第三次作业: 第三章 8、10、12、16、17、19

8、

设 $\{X_n\}$ 是一时齐马尔可夫链，状态空间为 $\{0,1,2\}$，一步转移矩阵为

$$P=\begin{bmatrix}\frac{1}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0\end{bmatrix}$$

设 $P(X_0=0)=P(X_0=1)=P(X_0)=2=\frac{1}{3}$.

(1) 计算 $P(X_2=0|X_0=0), P(X_0=0|X-2=0)$;

(2) 计算 $P(X_1=0), P(X_1=0,X_3=0,X_4=1,X_6=1)$;

(3) 计算 $f_{11}^{(n)}, f_{11}, \mu_1$

解:

(1)

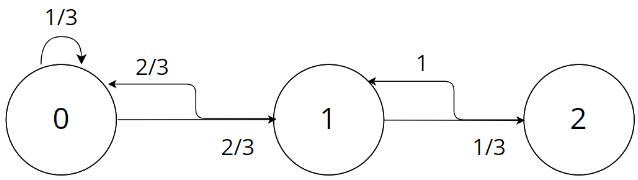
$$\begin{aligned}
&P(X_0=0)=\frac{1}{3}\\
&P(X_0=0,X_2=0)=\frac{1}{3}\times(\frac{1}{3}\times\frac{1}{3}+\frac{2}{3}\times\frac{2}{3}+0)=\frac{5}{27}\\
&P(X_2=0|X_0=0)=\frac{P(X_0=0,X_2=0)}{P(X_0=0)}=\frac{5}{9}\\
&P^2=\begin{bmatrix}\frac{5}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{7}{9} & 0 \\ \frac{2}{3} & 0 & \frac{1}{3}\end{bmatrix}\\
&P(X_2=0)=P(X_0=0)p_{00}^2+P(X_0=1)p_{10}^2+P(X_0=2)p_{20}^2\\
&\qquad\qquad=\frac{1}{3}\times\frac{5}{9}+\frac{1}{3}\times\frac{2}{9}+\frac{1}{3}\times\frac{2}{3}=\frac{13}{27}\\
&P(X_0=0|X_2=0)=\frac{P(X_2=0,X_0=0)}{X_2=0}=\frac{5}{13}
\end{aligned}$$

(2)

$$\begin{aligned}
&P(X_1=0)=\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{2}{3}=\frac{1}{3}\\
&P(X_1=0,X_3=0,X_4=1,X_6=1)=\frac{1}{3}\times\frac{5}{9}\times\frac{2}{3}\times\frac{7}{9}=\frac{70}{729}
\end{aligned}$$

(3)

状态图如下所示:



$$f_{11}^{(1)} = 0$$

$$f_{11}^{(2)} = \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times 1 = \frac{7}{9}$$

$$f_{11}^{(n)} = \frac{2}{3} \times \frac{2}{3} \times (\frac{1}{3})^{n-2} = 4(\frac{1}{3})^{n}, n \geqslant 3$$

$$f_{11} = \frac{7}{9} + \sum_{n=3}^{+\infty} 4(\frac{1}{3})^{n} = \frac{7}{9} + \frac{4/27}{1-1/3} = 1$$

$$\mu_1 = \frac{14}{9} + \sum_{n=3}^{+\infty} 4n(\frac{1}{3})^{n} = \frac{14}{9} + \frac{7}{9} = \frac{7}{3}$$

10、

设 $\{X_n\}$ 是一时齐马尔可夫链，状态空间为 $\{0,1,2,3\}$, 一步转移矩阵为

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

设 $P(X_0 = 0) = P(X_0 = 1) = P(X_0 = 3) = \frac{1}{3}$

(1) 计算 $P(X_1 = 1, X_3 = 2), P(X_2 = 1), P(X_{10} = 0)$;

(2) 求出各状态的常返性，并计算正常返态的平均回转时.

