正交坐标系下的梯度、散度、旋度表达式

1、拉梅系数

对于一个正交坐标系，因一个正交坐标q_i的微笑改变了所引起位移的微小该变量为$\boldsymbol{dr} = H_i dq_i \boldsymbol{e}_i$，$\boldsymbol{e}_i$为单位向量，指向$q_i$增速最快的方向，$H_i$为对应变量的拉梅系数。可以看到$H_i = \left|\frac{\partial \boldsymbol{r}}{\partial q_i}\right|$，若位移矢量$\boldsymbol{r}$原本是直角坐标系下的，即：$\boldsymbol{r} = (x, y, z)$，那么：$H_i = \sqrt{\left(\frac{\partial x}{\partial q_i}\right)^2 + \left(\frac{\partial y}{\partial q_i}\right)^2 + \left(\frac{\partial z}{\partial q_i}\right)^2}$。如在圆柱坐标系里：

$$\begin{cases} x = r\,cos\theta \\ y = r\,sin\theta \\ z = z \end{cases}$$

可得：$H_r = 1; H_\theta = r; H_z = 1$;同样，无穷小面积元就是：$dS = ds_i ds_j = H_i H_j dq_i dq_j$，无穷小体积元就是：$dV = ds_i ds_j ds_k = H_i H_j H_k dq_i dq_j dq_k$。

2、梯度

n方向的方向导数就是梯度在n方向上的分量：$\nabla u \cdot \boldsymbol{n} = \frac{du}{ds}$，$ds$是当$dq$改变时朝$n$方向的无穷小位移。当$n$与$\boldsymbol{e}_i$同向时，$ds_i = H_i dq_i$，此时的方向导数就是$\frac{du}{ds} = \frac{\partial u}{H_i \partial q_i}$，即：$\nabla u \cdot \boldsymbol{e}_i = \frac{\partial u}{H_i \partial q_i}$，也就是说：$\nabla u = \frac{1}{H_1}\frac{\partial u}{\partial q_1}\boldsymbol{e}_1 + \frac{1}{H_2}\frac{\partial u}{\partial q_2}\boldsymbol{e}_2 + \frac{1}{H_3}\frac{\partial u}{\partial q_3}\boldsymbol{e}_3$

3、散度

恒等式：$\nabla \cdot (\nabla\varphi_1 \times \nabla\varphi_2) = 0$。由已经得到的梯度表达式得：$\nabla q_2 = \frac{1}{H_2}\boldsymbol{e}_2$，$\nabla q_3 = \frac{1}{H_3}\boldsymbol{e}_3$。则：

$$\nabla \cdot \boldsymbol{A} = \nabla \cdot (A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3) = \nabla \cdot (A_1\boldsymbol{e}_1) + \nabla \cdot (A_2\boldsymbol{e}_2) + \nabla \cdot (A_3\boldsymbol{e}_3)$$

$$\nabla \cdot (A_1\boldsymbol{e}_1) = \nabla \cdot \left(\frac{A_1 H_2 H_3 \boldsymbol{e}_1}{H_2 H_3}\right) = A_1 H_2 H_3 \nabla \cdot \left(\frac{\boldsymbol{e}_1}{H_2 H_3}\right) + \frac{\boldsymbol{e}_1}{H_2 H_3} \cdot \nabla(\mathrm{A_1 H_2 H_3})$$

注意到：$\frac{\boldsymbol{e}_1}{H_2 H_3} = \nabla q_2 \times \nabla q_3$，则：$\nabla \cdot \left(\frac{A_1 H_2 H_3 \boldsymbol{e}_1}{H_2 H_3}\right) = A_1 H_2 H_3 \nabla \cdot (\nabla q_2 \times \nabla q_3) + \frac{\boldsymbol{e}_1}{H_2 H_3} \cdot \nabla(A_1 H_2 H_3)$。右边第一项为 0，由梯度表达式，右边第二项写成：$= \frac{\boldsymbol{e}_1}{H_2 H_3} \cdot \left(\frac{1}{H_1}\frac{\partial A_1 H_2 H_3}{\partial q_1}\boldsymbol{e}_1 + \frac{1}{H_2}\frac{\partial A_1 H_2 H_3}{\partial q_2}\boldsymbol{e}_2 + \frac{1}{H_3}\frac{\partial A_1 H_2 H_3}{\partial q_3}\boldsymbol{e}_3\right) = \frac{1}{H_1 H_2 H_3}\frac{\partial A_1 H_2 H_3}{\partial q_1}$，即：$\nabla \cdot (A_1\boldsymbol{e}_1) = \frac{1}{H_1 H_2 H_3}\frac{\partial A_1 H_2 H_3}{\partial q_1}$。也就是说：

