1}} - Z_0}{Z_{\text{in1}} + Z_0} = \frac{Z_1 Z_2 + Z_1 Z_0 - Z_2 Z_0 - Z_0^2}{Z_1 Z_2 + Z_1 Z_0 + Z_2 Z_0 + Z_0^2} \tag{E.36}$$

$$= \frac{Z_2 (Z_1 - Z_0) + Z_0 (Z_1 - Z_0)}{Z_2 (Z_1 + Z_0) + Z_0 (Z_1 + Z_0)} = \frac{(Z_2 + Z_0)(Z_1 - Z_0)}{(Z_2 + Z_0)(Z_1 + Z_0)} = \frac{Z_1 - Z_0}{Z_1 + Z_0} \tag{E.37}$$

Since the input impedance is $Z_{\text{in1}} = Z_1$ our result is correct.

Scattering matrix of a π-network

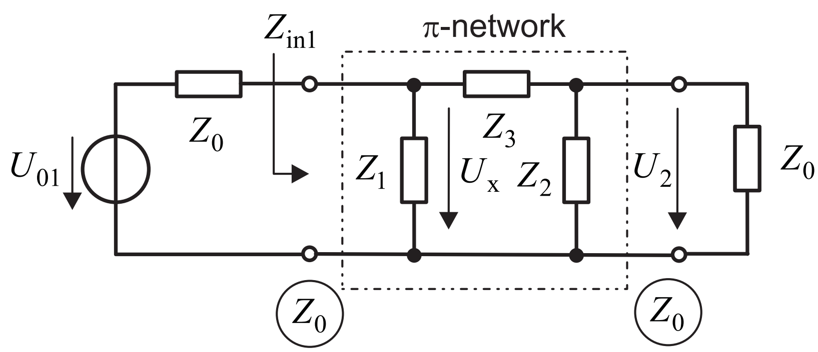


Figure E.9: Two-port circuit with a π-network

Figure E.9 shows the π-network. First, we calculate the input impedance.

$$Z_{\text{in1}} = Z_1 \parallel (Z_3 + Z_2 \parallel Z_0) \tag{E.38}$$

The input reflection coefficient becomes

$$s_{11} = \frac{Z_{\text{in1}} - Z_0}{Z_{\text{in1}} + Z_0} \tag{E.39}$$

$$= \frac{Z_1 Z_2 Z_3 + Z_1 Z_3 Z_0 - Z_2 Z_3 Z_0 - Z_1 Z_0^2 - Z_2 Z_0^2 - Z_3 Z_0^2}{Z_1 Z_2 Z_3 + 2 Z_1 Z_2 Z_0 + Z_1 Z_3 Z_0 + Z_2 Z_3 Z_0 + Z_1 Z_0^2 + Z_2 Z_0^2 + Z_3 Z_0^2} \tag{E.40}$$

In order to calculate the output reflection coefficient we swap indices 1 and 2.

$$s_{22} = \frac{Z_1 Z_2 Z_3 - Z_1 Z_3 Z_0 + Z_2 Z_3 Z_0 - Z_1 Z_0^2 - Z_2 Z_0^2 - Z_3 Z_0^2}{Z_1 Z_2 Z_3 + 2 Z_1 Z_2 Z_0 + Z_1 Z_3 Z_0 + Z_2 Z_3 Z_0 + Z_1 Z_0^2 + Z_2 Z_0^2 + Z_3 Z_0^2} \tag{E.41}$$

The calculation of the transmission factors involves the voltage ratio U_2/U_{01}. With an auxiliary quantity U_x we first determine the following voltage ratios.

$$\frac{U_2}{U_x} = \frac{Z_2 \parallel Z_0}{Z_3 + Z_2 \parallel Z_0} \quad \text{and} \quad \frac{U_x}{U_{01}} = \frac{Z_1 \parallel (Z_3 + Z_2 \parallel Z_0)}{Z_0 + Z_1 \parallel (Z_3 + Z_2 \parallel Z_0)} \tag{E.42}$$

Finally, we get

$$s_{21} = \frac{2U_2}{U_{01}}\sqrt{\frac{Z_0}{Z_0}} = 2\frac{U_2}{U_\mathrm{x}} \cdot \frac{U_\mathrm{x}}{U_{01}} \tag{E.43}$$

$$= 2\frac{Z_2 \parallel Z_0}{Z_3 + Z_2 \parallel Z_0} \cdot \frac{Z_1 \parallel (Z_3 + Z_2 \parallel Z_0)}{Z_0 + Z_1 \parallel (Z_3 + Z_2 \parallel Z_0)} \tag{E.44}$$

$$= \frac{2Z_1Z_2Z_0}{Z_1Z_2Z_3 + 2Z_1Z_2Z_0 + Z_1Z_3Z_0 + Z_2Z_3Z_0 + Z_1Z_0^2 + Z_2Z_0^2 + Z_3Z_0^2} = s_{12} \tag{E.45}$$

Due to reciprocity we find $s_{12} = s_{21}$.

Special case $Z_1 = 0$

Let us consider the case $Z_1 = 0$ (short circuit). Now, the input reflection coefficient in Equation E.40 becomes

$$s_{11} = \frac{-Z_2Z_3Z_0 - Z_2Z_0^2 - Z_3Z_0^2}{Z_2Z_3Z_0 + Z_2Z_0^2 + Z_3Z_0^2} = -1 \tag{E.46}$$

This represents the short circuit ($r = -1$) at port 1. The output reflection coefficient in Equation E.41 is now

$$s_{22} = \frac{Z_2Z_3 - Z_2Z_0 - Z_3Z_0}{Z_2Z_3 + Z_2Z_0 + Z_3Z_0} = \frac{Z_2Z_3 - (Z_2 + Z_3)\,Z_0}{Z_2Z_3 + (Z_2 + Z_3)\,Z_0} \tag{E.47}$$

$$= \frac{\dfrac{Z_2Z_3}{Z_2 + Z_3} - Z_0}{\dfrac{Z_2Z_3}{Z_2 + Z_3} + Z_0} = \frac{Z_2 \parallel Z_3 - Z_0}{Z_2 \parallel Z_3 + Z_0} \tag{E.48}$$

This result is correct since the output impedance is $Z_2 \parallel Z_3$. Finally, Equation E.45 gives us zero transmission which is correct.

E.4 Problem 5.4

Figure E.10 shows us the signal flow graph of the circuit in Figure 5.25 (book page 185).

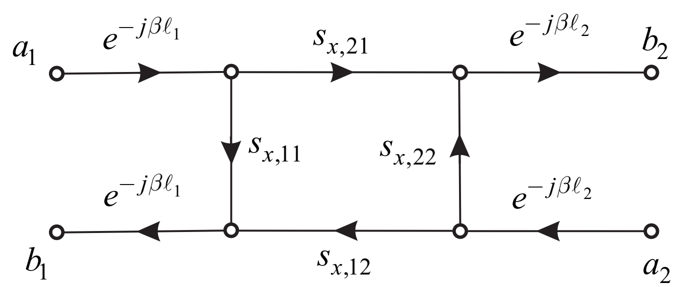


Figure E.10: Signal flow graph

The signal flow graph gives us the following scattering parameters.

$$s_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = s_{x,11}\mathrm{e}^{-j\beta(\ell_1+\ell_1)} = s_{x,11}\mathrm{e}^{-j2\beta\ell_1} \tag{E.49}$$

$$s_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} = s_{x,22}\mathrm{e}^{-j\beta(\ell_2+\ell_2)} = s_{x,22}\mathrm{e}^{-j2\beta\ell_2} \tag{E.50}$$

$$s_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = s_{x,21}\mathrm{e}^{-j\beta(\ell_1+\ell_2)} \tag{E.51}$$

$$s_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} = s_{x,12}\mathrm{e}^{-j\beta(\ell_2+\ell_1)} \tag{E.52}$$

The transmission lines only modify the phase of the scattering parameters. If the lengths of the lines are known the effect can be compensated.

E.5 Problem 5.5

We will derive the equation for the wave source in Figure 5.14 (book page 177). We repeat the representation in Figure E.11 for convenience.

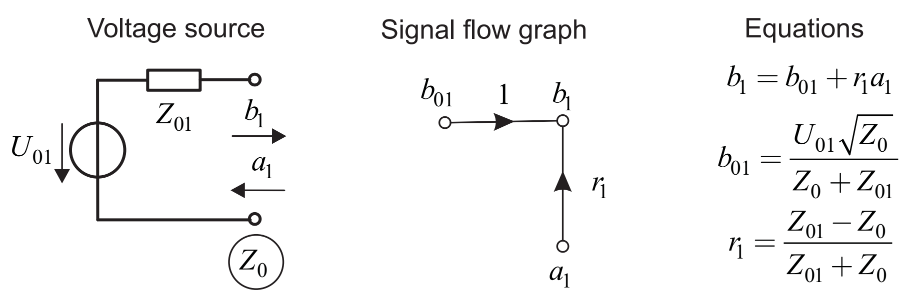


Figure E.11: Signal flow graph of a voltage source

Reflection coefficient r_1

The reflection coefficient is determined by the input impedance of the circuit. The ideal voltage source represents a short circuit. Therefore, the input impedance is Z_{01} and the reflection coefficient is

$$r_1 = \frac{Z_{01} - Z_0}{Z_{01} + Z_0} \tag{E.53}$$

Wave source

The circuit in Figure E.11 includes an ideal voltage source and is therefore an active circuit. Consequently, there is an outgoing wave b_1 even though there is no incoming wave a_1. This is expressed by the term b_{01} in the following equation.

$$b_1 = b_{01} + r_1 a_1 \tag{E.54}$$

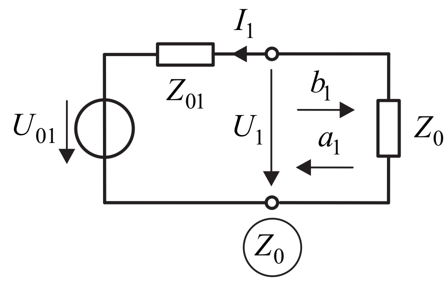


Figure E.12: Voltage, current and power waves

In order to calculate the term b_{01}, we consider the relation between voltage U and current I and power waves a and b (see Figure E.12).

The relation between power waves a and b and voltage U and current I is given as

$$b_1 = \frac{U_1 - Z_0 I_1}{2\sqrt{Z_0}} \qquad \text{and} \qquad a_1 = \frac{U_1 + Z_0 I_1}{2\sqrt{Z_0}} \tag{E.55}$$

The voltage is determined by the voltage divider rule.

$$U_1 = \frac{Z_0}{Z_0 + Z_{01}} U_{01} \tag{E.56}$$

The current is given by Ohm's law.

$$I_1 = -\frac{U_{01}}{Z_0 + Z_{01}} \tag{E.57}$$

Therefore, we get

$$a_1 = \frac{U_1 + Z_0 I_1}{2\sqrt{Z_0}} = \frac{Z_0 U_{01} + (-Z_0 U_{01})}{2\sqrt{Z_0}\,(Z_0 + Z_{01})} = 0 \tag{E.58}$$

and

$$b_1 = \frac{U_1 - Z_0 I_1}{2\sqrt{Z_0}} = \frac{Z_0 U_{01} - (-Z_0 U_{01})}{2\sqrt{Z_0}\,(Z_0 + Z_{01})} = \frac{\sqrt{Z_0} U01}{Z_0 + Z_{01}} = b_{01} \tag{E.59}$$

The term b_{01} represents the active part in the signal flow graph.

E.6 Problem 5.6

Figure E.13 shows a network with two two-port networks: the first network has a forward gain G_f and the feedback network has a gain of G_r.

From Figure E.13 we derive the following two relations.

$$X' = X + G_\mathrm{r} Y \qquad \text{and} \qquad Y = G_\mathrm{f} X' \tag{E.60}$$

This gives us

$$Y = \frac{G_\mathrm{f}}{1 - G_\mathrm{cl}} X \tag{E.61}$$

where $G_\mathrm{cl} = G_\mathrm{f} G_\mathrm{r}$ is the *closed loop gain.*

If we apply the result to the signal flow graph in Figure 5.12 (book page 175), we get

$$b = \frac{s_x}{1 - s_x s_y} a \tag{E.62}$$

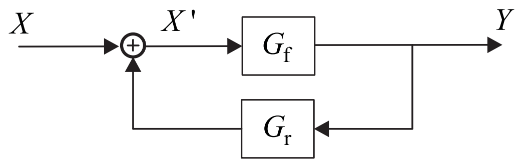


Figure E.13: Network with feedback

E.7 Problem 5.7

We will calculate reflection coefficients and impedances of the circuit in Figure E.14. We assume the following parameters.

$$f = 1\,\mathrm{GHz} \tag{E.63}$$
$$Z_0 = 50\,\Omega \qquad \text{(Port reference impedance)} \tag{E.64}$$
$$Z_\mathrm{A} = 100\,\Omega + j\omega L = 100\,\Omega + j100\,\Omega \qquad \text{where} \qquad L = 15.92\,\mathrm{nH} \tag{E.65}$$
$$Z = \frac{1}{j\omega C} \qquad \text{where} \quad C = 5\,\mathrm{pF} \tag{E.66}$$

The input impedance Z_in1 is

$$Z_\mathrm{in1} = Z_0 \parallel \frac{1}{j\omega C} = \frac{Z_0(1 - j\omega C Z_0)}{1 + (\omega C Z_0)^2} = (14.42 - j22.65)\ \Omega \tag{E.67}$$

The reflection coefficient is

$$r_1 = \frac{Z_\mathrm{in1} - Z_0}{Z_\mathrm{in1} + Z_0} = \frac{\mathrm{Re}\{Z_\mathrm{in1}\}^2 + \mathrm{Im}\{Z_\mathrm{in1}\}^2 - Z_0^2 + j2\,\mathrm{Im}\{Z_\mathrm{in1}\}\,Z_0}{(\mathrm{Re}\{Z_\mathrm{in1}\} + Z_0)^2 + \mathrm{Im}\{Z_\mathrm{in1}\}^2} \tag{E.68}$$
$$= 0.6176\,\mathrm{e}^{-128.15^\circ} \tag{E.69}$$

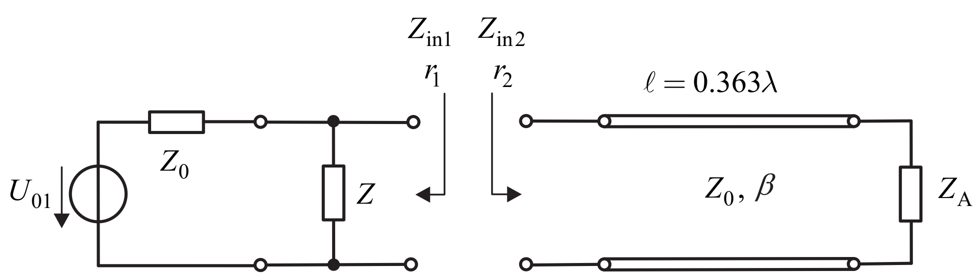


Figure E.14: Network

In order to determine the reflection coefficient r_2 we first consider the reflection coefficient r_2' at the end of the line.

$$r_2' = \frac{Z_\mathrm{A} - Z_0}{Z_\mathrm{A} + Z_0} \tag{E.70}$$
$$= \frac{\mathrm{Re}\{Z_\mathrm{A}\}^2 + \mathrm{Im}\{Z_\mathrm{A}\}^2 - Z_0^2 + j2\,\mathrm{Im}\{Z_\mathrm{A}\}\,Z_0}{(\mathrm{Re}\{Z_\mathrm{A}\} + Z_0)^2 + \mathrm{Im}\{Z_\mathrm{A}\}^2} = 0.62\,\mathrm{e}^{j29.75^\circ} \tag{E.71}$$

At the input of the transmission line we get

$$r_2 = r_2'\,\mathrm{e}^{-j2\beta\ell} = 0.62\,\mathrm{e}^{j29.75^\circ}\,\mathrm{e}^{j98.64^\circ} = 0.62\,\mathrm{e}^{j128.4^\circ} = r_1^* \tag{E.72}$$

If we compare the reflection coefficients r_1 and r_2 we see that $r_2 = r_1^*$ (complex conjugate matching).

Using a Smith-Chart gives us the input impedance Z_{in2} (see Figure E.15). We apply a commercial circuit simulator (ADS by Agilent, inc.). The electrical line length is $0.363\lambda = 130.68^\circ$. The load impedance is represented by a black diamond the input impedance is depicted by a red square. We read from the diagram

$$Z_{\text{in2}} = (14.28 + j22.56)\,\Omega = Z_{\text{in1}}^* \tag{E.73}$$

Impedances Z_{in1} and Z_{in2} – as well as reflection coefficients r_1 and r_2 – show complex conjugate values.

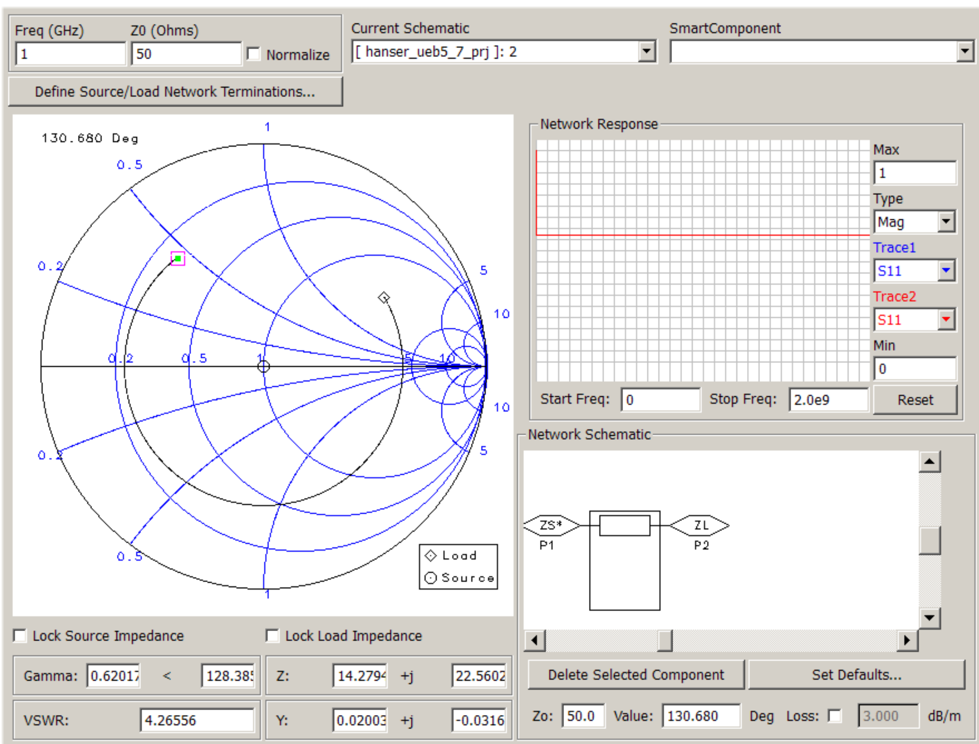


Figure E.15: Smith chart and input impedance Z_{in2}

Circuit simulation

We will validate our result by performing a circuit simulation. In our previous calculation we considered a frequeny of $f = 1\,\text{GHz}$. Now we extend the frequency range from 100 MHz to 2 GHz. Figure E.16 shows the schematic. The results are given in Figure E.17.

The upper diagrams in Figure E.17 show reflection coefficients (magnitude and phase) as well as input impedances (real and imaginary part). In order to compare the results with our simulation we have to look at a frequency of $f = 1\,\text{GHz}$ (indicated by markers). The simulation results agree with our previously calculated values.

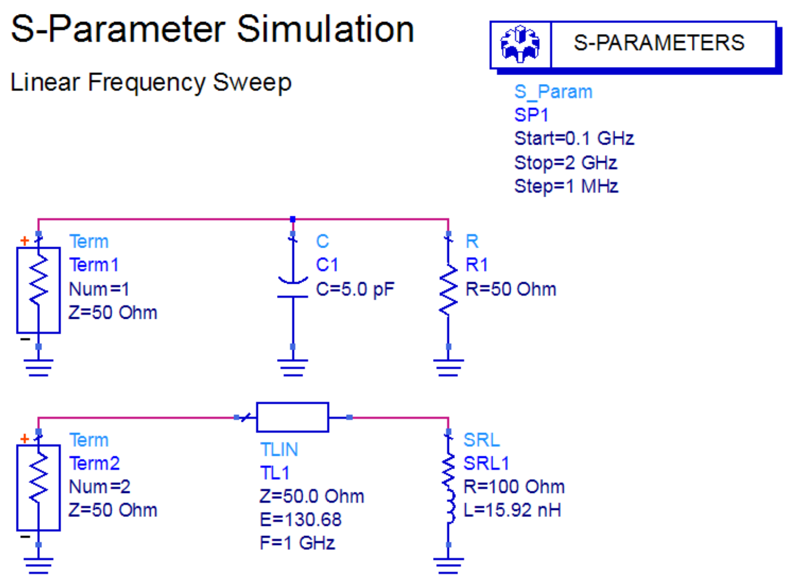


Figure E.16: Schematic

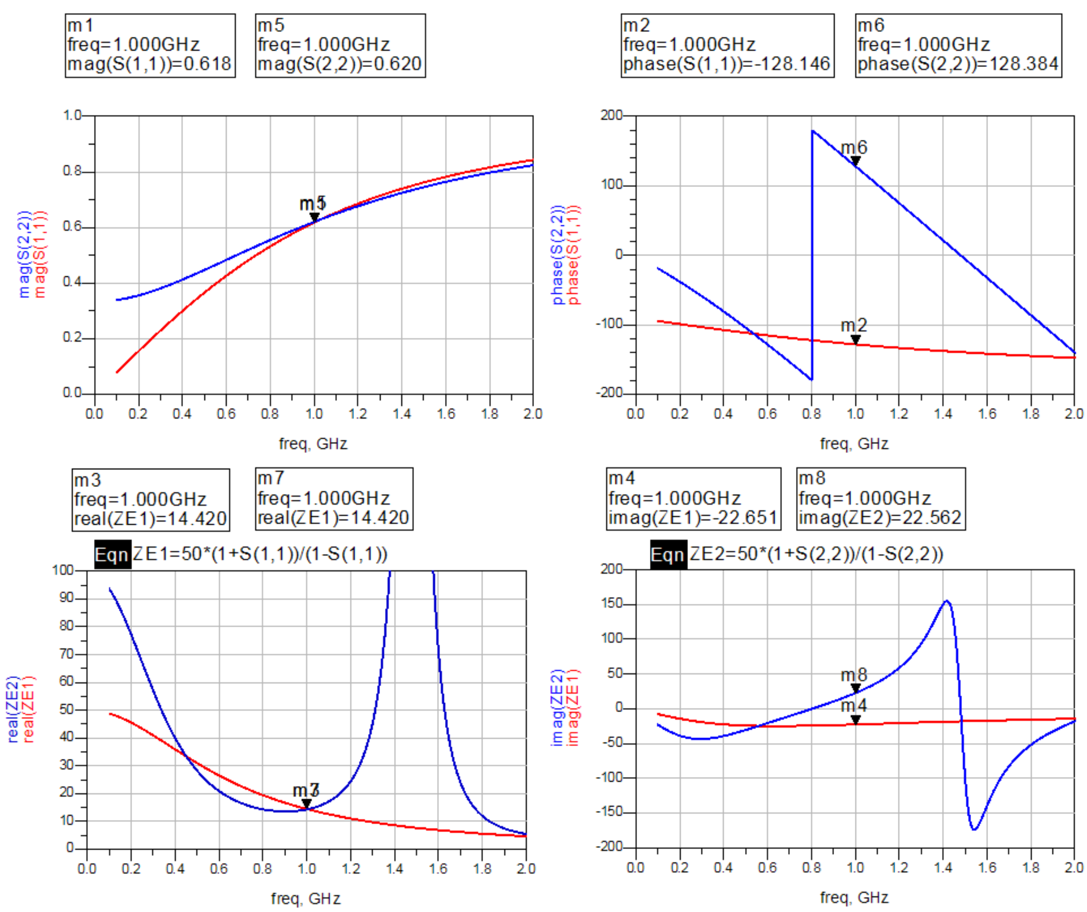


Figure E.17: Simulation results: s-parameters and impedances

E.8 Problem 5.8

A three-port network shall have the following three properties:

- matching at all ports ($s_{ii} = 0$),
- reciprocity ($s_{ij} = s_{ji}$) and
- no losses ($\mathbf{S}^{\mathrm{T}}\mathbf{S}^* = \mathbf{I}$).