解:

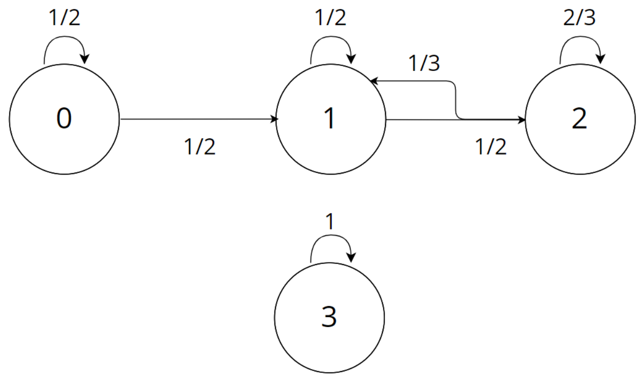
(1)

$$P^2 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{5}{12} & \frac{7}{12} & 0 \\ 0 & \frac{7}{18} & \frac{11}{18} & 0 \\ 0 & \frac{7}{18} & \frac{11}{18} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$P(X_1 = 1, X_3 = 2) = (\frac{1}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}) \times \frac{7}{12} = \frac{7}{36}$$

$P(X_2=1)=\frac{1}{3}\times\frac{1}{2}+\frac{1}{3}\times\frac{5}{12}=\frac{11}{36}$

状态图如下所示：



可知 1、2 为互达等价类，而 0 不封闭，仅有 0 能到达 0

$P(X_{10}=0)=\frac{1}{3}\times(\frac{1}{2})^{10}=\frac{1}{3\cdot 2^{10}}$

(2)

0 暂留;1、2 为互达等价类

$f_{11}^{(1)}=\frac{1}{2}.f_{11}^{(2)}=\frac{1}{6},f_{11}^{(3)}=\frac{1}{9}$

$d(1)=1$,1 为非周期正常返，2 和 1 周期性常返性相同。

对于 1、2 有 $P=\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}$

有平稳分布：$\begin{cases}\pi_1+\pi_2=1\\ \pi_1=\frac{1}{2}\pi_1+\frac{1}{3}\pi_2\end{cases}$

$\pi_1=\frac{2}{5},\pi_2=\frac{3}{5}$

$\mu_1=\frac{5}{2},\mu_2=\frac{5}{3}$

3 为吸收态，非周期正常返，$\mu_3=1$

12、

求第 8 题中 $\{X_n\}$ 的平稳分布.

解:

设平稳分布为 (π_0, π_1, π_2)

$$\begin{cases} \pi_0 + \pi_1 + \pi_2 = 1 \\ \pi_2 = \frac{1}{3}\pi_1 \\ \pi_1 = \frac{2}{3}\pi_0 + \pi_2 \end{cases}$$

$\pi_0 = \frac{3}{7}, \pi_1 = \frac{3}{7}, \pi_2 = \frac{1}{7}$

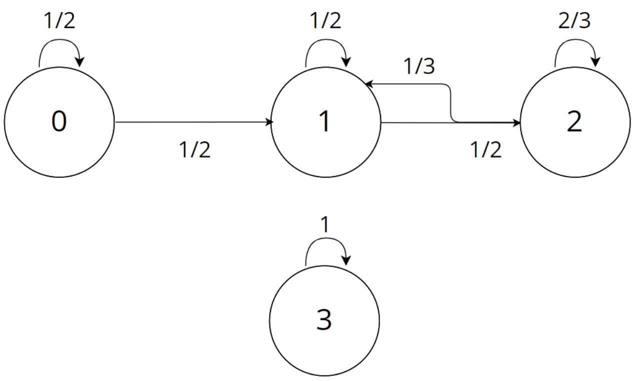
平稳分布为 $(\frac{3}{7}, \frac{3}{7}, \frac{1}{7})$

16、

在第 10 题中对 $i = 0, 1, 2, 3$, 计算 $\lim_{n\to\infty} P(X_n = i)$

解:

状态图如下所示:



