《量子信息基础》2022.3.29 随堂作业：

1. In a quantum system, the Eigen energy and wavefunctions are $E_n, \psi_n, n = 0,1,2,\cdots$. When $t \le 0$, the system is in the ground state of E_0, ψ_0. A perturbation occurs for $t \ge 0$, which is $H'(t) = Fe^{-t/\tau}$. Calculate the probabilities for the system evolves into the state of E_n, ψ_n when $t \ge 0$.

Let's define

$$F_{n0} = \langle \psi_n | F(x) | \psi_0 \rangle$$

$$c_n^{(1)}(t) = -\frac{i}{\hbar}\int_0^t H'_{n0}(t')e^{i\omega_{n0}t'}dt' = -\frac{i}{\hbar}\int_0^t F_{n0}e^{-\frac{t'}{\tau}}e^{i\omega_{n0}t'}dt' = -\frac{iF_{n0}}{\hbar}\int_0^t e^{i\omega_{n0}t'-\frac{t'}{\tau}}dt'$$

$$= -\frac{iF_{n0}}{\hbar}\int_0^t e^{i\omega_{n0}t'-\frac{t'}{\tau}}dt' = \frac{F_{n0}\left(e^{i\omega_{n0}t-t/\tau}-1\right)}{-\hbar\omega_{n0}-i\hbar/\tau}$$

$$\left|c_n^{(1)}(t)\right|^2 = \frac{|F_{n0}|^2\left|e^{i\omega_{n0}t-\frac{t}{\tau}}-1\right|^2}{(\hbar\omega_{n0})^2+(\hbar/\tau)^2}$$

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$$t \gg \tau$$

$$\left|c_n^{(1)}(t)\right|^2 = \frac{|F_{n0}|^2}{(\hbar\omega_{n0})^2+(\hbar/\tau)^2}$$

2. (Text book* Problem 11.3)
Solve Equation 11.17 for the case of a *time-independent* perturbation, assuming that $c_a(0) = 1$ and $c_b(0) = 0$. Check that $|c_a(t)|^2 + |c_b(t)|^2 = 1$. Comment: Ostensibly, this system oscillates between "pure ψ_a" and "some ψ_b". Doesn't this contradict my general assertion that no transitions occur for time-independent perturbations? No, but the reason is rather subtle: In this case ψ_a and ψ_b are not, and never were, eigenstates of the Hamiltonian—a measurement of the energy *never* yields E_a or E_b. In time-dependent perturbation theory we typically contemplate turning *on* the perturbation for a while, and then turning it *off* again, in order to examine the system. At the beginning, and at the end, ψ_a and ψ_b are eigenstates of the exact Hamiltonian, and only in this context does it make sense to say that the system underwent a transition from one to the other. For the present problem, then, assume that the perturbation was turned on at time $t = 0$, and off again at time t—this doesn't affect the *calculations*, but it allows for a more sensible interpretation of the result.

$$\boxed{\begin{cases} \dot{c}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} c_b \\ \dot{c}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} c_a \end{cases}}$$

$$\ddot{c}_a = -i\omega_0 \dot{c}_a - \frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} \dot{c}_b = -i\omega_0 \dot{c}_a - \frac{1}{\hbar^2} H'_{ab} H'_{ba} c_a$$

$$\ddot{c}_a + i\omega_0 \dot{c}_a + \frac{1}{\hbar^2} H'_{ab} H'_{ba} c_a = 0$$

Assume

$$c_a = e^{\lambda t}$$

We get

$$\lambda^2 + i\omega_0 \lambda + \frac{|H'_{ab}|^2}{\hbar^2} = 0$$

$$\lambda = \frac{1}{2}\left(-i\omega_0 \pm i\sqrt{-\omega_0^2 - \frac{4}{\hbar^2}\left|H'_{ab}\right|^2}\right) = \frac{i}{2}(-\omega_0 \pm \omega)$$

where

$$\omega \equiv \sqrt{-\omega_0^2 - \frac{4}{\hbar^2}\left|H'_{ab}\right|^2}$$

The general solution is

$$c_a(t) = C_1 e^{-\frac{i}{2}(\omega_0 - \omega)t} + C_2 e^{-\frac{i}{2}(\omega_0 + \omega)t} = e^{-\frac{i}{2}\omega_0 t}\left[C_3 \cos\left(\frac{\omega t}{2}\right) + C_4 \sin\left(\frac{\omega t}{2}\right)\right]$$

$$\because \ c_a(0) = 1, \therefore C_3 = 1$$

$$c_a(t) = e^{-\frac{i}{2}\omega_0 t}\left[\cos\left(\frac{\omega t}{2}\right) + C_4 \sin\left(\frac{\omega t}{2}\right)\right]$$

$$\begin{aligned} \dot{c}_a(t) &= -\frac{i\omega_0}{2} e^{-\frac{i}{2}\omega_0 t}\left[\cos\left(\frac{\omega t}{2}\right) + C_4 \sin\left(\frac{\omega t}{2}\right)\right] + e^{-\frac{i}{2}\omega_0 t}\left[-\frac{\omega}{2}\sin\left(\frac{\omega t}{2}\right) + C_4 \frac{\omega}{2}\cos\left(\frac{\omega t}{2}\right)\right] \\ &= e^{-\frac{i}{2}\omega_0 t}\left[\left(C_4 \frac{\omega}{2} - \frac{i\omega_0}{2}\right)\cos\left(\frac{\omega t}{2}\right) - \left(C_4 \frac{i\omega_0}{2} + \frac{\omega}{2}\right)\sin\left(\frac{\omega t}{2}\right)\right] \end{aligned}$$

$$c_b(t) = -\frac{\hbar}{iH'_{ab}} e^{i\omega_0 t} \dot{c}_a = -\frac{\hbar}{iH'_{ab}} e^{\frac{i}{2}\omega_0 t}\left[\left(C_4 \frac{\omega}{2} - \frac{i\omega_0}{2}\right)\cos\left(\frac{\omega t}{2}\right) - \left(C_4 \frac{i\omega_0}{2} + \frac{\omega}{2}\right)\sin\left(\frac{\omega t}{2}\right)\right]$$

$$\because \ c_b(0) = 1, \therefore C_4 = \frac{i\omega_0}{\omega}$$

$$c_a(t) = e^{-\frac{i}{2}\omega_0 t}\left[\cos\left(\frac{\omega t}{2}\right) + \frac{i\omega_0}{\omega}\sin\left(\frac{\omega t}{2}\right)\right]$$

$$c_b(t) = \frac{\hbar}{iH'_{ab}} e^{\frac{i}{2}\omega_0 t}\left(-\frac{\omega_0^2}{2\omega} + \frac{\omega}{2}\right)\sin\left(\frac{\omega t}{2}\right) = \frac{2H'_{ba}}{i\hbar\omega} e^{\frac{i}{2}\omega_0 t} \sin\left(\frac{\omega t}{2}\right)$$

$$|c_a(t)|^2 + |c_b(t)|^2 = \cos^2\left(\frac{\omega t}{2}\right) + \left(\frac{\omega_0}{\omega}\right)^2 \sin^2\left(\frac{\omega t}{2}\right) + \frac{4|H'_{ab}|^2}{\hbar^2\omega^2}\sin^2\left(\frac{\omega t}{2}\right)$$
$$= \cos^2\left(\frac{\omega t}{2}\right) + \left(\frac{\omega_0}{\omega}\right)^2 + \frac{\omega^2 - \omega_0^2}{\omega^2}\sin^2\left(\frac{\omega t}{2}\right) = 1$$

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* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).