From these requirements the matrix equation

$$\begin{pmatrix} 0 & s_{12} & s_{13} \\ s_{12} & 0 & s_{23} \\ s_{13} & s_{23} & 0 \end{pmatrix} \begin{pmatrix} 0 & s_{12}^* & s_{13}^* \\ s_{12}^* & 0 & s_{23}^* \\ s_{13}^* & s_{23}^* & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{E.74}$$

gives us the following set of equations

$$|s_{12}|^2 + |s_{13}|^2 = 1 \tag{E.75}$$
$$s_{13}s_{23}^* = 0 \tag{E.76}$$
$$s_{12}s_{23}^* = 0 \tag{E.77}$$
$$s_{23}s_{13}^* = 0 \tag{E.78}$$
$$|s_{12}|^2 + |s_{23}|^2 = 1 \tag{E.79}$$
$$s_{12}s_{13}^* = 0 \tag{E.80}$$
$$s_{23}s_{12}^* = 0 \tag{E.81}$$
$$s_{13}s_{12}^* = 0 \tag{E.82}$$
$$|s_{13}|^2 + |s_{23}|^2 = 1 \tag{E.83}$$

From Equations E.76 and E.77 we get (for $s_{23} \neq 0$):

$$s_{12} = s_{13} \tag{E.84}$$

From Equation E.80 we then conclude:

$$s_{12} = s_{13} = 0 \tag{E.85}$$

This result contradicts Equation E.75 since

$$|s_{12}|^2 + |s_{13}|^2 = 0 \neq 1 \qquad \text{(Conflict for } s_{23} \neq 0\text{)} \tag{E.86}$$

The conflict can be solved for $s_{23} = 0$. This implies (Equation E.79)

$$|s_{12}| = 1 \tag{E.87}$$

and (Equation E.83)

$$|s_{13}| = 1 \tag{E.88}$$

This result contradicts Equation E.75 since

$$|s_{12}|^2 + |s_{13}|^2 = 2 \neq 1 \qquad \text{(Conflict for } s_{23} = 0\text{)} \tag{E.89}$$

The set of equations shows that a three-port network cannot satisfy the following conditions simultaneously

- matching at all ports ($s_{ii} = 0$),
- reciprocity ($s_{ij} = s_{ji}$) and
- losslessness.

(Last modified: 02.01.2013)

Appendix F

Solutions to Problems in Chapter 6

F.1 Problem 6.1

Short-circuited transmission lines

Section 6.2.1 (book page 193) describes the method to determine the overall length of the transmission line resonator with open-circuited lines. According to our considerations in Chapter 3 a lossless short-circuited transmission line exhibits a reactive input impedance (Equation 3.101, book page 77).

$$Z_{\text{in}} = jZ_0 \tan(\beta\ell) = jX_{\text{in}} \tag{F.1}$$

where ℓ is the length of the transmission line.

In order to design a transmission line resonator we connect two transmission lines in parallel (see Fig. F.1) where ℓ_1 and ℓ_2 are the individual lengths of the lines. The overall length of the resonator is $\ell = \ell_1 + \ell_2$.

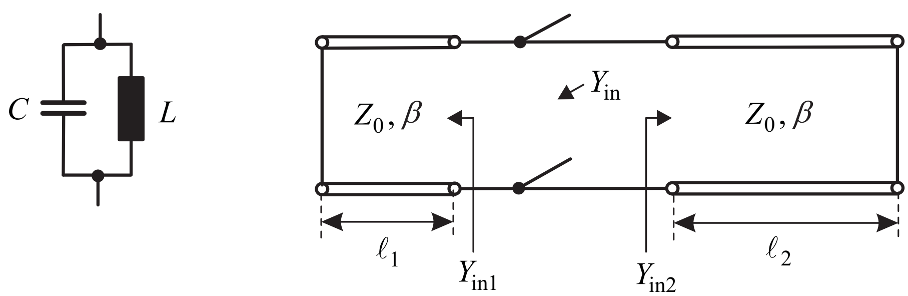


Figure F.1: Transmission line resonator with short-circuited lines

At the input terminal we see the following admittance

$$Y_{\text{in}} = Y_{\text{in1}} + Y_{\text{in2}} = \frac{1}{jZ_0}\left(\frac{1}{\tan(\beta\ell_1)} + \frac{1}{\tan(\beta\ell_2)}\right) = \frac{1}{jZ_0}\left(\cot(\beta\ell_1) + \cot(\beta\ell_2)\right) \tag{F.2}$$

At parallel resonance the imaginary part of the admittance is zero. Therefore, we rewrite the sum of cotangent functions as

$$\cot x \pm \cot y = \pm\frac{\sin(x \pm y)}{\sin x \sin y} \tag{F.3}$$

We get

$$\mathrm{Im}\left\{Y_{\mathrm{in}}\right\} = -\frac{1}{Z_0} \cdot \frac{\sin(\beta(\ell_1 + \ell_2))}{\sin(\beta\ell_1)\sin(\beta\ell_2)} = 0 \qquad \text{(at resonance)} \tag{F.4}$$

The sine function in the numerator determines the zeros of the expression.

$$\beta(\ell_1 + \ell_2) = n\pi \qquad \text{where} \qquad n \in \mathbb{Z} \tag{F.5}$$

Therefore, the overall length of the resonator is

$$\boxed{\ell = \ell_1 + \ell_2 = \frac{n\pi}{\beta} = \frac{n\pi\lambda}{2\pi} = n\frac{\lambda}{2}} \tag{F.6}$$

A transmission line with short-circuited terminations at both ends shows parallel resonance for a line length equal to a half wavelength.

Resonator with a short-circuited and an open-circuited line

Let us consider the configuration in Figure F.2.

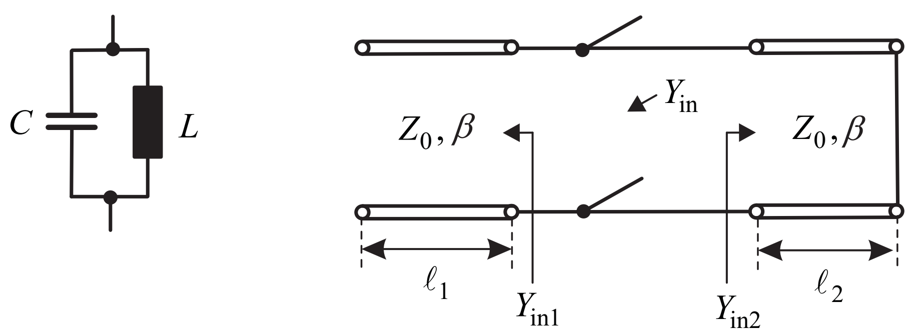


Figure F.2: Transmission line resonator with short-circuited and open-circuited line

The input impedance of the open-circuited line is

$$Z_{\mathrm{in1}} = -jZ_0 \cot(\beta\ell_1) = jX_{\mathrm{in1}} \tag{F.7}$$

The input impedance of the short-circuited line is

$$Z_{\mathrm{in2}} = jZ_0 \tan(\beta\ell_2) = jX_{\mathrm{in2}} \tag{F.8}$$

The admittance at the input terminal becomes

$$Y_{\mathrm{in}} = Y_{\mathrm{in1}} + Y_{\mathrm{in2}} = \frac{1}{jZ_0}\left(-\frac{1}{\cot(\beta\ell_1)} + \frac{1}{\tan(\beta\ell_2)}\right) = \frac{1}{jZ_0}\left(\cot(\beta\ell_2) - \tan(\beta\ell_1)\right) \tag{F.9}$$

At parallel resonance the imaginary part of the admittance is zero. Therefore, we rewrite the expression in the numerator as

$$\cot x - \tan y = \frac{\cos(x+y)}{\sin x \cos y} \tag{F.10}$$

and get the following condition

$$\mathrm{Im}\{Y_{\mathrm{in}}\} = -\frac{1}{Z_0} \cdot \frac{\cos(\beta(\ell_1 + \ell_2))}{\sin(\beta\ell_2)\sin(\beta\ell_1)} = 0 \qquad \text{(at resonance)} \tag{F.11}$$

We find the first zero of the cosine function at $\pi/2$.

$$\beta(\ell_1 + \ell_2) = \frac{\pi}{2} \qquad \rightarrow \qquad \ell = \ell_1 + \ell_2 = \frac{\pi}{2} \cdot \frac{1}{\beta} = \frac{\pi}{2} \cdot \frac{\lambda}{2\pi} = \frac{\lambda}{4} \tag{F.12}$$

Therefore, the shortest line length ℓ is a quarter of a wavelength.

F.2 Problem 6.2

a) Resistive termination $330\,\Omega$

A resistive load termination $Z_{\mathrm{A1}} = 330\,\Omega$ shall be matched to a reference impedance of $Z_{\mathrm{in}} = 50\,\Omega$ by a LC-network at a frequency $f = 900\,\mathrm{MHz}$. According to Section 6.3.1 low-pass and high-pass realisations exist (book page 197).

Figure F.3 shows the corresponding circuits and Smith chart transformations using ADS circuit simulator from Agilent. The Smith diagram shows red circles (admittances) and blue circles (impedances). The matching point (zero reflection coefficient) is located at the center of the diagram. The transformation paths (at $f = 900\,\mathrm{MHz}$) are show by coloured arrows. Furthermore, Figure F.3 shows the magnitude of the reflection coefficient in the frequency range from DC to $f = 1800\,\mathrm{MHz}$. As expected the reflection coefficient is practically zero at $f = 900\,\mathrm{MHz}$.

For comparison we calculate the component values using the formulas given in Section 6.3.1. For $R_{\mathrm{A}} > R_{\mathrm{I}}$ we get

$$\begin{aligned} |X_1| &= R_{\mathrm{I}} \cdot Q = 118.3\,\Omega \quad \text{and} \\ |X_2| &= R_{\mathrm{A}}/Q = 139.5\,\Omega \quad \text{where} \quad Q = \sqrt{\frac{R_{\mathrm{A}}}{R_{\mathrm{I}}} - 1} = 2.366 \end{aligned} \tag{F.13}$$

The reactances X_1 and X_2 represent reactive components, that are either inductive ($X = \omega L$) or capacitive $X = -1/(\omega C)$. High-pass and low-pass configurations consist of two different components: one is a capacitance and one is an inductance (Figure 6.12). The component values for a given frequency f_0 are determined by the following equations and summarized in Table F.1. The results are in good agreement. Minor differences are caused by the graphical evaluation of the Smith chart.

$$L_{1,2} = \frac{|X_{1,2}|}{2\pi f_0} \qquad \text{and} \qquad C_{1,2} = \frac{1}{|X_{1,2}| 2\pi f_0} \tag{F.14}$$

	Series component	Shunt component
Low-pass design	$L_1 = 20.92\,\mathrm{nH}$ ($20.82\,\mathrm{nH}$)	$C_2 = 1.268\,\mathrm{pF}$ ($1.277\,\mathrm{pF}$)
High-pass design	$C_1 = 1.495\,\mathrm{pF}$ ($1.498\,\mathrm{pF}$)	$L_2 = 24.67\,\mathrm{nH}$ ($24.58\,\mathrm{nH}$)

Table F.1: Component values for $Z_{\mathrm{A}} = 330\,\Omega$ (Smith chart results in parentheses)

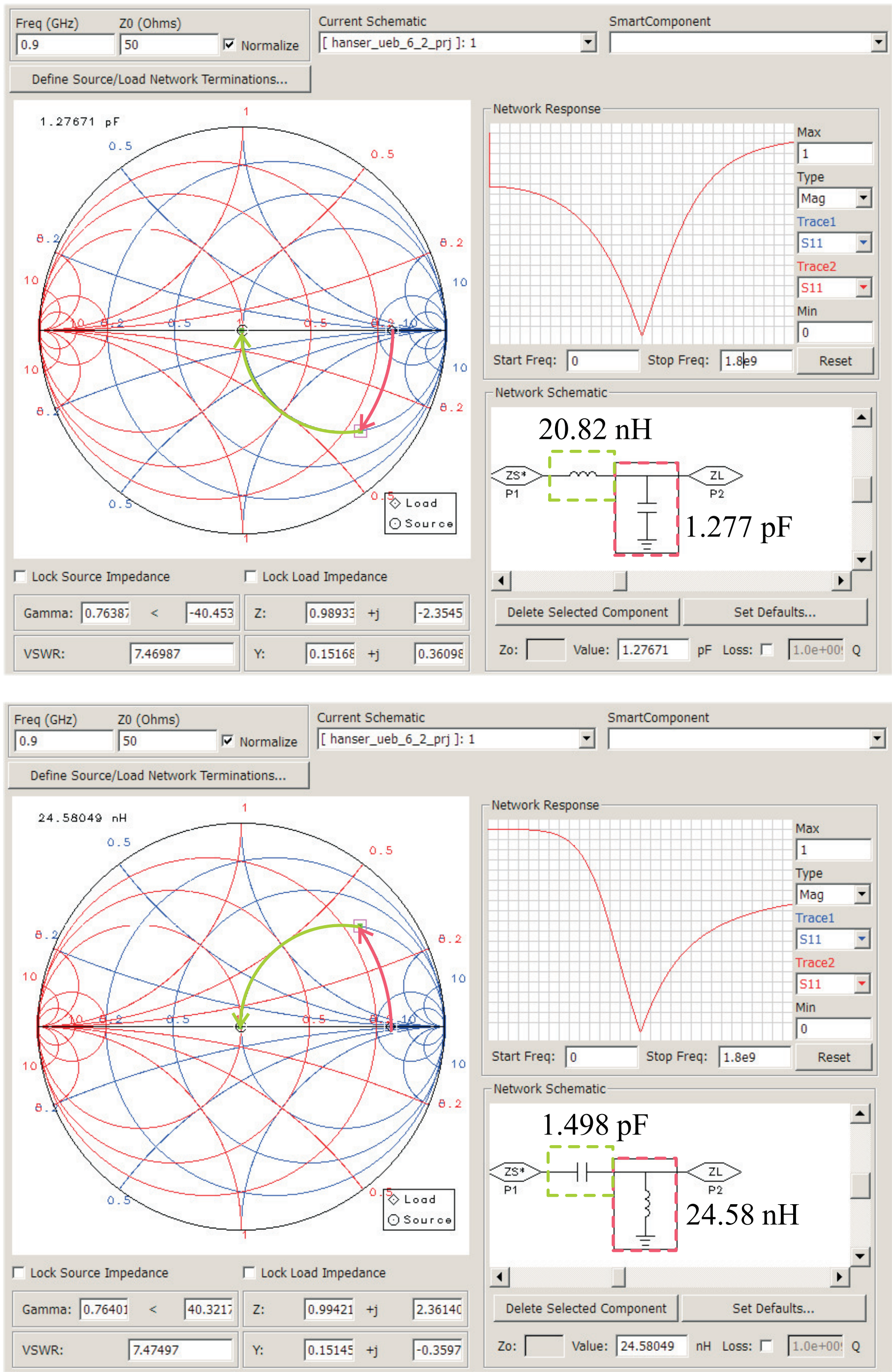


Figure F.3: Solution to Problem 6.2a (low-pass (top), high-pass (bottom))

b) Resistive termination $10\,\Omega$

We follow the method in the previous paragraph and use equations from Section 6.3.1. Now we have $R_\mathrm{A} < R_\mathrm{I}$. So we get

$$\begin{aligned} &|X_1| = R_\mathrm{I}/Q = 25\,\Omega \quad \text{and} \\ &|X_2| = R_\mathrm{A} \cdot Q = 20\,\Omega \quad \text{where} \quad Q = \sqrt{\frac{R_\mathrm{I}}{R_\mathrm{A}} - 1} = 2 \end{aligned} \tag{F.15}$$

Component values are given as

$$L_{1,2} = \frac{|X_{1,2}|}{2\pi f_0} \qquad \text{and} \qquad C_{1,2} = \frac{1}{|X_{1,2}| 2\pi f_0} \tag{F.16}$$

Table F.2 lists the values for high-pass and low-pass design. Figure F.4 shows the design using a Smith chart tool.

	Series component	Shunt component
High-pass design	$L_1 = 4.42\,\mathrm{nH}$ $(4.44\,\mathrm{nH})$	$C_2 = 8.84\,\mathrm{pF}$ $(8.84\,\mathrm{pF})$
Low-pass design	$C_1 = 7.07\,\mathrm{pF}$ $(7.12\,\mathrm{pF})$	$L_2 = 3.54\,\mathrm{nH}$ $(3.53\,\mathrm{nH})$

Table F.2: Component values for $Z_\mathrm{A} = 10\,\Omega$ (Smith chart results in parentheses)

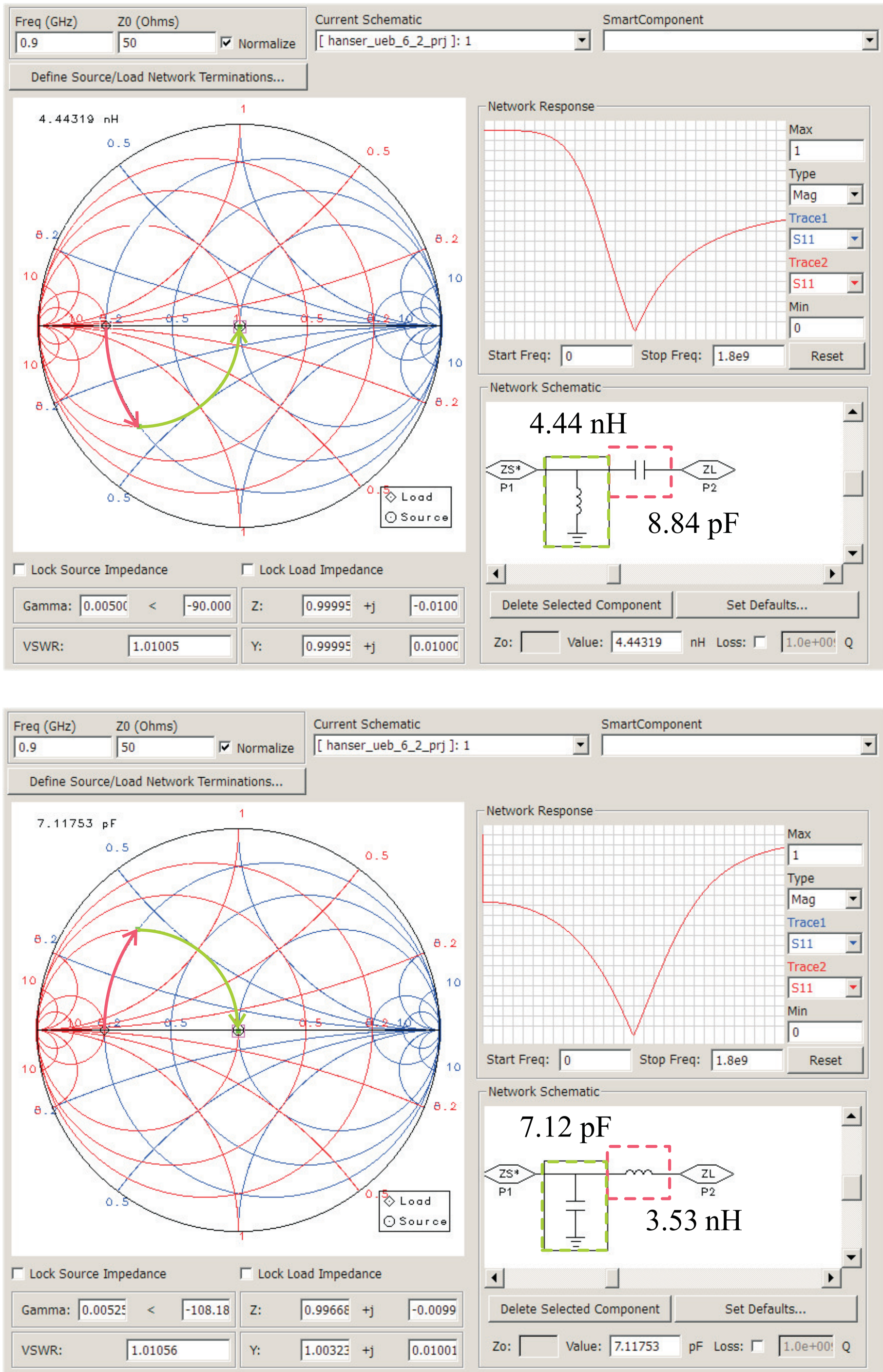
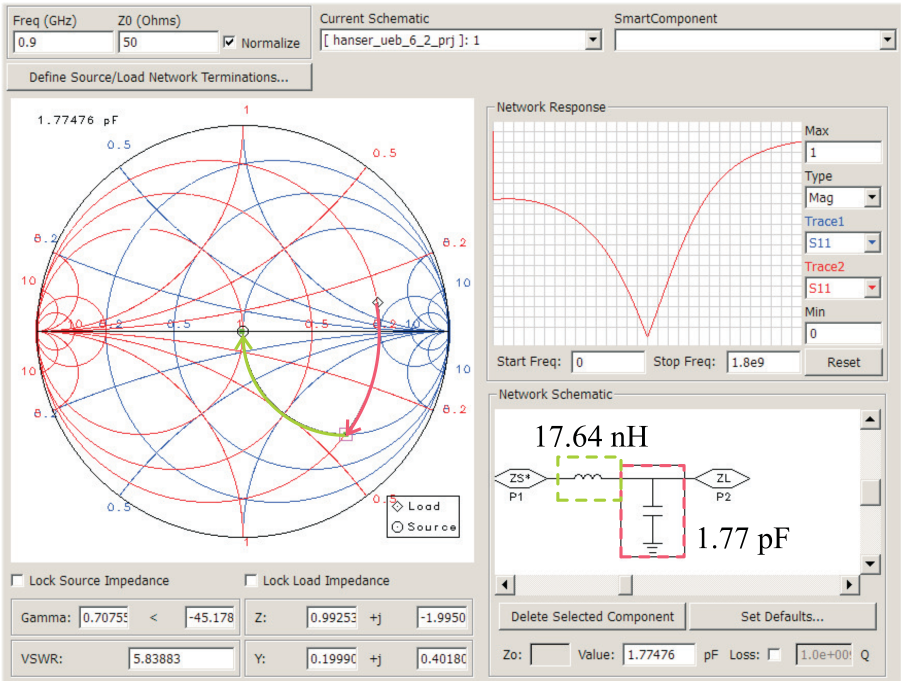


Figure F.4: Solution to Problem 6.2b

c) Complex termination $(200 + j100)\,\Omega$

Figure F.5 shows the transformation paths in the Smith diagram.