$i = 0$,0 为暂留态，有 $\lim_{n\to\infty} P(X_n = 0) = 0$

$i = 3$,3 为非周期正常返，且仅有 3 能进入 3 态，$\lim_{n\to\infty}(X_n = 3) = P(X_0 = 3) = \frac{1}{3}$

$i = 1$, 1 为非周期正常返，有 $\lim_{n\to\infty}(X_n = 1) = \lim_{n\to\infty} \sum_{i=1}^{3} P(X_n = 1|X_0 = i)P(X_0 = i)$

$= \frac{1}{3} \lim_{n\to\infty} P(X_n = 1|X_0 = 0) + \frac{1}{3} \lim_{n\to\infty} P(X_n = 1|X_0 = 1) = \frac{1}{3}\pi_1 + \frac{1}{3}\pi_1 = \frac{4}{15}$

$i = 2$, 2 为非周期正常返

$\lim_{n\to\infty} P(X_n = 2) = \lim_{n\to\infty} \sum_{i=1}^{3} P(X_n = 2|X_0 = i)P(X_0 = i)$

$= \frac{1}{3} \lim_{n\to\infty} P(X_n = 1|X_0 = 0) + \frac{1}{3} \lim_{n\to\infty} P(X_n = 1|X_0 = 1) = \frac{2}{5}$

17、

设 $\{X_n\}$ 是一时齐马尔可夫链，状态空间为 $\{0,1,2,3,4,5,6,7\}$，一步转移概率为 $p_{01}=p_{32}=p_{67}=1, p_{10}=p_{12}=p_{21}=p_{23}=p_{54}=p_{56}=p_{76}=p_{77}=0.5, p_{43}=p_{44}=p_{45}=\dfrac{1}{3}$.

(1) 写出所有互达等价类，并判断哪些是闭的?

(2) 求出各状态的周期和常返性，并计算正常返态的平均回转时;

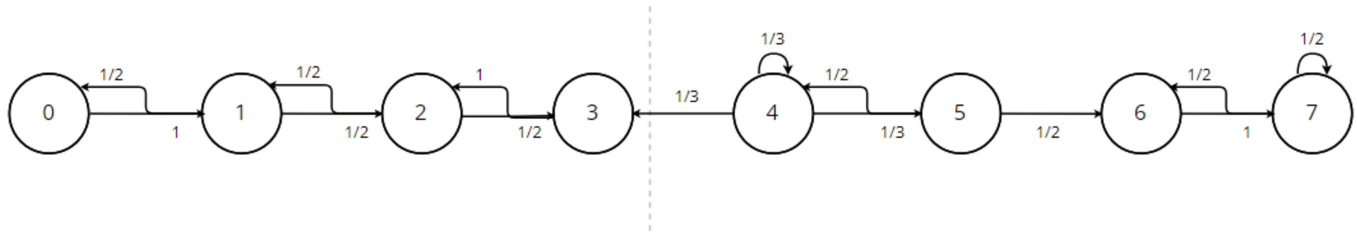
(3) 计算 $\lim\limits_{n\to\infty} p_{45}^{(n)}$, $\lim\limits_{n\to\infty} p_{67}^{(n)}$;

(4) 若 $P(X_0=3)=P(X_0=4)=\dfrac{1}{2}$, 对 $i=4,5,6,7$, 计算 $\lim\limits_{n\to\infty} P(X_n=i)$.

解:

(1)

状态图如下所示:



$\{0,1,2,3\}$ $\{4,5\}$ $\{6,7\}$ 为互达等价类，且 $\{0,1,2,3\}$ $\{6,7\}$ 为闭等价类

(2)

对于 0 而言，$\{0,1,2,3\}$ 有限且闭，则 0 正常返

从状态转换图可以看出从 0 态回到 0 态必须经过偶数步，$d(0)=2$, 且 1、2、3 态周期性常返性与 0 相同。

对于 4、5 而言，非闭等价类，暂留

对于 7 而言，有限闭等价类，7 正常返

从状态转换图可以看出 7 存在自循环，所以 $d(7)=1$, 且 6 周期性常返性和 7 相同

对正常返求平均回转时

$\{0,1,2,3\}$ 平稳分布有

$$(\pi_0,\pi_1,\pi_2,\pi_3)=(\pi_0,\pi_1,\pi_2,\pi_3)\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$\pi_0=\dfrac{1}{6}, \pi_1=\dfrac{1}{3}, \pi_2=\dfrac{1}{3}, \pi_3=\dfrac{1}{6}$

