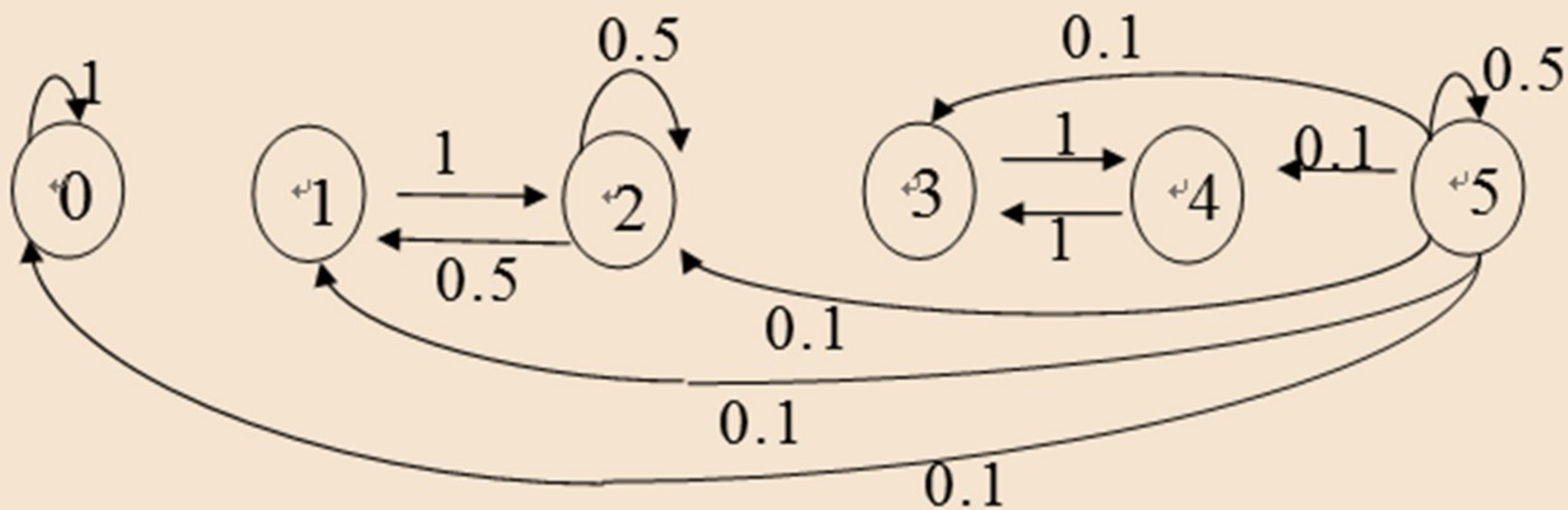
已知 $P(X_0=i)=\frac{1}{6}, i=0,1,\cdots,5.$ 计算

$\lim_{n\to\infty} P(X_n=i), i=0,1,\cdots,5.$



有四个等价类$C_1=\{0\},C_2=\{1,2\}$，$C_3=\{3,4\}$和$\{5\}$

只有$\{5\}$不闭。$\therefore$ 0,1,2,3,4正常返，5暂留，

$\lim_{n\to\infty} P(X_n = 5) = 0$

记 $h_{5,k} = \lim_{n\to\infty} P(X_n = k | X_0 = 5)$

$$\lim_{n\to\infty} P(X_n = 0) = \lim_{n\to\infty} \sum_{i=0}^{5} P(X_n = 0 | X_0 = i) P(X_0 = i)$$

$$= \frac{1}{6}\left(1 + \lim_{n\to\infty} P(X_n = 0 | X_0 = 5)\right) = \frac{1}{6}(1 + h_{5,0})$$

$$h_{5,0}=\lim_{n\to\infty}\sum_{i=0}^{5}P(X_n=0|X_1=i,X_0=5)\,P(X_1=i|X_0=5)$$

$$=\sum_{i=0}^{5}\lim_{n\to\infty}P(X_n=0|X_1=i)\,P(X_1=i|X_0=5)$$

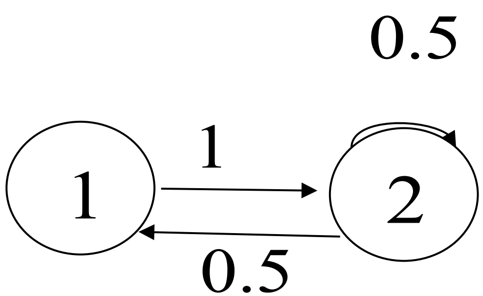
$$=\sum_{i=0}^{5}\lim_{n\to\infty}P(X_n=0|X_0=i)\,P(X_1=i|X_0=5)=\frac{1}{10}+\frac{1}{2}h_{5,0})$$