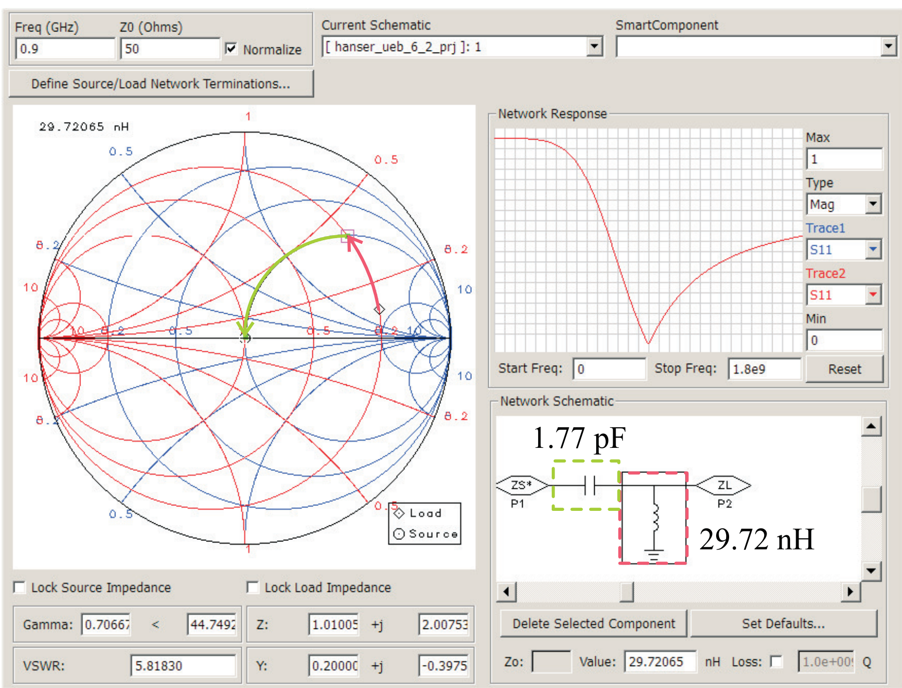


Figure F.5: Solution to Problem 6.2c

d) Complex termination $(15 - j75)\,\Omega$

Figure F.6 shows the transformation paths in the Smith diagram.

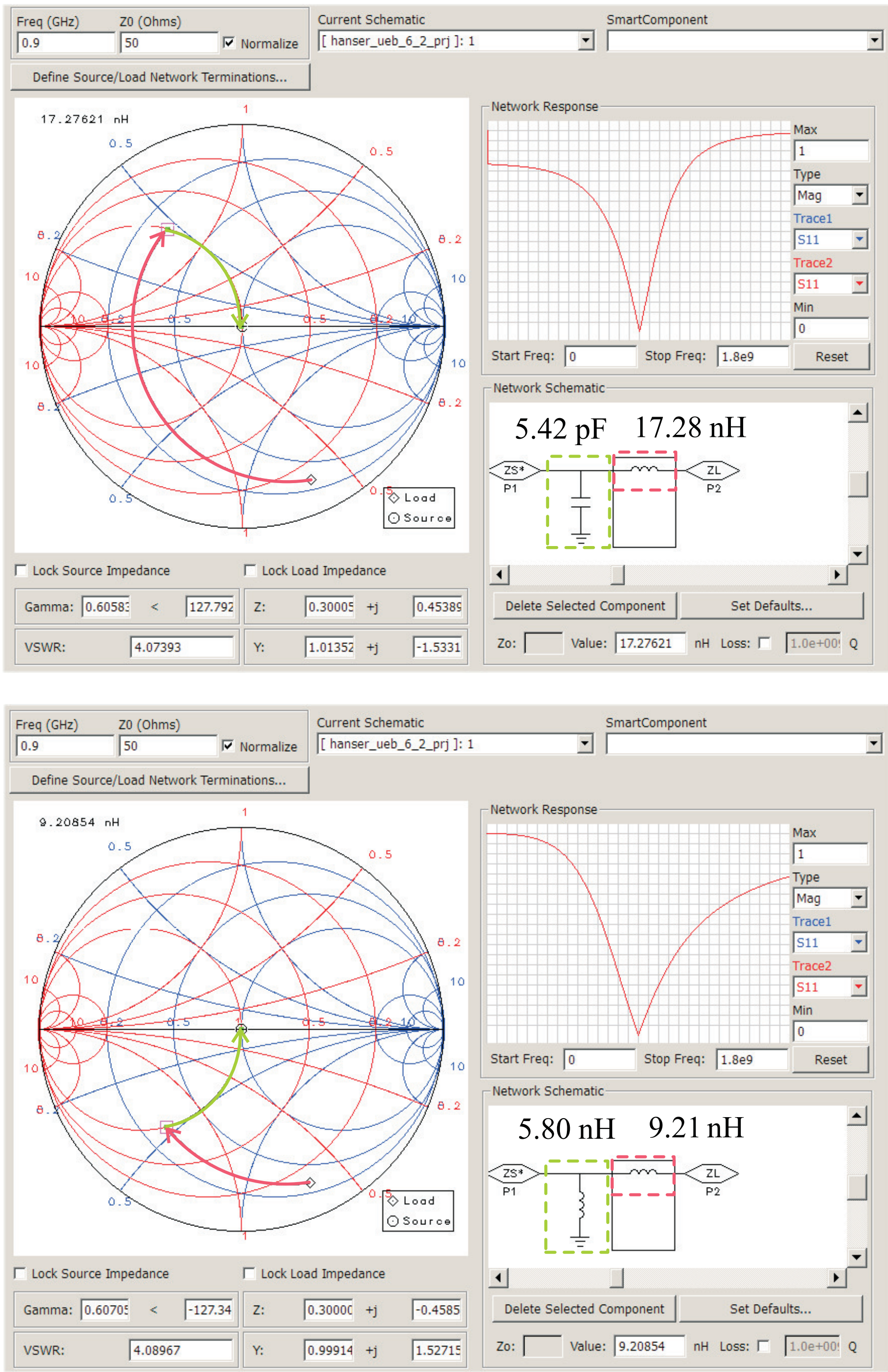


Figure F.6: Solution to Problem 6.2d

F.3 Problem 6.3

A load resistance of $R_A = 1100\,\Omega$ shall be matched to a reference impedance of $R_I = 50\,\Omega$ at a frequency of $f = 1\,\text{GHz}$. We will investigate the bandwidth of single-stage, two-stage and three-stage networks.

1. Single-stage LC-matching network

We determine the component values by using the equations of Section 6.3.1.

$$Q = \sqrt{\frac{R_A}{R_I} - 1} = 4.5826 \tag{F.17}$$

$$X_1 = R_I \cdot Q = 229.13\,\Omega \qquad \text{and} \qquad X_2 = R_A/Q = 240.04\,\Omega \tag{F.18}$$

We choose a high-pass design and get

$$C_1 = \frac{1}{\omega X_1} = 0.69\,\text{pF} \qquad \text{and} \qquad L_2 = \frac{X_2}{\omega} = 38.2\,\text{nH} \tag{F.19}$$

Figure F.7 shows the schematic (ADS) and Figure F.8 illustrates the transformation path in the Smith chart. The reflection coefficient in the frequency range from 0.9 GHz to 1.1 GHz is given in Figure F.9 (red curve).

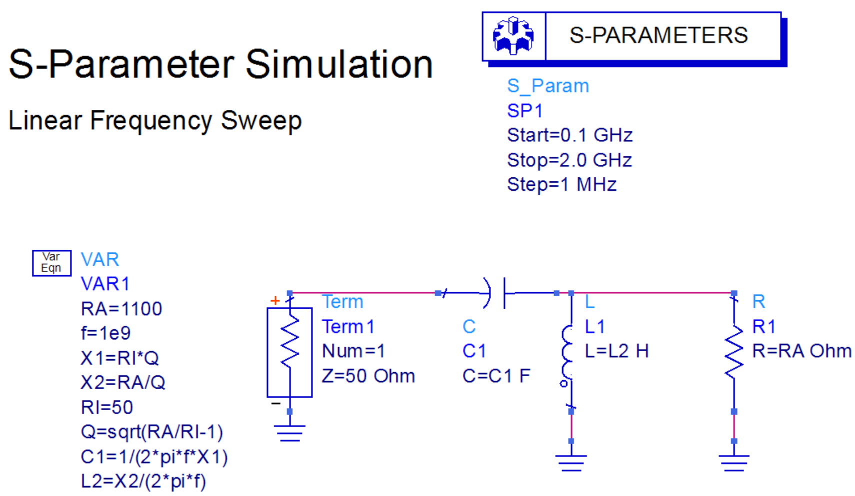


Figure F.7: Single-stage LC-matching network (high-pass design)

2. Two-stage LC-matching network

Now matching is achieved in two steps. First, we transform the load impedance to an intermediate resistance value

$$R_m = \sqrt{R_A R_I} \tag{F.20}$$

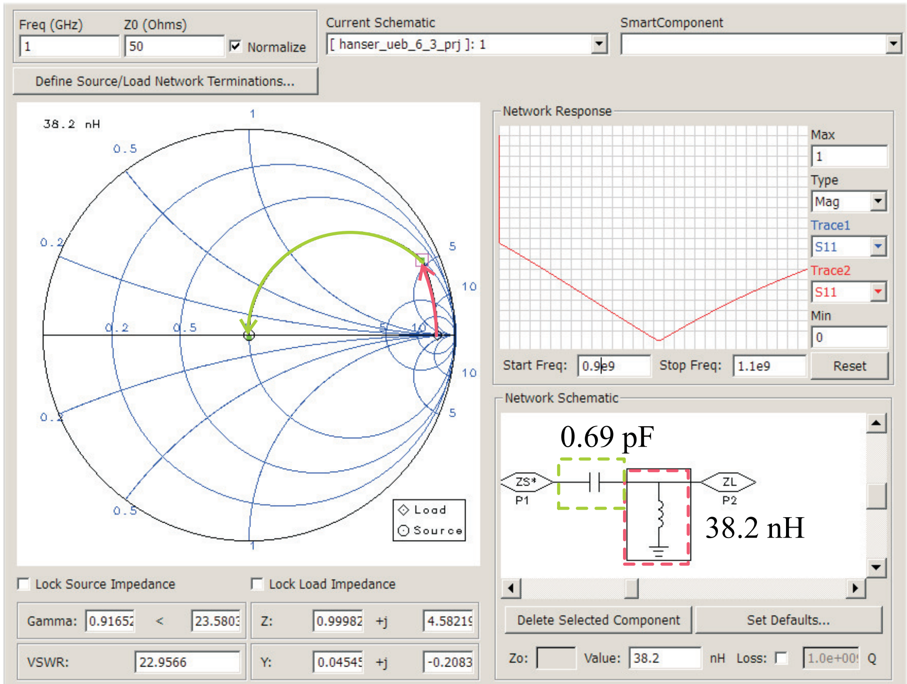


Figure F.8: Smith chart for single LC-matching network (high-pass design)

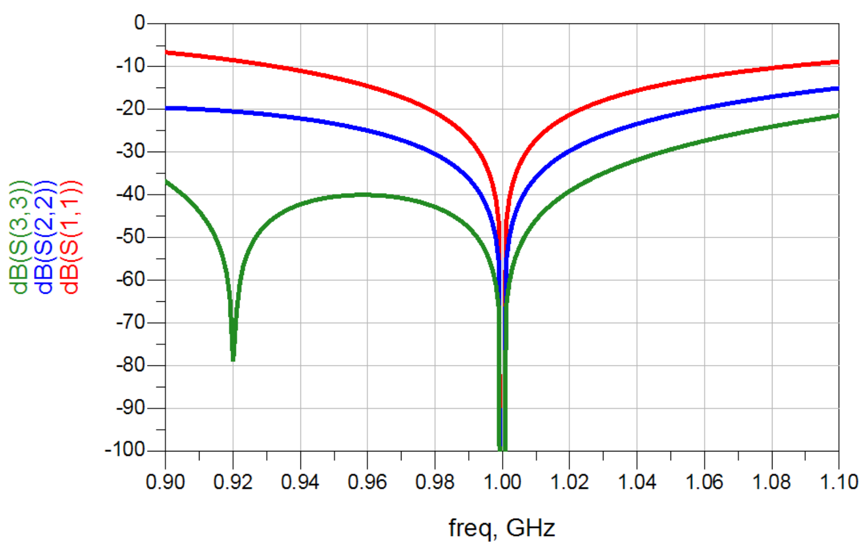


Figure F.9: Scattering parameters for single LC-matching network (red), two-stage network (blue) and three-stage network (green)

Using our formulas from Section 6.3.1 the component values of the two LC-networks ($R_\mathrm{A} \rightarrow R_\mathrm{m}$ and $R_\mathrm{m} \rightarrow R_\mathrm{I}$) are

$$Q = Q_1 = \sqrt{\frac{R_\mathrm{m}}{R_\mathrm{I}} - 1} = Q_2 = \sqrt{\frac{R_\mathrm{A}}{R_\mathrm{m}} - 1} = 1.921 \tag{F.21}$$

$$X_{11} = R_\mathrm{I} \cdot Q = 96.05\,\Omega \qquad \text{and} \qquad X_{21} = R_\mathrm{m} \cdot Q = 450.51\,\Omega \tag{F.22}$$

$$X_{12} = R_\mathrm{m}/Q = 122.08\,\Omega \qquad \text{and} \qquad X_{22} = R_\mathrm{A}/Q = 572.6\,\Omega \tag{F.23}$$

$$\tag{F.24}$$

Choosing a high-pass design we get

$$C_{11} = \frac{1}{\omega X_{11}} = 1.657\,\text{pF} \quad \text{and} \quad L_{12} = \frac{X_{12}}{\omega} = 19.4\,\text{nH} \tag{F.25}$$

$$C_{21} = \frac{1}{\omega X_{21}} = 0.35\,\text{pF} \quad \text{and} \quad L_{22} = \frac{X_{22}}{\omega} = 91.0\,\text{nH} \tag{F.26}$$

Figure F.10 shows the schematic (ADS) and Figure F.11 illustrates the transformation path in the Smith chart. The reflection coefficient in the frequency range from 0.9 GHz to 1.1 GHz is given in Figure F.9 (blue curve). The two-stage network shows an extended bandwidth.

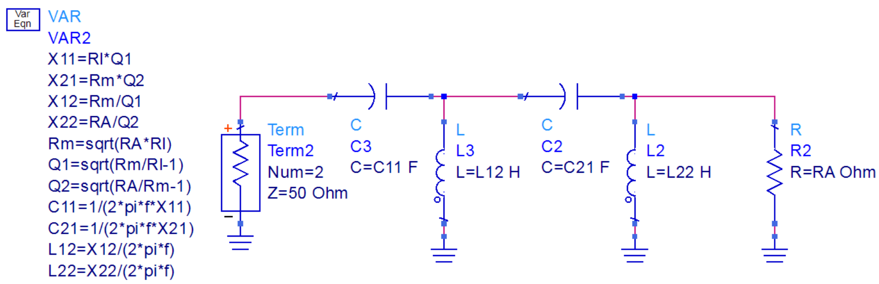


Figure F.10: Two-stage LC-matching network (high-pass design)

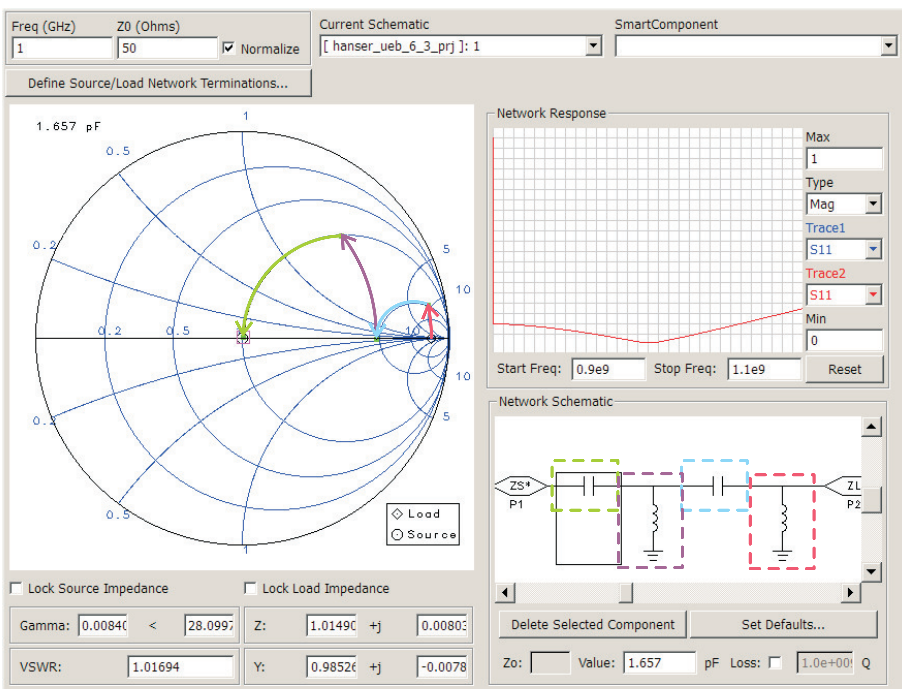


Figure F.11: Smith chart for a two-stage LC-matching network (high-pass design)

3. Three-stage LC-matching network

Now matching is achieved in three steps. We transform to intermediate resistance values of

$$R_{\mathrm{mo}} = \sqrt{R_{\mathrm{A}}^2 R_{\mathrm{I}}} \qquad \text{and} \qquad R_{\mathrm{mu}} = \sqrt{R_{\mathrm{A}} R_{\mathrm{I}}^2} \tag{F.27}$$

Using our formulas from Section 6.3.1 the component values of the three LC-networks ($R_{\mathrm{A}} \to R_{\mathrm{mo}}$ and $R_{\mathrm{mo}} \to R_{\mathrm{mu}}$ and $R_{\mathrm{mu}} \to R_{\mathrm{I}}$) are

$$Q = Q_1 = \sqrt{\frac{R_{\mathrm{mu}}}{R_{\mathrm{I}}} - 1} = Q_2 = \sqrt{\frac{R_{\mathrm{mo}}}{R_{\mathrm{mu}}} - 1} = Q_3 = \sqrt{\frac{R_{\mathrm{A}}}{R_{\mathrm{mo}}} - 1} = 1.342 \tag{F.28}$$

$$X_{011} = R_{\mathrm{I}} \cdot Q = 67.1\,\Omega \quad ; \quad X_{021} = R_{\mathrm{mu}} \cdot Q = 188.0\,\Omega \quad ; \quad X_{031} = R_{\mathrm{mo}} \cdot Q = 526.8\,\Omega \tag{F.29}$$

$$X_{012} = R_{\mathrm{mu}}/Q = 105.82\,\Omega \quad ; \quad X_{022} = R_{\mathrm{mo}}/Q = 292.53\,\Omega \quad ; \quad X_{032} = R_{\mathrm{A}}/Q = 819.67\,\Omega \tag{F.30}$$

Choosing a high-pass design we get

$$C_{011} = \frac{1}{\omega X_{011}} = 2.37\,\mathrm{pF} \qquad \text{and} \qquad L_{012} = \frac{X_{012}}{\omega} = 16.8\,\mathrm{nH} \tag{F.31}$$

$$C_{021} = \frac{1}{\omega X_{021}} = 0.847\,\mathrm{pF} \qquad \text{and} \qquad L_{022} = \frac{X_{022}}{\omega} = 46.6\,\mathrm{nH} \tag{F.32}$$

$$C_{031} = \frac{1}{\omega X_{031}} = 0.302\,\mathrm{pF} \qquad \text{and} \qquad L_{032} = \frac{X_{032}}{\omega} = 130.4\,\mathrm{nH} \tag{F.33}$$

Figure F.12 shows the schematic (ADS) and Figure F.13 illustrates the transformation path in the Smith chart. The reflection coefficient in the frequency range from 0.9 GHz to 1.1 GHz is given in Figure F.9 (green curve). The three-stage network shows an extended bandwidth in comparison to the two-stage network.

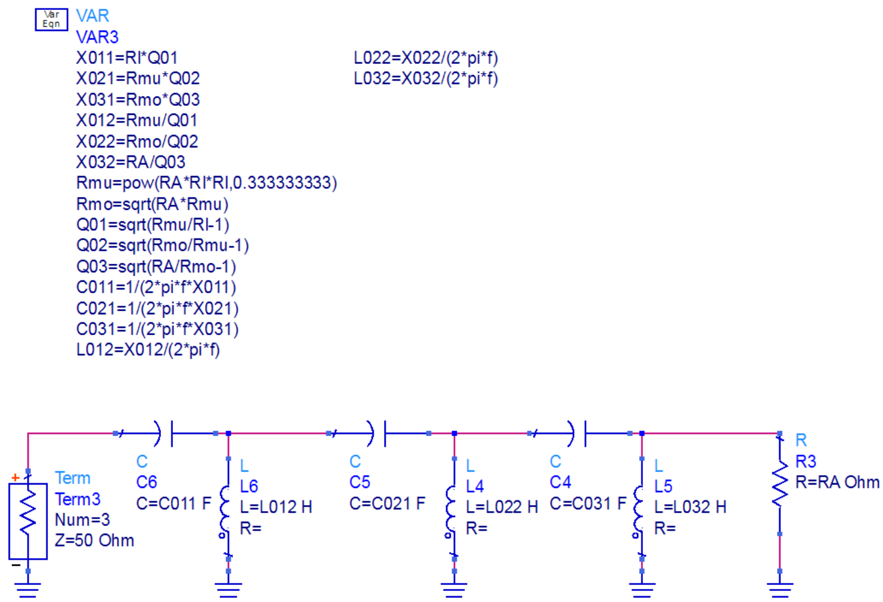


Figure F.12: Three-stage LC-matching network (high-pass design)

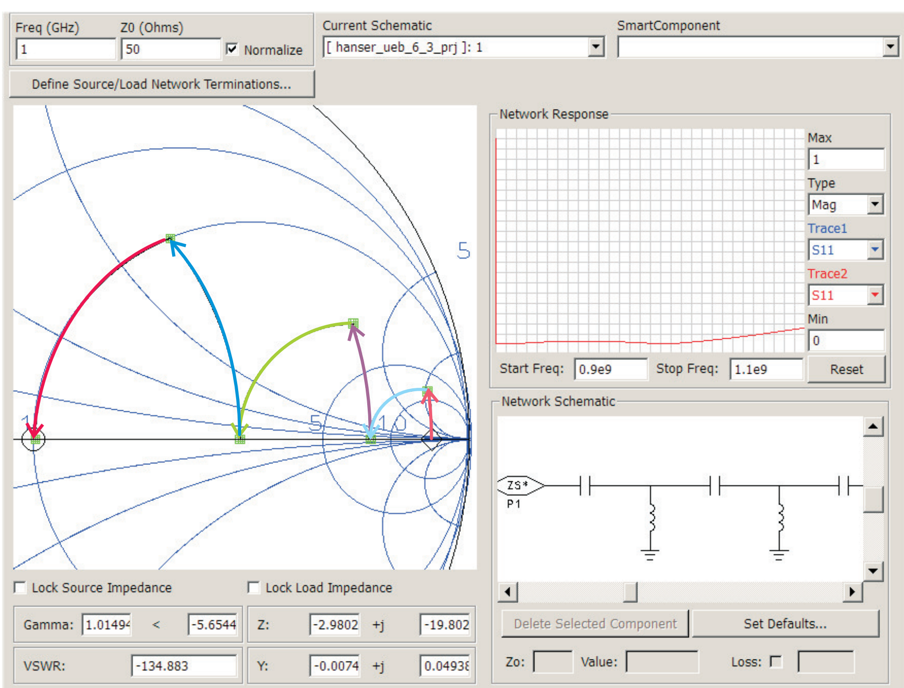


Figure F.13: Smith chart for three-stage LC-matching network (high-pass design)

F.4 Problem 6.4

In Section 6.3.2.1. (book page 200) we introduced a simple microstrip power divider (substrate material: alumina; relative permittivity $\varepsilon_r = 9.8$; substrate height $h = 635\,\mu\text{m}$). A quarter-wave transformer is used to match port 1 to $50\,\Omega$ at a frequency of $f = 5\,\text{GHz}$. In the following we design and compare different circuits.