$\mu_0=6, \mu_1=3, \mu_2=3\mu_3=6$

$\{6,7\}$ 平稳分布有 $(\pi_6,\pi_7)=(\pi_6,\pi_7)\begin{bmatrix}0 & 1\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$

$\pi_6=\frac{1}{3},\pi_7=\frac{2}{3},\mu_6=3,\mu_7=\frac{3}{2}$

(3)

4、5 暂留，所以 $\lim\limits_{n\to\infty}p_{45}^{(n)}=0$

6、7 非周期正常返，所以 $\lim\limits_{n\to\infty}p_{67}^{n}=\pi_7=\frac{2}{3}$

(4)

i=0,(利用“走一步”)

$\lim\limits_{n\to\infty}P(X_n=0)=\frac{1}{2}\lim\limits_{n\to\infty}P(X_n=0|X_0=3)+\frac{1}{2}\lim\limits_{n\to\infty}P(X_n=0|X_0=4)$

$\lim\limits_{n\to\infty}P(X_n=0,X_0=4)=\frac{1}{3}h_{40}+\frac{1}{3}h_{30}+\frac{1}{3}h_{50}$

$h_{50}=\frac{1}{2}h_{40}+\frac{1}{2}h_{60}$

$h_{40}=\frac{1}{9}$

$\lim\limits_{n\to\infty}P(X_n=0)=\frac{1}{2}\pi_0+\frac{1}{2}h_{40}=\frac{5}{36}$

其他情况同理 (敲不动了)

i=1

$\lim\limits_{n\to\infty}P(X_n=1)=\frac{1}{2}\pi_3+\frac{1}{2}h_{41}=\frac{5}{18}$

i=2

$\lim\limits_{n\to\infty}P(X_n=2)=\lim\limits_{n\to\infty}P(X_n=1)=\frac{5}{18}$

i=3

$\lim\limits_{n\to\infty}P(X_n=3)=\frac{5}{36}$

i=4、5 暂留

$\lim\limits_{n\to\infty}P(X_n=4)=\lim\limits_{n\to\infty}P(X_n=5)=0$

i=6

$\lim\limits_{n\to\infty}P(X_n=6)=\frac{1}{2}h_{46}=\frac{1}{18}$

i=7

$\lim\limits_{n\to\infty}P(X_n=7)=\frac{1}{2}h_{47}=\frac{1}{9}$

19、

设 $\{X_n; n=0,1,2\cdots\}$ 是一时齐马尔可夫链，状态空间 $I=\{1,2,3,4\}$, 一步转移矩阵

$$P=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{bmatrix}$$

令 $T_1 = inf\{n \geqslant 0; X_n = 1\}$, 计算 $P(T_1 < \infty | X_0 = 3)$

解:

$$P(T_1<\infty|X_0=3)=\lim_{n\to\infty}P(X_n=1|X_0=3)$$
$$=\frac{1}{4}+\frac{1}{4}h_{31}+\frac{1}{4}h_{41}$$
$$\begin{cases} h_{31}=\frac{1}{4}+\frac{1}{4}h_{31}+\frac{1}{4}h_{41} \\ h_{41}=\frac{1}{8}+\frac{3}{8}h_{31}+\frac{1}{8}h_{41} \end{cases}$$
$$h_{31}=\frac{4}{9}, h_{41}=\frac{1}{3}$$
$$P(T_1<\infty|X_0=3)=\lim_{n\to\infty}P(X_n=1|X-0=3)=\frac{1}{4}+\frac{1}{9}+\frac{1}{12}=\frac{4}{9}$$