C2-4

1. (1) Prove that in the infinite square well, the wave function ψ_n satisfy the orthogonal condition

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \delta_{mn}$$

and write down the expansion formula for an arbitrary function *f*(*x*) (text book* Page 51).

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \frac{2}{a}\int_0^a \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{a}x\right)dx$$

$$= \frac{2}{a}\int_0^a \cos\left(\frac{m-n}{a}\pi x\right) - \cos\left(\frac{m+n}{a}\pi x\right)dx$$

$$= \left\{\frac{1}{(m-n)\pi}\sin\left(\frac{m-n}{a}\pi x\right) - \frac{1}{(m+n)\pi}\sin\left(\frac{m+n}{a}\pi x\right)\right\}\Bigg|_0^a$$

$$= \frac{1}{\pi}\left\{\frac{\sin\big((m-n)\pi\big)}{(m-n)} - \frac{\sin\big((m+n)\pi\big)}{(m+n)}\right\}$$

If m=n, $\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = 1$

If m≠n, $\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = 0$

推导出正确结果给 10 分，只有推导或者只有结果给 5 分

(2) <text book* Problem 2.37>

A particle in the infinite square well has the initial wave function

$$\Psi(x,0) = A\sin^3(\pi x/a) \qquad (0 \le x \le a).$$

Determine A, find $\Psi(x,t)$, and calculate $\langle x\rangle$, as a function of time. What is the expectation value of the energy? *Hint:* $\sin^n\theta$ and $\cos^n\theta$ can be reduced, by repeated application of the trigonometric sum formulas, to linear combinations of $\sin(m\theta)$ and $\cos(m\theta)$, with $m = 0, 1, 2, \ldots, n$.

$$\sin 3\theta = \sin\theta\cos 2\theta + \sin 2\theta\cos\theta = \sin\theta\,(1 - 2\sin^2\theta) + 2\sin\theta\,(1-\sin^2\theta)$$
$$= 3\sin\theta - 4\sin^3\theta$$

$$\psi_n(x) = \sqrt{\frac{a}{2}}\sin\left(\frac{n\pi x}{a}\right)$$

$$\Psi(x,0) = A\sin^3\left(\frac{\pi x}{a}\right) = A\left[\frac{3}{4}\sin\left(\frac{\pi x}{a}\right) - \frac{1}{4}\sin\left(\frac{3\pi x}{a}\right)\right]$$

$$= A\sqrt{\frac{a}{2}}\left[\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right]$$

$$\int_0^a |\Psi(x,0)|^2 dx = |A|^2\frac{a}{2}\int_0^a \left|\frac{3}{4}\psi_1(x) - \frac{1}{4}\psi_3(x)\right|^2 dx = |A|^2\frac{a}{2}\left(\frac{9}{16}+\frac{1}{16}\right) = 1$$

$$\therefore A = \sqrt{\frac{16}{5a}}$$

推导出 A 的正确结果给 10 分，只有推导或者只有结果给 5 分

$$\Psi(x,0) = \frac{1}{\sqrt{10}}[3\psi_1(x) - \psi_3(x)]$$

$$\Psi(x,t) = \frac{1}{\sqrt{10}}\left[3\psi_1(x)e^{-iE_1t/\hbar} - \psi_3(x)e^{-iE_3t/\hbar}\right]$$

$$\text{其中 } E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

推导出$\Psi(x,t)$的正确结果给 10 分，只有推导或者只有结果给 5 分

$$\langle x\rangle = \int_0^a x|\Psi(x,t)|^2dx = \frac{9}{10}\int_0^a x\,\psi_1^2dx + \frac{1}{10}\int_0^a x\,\psi_3^2dx - \frac{3}{5}\cos(\omega t)\int_0^a x\,\psi_1\psi_3dx$$
$$= \frac{9}{10}\langle x\rangle_1 + \frac{1}{10}\langle x\rangle_3 - \frac{3}{5}\cos(\omega t)\int_0^a x\,\psi_1\psi_3dx$$

$$\langle x\rangle_n = \int_0^a x|\psi_n(x)|^2dx = \frac{a}{2}$$
$$\int_0^a x\,\psi_1\psi_3dx = \frac{2}{a}\int_0^a x\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi x}{a}\right)dx$$
$$= \frac{2}{a}\int_0^a x\left[\cos\left(\frac{2\pi x}{a}\right) - \cos\left(\frac{4\pi x}{a}\right)\right]dx = 0$$
$$\therefore\ \langle x\rangle = \frac{9}{10}\frac{a}{2} + \frac{1}{10}\frac{a}{2} - 0 = \frac{a}{2}$$