1. Single-stage quarter-wave transformer

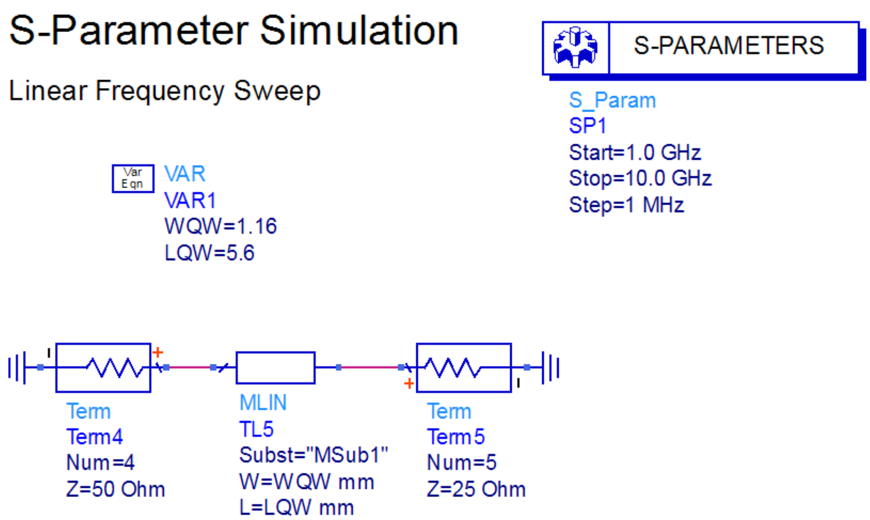


Figure F.14: Quarter-wave transformer ($25\,\Omega \rightarrow 50\,\Omega$)

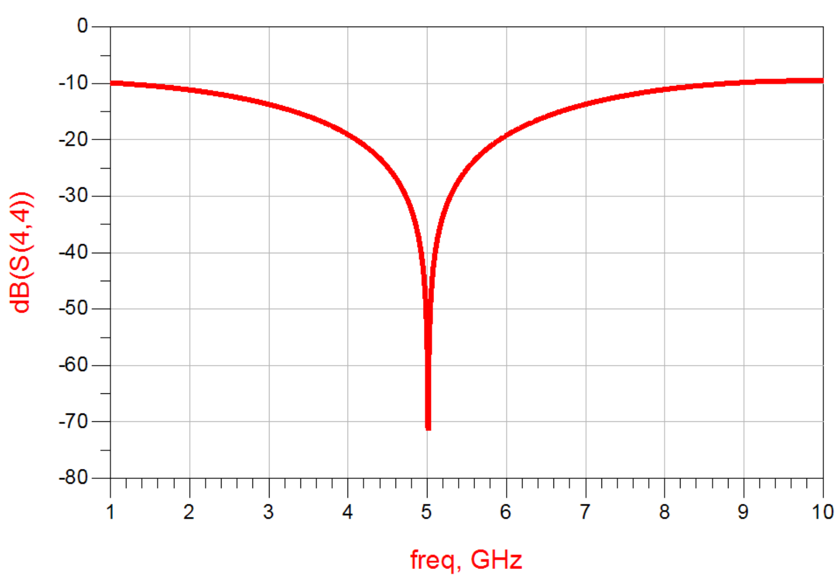


Figure F.15: Reflection coefficient at port 1 for the quarter-wave transformer ($25\,\Omega \rightarrow 50\,\Omega$)

The parallel circuit of the two outgoing lines (Figure 6.14) has an impedance of 25 Ω. Therefore, the line impedance of the quarter-wave transformer is

$$Z_0 = \sqrt{50\,\Omega\,25\,\Omega} = 35.4\,\Omega \tag{F.34}$$

Using a transmission line calculator (e.g. TX-Line from AWR or LineCalc from Agilent) gives us the following microstrip line dimensions

$$L = 5.6\,\text{mm} \qquad \text{and} \qquad W = 1.16\,\text{mm} \tag{F.35}$$

Figure F.14 shows the schematic. The reflection coefficient is given in Figure F.15.

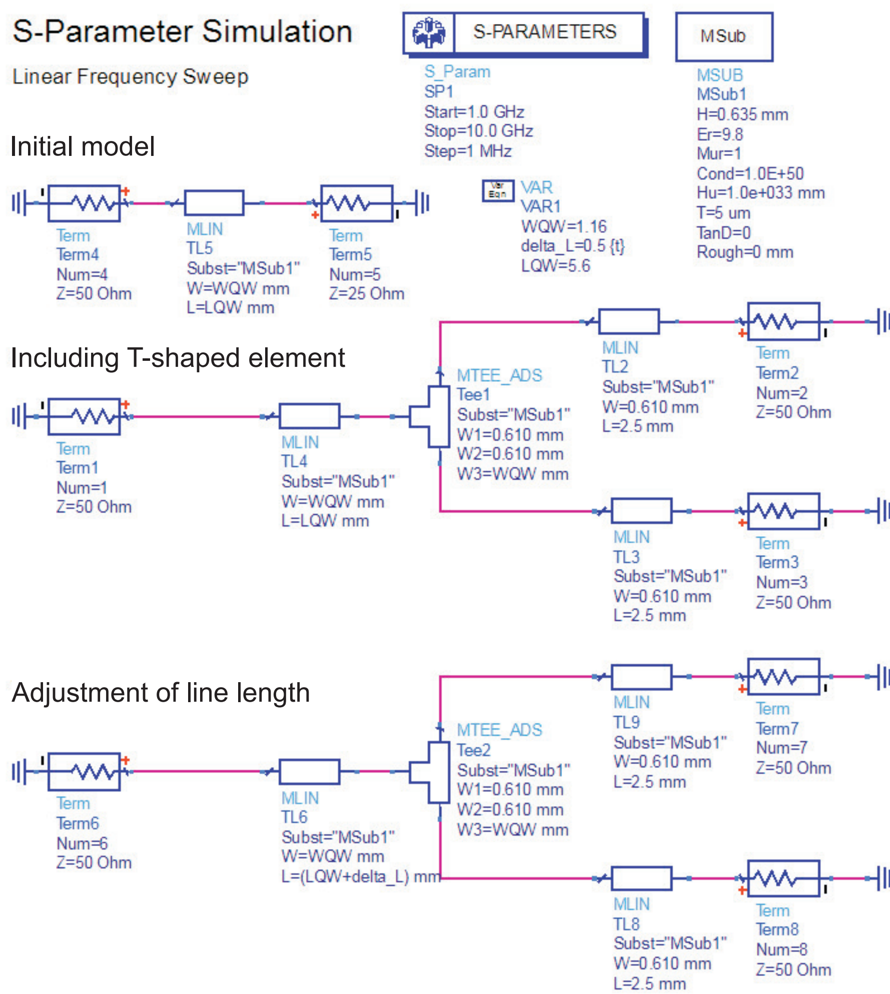


Figure F.16: Adjusting line length to compensate for T-shaped element effect

In our example (Section 6.3.2.1) the outgoing lines are connected by a T-shaped element (a rectangular metallic plate that connects outgoing lines and quarter-wave transformer). This T-shaped element alters the impedance slightly, so that we no longer see exactly the expected impedance of 25 Ω at the end of the quarter-wave transformer. By adjusting the length of the quarter-wave transformer matching is restored. Figure F.16 shows the schematics. The reflection coefficients are given in Figure F.17.

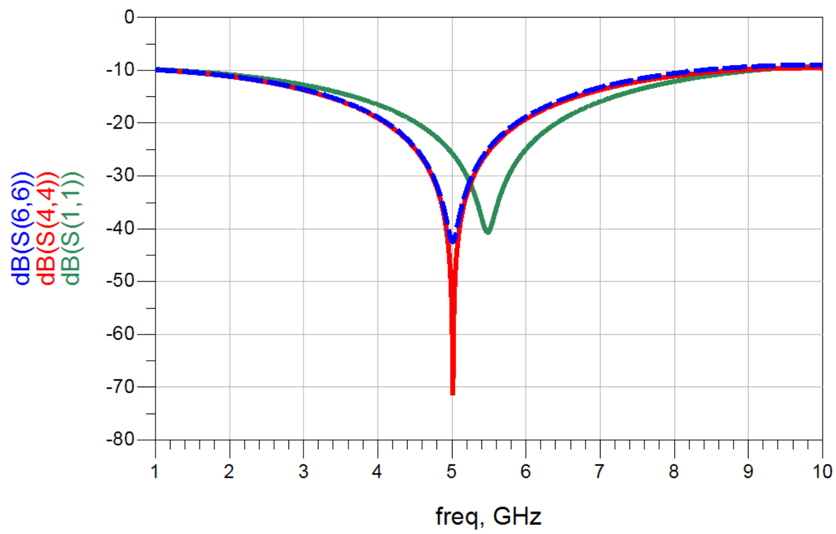


Figure F.17: Reflection coefficients: Initial design ($25\,\Omega \rightarrow 50\,\Omega$) (red); after introducing real microstrip lines and T-shaped element (green); adjusting line length to compensate for T-shaped element effect (blue)

2. Quarter-wave transformer (two-stage design)

Now we will investigate the effect of a two-stage design on the resulting bandwidth. As in Problem 6.3 we transform in a first step to an intermediate resistance.

$$R_{\mathrm{m}} = \sqrt{50\,\Omega\,25\,\Omega} = 35.4\,\Omega \tag{F.36}$$

The line impedances of the quarter-wave transformers are

$$Z_{01} = \sqrt{50\,\Omega\,R_{\mathrm{m}}} = 42.1\,\Omega \qquad \text{and} \qquad Z_{02} = \sqrt{R_{\mathrm{m}}\,25\,\Omega} = 29.7\,\Omega \tag{F.37}$$

By using a transmission line calculator the microstrip dimensions become

$$L_1 = 5.7\,\mathrm{mm} \qquad \text{and} \qquad W_1 = 0.86\,\mathrm{mm} \tag{F.38}$$
$$L_2 = 5.51\,\mathrm{mm} \qquad \text{and} \qquad W_2 = 1.54\,\mathrm{mm} \tag{F.39}$$

The schematic is given in Figure F.18. The reflection coefficient is shown in Figure F.19 and compared to the single-stage design. The two-stage design has larger bandwidth.

Finally, we will investigate the effect of the T-shaped element in our microstrip design. This can be easily done using a circuit simulator (Figure F.20). The reflection coefficients are shown in Figure F.21.

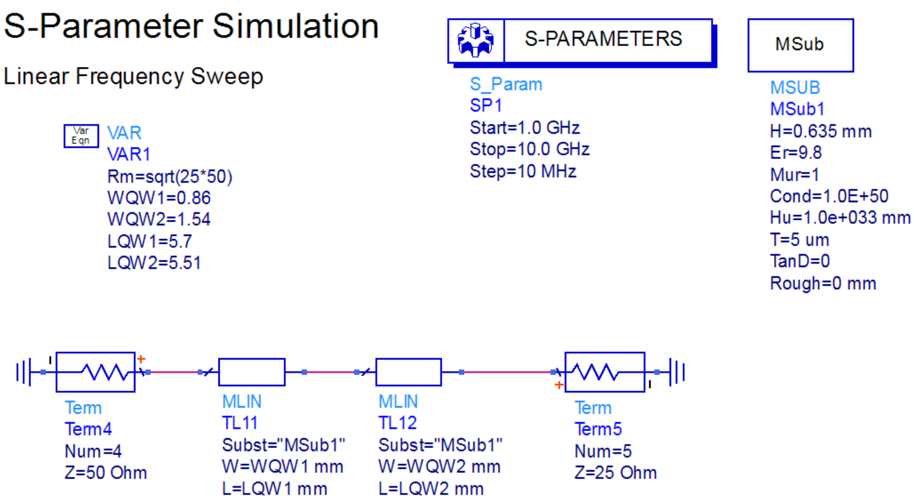


Figure F.18: Two-stage quarter-wave transformer

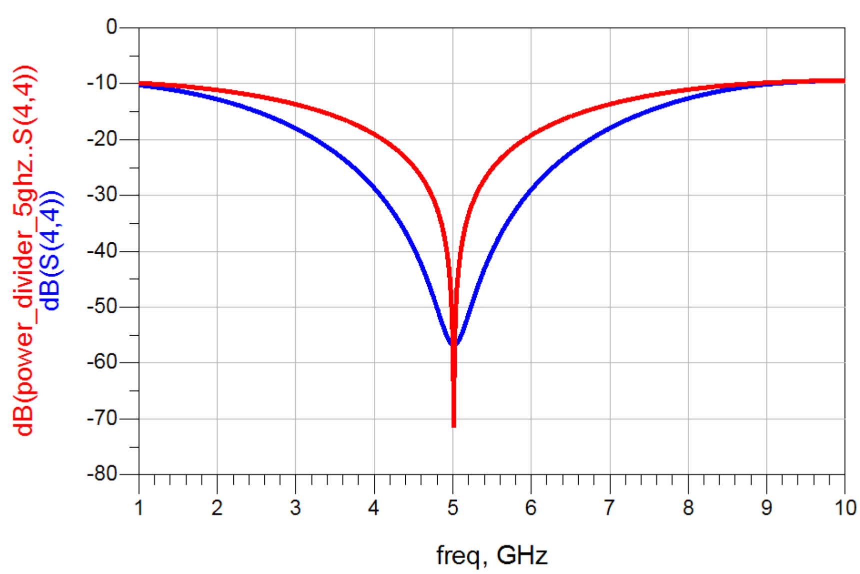


Figure F.19: Reflection coefficients of single-stage (red) and two-stage (blue) quarter-wave transformer (load resistance $25\,\Omega$)

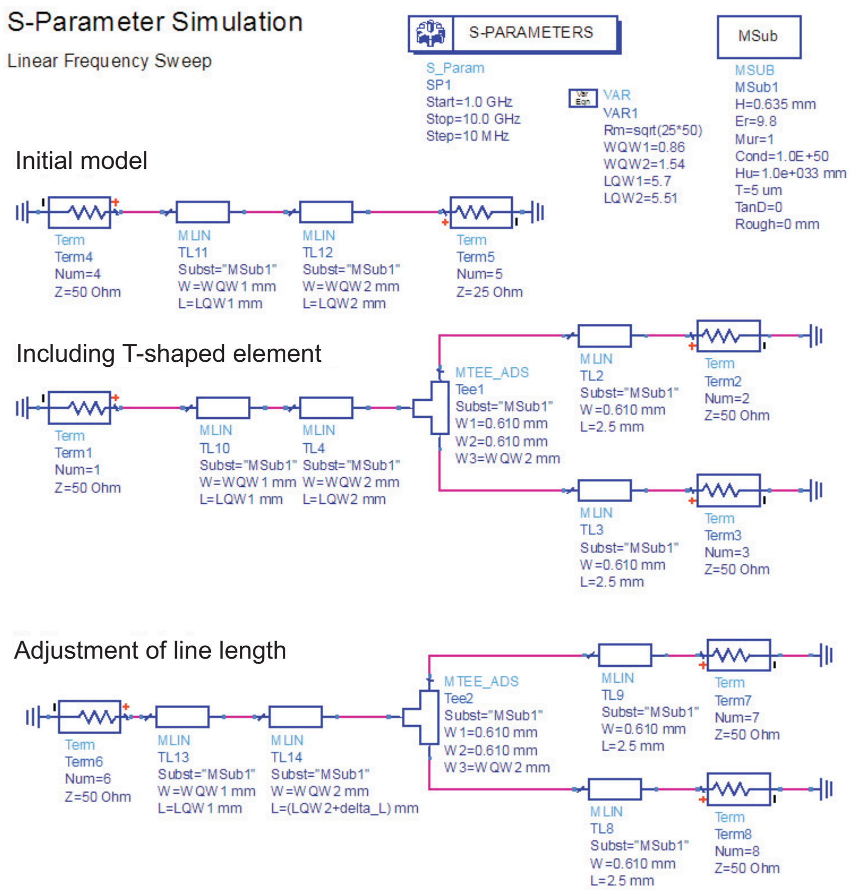


Figure F.20: Investigating and compensating the effect of the T-shaped element

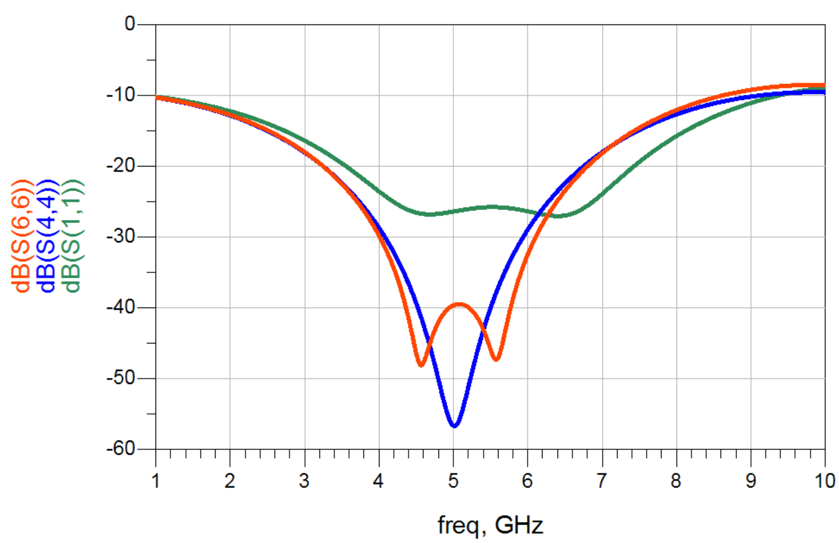


Figure F.21: Reflection coefficients for the two-stage-design: with ideal load of 25 Ω (blue); with outgoing 50 Ω-lines and T-element as load (green); with outgoing 50 Ω-lines and T-element as load (red) but adjusted length of quarter-wave transformer

F.5 Problem 6.5

Figure F.22 shows the design of the matching circuit using the Smith chart tool in ADS. The resulting electric line length is 131.2° for the serial line and 37.2° for the stub line. The dimensions of the microstrip lines are determined by TX-Line. Figure F.23 shows the graphical user interface and the dimensions of the lines.

$$w_\mathrm{s} = 0.3\,\mathrm{mm} \quad \text{and} \quad \ell_\mathrm{s} = 4.69\,\mathrm{mm} \quad \text{(serial line)} \tag{F.40}$$

$$w_\mathrm{p} = 0.3\,\mathrm{mm} \quad \text{and} \quad \ell_\mathrm{p} = 1.33\,\mathrm{mm} \quad \text{(stub (parallel) line)} \tag{F.41}$$

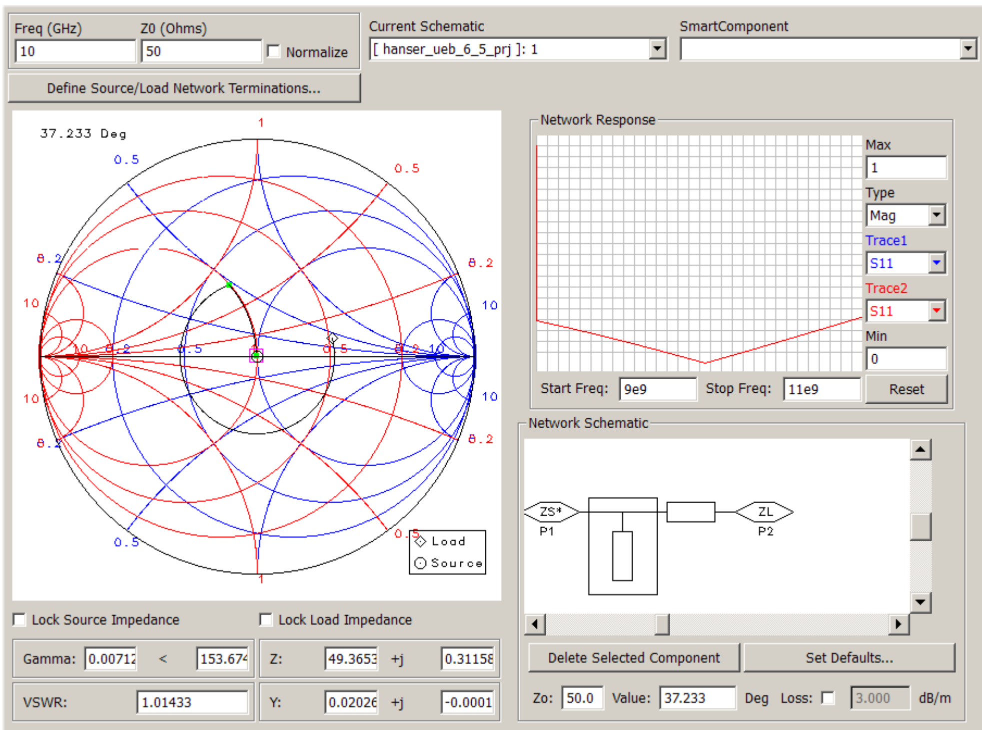


Figure F.22: Design using Smith chart tool in ADS

In order to validate our results we set up a circuit simulation (Figure F.24). First, we use *ideal* lines. TL4 represents a serial line and TL5 an open ended stub line in parallel. The open end is modelled by a very high resistance value ($1\,\mathrm{T}\Omega$). The green curve in Figure F.25 shows the resulting reflection coefficient with matching at 10 GHz.

Next, we consider microstrip lines. The blue curve in Figure F.25 shows that matching is off the intended frequency. Our initial model did not take into account

- capacitive open end effect of the stub line and
- influence of the connecting element (MTEE), that has to be included for geometrical reasons.

By tuning the line lengths manually we achieve matching at 10 GHz (red curve). The geometry of the final design is shown in Figure F.26.