则 $h_{5,0}=\frac{1}{5},\lim_{n\to\infty}P(X_n=0)=\frac{1}{5}$

计算 $\lim_{n\to\propto} P(X_n = k), \quad k = 1,2$

把$\{X_n\}$限制在C_2上得到一个遍历Markov链，

状态空间为C_2



则 $\begin{cases} \pi_1 + \pi_2 = 1 \\ \pi_1 = 0.5\pi_2 \end{cases}$

平稳分布 $(\pi_1, \pi_2) = (\frac{1}{3}, \frac{2}{3})$

$$\lim_{n\to\infty} P(X_n = 1) = \lim_{n\to\infty} \sum_{i=0}^{5} P(X_n = 1 | X_0 = i)\, P(X_0 = i)$$

$$= \frac{1}{6}\Big(\lim_{n\to\infty} P(X_n = 1 | X_0 = 1) + \lim_{n\to\infty} P(X_n = 1 | X_0 = 2)$$

$$+ \lim_{n\to\infty} P(X_n = 1 | X_0 = 5)\Big) = \frac{1}{6}(\pi_1 + \pi_1 + h_{5,1})$$

$$h_{5,1} = \lim_{n\to\infty} \sum_{i=0}^{5} P(X_n = 1|X_1 = i, X_0 = 5)\, P(X_1 = i|X_0 = 5)$$

$$= \sum_{i=0}^{5} \lim_{n\to\infty} P(X_n = 1|X_1 = i)\, P(X_1 = i|X_0 = 5)$$

$$= \sum_{i=0}^{5} \lim_{n\to\infty} P(X_n = 1|X_0 = i)\, P(X_1 = i|X_0 = 5)$$

$$= \frac{1}{10} \times 2\pi_1 + \frac{1}{2} h_{5,1}$$

则 $h_{5,1} = \frac{2}{15}, \lim_{n\to\infty} P(X_n = 1) = \frac{2}{15}$

同理 $h_{5,2} = \frac{4}{15}, \lim_{n\to\infty} P(X_n = 2) = \frac{4}{15}$

计算 $\lim_{n\to\infty} P\{X_n \in C_k\}, k = 1,2,3$

记 $h_{i,k} = \lim\limits_{n\to\infty} P\{X_n \in C_k | X_0 = i\}, k = 1,2,3, i = 0,1,\cdots,5$

$h_{i,1}$: $h_{0,1} = 1, h_{1,1} = h_{2,1} = h_{3,1} = h_{4,1} = 0$

$$h_{5,1} = \frac{1}{6}\sum_{i=0}^{5} h_{i,1} = = \frac{1}{6}(1 + h_{5,1})$$

$\therefore \quad h_{5,1} = \frac{1}{5}$

$$\lim_{n\to\infty} P\{X_n \in C_1\} = \sum_{i=0}^{5} P\{X_0 = i\} h_{i,1} = = \frac{1}{6}(1 + h_{5,1}) = \frac{1}{5}$$

$\mathbf{h_{i,2}}$: $h_{1,2} = h_{2,2} = 1, h_{0,2} = h_{3,2} = h_{4,2} = 0$

$$h_{5,2} = \frac{1}{6}\sum_{i=0}^{5} h_{i,2} = = \frac{1}{6}(2 + h_{5,2})$$

$$\therefore \quad h_{5,2} = \frac{2}{5}$$

$$\lim_{n\to\infty} P\{X_n \in C_2\} = \sum_{i=0}^{5} P\{X_0 = i\}h_{i,2} = = \frac{1}{6}(2 + h_{5,2}) = \frac{2}{5}$$

$\mathbf{h_{i,3}}$: $h_{3,3} = h_{4,3} = 1, h_{0,3} = h_{1,3} = h_{2,3} = 0$

$$h_{5,3} = \frac{1}{6}\sum_{i=0}^{5} h_{i,3} = = \frac{1}{6}(2 + h_{5,3})$$

$$\therefore \quad h_{5,3} = \frac{2}{5}$$

$$\lim_{n\to\infty} P\{X_n \in C_3\} = \sum_{i=0}^{5} P\{X_0 = i\}h_{i,3} = = \frac{1}{6}(2 + h_{5,3}) = \frac{2}{5}$$

注：也可以利用这计算 $\lim_{n\to\infty} P(X_n = k)$， $k = 0, \cdots, 4$

$$C_0 = \{5\}$$

例如：$\lim_{n\to\infty} P(X_n = 1) = \lim_{m\to\infty} \lim_{n\to\propto} P(X_{m+n} = 1)$

$$= \lim_{m\to\infty} \lim_{n\to\propto} \sum_{k=0}^{3} P(X_{m+n} = 1 | X_n \in C_k) P(X_n \in C_k)$$

$$= \lim_{m\to\infty} \lim_{n\to\propto} \sum_{k=0}^{3} P(X_m = 1 | X_0 \in C_k) P(X_n \in C_k)$$

$$= \sum_{k=0}^{3} \lim_{m\to\infty} P(X_m = 1 | X_0 \in C_k) \times \lim_{n\to\propto} P(X_n \in C_k) = \frac{2}{15}$$