$$\nabla \cdot \boldsymbol{A} = \frac{1}{H_1 H_2 H_3}\frac{\partial A_1 H_2 H_3}{\partial q_1} + \frac{1}{H_1 H_2 H_3}\frac{\partial A_2 H_1 H_3}{\partial q_2} + \frac{1}{H_1 H_2 H_3}\frac{\partial A_3 H_1 H_2}{\partial q_3}$$

4、旋度

$\nabla \times \boldsymbol{A} = \nabla \times (A_1\boldsymbol{e}_1 + A_2\boldsymbol{e}_2 + A_3\boldsymbol{e}_3) = \nabla \times (A_1\boldsymbol{e}_1) + \nabla \times (A_2\boldsymbol{e}_2) + \nabla \times (A_3\boldsymbol{e}_3)$。右边第一项：$\nabla \times (A_1\boldsymbol{e}_1) = \nabla \times \left(\frac{H_1 A_1 \boldsymbol{e}_1}{H_1}\right) = H_1 A_1 \left(\nabla \times \frac{\boldsymbol{e}_1}{H_1}\right) + (\nabla H_1 A_1) \times \frac{\boldsymbol{e}_1}{H_1}$。注意到：$\nabla q_1 = \frac{\boldsymbol{e}_1}{H_1}$，故第一项：$H_1 A_1 \left(\nabla \times \frac{\boldsymbol{e}_1}{H_1}\right) = H_1 A_1 (\nabla \times \nabla q_1) = 0$。第二项由$\nabla H_1 A_1 = \nabla u = \frac{1}{H_1}\frac{\partial H_1 A_1}{\partial q_1}\boldsymbol{e}_1 + \frac{1}{H_2}\frac{\partial H_1 A_1}{\partial q_2}\boldsymbol{e}_2 + \frac{1}{H_3}\frac{\partial H_1 A_1}{\partial q_3}\boldsymbol{e}_3$ 以及 $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = 0$，那么：$\nabla \times \left(\frac{H_1 A_1 \boldsymbol{e}_1}{H_1}\right) = \left(\frac{1}{H_2}\frac{\partial H_1 A_1}{\partial q_2}\boldsymbol{e}_2 + \frac{1}{H_3}\frac{\partial H_1 A_1}{\partial q_3}\boldsymbol{e}_3\right) \times \frac{\boldsymbol{e}_1}{H_1} = \frac{\boldsymbol{e}_2}{H_1 H_3}\frac{\partial H_1 A_1}{\partial q_3} - \frac{\boldsymbol{e}_3}{H_1 H_2}\frac{\partial H_1 A_1}{\partial q_2}$。即：$\nabla \times (A_1\boldsymbol{e}_1) = \frac{\boldsymbol{e}_2}{H_1 H_3}\frac{\partial H_1 A_1}{\partial q_3} - \frac{\boldsymbol{e}_3}{H_1 H_2}\frac{\partial H_1 A_1}{\partial q_2}$。综合可得到：

$$\nabla \times \boldsymbol{A} = \frac{1}{H_1 H_2 H_3} \begin{vmatrix} H_1 \boldsymbol{e}_1 & H_2 \boldsymbol{e}_2 & H_3 \boldsymbol{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ H_1 A_1 & H_2 A_2 & H_3 A_3 \end{vmatrix}$$

5、拉普拉斯算子

综合梯度、散度公式，可得到：

$$\nabla^2 \Phi = \frac{1}{H_1 H_2 H_3} [\frac{\partial}{\partial q_1}\left(\frac{H_2 H_3}{H_1}\frac{\partial \Phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1 H_3}{H_2}\frac{\partial \Phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}(\frac{H_1 H_2}{H_3}\frac{\partial \Phi}{\partial q_3})]$$

6、圆柱坐标和球坐标系下的表达式

圆柱坐标系中，$H_r = 1; H_\theta = r; H_z = 1$。球坐标系中，$H_\rho = 1; H_\theta = r; H_\varphi = rsin\theta$