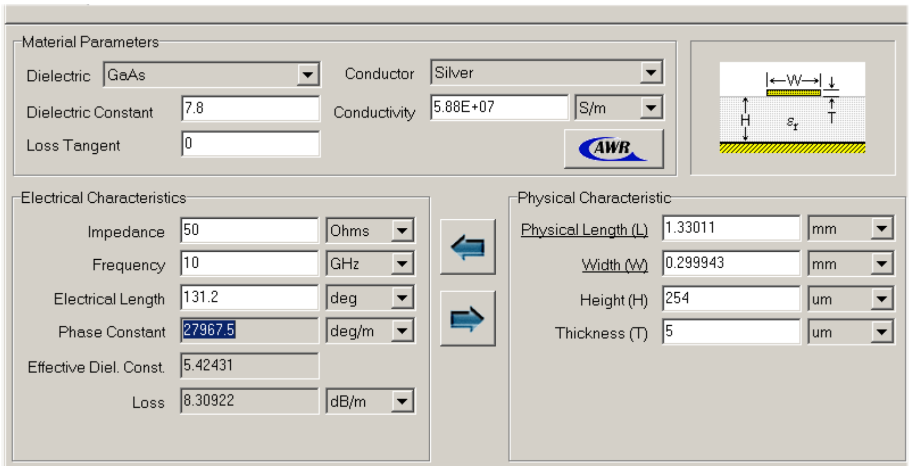


Figure F.23: Design of microstrip lines using TX-Line

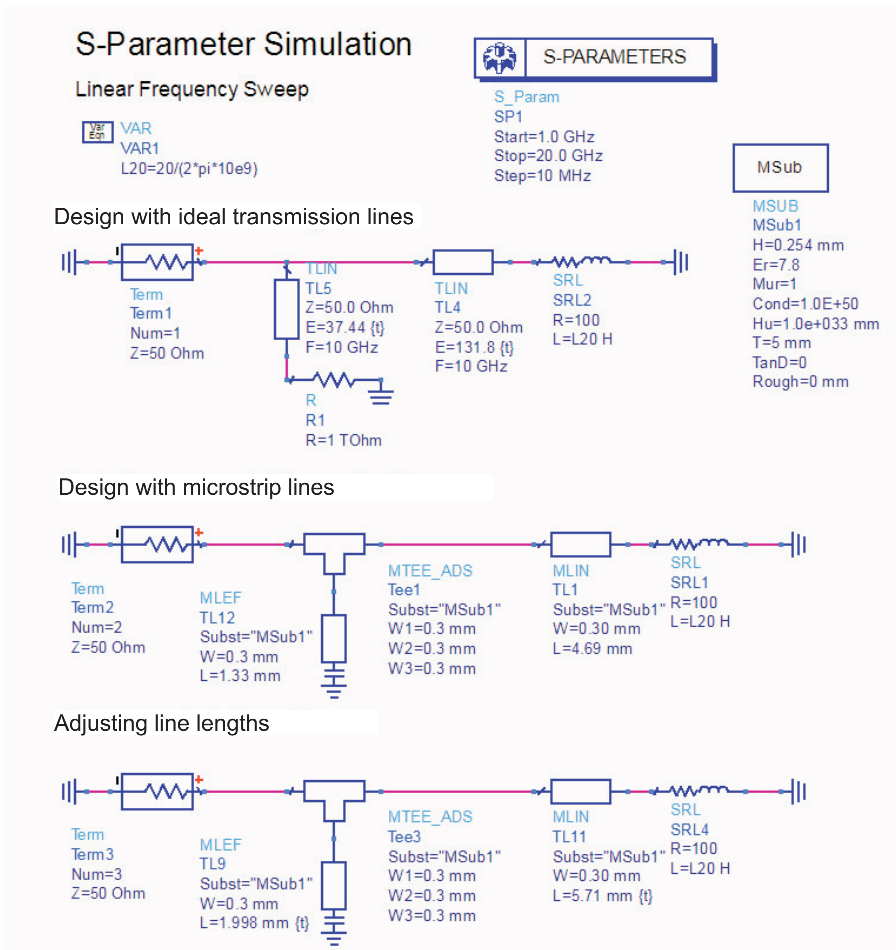


Figure F.24: Circuit simulation with ideal lines and microstrip lines

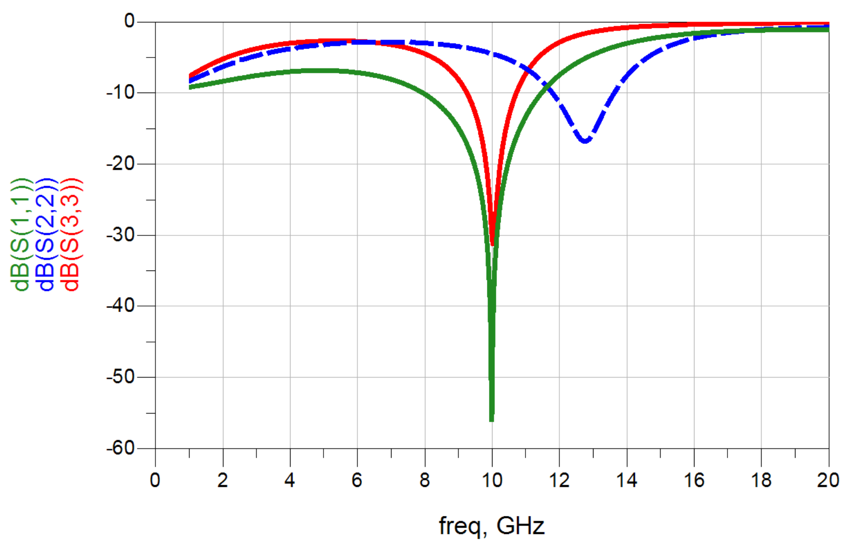


Figure F.25: Simulation results (reflection coefficients)

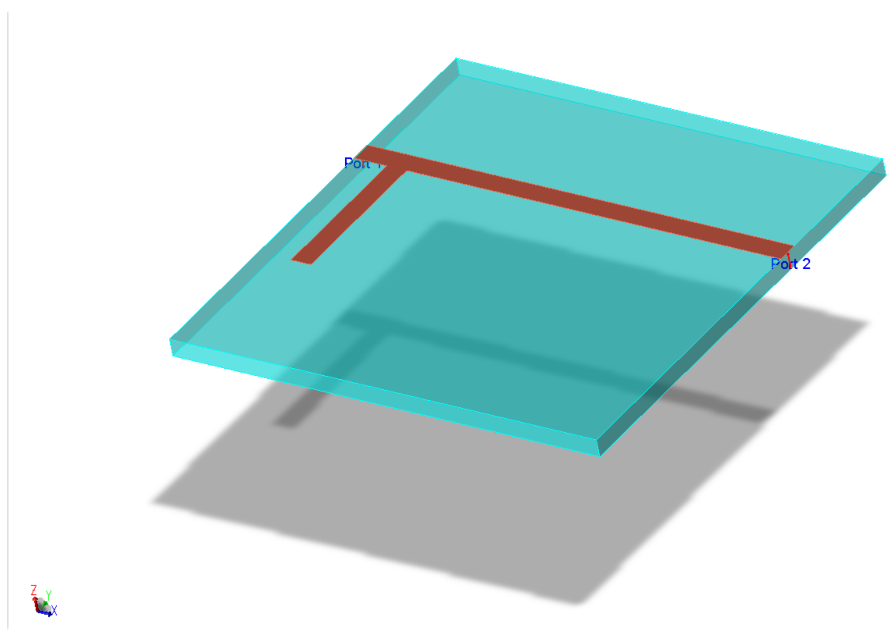


Figure F.26: Microstrip layout (Port 1 = 50 Ω; Port 2 = load impedance)

F.6 Problem 6.6

The scattering matrix of a Wilkinson power divider is given as

$$\mathbf{S}_{\text{Wilkinson}} = \frac{-j}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{F.42}$$

The scattering matrix of a *lossless* component fulfils the unitary matrix condition: $\mathbf{S}^{\mathrm{T}}\mathbf{S}^{*} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. We will show, that the scattering matrix of a Wilkinson power divider does not fulfil the unitary matrix condition.

The Wilkinson power divider is reciprocal ($s_{ij} = s_{ji}$), so we write

$$\begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{12} & s_{22} & s_{23} \\ s_{13} & s_{23} & s_{33} \end{pmatrix} \begin{pmatrix} s_{11}^{*} & s_{12}^{*} & s_{13}^{*} \\ s_{12}^{*} & s_{22}^{*} & s_{23}^{*} \\ s_{13}^{*} & s_{23}^{*} & s_{33}^{*} \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{F.43}$$

The first equation we derive from the matrix is

$$|s_{11}|^2 + |s_{12}|^2 + |s_{13}|^2 = 1 \tag{F.44}$$

Putting in the numbers from the scattering matrix of the Wilkinson divider yields

$$|0|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2} + \frac{1}{2} = 1 \tag{F.45}$$

The next equation we get reads

$$\underbrace{s_{11}}_{=0} s_{12}^{*} + s_{12} \underbrace{s_{22}^{*}}_{=0} + s_{13}s_{23}^{*} = 0 \tag{F.46}$$

We end up with

$$s_{13}s_{23}^{*} = \left(\frac{-j}{\sqrt{2}}\right)\left(\frac{j}{\sqrt{2}}\right) \neq 0 \tag{F.47}$$

The equation is not satisfied. Therefore, the scattering matrix of the Wilkinson divider is no unitary matrix.

F.7 Problem 6.7

The design formulas for a branchline coupler with unequal power split are given in Section 6.8.2. A power ratio of 3:1 means that 75% of the incoming power P_{a1} at port 1 is transferred to port 2 and 25% is transferred to port 3.

The value of k is

$$k = \left|\frac{s_{31}}{s_{21}}\right| = \sqrt{\frac{P_{\mathrm{b3}}/P_{\mathrm{a1}}}{P_{\mathrm{b2}}/P_{\mathrm{a1}}}} = \sqrt{\frac{0.25}{0.75}} = \frac{1}{\sqrt{3}} \tag{F.48}$$

where

$$P_{\mathrm{b2}} = |s_{21}|^2 P_{\mathrm{a1}} \qquad \text{and} \qquad P_{\mathrm{b3}} = |s_{31}|^2 P_{\mathrm{a1}} \tag{F.49}$$

The absolute values of the transmission coefficients are

$$|s_{21}| = \sqrt{0.75} = -1.249\,\mathrm{dB} \qquad \text{and} \qquad |s_{31}| = \sqrt{0.25} = -6.021\,\mathrm{dB} \tag{F.50}$$

All port reference impedances are equal to $50\,\Omega$:

$$Z_{01} = Z_{02} = Z_0 = 50\,\Omega \tag{F.51}$$

So, the line impedances become

$$Z_{0\mathrm{a}} = \frac{Z_{01}}{k} = 86.6\,\Omega \tag{F.52}$$

$$Z_{0\mathrm{b}} = \frac{Z_0}{\sqrt{1+k^2}} = 43.3\,\Omega \tag{F.53}$$

$$Z_{0\mathrm{c}} = \frac{Z_{02}}{k} = 86.6\,\Omega \tag{F.54}$$

Each line length corresponds to a quarter wave length at $f = 6\,\mathrm{GHz}$ (electrical line lenth is 90°). Figure F.27 shows a circuit simulation using ideal lines. The corresponding scattering parameters are given in Figure F.28.

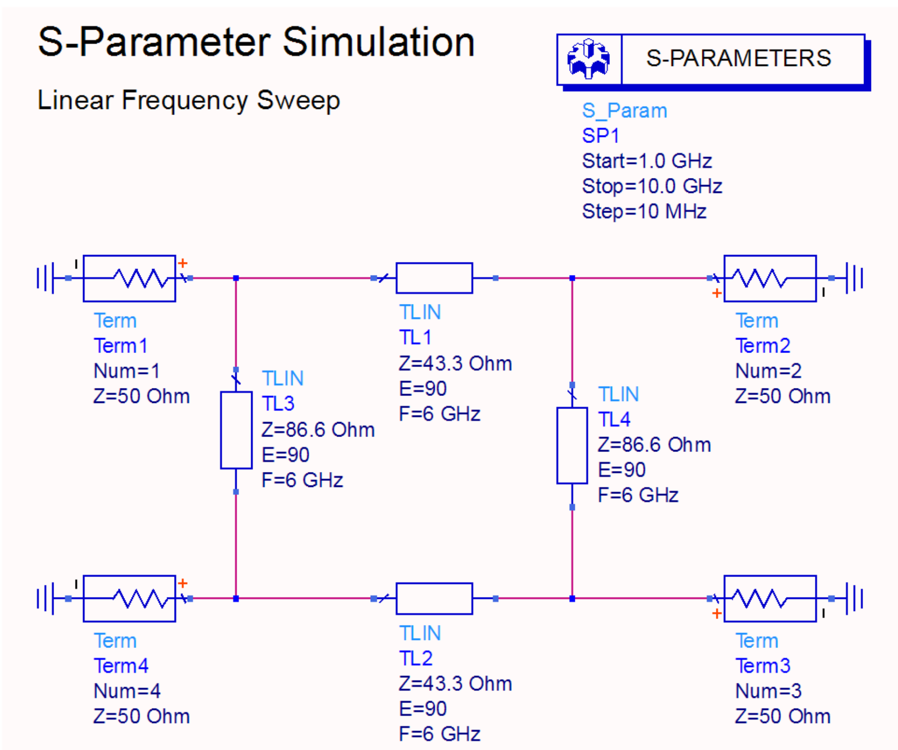


Figure F.27: Unequal split branchline coupler with ideal lines

Finally, we will look at a design with microstrip lines. Choosing substrate parameters ($h = 635\,\mu\mathrm{m}$ and $\varepsilon_\mathrm{r} = 9.8$) we can calculate line lengths and widths with TX-Line.

$$w_\mathrm{a} = w_\mathrm{c} = 0.140\,\mathrm{mm} \qquad \text{and} \qquad \ell_\mathrm{a} = \ell_\mathrm{c} = 5.066\,\mathrm{mm} \tag{F.55}$$

$$w_\mathrm{b} = 0.814\,\mathrm{mm} \qquad \text{and} \qquad \ell_\mathrm{b} = 4.75\,\mathrm{mm} \tag{F.56}$$

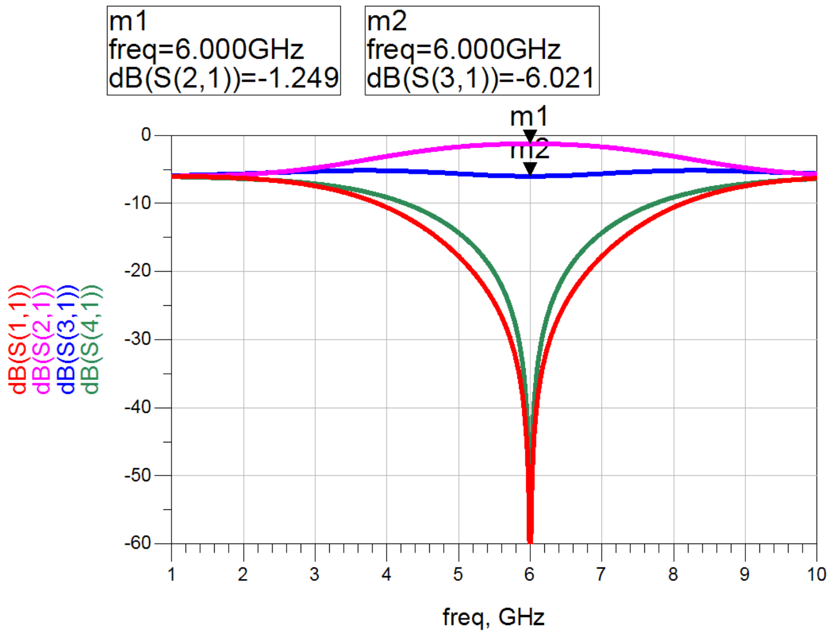


Figure F.28: Scattering parameters of unequal split branchline coupler with ideal lines

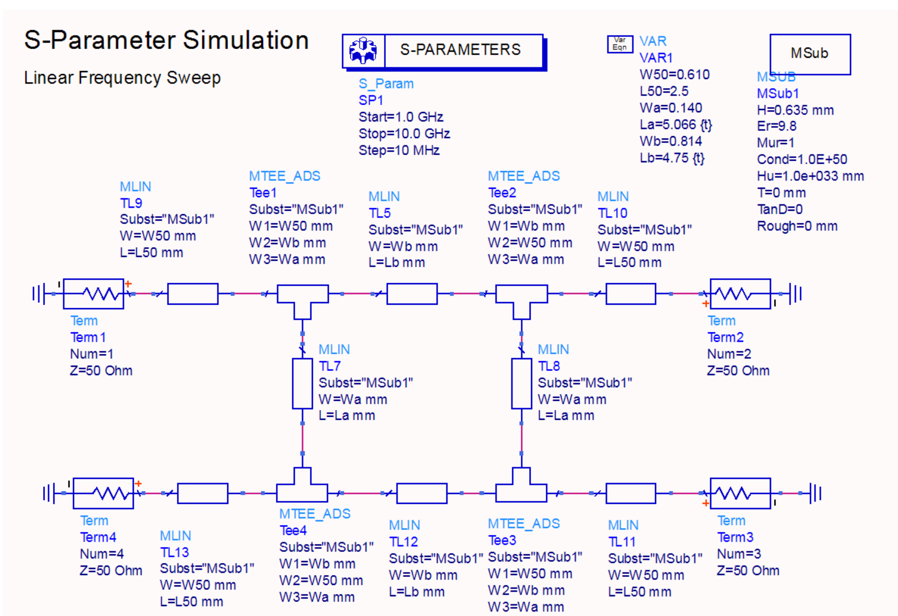


Figure F.29: Unequal split branchline coupler with microstrip lines

Figure F.29 shows a circuit simulation using microstrip lines. The corresponding scattering parameters are given in Figure F.30. The layout is shown in Figure F.31.

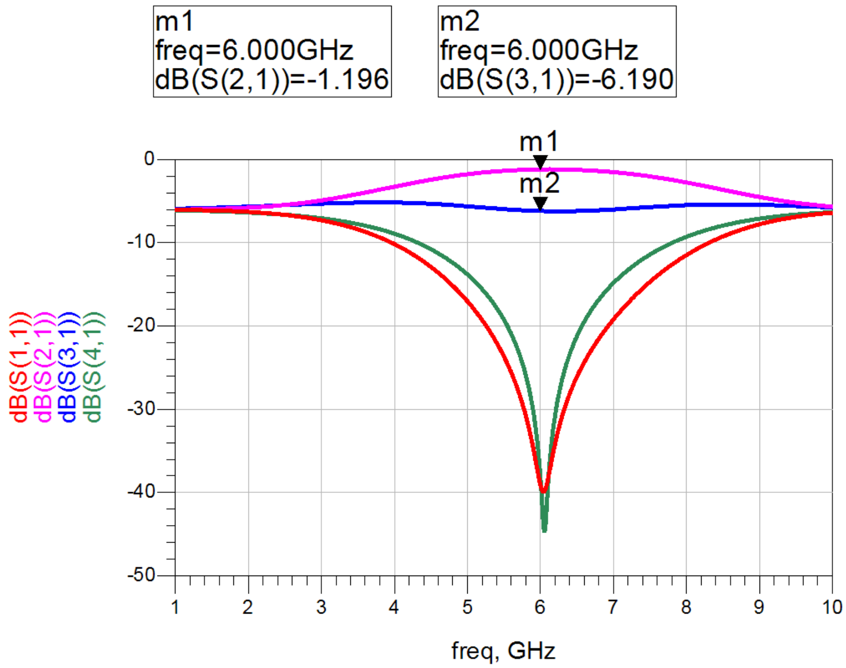


Figure F.30: Scattering parameters of unequal split branchline coupler with microstrip lines

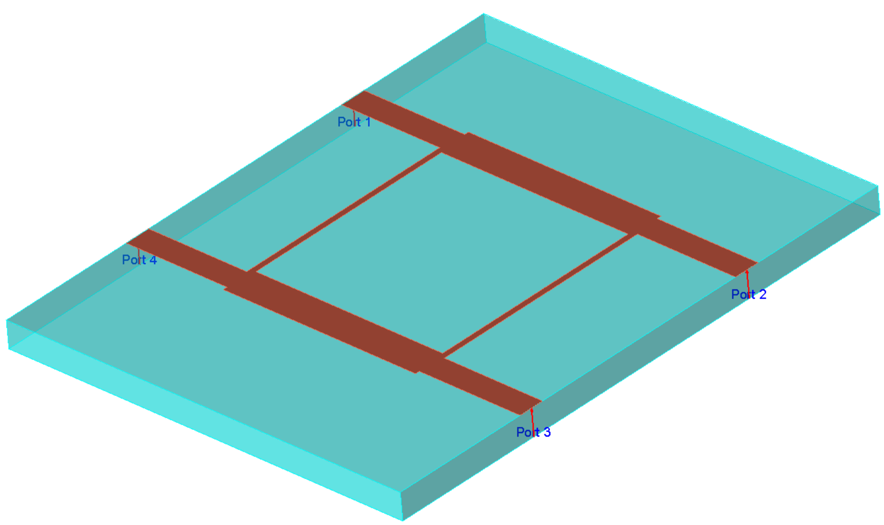


Figure F.31: Layout of unequal split branchline coupler with microstrip lines

F.8 Problem 6.8

We follow the design procedure described in Section 6.4.2.1. The order $n = 3$ is derived from Table 6.2 (book page 205) in order to achieve the desired attenuation at twice the cut-off frequency. The Butterworth filter coefficients are given by Equation 6.26.

$$a_1 = 1 \qquad a_2 = 2 \qquad a_3 = 1 \tag{F.57}$$

Our first design starts with a shunt capacitance. We get

$$C_1 = 2.27\,\mathrm{pF} \qquad L_2 = 45.47\,\mathrm{nF} \qquad C_3 = C_1 \tag{F.58}$$

The second design start with a series inductance. The component values are

$$L_1 = 22.74\,\mathrm{nH} \qquad C_2 = 4.547\,\mathrm{pF} \qquad L_3 = L_1 \tag{F.59}$$

Figure F.32 shows the schematics and Figure F.33 displays the transmission coefficients. For comparison we included an automatic design guide (a software assistant). The response type "maximally flat" indicates a Butterworth filter. Figure F.34 shows the circuit automatically designed by ADS design guide. The circuit corresponds to our manual design.

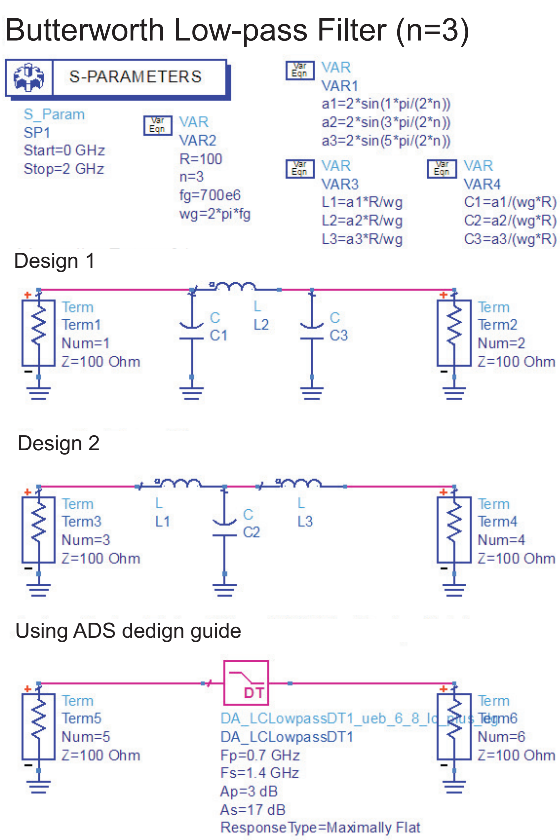


Figure F.32: Butterworth Filter

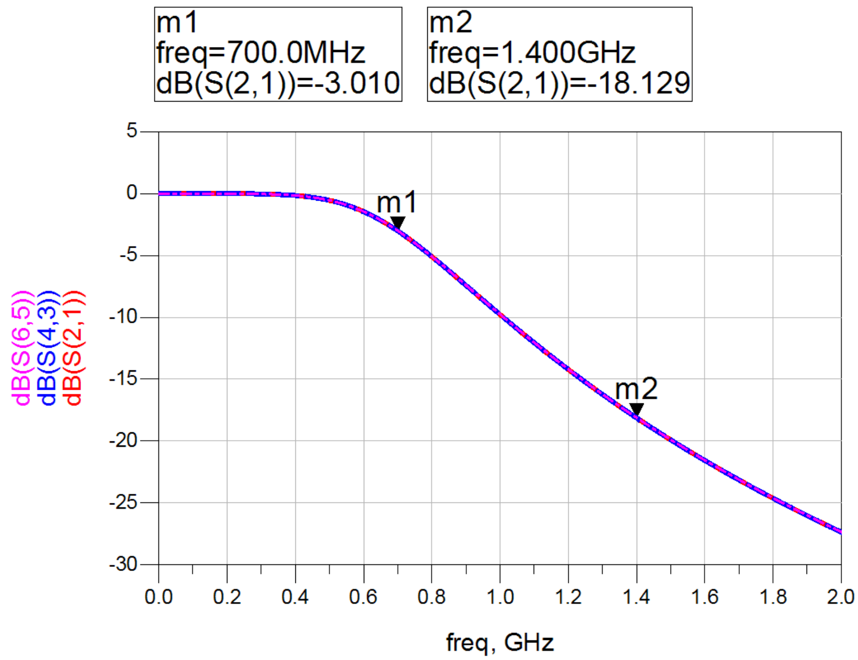


Figure F.33: Transmission coefficient of Butterworth Filter

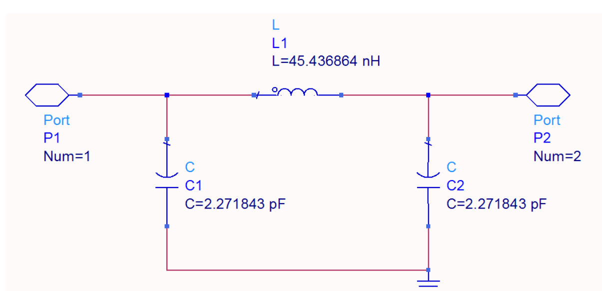


Figure F.34: Circuit automatically created by ADS design guide

F.9 Problem 6.9

A band-pass filter can be derived from a low-pass prototype (Section 6.4.2.3, book page 210). Equation 6.33 yields

$$\frac{f_{\mathrm{s,LP}}}{f_{\mathrm{c}}} = 2\frac{f_{\mathrm{s2}} - f_0}{f_{\mathrm{p2}} - f_{\mathrm{p1}}} = 3 \tag{F.60}$$

The stop frequency $f_{\mathrm{s,LP}}$ of the corresponding low-pass prototype filter is tree-times the value of the cut-off frequency f_{c}. Table 6.2 (book page 205) only lists attenuation values of up to two-fold values of the cut-off frequency. Therefore, we apply Equation 6.25 to determine the necessary filter order: An order of $n = 3$ yields an attenuation of 28.6 dB.