推导出$\langle x\rangle$的正确结果给 10 分，只有推导或者只有结果给 5 分

2. Prove that for wave functions ψ, ϕ and hermitian operator A, the following two conditions hold.

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle^*$$

$$|\psi\rangle = \begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\\ \cdots\\ \psi_N\end{bmatrix}$$

$$\langle\psi| = [\psi_1^*\quad \psi_2^*\quad \psi_3^*\quad \psi_4^*\quad \ldots\quad \psi_N^*]$$

$$|\varphi\rangle = \begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\\ \cdots\\ \varphi_N\end{bmatrix}$$

$$\langle\varphi| = [\varphi_1^*\quad \varphi_2^*\quad \varphi_3^*\quad \varphi_4^*\quad \ldots\quad \varphi_N^*]$$

$$\langle\psi|\phi\rangle = \psi_1^*\varphi_1 + \psi_2^*\varphi_2 + \psi_3^*\varphi_3 + \cdots + \psi_N^*\varphi_N$$
$$= (\varphi_1^*\psi_1 + \varphi_2^*\psi_2 + \varphi_3^*\psi_3 + \cdots + \varphi_N^*\psi_N)^* = \langle\phi|\psi\rangle^*$$

给出证明过程正确给 10 分，只有部分推导给 5 分

$$\langle\psi|A^+|\phi\rangle = (\langle\phi|A|\psi\rangle)^+ = \langle\phi|A|\psi\rangle^*$$

给出证明过程正确给 10 分，只有部分推导给 5 分

3. (Ref to text book* Problem 3.39)
Find the matrix elements $\langle n|x|n'\rangle$ and $\langle n|p|n'\rangle$ in the orthonormal basis of stationary states for the harmonic oscillator $|n\rangle \equiv \psi_n(x)$. Construct the corresponding matrix a_+ and a_-, and construct the corresponding matrix $\hat{n}$ from the matrix a_+ and a_-.

$$a_+|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a_-|n\rangle = \sqrt{n}|n-1\rangle$$
$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$$
$$p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$\langle n|x|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|(a_+ + a_-)|n'\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle n|a_+|n'\rangle + \sqrt{\frac{\hbar}{2m\omega}}\langle n|a_-|n'\rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta(n,n'+1) + \sqrt{n'}\delta(n,n'-1)\right)$$

推导出$\langle n|x|n'\rangle$正确结果给 10 分，只有推导或者只有结果给 5 分

$$\langle n|p|n'\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\langle n|(a_+ - a_-)|n'\rangle = \mathrm{i}\sqrt{\frac{\hbar m\omega}{2}}\langle n|a_+|n'\rangle - i\sqrt{\frac{\hbar m\omega}{2}}\langle n|a_-|n'\rangle$$
$$= i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n}\delta(n,n'+1) - \sqrt{n'}\delta(n,n'-1)\right)$$

推导出$\langle n|p|n'\rangle$正确结果给 10 分，只有推导或者只有结果给 5 分

$$X = \sqrt{\frac{\hbar}{2m\omega}}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \\ 1 & 0 & \sqrt{2} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \\ & & \cdots & & & \end{bmatrix}$$

$$P = i\sqrt{\frac{\hbar m\omega}{2}}\begin{bmatrix} 0 & -1 & 0 & 0 & 0 & \\ 1 & 0 & -\sqrt{2} & 0 & 0 & \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \\ & & \cdots & & & \end{bmatrix}$$

$$A_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-iP + m\omega X) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0 & \\ 0 & \sqrt{2} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & 0 & \\ 0 & 0 & 0 & \sqrt{4} & 0 & \\ & & \cdots & & & \end{bmatrix}$$

$$A_- = \frac{1}{\sqrt{2\hbar m\omega}}(iP + m\omega X) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & \sqrt{2} & 0 & 0 & \\ 0 & 0 & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & 0 & 0 & \sqrt{4} & \\ 0 & 0 & 0 & 0 & 0 & \\ & & \cdots & & & \end{bmatrix}$$

推导出A_+和A_-正确结果给 10 分，只有推导或者只有结果给 5 分，只推出一项给 5 分

$$\hat{n} = A_+ A_- = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \\ 0 & 0 & 0 & 0 & 4 & \\ & & \cdots & & & \end{bmatrix}$$

推导出$\hat{n}$正确结果给 10 分，只有推导或者只有结果给 5 分

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).