The Butterworth filter coefficients are given by Equation 6.26.

$$a_1 = 1 \qquad a_2 = 2 \qquad a_3 = 1 \tag{F.61}$$

Furthermore, we get

$$a_0 = 1 \qquad a_4 = 1 \tag{F.62}$$

Using Equations 6.30 to 6.32 yields

$$f_0 = 8.25\,\mathrm{GHz} \qquad \text{and} \qquad \omega_0 = 51.836\,\mathrm{GHz} \tag{F.63}$$

$$B = 0.5\,\mathrm{GHz} \qquad \text{and} \qquad BW = 3.1416\,\mathrm{GHz} \tag{F.64}$$

Next, we calculate the auxiliary values

$$J_{01} = J_{34} = 6.171 \cdot 10^{-3}\frac{1}{\Omega} \quad \text{and} \quad J_{12} = J_{23} = 1.346 \cdot 10^{-3}\frac{1}{\Omega} \tag{F.65}$$

The even-mode and odd-mode impedances are listed in Table F.3. With TX-Line we determine the line widths w, line lengths ℓ and distances (spacing) s between the lines.

a_i	Z_{0e}/Ω	Z_{0e}/Ω	$w/\mu\mathrm{m}$	$s/\mu\mathrm{m}$
$a_0 = a_4 = 1$	$Z_{0e,1} = Z_{0e,4} = 70.19$	$Z_{0e,1} = Z_{0e,4} = 39.33$	$w_1 = w_4 = 363.2$	$s_1 = s_4 = 161.4$
$a_1 = a_3 = 1$	$Z_{0e,2} = Z_{0e,3} = 53.59$	$Z_{0e,2} = Z_{0e,3} = 46.86$	$w_2 = w_3 = 456.4$	$s_2 = s_3 = 778.5$
$a_2 = 2$				

Table F.3: Filter parameters and dimensions

The line length do not consider open-end and coupling effects. Therefore, we introduce a term to tune the line lengths in order to achieve the desired center frequency f_0. ADS provides sliders to perform this design step interactively. The final circuit is given in Figure F.35. The transmission coefficient in Figure F.36 shows good agreement with the specifications. The layout is given Figure F.37.

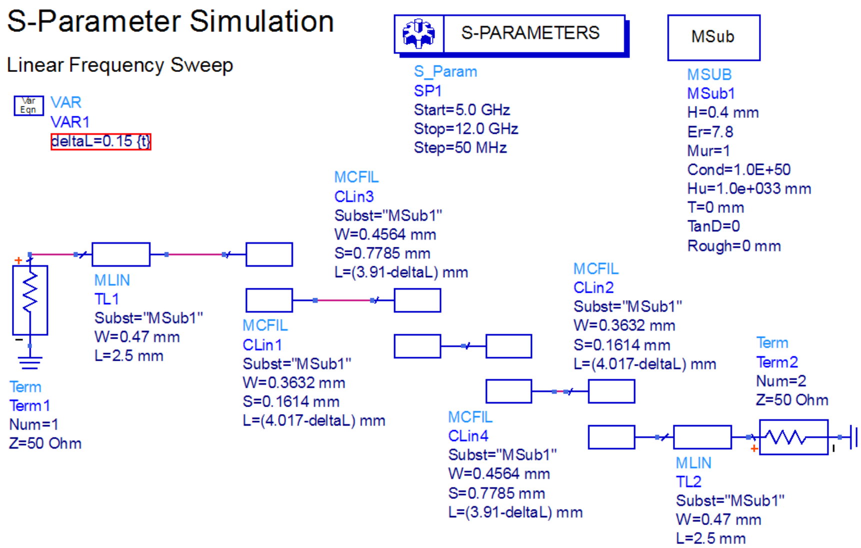


Figure F.35: Schematic of microstrip band-pass filter

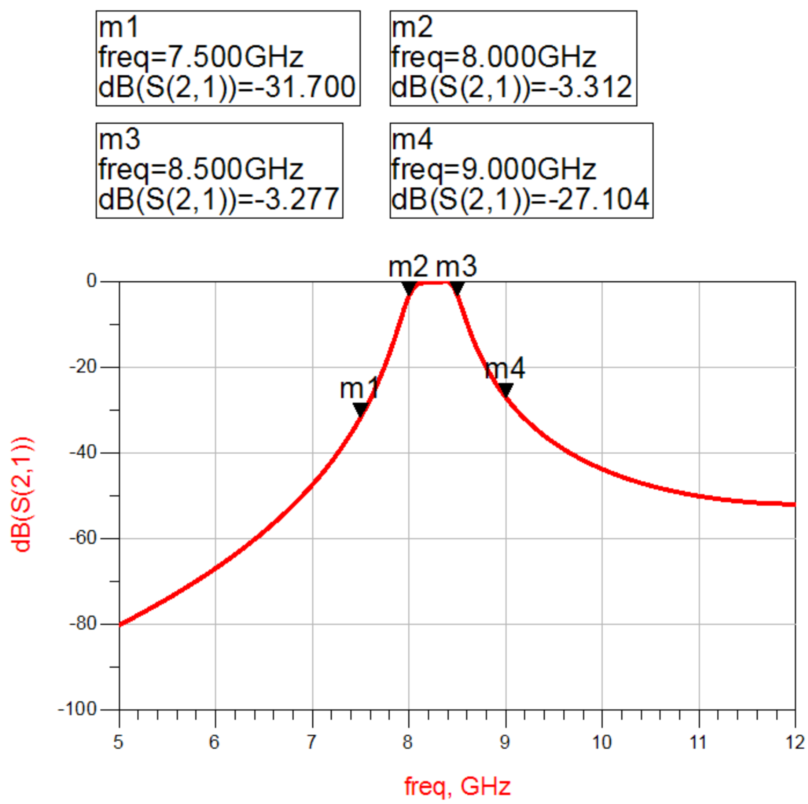


Figure F.36: Transmission coefficient of microstrip band-pass filter

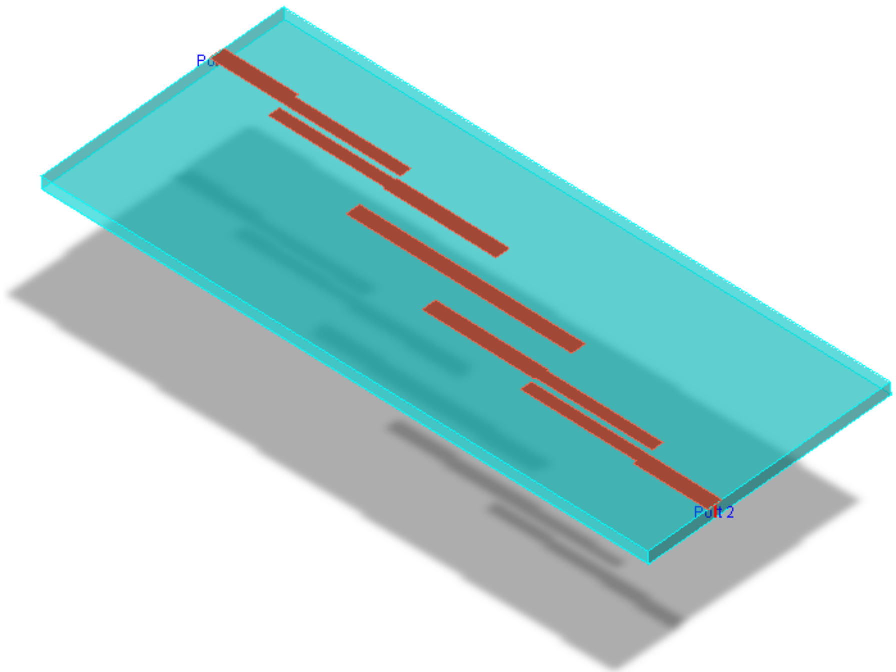


Figure F.37: Layout of microstrip band-pass filter

(Last modified: 26.02.2013)

Appendix G

Solutions to Problems in Chapter 7

G.1 Problem 7.1

The maximum radiated power is

$$EIRP = GP = 100\,\mathrm{mW} = 20\,\mathrm{dBm} \tag{G.1}$$

First, we determine the transmit power of the initial configuration (line loss $a_1 = 2.5\,\mathrm{dB}$ and antenna gain $G_1 = 5\,\mathrm{dBi}$).

$$P_{\mathrm{TX1}} = EIRP - G_1 + a_1 = 20\,\mathrm{dBm} - 5\,\mathrm{dBi} + 2.5\,\mathrm{dB} = 17.5\,\mathrm{dBm} \tag{G.2}$$

For the modified configuration (line loss $a_2 = 1\,\mathrm{dB}$ and antenna gain $G_2 = 15\,\mathrm{dBi}$) the transmit power is

$$P_{\mathrm{TX2}} = EIRP - G_2 + a_2 = 20\,\mathrm{dBm} - 15\,\mathrm{dBi} + 1\,\mathrm{dB} = 6\,\mathrm{dBm} \tag{G.3}$$

When the second configuration acts as a receiver the larger antenna gain and reduced line loss lead to increased receive power and increased range.

G.2 Problem 7.2

We will simulate three antenna structures with a commercial EM simulation program (Empire from IMST).

- Monopole antenna
- Top loaded monopole antenna
- Inverted-F antenna

The operational frequency shall be 2.45 GHz.

Monopole antenna

Figure G.1 shows the monopole over conducting ground. The geometrical length ($\ell = 2.8\,\mathrm{cm}$) is slightly smaller than a quarter wavelength ($\lambda/4 = 3.06\,\mathrm{cm}$). Figure G.2 shows the reflection coefficient for a port reference impedance of $Z_0 = 50\,\Omega$. At 2.45 GHz we observe low reflection.

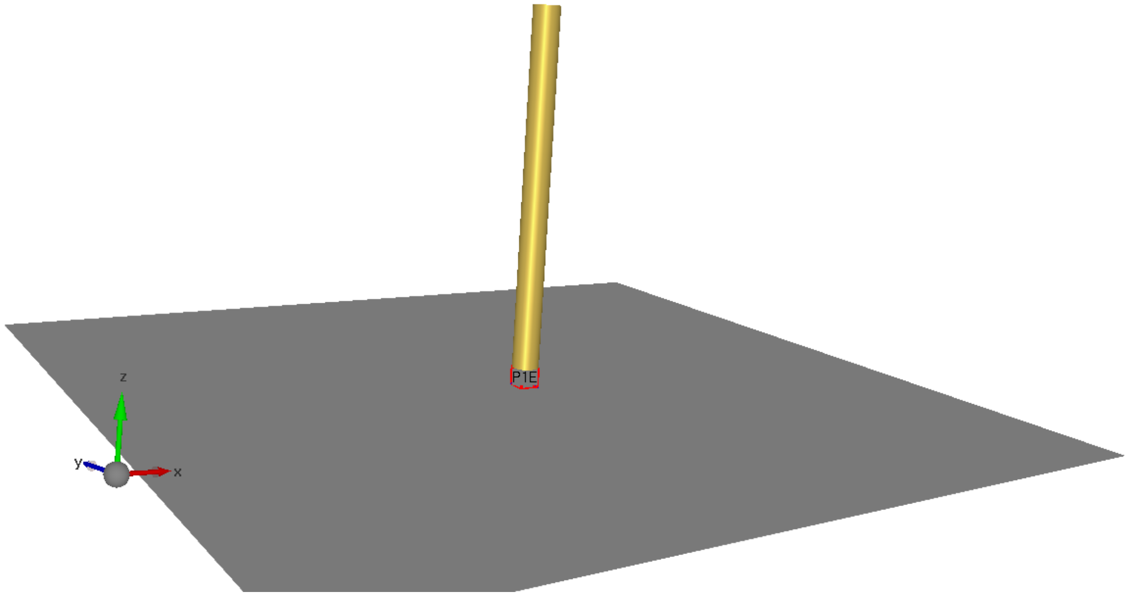


Figure G.1: Monopole antenna over conducting ground

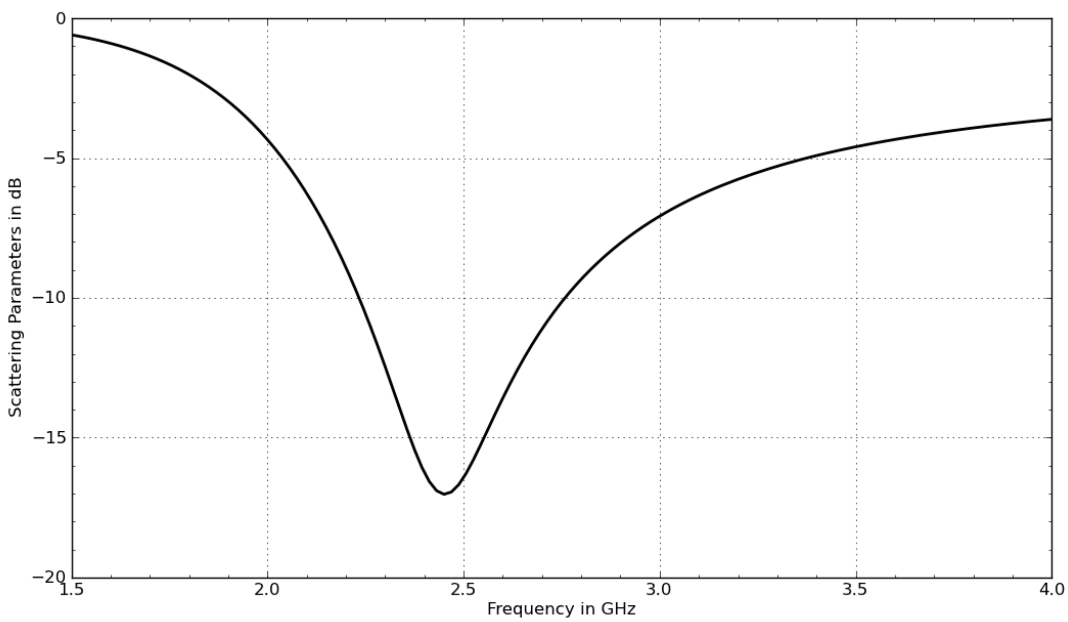


Figure G.2: Reflection coefficient of monopole antenna

The input impedance Z_{in} is given in Figure G.3. At a frequency of 2.45 GHz the imaginary part is close to zero and the real part is approximately $36\,\Omega$. Figure G.4 shows the radiation pattern of the monopole antenna. The simulated directivity of $D = 5.19\,\text{dBi}$ is close to the theoretical value of 5.15 dBi.

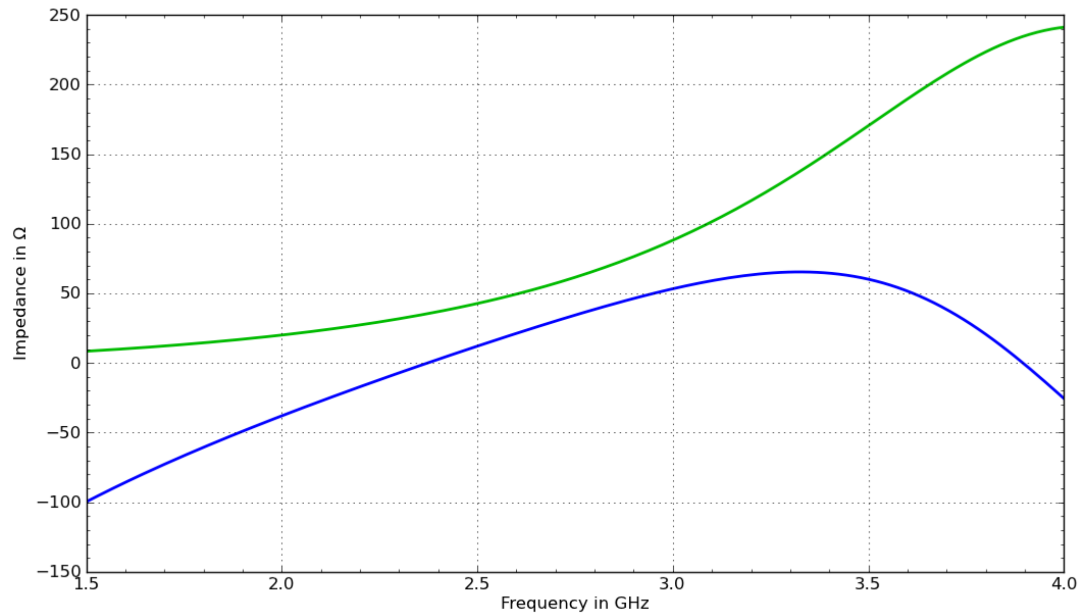


Figure G.3: Real (green line) and imaginary (blue line) parts of input impedance (monopole antenna)

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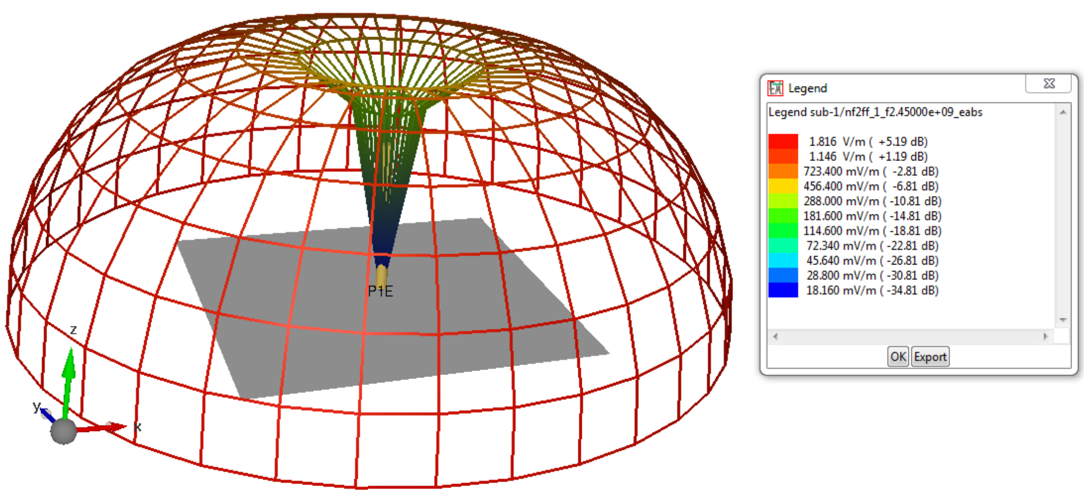

Figure G.4: Radiation pattern of monopole antenna

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Top loaded monopole antenna

Figure G.5 shows a top loaded monopole antenna. The cylindrical structure at the top of the monopole has a diameter of 12 mm. The overall height of the antenna is $h = 16$ mm. The height is significantly reduced compared to the initial length ℓ of the quarterwave monopole antenna

($h = 16\,\mathrm{mm} < \ell = 28\,\mathrm{mm} \approx \lambda/4$). Figure G.6 shows the reflection coefficient of the top loaded

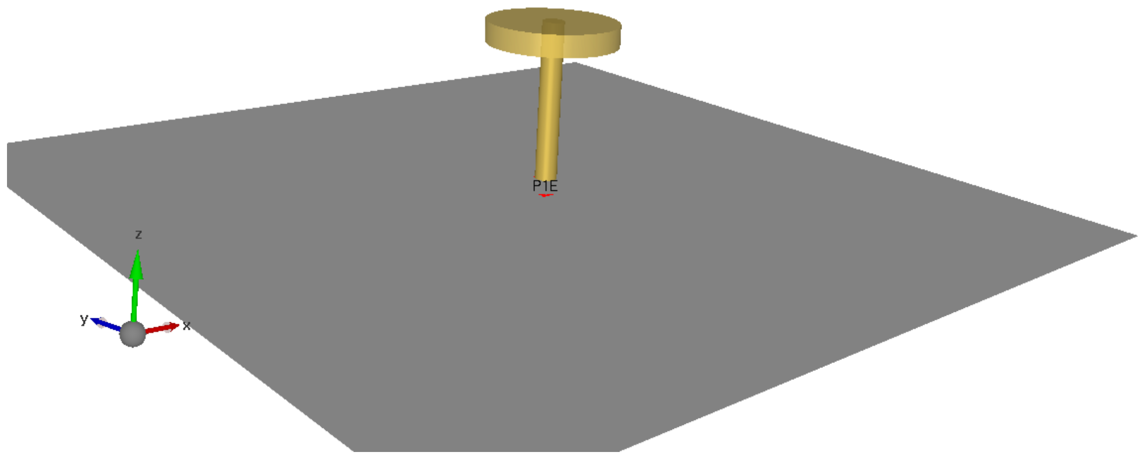

Figure G.5: Top loaded monopole antenna

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monopole antenna. Due to a reduced real part of the input impedance the matching is poor compared to the initial quarterwave monopole (see Figure G.7).

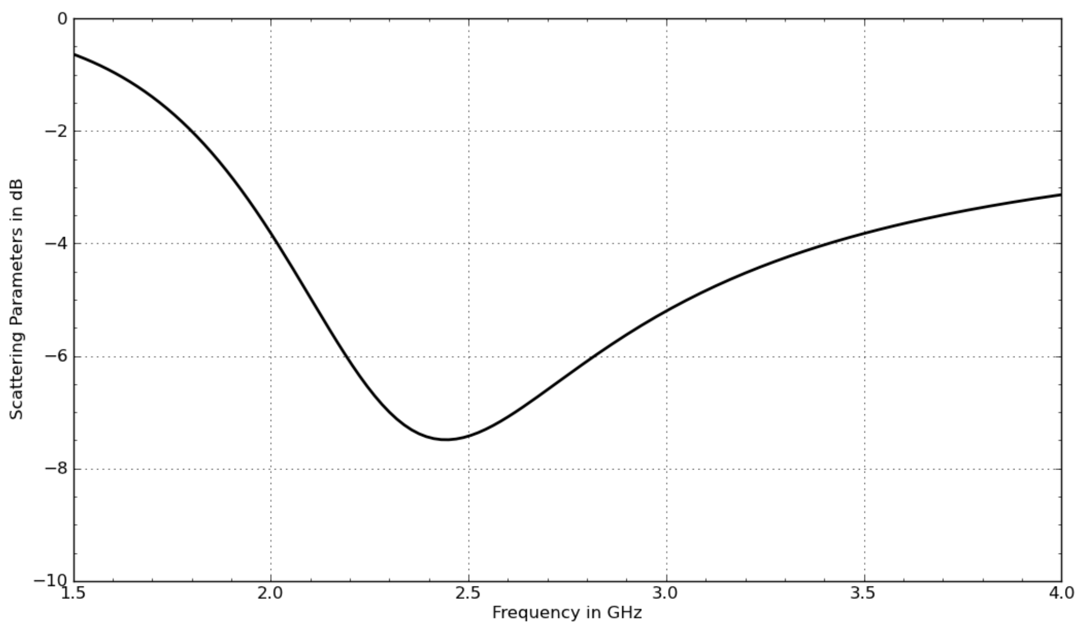

Figure G.6: Reflection coefficient of top loaded monopole antenna

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The radiation pattern in Figure G.8 shows only minor changes compared to the initial monopole antenna.

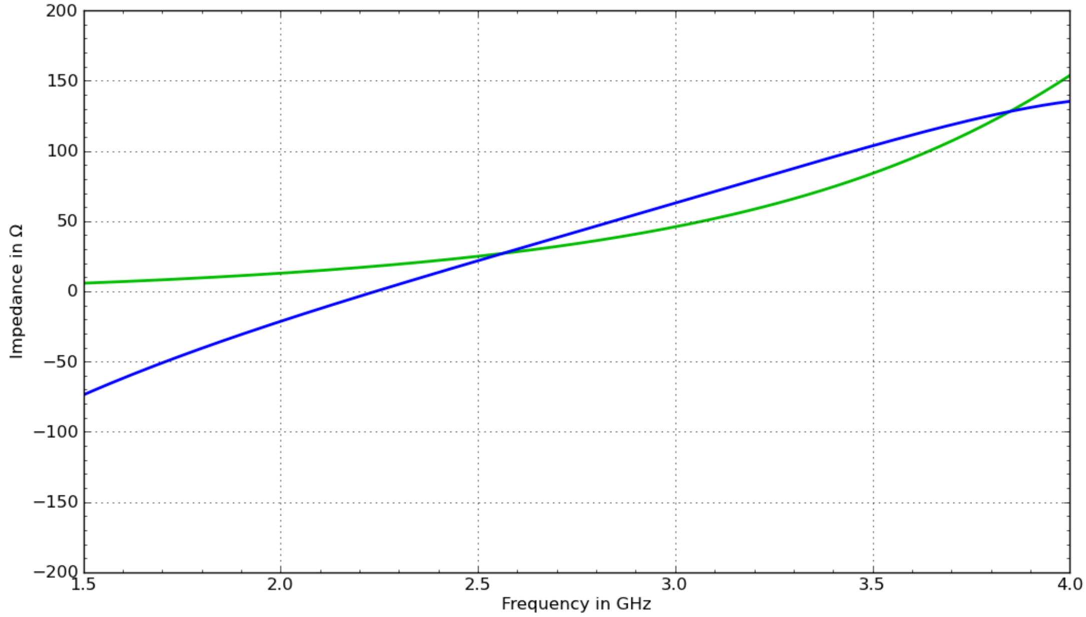

Figure G.7: Real (green line) and imaginary (blue line) parts of input impedance (top loaded monopole antenna)

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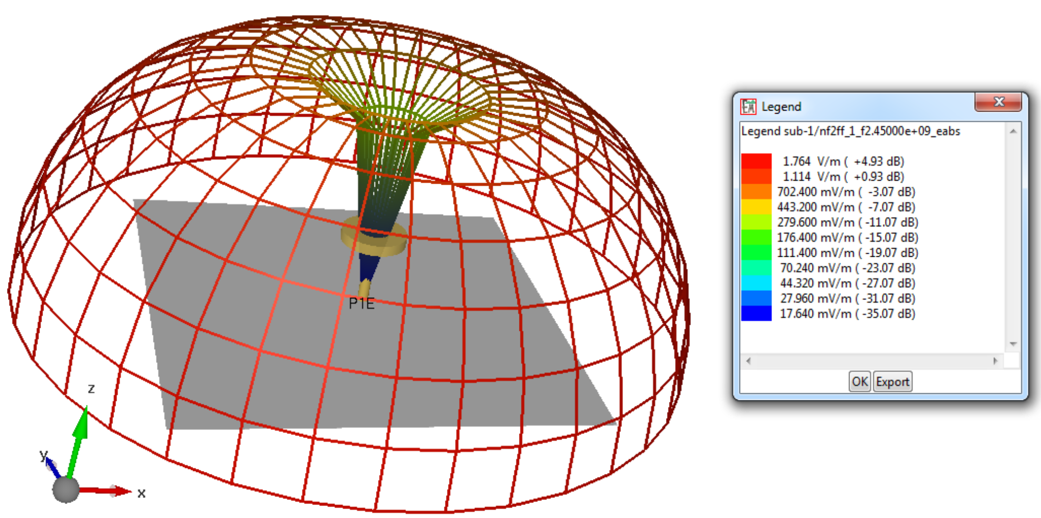

Figure G.8: Radiation pattern of top loaded monopole antenna

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Inverted-F antenna

Figure G.9 shows an inverted-F antenna over conducting ground. The height of $h_\mathrm{F} = 14\,\mathrm{mm}$ is only half of the initial monopole length ($h_\mathrm{F} = \ell/2$). The feedpoint can be adjusted to achieve a good matching to $50\,\Omega$ as shown in Figure G.10. The frequency dependence of the input

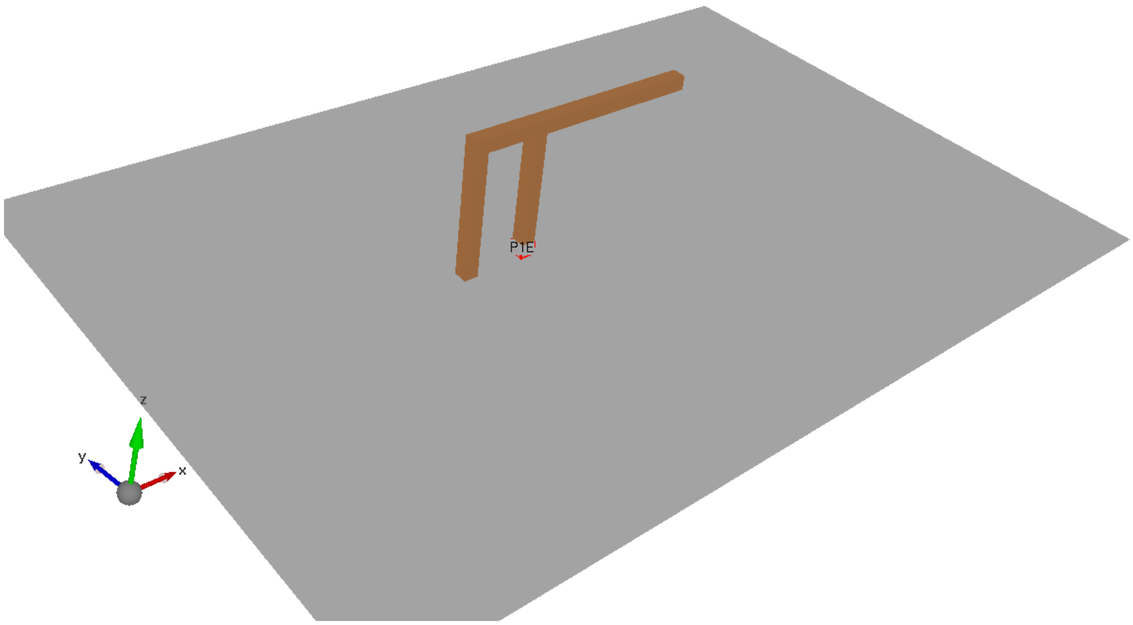

Figure G.9: Inverted-F antenna

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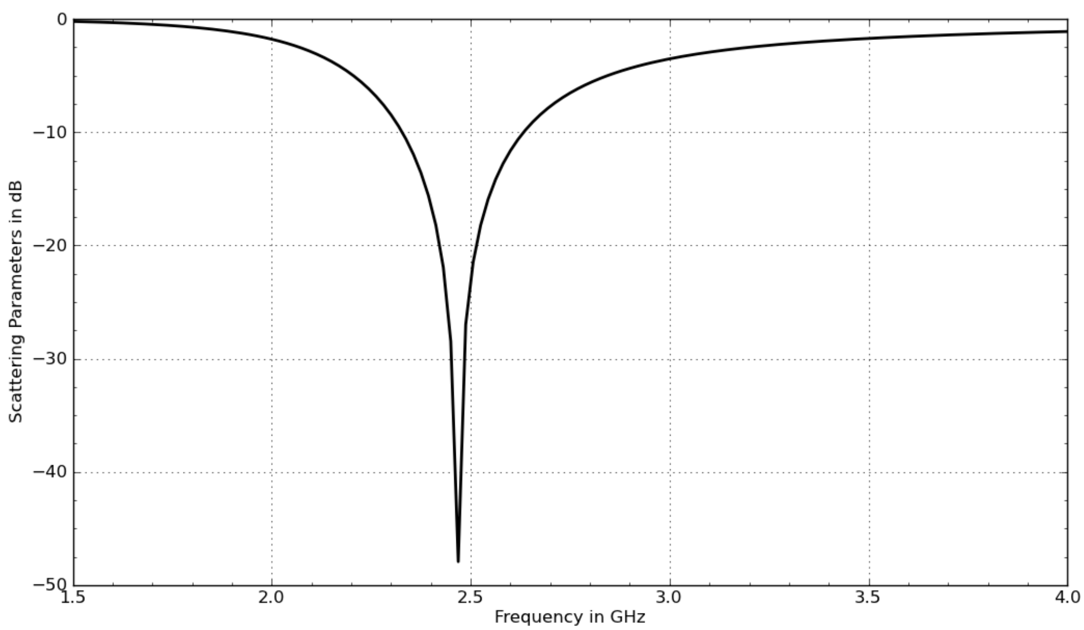

Figure G.10: Reflection coefficient of inverted-F antenna

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impedance given in Figure G.11 is quite different from the input impedance of a monopole. For low frequencies a monopole shows capacitive behaviour (negative imaginary part) whereas an inverted-F antenna shows inductive behaviour (positive imaginary part). As opposed to a

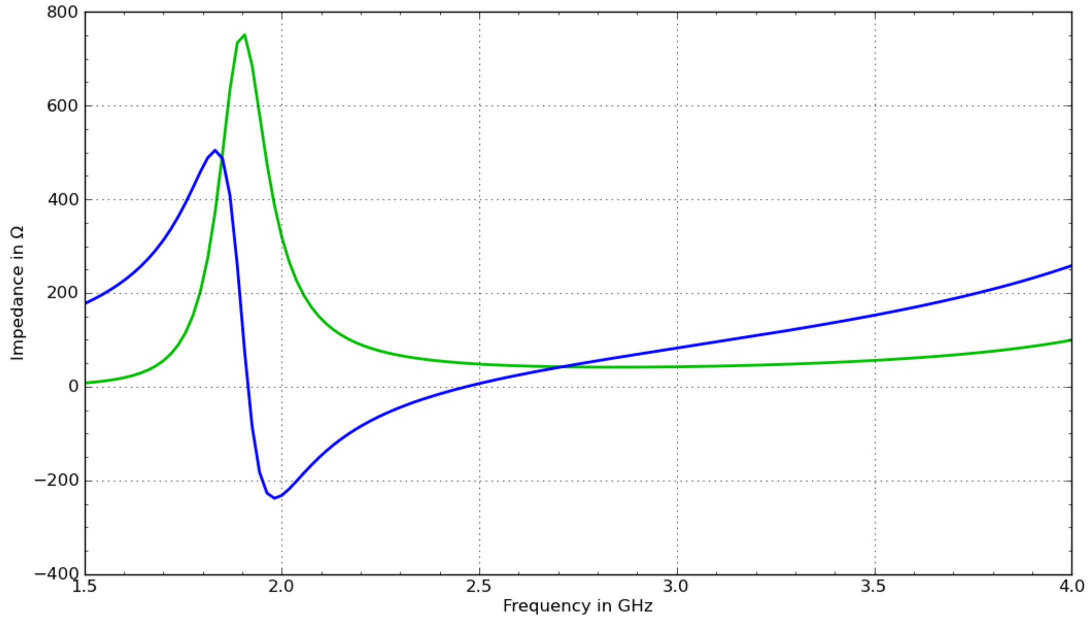

Figure G.11: Real (green line) and imaginary (blue line) parts of input impedance (inverted-F antenna)

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monopole antenna an inverted-F antenna shows radiation also in vertical direction (see Figure G.12).

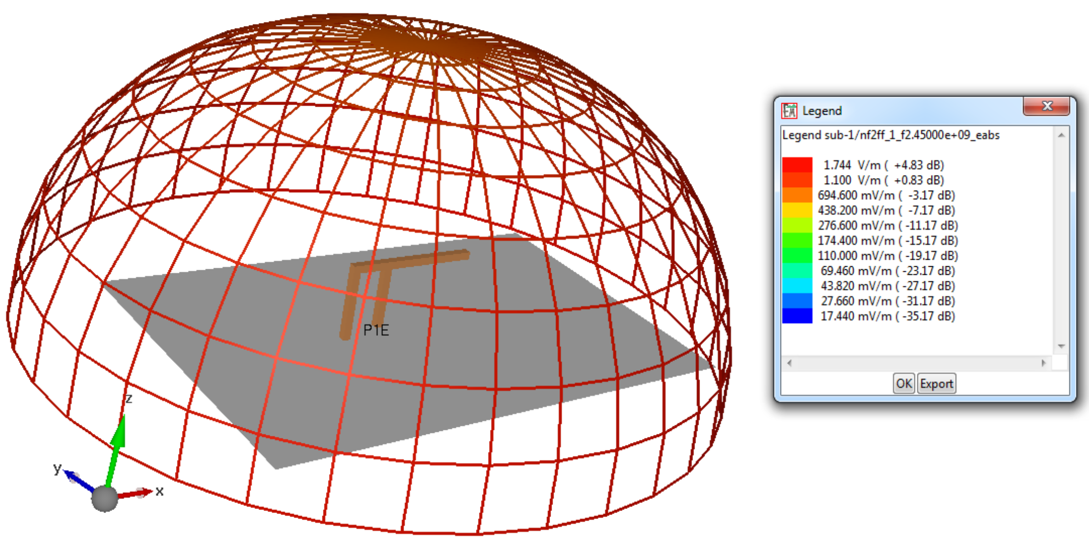

Figure G.12: Radiation pattern of inverted-F antenna

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G.3 Problem 7.3

We use Equation 7.49 and 7.50 (book page 273) to estimate the patch length L. Since the equations cannot be solved directly for L, we try values numerically and get a length of $L = 19.6\,\text{mm}$. Using the given relation of $W = 1.5L$ the width is $W = 29.4\,\text{mm}$.

The feedpoint location is given by Equation 7.52 and 7.53 (book page 275). We get $x_\text{f} = 5.6\,\text{mm}$ and $y_\text{f} = 14.7\,\text{mm}$. Figure G.13 shows the patch (red) over a substrate (grey) as well as the feedpoint location (light red). Furthermore the radiation pattern is given. The directivity is $D = 6.44\,\text{dBi}$. (The simulations have been performed with the EM simulation software Empire from IMST.)

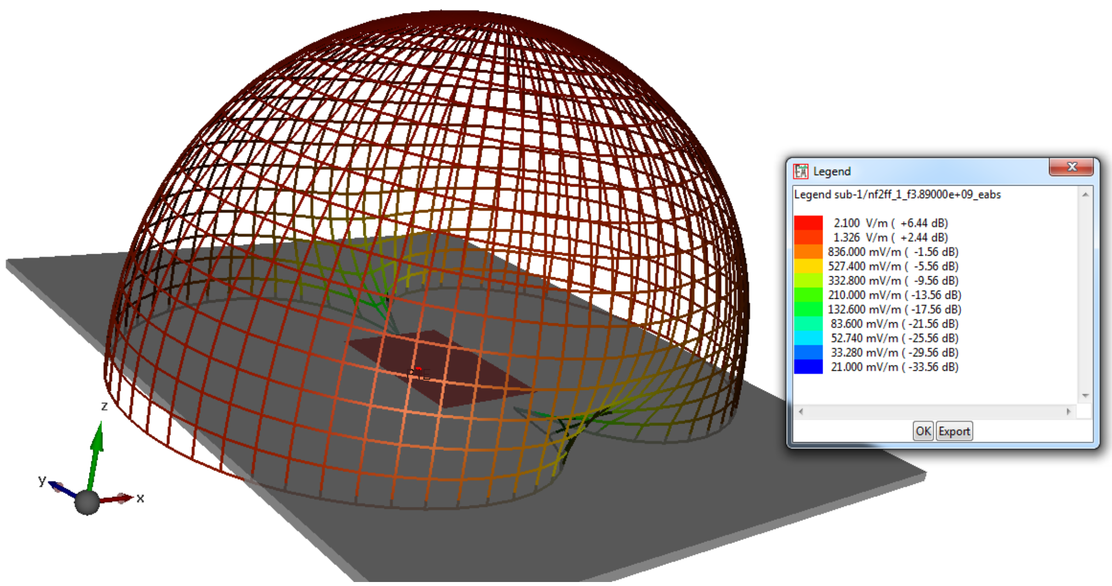

Figure G.13: Radiation pattern and geometry of the designed patch antenna

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The reflection coefficient is shown in Figure G.14. The initial design shows matching at a frequency of 3.89 GHz (about 3% lower than the specified frequency of 4 GHz). In order to achieve matching at the specified frequency of 4 GHz we reduce the length of the patch slightly from 19.6 mm to 19.0 mm and change the feedpoint location from $f_\text{f} = 5.6\,\text{mm}$ to $f_\text{f} = 5.4\,\text{mm}$. The resulting reflection coefficient is shown in Figure G.15. The initial design provided a good starting point for the subsequent optimization.

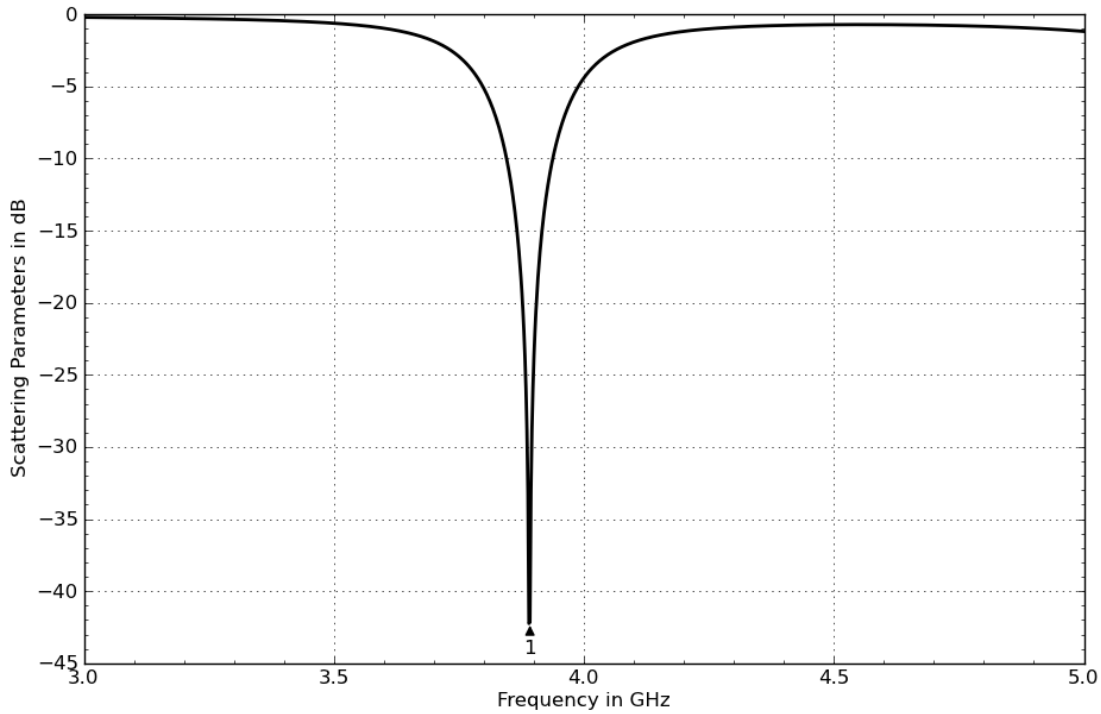

Figure G.14: Reflection coefficient of the designed patch antenna (initial design)

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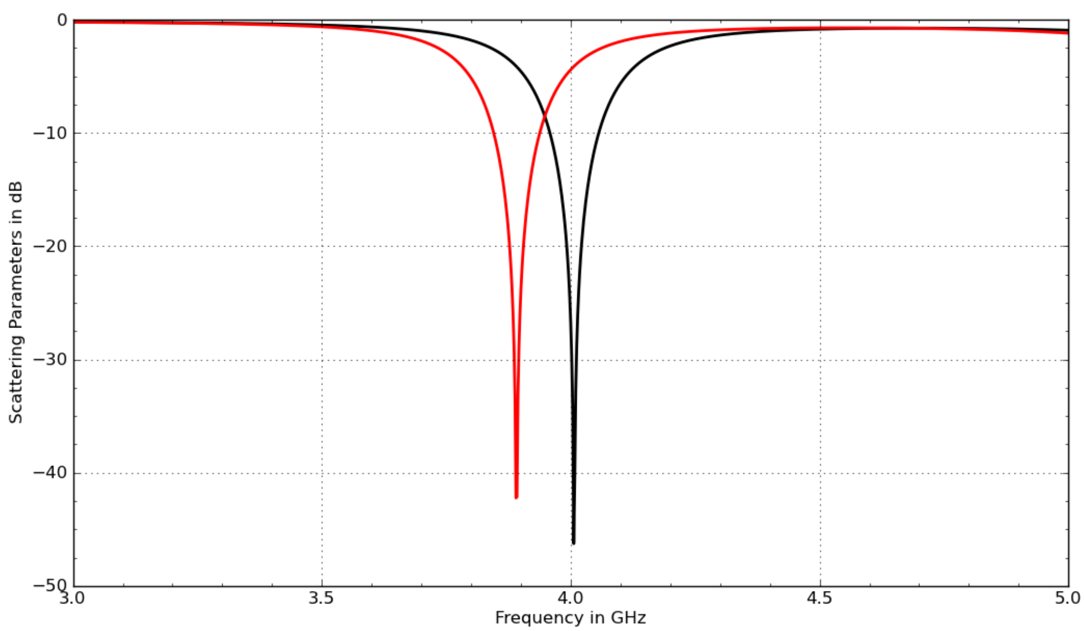

Figure G.15: Reflection coefficient of the patch antenna (initial design (red line)and optimized design (black line))

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G.4 Problem 7.4

The magnetic vector potential $\vec{A}$ (see Equation (7.29) on book page 263) is given as

$$\vec{A} = \frac{\mu_0 I\ell}{4\pi} \cdot \frac{\mathrm{e}^{-jkr}}{r} \vec{e}_z \tag{G.4}$$

In Equation G.4 we use mixed Cartesian and spherical coordinates. In order to perform the calculation in spherical coordinates we express the unit vector $\vec{e}_z$ in spherical coordinates (see Equation (A.14) on book page 324).

$$\vec{e}_z = \vec{e}_r \cos\vartheta - \vec{e}_\vartheta \sin\vartheta \tag{G.5}$$

Hence, no φ-component exists ($A_\varphi = 0$).

We determine the magnetic field strength $\vec{H}$ using Equation (7.23).

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A} = \frac{I\ell}{4\pi} \nabla \times \left(\frac{\mathrm{e}^{-jkr}}{r} \left[\vec{e}_r \cos\vartheta - \vec{e}_\vartheta \sin\vartheta \right] \right) \tag{G.6}$$

The curl operator in spherical coordinates reads

$$\begin{aligned} \nabla \times \vec{A} = & \frac{1}{r\sin\vartheta} \left(\frac{\partial \left(A_\varphi \sin\vartheta \right)}{\partial\vartheta} - \frac{\partial A_\vartheta}{\partial\varphi} \right) \vec{e}_r \\ & + \frac{1}{r} \left(\frac{1}{\sin\vartheta} \frac{\partial A_r}{\partial\varphi} - \frac{\partial \left(r A_\varphi \right)}{\partial r} \right) \vec{e}_\vartheta \\ & + \frac{1}{r} \left(\frac{\partial \left(r A_\vartheta \right)}{\partial r} - \frac{\partial A_r}{\partial\vartheta} \right) \vec{e}_\varphi \end{aligned} \tag{G.7}$$

Since there is no φ-component ($A_\varphi = 0$) and the remaining components are no functions of φ ($A_r; A_\vartheta \neq \mathrm{fct}(\varphi)$) the expression is simplified to

$$\vec{H} = \frac{I\ell}{4\pi} \frac{1}{r} \underbrace{\left(\frac{\partial}{\partial r} \left[r \left(-\frac{\mathrm{e}^{-jkr}}{r} \right) \sin\vartheta \right] \right)}_{jk\mathrm{e}^{-jkr}\sin\vartheta} \vec{e}_\varphi - \frac{I\ell}{4\pi} \frac{1}{r} \underbrace{\frac{\partial}{\partial\vartheta} \left(\frac{\mathrm{e}^{-jkr}}{r} \cos\vartheta \right)}_{\frac{e^{-jkr}}{r}(-\sin\vartheta)} \vec{e}_\varphi \tag{G.8}$$

So we end up with the following result (see Equation (7.34)).

$$\boxed{\vec{H} = \frac{I\ell}{4\pi} \frac{\mathrm{e}^{-jkr}}{r^2} (1 + jkr) \sin\vartheta \, \vec{e}_\varphi} \tag{G.9}$$

The electrical field strength $\vec{E}$ is given by Equation (7.24) (see book page 263).

$$\vec{E} = \frac{\nabla(\nabla \cdot \vec{A})}{j\omega\mu_0\varepsilon_0} - j\omega\vec{A} \tag{G.10}$$

The divergence operator in spherical coordinates reads:

$$\nabla \cdot \vec{A} = \frac{1}{r^2} \frac{\partial \left(r^2 A_r \right)}{\partial r} + \frac{1}{r\sin\vartheta} \frac{\partial \left(A_\vartheta \sin\vartheta \right)}{\partial\vartheta} + \frac{1}{r\sin\vartheta} \frac{\partial A_\varphi}{\partial\varphi} \tag{G.11}$$

We get

$$\nabla\cdot\vec{A} = \frac{\mu_0 I\ell}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\mathrm{e}^{-jkr}}{r}\cos\vartheta\right) + \frac{\mu_0 I\ell}{4\pi}\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\frac{\mathrm{e}^{-jkr}}{r}(-\sin^2\vartheta)\right) \tag{G.12}$$

$$= \frac{\mu_0 I\ell}{4\pi}\left[\cos\vartheta\frac{1}{r^2}\left(r(-jk)\mathrm{e}^{-jkr}+\mathrm{e}^{-jkr}\right) + \frac{1}{r\sin\vartheta}\frac{\mathrm{e}^{-jkr}}{r}\left(-2\sin\vartheta\,\cos\vartheta\right)\right] \tag{G.13}$$

Combining all terms yields

$$\nabla\cdot\vec{A} = \frac{\mu_0 I\ell}{4\pi}\frac{\mathrm{e}^{-jkr}}{r^2}(1+jkr)\cos\vartheta \tag{G.14}$$

Now we apply the gradient operator which is given as

$$\nabla\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\vartheta}\vec{e}_\vartheta + \frac{1}{r\sin\vartheta}\frac{\partial\phi}{\partial\varphi}\vec{e}_\varphi \qquad \text{where} \qquad \phi = \nabla\cdot\vec{A} \tag{G.15}$$

Since $\nabla\cdot\vec{A}$ is no function of φ the last term is zero. So we consider first the derivative with respect to r:

$$\frac{\partial(\nabla\cdot\vec{A})}{\partial r} = -\frac{\mu_0 I\ell}{4\pi}\cos\vartheta\frac{\partial}{\partial r}\left[\frac{\mathrm{e}^{-jkr}}{r^2} + jk\frac{\mathrm{e}^{-jkr}}{r}\right] \tag{G.16}$$

The quotient rule

$$\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2} \tag{G.17}$$

gives us

$$\frac{\partial(\nabla\cdot\vec{A})}{\partial r} = -\frac{\mu_0 I\ell}{4\pi}\cos\vartheta\left[\frac{-jk\mathrm{e}^{-jkr}r^2 - \mathrm{e}^{-jkr}2r}{r^4} + jk\frac{(-jk)\mathrm{e}^{-jkr}r - \mathrm{e}^{-jkr}}{r^2}\right] \tag{G.18}$$

$$= -\frac{\mu_0 I\ell}{4\pi}\cos\vartheta\left[-2jk\frac{\mathrm{e}^{-jkr}}{r^2} - 2\frac{\mathrm{e}^{-jkr}}{r^3} + k^2\frac{\mathrm{e}^{-jkr}}{r}\right] \tag{G.19}$$

Hence, the radial component of the electric field strength yields

$$\begin{aligned} E_r =& \frac{1}{j\omega\mu_0\varepsilon_0}\left(-\frac{\mu_0 I\ell}{4\pi}\right)\cos\vartheta\left[-2jk\frac{\mathrm{e}^{-jkr}}{r^2} - 2\frac{\mathrm{e}^{-jkr}}{r^3} + k^2\frac{\mathrm{e}^{-jkr}}{r}\right] \\ &- j\omega\underbrace{\left(\frac{\mu_0 I\ell}{4\pi}\frac{\mathrm{e}^{-jkr}}{r}\cos\vartheta\right)}_{A_r} \end{aligned} \tag{G.20}$$

With

$$k^2 = \omega^2\varepsilon_0\mu_0 \tag{G.21}$$

we get

$$E_r = \frac{\mu_0 I\ell}{4\pi}\cos\vartheta\,\mathrm{e}^{-jkr}\left[\frac{2k}{\omega\mu_0\varepsilon_0 r^2} + \frac{2}{j\omega\mu_0\varepsilon_0 r^3}\right] \tag{G.22}$$

Our final result for the radial component of the electrical field strength is

$$\boxed{E_r = \frac{I\ell}{j2\pi\omega\varepsilon_0}\cos\vartheta\frac{\mathrm{e}^{-jkr}}{r^3}(1+jkr)} \tag{G.23}$$

Finally, we consider the ϑ-component of the electrical field strength.

$$E_\vartheta = \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{r}\frac{\partial(\nabla\cdot\vec{A})}{\partial\vartheta} - j\omega A_\vartheta \tag{G.24}$$

$$= \frac{1}{j\omega\mu_0\varepsilon_0}\frac{1}{r}\left(-\frac{\mu_0 I\ell}{4\pi}\cdot\frac{\mathrm{e}^{-jkr}}{r^2}(1+jkr)(-\sin\vartheta)\right) - j\omega\left(-\frac{\mu_0 I\ell}{4\pi}\cdot\frac{\mathrm{e}^{-jkr}}{r}\sin\vartheta\right) \tag{G.25}$$

With

$$k^2 = \omega^2\varepsilon_0\mu_0 \qquad \rightarrow \qquad \omega = \frac{k^2}{\omega\varepsilon_0\mu_0} \tag{G.26}$$

we get

$$\boxed{E_\vartheta = \frac{I\ell}{j4\pi\omega\varepsilon_0}\frac{\mathrm{e}^{-jkr}}{r^3}\sin\vartheta(1+jkr-(kr)^2)} \tag{G.27}$$

G.5 Problem 7.5

The elements of a two-dimensional array antenna are located in xy-plane ($z = 0$). Figure 7.28 (book page 288) shows the arrangement. The lower left element is positioned at $x = y = 0$, the corresponding indexes are $m = n = 0$.

Exciting all elements with equal phase results in a main beam direction perpendicular to the antenna plane, i.e. the main lobe points into z-direction ($\vartheta = 0°$). By changing the phase the main lobe may be tilted into another direction. The direction may be described either by the angles φ_0 and ϑ_0 or by the normal vector $\vec{n}$.

$$\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} = \begin{pmatrix} \cos\varphi_0\sin\vartheta_0 \\ \sin\varphi_0\sin\vartheta_0 \\ \cos\varphi_0 \end{pmatrix} \tag{G.28}$$

A plane wave that travels in the direction of the normal vector has planes of equal phase that are perpendicular to the direction of propagation. Let us consider such a plane that includes the origin of the coordinate system.

$$n_x x + n_y y + n_z z = 0 \tag{G.29}$$

In order to produce constructive superposition in the direction of the normal vector the phase (or delay time) of each individual antenna element has to be adjusted. The antenna elements are located at the following points in space.

$$P = \begin{pmatrix} m\,d_x \\ n\,d_y \\ 0 \end{pmatrix} \tag{G.30}$$

The distance d_{mn} between antenna element and plane of equal phase is given by (Hesse normal form)

$$d_{mn} = \frac{n_x x + n_y y + n_z z - 0}{\sqrt{n_x^2 + n_y^2 + n_z^2}} = m d_x \cos\varphi_0 \sin\vartheta_0 + n d_y \sin\varphi_0 \sin\vartheta_0 \tag{G.31}$$

Due to the speed of propagation c_0 we determine the delay times as

$$\Delta t_{mn} = \frac{d_{mn}}{c_0} \tag{G.32}$$

For a given frequency $f_0 = c_0/\lambda_0$ the phases are

$$\Delta\Phi_{mn} = \Delta t_{mn} \frac{c_0\, 360^\circ}{\lambda_0} = \frac{d_{mn}}{c_0} \cdot \frac{c_0\, 360^\circ}{\lambda_0} \tag{G.33}$$

$$= \frac{360^\circ}{\lambda_0} \left[m d_x \cos\varphi_0 \sin\vartheta_0 + n d_y \sin\varphi_0 \sin\vartheta_0 \right] \tag{G.34}$$

Remember: A *negative* phase represents a *delay* in the time domain.

(Last modified: 14.02.2013)

Appendix H

Solutions to Problems in Chapter 8

H.1 Problem 8.1

We will derive the path loss for the plane earth model shown in Figure H.1.

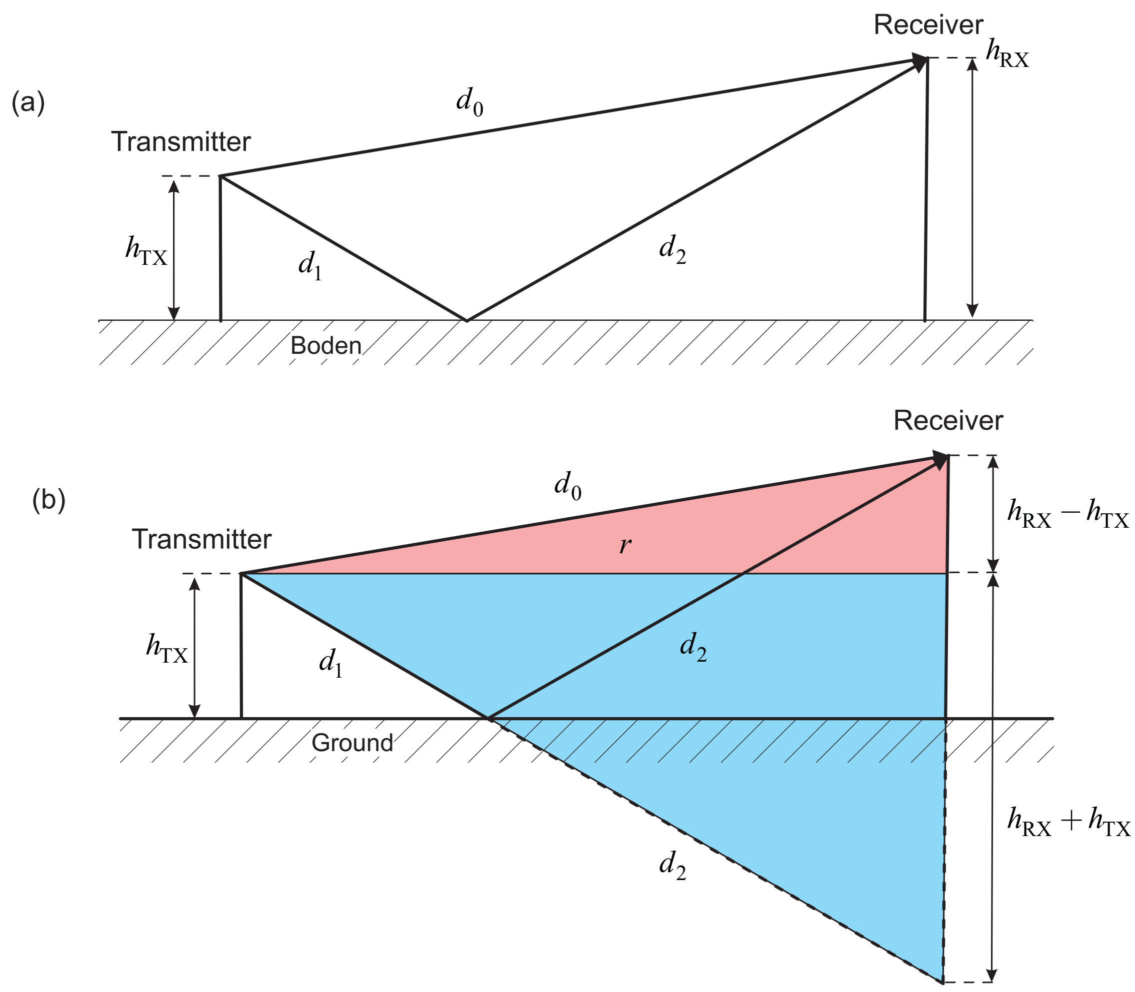

Figure H.1: (a) Plane earth model (b) geometric definitions

There are two paths to consider between transmitter (antenna height h_{TX}) and receiver (antenna height h_{RX}).

- *direct* path (path length d_0) and
- *indirect* path (path length $d_1 + d_2$).

The horizontal distance is r. At the receiver location both waves add up.

$$\vec{E}_{\mathrm{RX}} = \vec{E}_{\mathrm{direct}} + \vec{E}_{\mathrm{indirect}} \tag{H.1}$$

In practical applications (macrocells in cellular communication, directional radio links) the antenna heights h_{TX} and h_{RX} are small compared to the horizontal distance r.

$$r \gg h_{\mathrm{TX}}, h_{\mathrm{RX}} \tag{H.2}$$

First, we consider the *magnitude* of the electric field components. For $r \gg h_{\mathrm{TX}}, h_{\mathrm{RX}}$ we get

$$r \approx d_0 \approx d_1 + d_2 \tag{H.3}$$

For an isotropic radiator the power density is

$$S = \frac{P}{4\pi r^2} = \frac{E_0^2}{Z_0} \tag{H.4}$$

If we solve Equation H.4 for the electric field strength E_0 we get

$$E_0 = \sqrt{\frac{P_{\mathrm{TX}} Z_0}{4\pi r^2}} = \sqrt{\frac{P_{\mathrm{TX}} Z_0}{4\pi}}\frac{1}{r} \tag{H.5}$$

If we include the *phase* – due to propagation delay – we get for the *direct* path

$$E_{\mathrm{direkt}} = E_0 \mathrm{e}^{-jkd_0} \tag{H.6}$$

The *indirect* path includes the ground reflection ($r_{\mathrm{Ground}} = -1$ for conducting ground)

$$E_{\mathrm{indirekt}} = (-1) \cdot E_0 \mathrm{e}^{-jk(d_1+d_2)} \tag{H.7}$$

Superposition of both electric fields at the receiver point yields

$$E_{\mathrm{RX}} = E_{\mathrm{direct}} + E_{\mathrm{indirect}} = E_0 \mathrm{e}^{-jkd_0} - E_0 \mathrm{e}^{-jk(d_1+d_2)} = E_0 \mathrm{e}^{-jkd_0}\left(1 - \mathrm{e}^{-jk(d_1+d_2-d_0)}\right) \tag{H.8}$$

Depending on the phase difference we see constructive or destructive interference. The phase difference depends on the path length difference:

$$\Delta d = (d_1 + d_2) - d_0 \tag{H.9}$$

The path length difference can be derived from Figure H.1b. The Pythagorean theorem (red triangle) gives us

$$r^2 + (h_{\mathrm{RX}} - h_{\mathrm{TX}})^2 = d_0^2 \tag{H.10}$$

Furthermore, the Pythagorean theorem (blue triangle) gives us

$$r^2 + (h_{\mathrm{RX}} + h_{\mathrm{TX}})^2 = (d_1 + d_2)^2 \tag{H.11}$$

Next, we solve the equation for the ratio path length over horizontal distance. Furthermore, we approximate the square root by the following relation:

$$\sqrt{1+x} \approx 1 + \frac{x}{2} \qquad \text{for} \qquad x \ll 1 \tag{H.12}$$

So, we get:

$$\frac{d_0}{r} = \sqrt{1 + \frac{(h_{\mathrm{RX}} - h_{\mathrm{TX}})^2}{r^2}} \approx 1 + \frac{(h_{\mathrm{RX}} - h_{\mathrm{TX}})^2}{2r^2} \tag{H.13}$$

$$\frac{d_1 + d_2}{r} = \sqrt{1 + \frac{(h_{\mathrm{RX}} + h_{\mathrm{TX}})^2}{r^2}} \approx 1 + \frac{(h_{\mathrm{RX}} + h_{\mathrm{TX}})^2}{2r^2} \tag{H.14}$$

The path length difference is

$$d_1 + d_2 - d_0 \approx r \left[1 + \frac{(h_{\mathrm{RX}} + h_{\mathrm{TX}})^2}{2r^2} - 1 - \frac{(h_{\mathrm{RX}} - h_{\mathrm{TX}})^2}{2r^2} \right] = \frac{2 h_{\mathrm{RX}} h_{\mathrm{TX}}}{r} \tag{H.15}$$

So, the magnitude of the electric field strength becomes

$$|E_{\mathrm{RX}}| = E_0 \left| 1 - \mathrm{e}^{-jk(d_1 + d_2 - d_0)} \right| = E_0 \left| 1 - \mathrm{e}^{-jk \frac{2 h_{\mathrm{RX}} h_{\mathrm{TX}}}{r}} \right| \tag{H.16}$$

$$= \sqrt{\frac{P_{\mathrm{TX}} Z_0}{4\pi}} \frac{1}{r} \left| 1 - \mathrm{e}^{-jk \frac{2 h_{\mathrm{RX}} h_{\mathrm{TX}}}{r}} \right| \tag{H.17}$$

Let us consider an isotropic radiator as a receiving antenna. The received power then is

$$P_{\mathrm{RX}} = A_{\mathrm{eff}} \cdot S = \frac{\lambda_0^2}{4\pi} \cdot \frac{E_{\mathrm{RX}}^2}{Z_0} = \frac{\lambda_0^2}{4\pi} \cdot \frac{P_{\mathrm{TX}} Z_0}{Z_0 4\pi r^2} \left| 1 - \mathrm{e}^{-jk \frac{2 h_{\mathrm{RX}} h_{\mathrm{TX}}}{r}} \right|^2 \tag{H.18}$$

where λ_0 is the free space wavelength. Hence, the path loss becomes

$$\frac{1}{L_{\mathrm{PEL}}} = \frac{P_{\mathrm{RX}}}{P_{\mathrm{TX}}} = \left(\frac{\lambda_0}{4\pi r} \right)^2 \left| 1 - \mathrm{e}^{-jk \frac{2 h_{\mathrm{RX}} h_{\mathrm{TX}}}{r}} \right|^2 \tag{H.19}$$

With $k = 2\pi/\lambda$ we get

$$\frac{1}{L_{\mathrm{PEL}}} = \frac{P_{\mathrm{RX}}}{P_{\mathrm{TX}}} = \left(\frac{\lambda_0}{4\pi r} \right)^2 \left| 1 - \mathrm{e}^{-j \frac{4\pi h_{\mathrm{RX}} h_{\mathrm{TX}}}{\lambda_0 r}} \right|^2 \tag{H.20}$$

Finally, we rewrite the term by using the following relation

$$\mathrm{e}^{j\alpha} = \cos\alpha + j \sin\alpha \tag{H.21}$$

This gives us

$$\left| 1 - \mathrm{e}^{j\alpha} \right|^2 = |1 - \cos\alpha - j \sin\alpha|^2 = (1 - \cos\alpha)^2 + (\sin\alpha)^2 \tag{H.22}$$

$$= 1 - 2\cos\alpha + \underbrace{(\cos\alpha)^2 + (\sin\alpha)^2}_{=1} = 2(1 - \cos\alpha) \tag{H.23}$$

So, the path loss is

$$\frac{1}{L_{\mathrm{PEL}}} = \frac{P_{\mathrm{RX}}}{P_{\mathrm{TX}}} = 2 \left(\frac{\lambda_0}{4\pi r} \right)^2 \left(1 - \cos\left(\frac{4\pi h_{\mathrm{RX}} h_{\mathrm{TX}}}{\lambda_0 r} \right) \right) \tag{H.24}$$

H.2 Problem 8.2

We consider a radio link with

- operational frequency $f = 400\,\mathrm{MHz}$,
- height of transmit antenna h_{TX}=5 m,
- height of receive antenna h_{RX1}=5 m,
- horizontal distance between antennas $r_1 = 2\,\mathrm{km}$.

The path loss is given in Equation 8.13 (book page 308).

$$\frac{1}{L_1} = \frac{P_{\mathrm{RX}}}{P_{\mathrm{TX}}} = 2\left(\frac{\lambda_0}{4\pi r_1}\right)^2 \left(1 - \cos\left(\frac{4\pi h_{\mathrm{RX1}} h_{\mathrm{TX}}}{\lambda_0 r_1}\right)\right) \approx 3.892 \cdot 10^{-11} \tag{H.25}$$

where the wavelength is $\lambda_0 = c/f = 0.75\,\mathrm{m}$.

In our example, the distance r is greater than the *break point distance*, i.e. $r > d_{\mathrm{break}}$ (Equation 8.15).

$$d_{\mathrm{break}} = \frac{4 h_{\mathrm{TX}} h_{\mathrm{RX}}}{\lambda_0} = 133.3\,\mathrm{m} \tag{H.26}$$

Therefore, we can use Equation 8.14 as an approximation.

$$\frac{1}{L_1} = \frac{h_{\mathrm{RX1}}^2 h_{\mathrm{TX}}^2}{r_1^4} \approx 3.90625 \cdot 10^{-11} \tag{H.27}$$

In logarithmic scale the path loss is

$$L_1 = 2.56 \cdot 10^{10} \mathrel{\widehat{=}} 10\lg(2.56 \cdot 10^{10})\,\mathrm{dB} = 104.08\,\mathrm{dB} \tag{H.28}$$

Now, we increase the distance between the antennas to $r_2 = 3\,\mathrm{km}$. In order to compensate for the increase in path loss the height of the receiver antenna shall be adjusted. Therefore, we get

$$\frac{1}{L_1} = \frac{h_{\mathrm{RX1}}^2 h_{\mathrm{TX}}^2}{r_1^4} = \frac{1}{L_2} = \frac{h_{\mathrm{RX2}}^2 h_{\mathrm{TX}}^2}{r_2^4} \tag{H.29}$$

Consequently, the new antenna height is

$$h_{\mathrm{RX2}} = \left(\frac{r_2}{r_1}\right)^2 h_{\mathrm{RX1}} = 11.25\,\mathrm{m} \tag{H.30}$$

H.3 Problem 8.3

Let us consider a satellite communication link with

- distance $r = 36\,000\,\mathrm{km}$,
- frequency $f = 10\,\mathrm{GHz}$,
- ground station antenna gain $G_{\mathrm{gs}} = 30\,\mathrm{dBi}$,
- satellite antenna gain $G_{\mathrm{sat}} = 20\,\mathrm{dBi}$.

According to Equation 8.7 (book page 303) the *free space loss* is

$$L_{\mathrm{F0}} = 20\lg\left(\frac{4\pi r f}{c_0}\right) = 203.57\,\mathrm{dB} \tag{H.31}$$

With Equation 8.6 we determine the power at the receiver as

$$\frac{P_{\mathrm{RX}}}{\mathrm{dBm}} = \frac{P_{\mathrm{TX}}}{\mathrm{dBm}} + \frac{G_{\mathrm{gs}}}{\mathrm{dBi}} + \frac{G_{\mathrm{sat}}}{\mathrm{dBi}} - \frac{L_{\mathrm{F0}}}{\mathrm{dB}} \tag{H.32}$$

Finally, we get

$$P_{\mathrm{RX}} = \underbrace{30\,\mathrm{dBm}}_{1\,\mathrm{W}} + 30\,\mathrm{dBi} + 20\,\mathrm{dBi} - 203.56\,\mathrm{dB} = -123.56\,\mathrm{dBm} = 4.4\cdot 10^{-16}\,\mathrm{W} \tag{H.33}$$

(Last modified: 08.01.2013)