# 浙江大学 2019-2020 学年春夏学期

## 《微积分(甲)II》课程期末考试试卷(A卷)

(10分) 把 $z$ 看成自变量为 $r,\theta$ 的函数试将方程 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0,(x,y)\neq(0,0)$ 变换为极坐标 $(r,\theta)$ 的形式.

解: $dz=\frac{\partial z}{\partial r}dr+\frac{\partial z}{\partial \theta}d\theta=\frac{\partial z}{\partial r}\cdot\frac{xdx+ydy}{r}+\frac{\partial z}{\partial \theta}\cdot\frac{xdy-ydx}{x^2+y^2}$

$=\frac{1}{r}\left(x\frac{\partial z}{\partial r}-\frac{y}{r}\frac{\partial z}{\partial \theta}\right)dx+\frac{1}{r}\left(y\frac{\partial z}{\partial r}+\frac{x}{r}\frac{\partial z}{\partial \theta}\right)dy$

$\therefore \frac{\partial z}{\partial x}=\frac{1}{r}\left(x\frac{\partial z}{\partial r}-\frac{y}{r}\frac{\partial z}{\partial \theta}\right)$

$\frac{\partial z}{\partial y}=\frac{1}{r}\left(y\frac{\partial z}{\partial r}+\frac{x}{r}\frac{\partial z}{\partial \theta}\right)$

$\therefore x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{r}\left((x^2+y^2)\frac{\partial z}{\partial r}\right)=\frac{\partial z}{\partial r}=0$ 即 $\frac{\partial z}{\partial r}=0$

| 得分 |
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一. 以下各题必须写出解题过程.

1. (7分) 设 $\alpha\in(-1,0)$, 试求幂级数 $\sum_{n=1}^{+\infty}(-1)^n\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{(2^n+3^n)n!}x^n$ 的收敛半径.

解: $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|=\lim_{n\to\infty}\frac{(2^{n+1}+3^{n+1})(n+1)}{(2^n+3^n)(\alpha-n)}=3\cdot\lim_{n\to\infty}\frac{((\frac{2}{3})^{n+1}+1)(n+1)}{((\frac{2}{3})^n+1)(n-\alpha)}$

$=3$

$\therefore R=3$

2. (8分) 试将函数 $f(x)=\arctan x,x\in(-1,1)$ 展开成幂级数 $\sum_{n=0}^{+\infty}a_nx^n$.

解: $f'(x)=\frac{1}{1+x^2}=\sum_{n=0}^{\infty}(-1)^nx^{2n}$ . $f(0)=0$

$\because f(x)=f(0)+\int_0^x f'(x)dx=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$

$x\in(-1,1)$

| 得分 |
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二. 以下各题必须写出解题过程.

3. (10分) 设 $f(x,y)=x^{\frac{2}{3}}y^{\frac{1}{3}}$, 试求 $f$ 在点 $(0,0)$ 处沿方向 $\vec{l}=(\cos\alpha,\sin\alpha)$ (其中 $\alpha\in[0,2\pi)$ )的方向导数 $\frac{\partial f}{\partial l}(0,0)$; 并问当 $\sin\alpha$ 取何值时, 该方向导数 $\frac{\partial f}{\partial l}(0,0)$ 取到最大值?

解: $f_l'(0,0)=\lim_{\rho\to0}\frac{f(\rho\cos\theta,\rho\sin\theta)-f(0,0)}{\rho}=\lim_{\rho\to0}\frac{\rho\cos^{\frac{2}{3}}\theta\sin^{\frac{1}{3}}\theta}{\rho}$

$=\cos^{\frac{2}{3}}\theta\sin^{\frac{1}{3}}\theta=((1-\sin^2\theta)\sin\theta)^{\frac{1}{3}}$

$\therefore \sin\theta=\frac{\sqrt{3}}{3}$ 时 $f_l'(0,0)$ 取到最大值

(10分) 利用傅里叶级数的理论证明: $\forall x\in(0,2\pi),\sum_{k=1}^{+\infty}\frac{\sin(kx)}{k}=\frac{\pi-x}{2}$.

解. $\frac{\pi-x}{2}$, $x\in(0,2\pi)$ 延拓 $a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\cos nx\,dx$

$\overset{x\to\pi-x}{=}\frac{1}{2\pi}\int_{-\pi}^{\pi}x(-1)^n\cos nx\,dx=0$

$b_n=\frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\sin nx\,dx=\frac{1}{2\pi}\int_{-\pi}^{\pi}x(-1)^{n-1}\sin nx\,dx=\frac{(-1)^{n-1}}{\pi}\int_0^{\pi}x\sin nx\,dx$

$=\frac{(-1)^{n-1}}{\pi}\left[-\frac{x\cos nx}{n}+\frac{\sin nx}{n^2}\right]_0^{\pi}$

$=\frac{1}{n}$

$\therefore \frac{\pi-x}{2}=\sum_{n=1}^{\infty}\frac{\sin nx}{n}$, $x\in(0,2\pi)$

4. (10分) 设 $f(x,y)=\begin{cases}\frac{xy^3}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$ 试求 $\frac{\partial f}{\partial x}(x,y)$, $\frac{\partial^2 f}{\partial x\partial y}(0,0)$.

$f(x,0)=0$ $\therefore f_x'(0,0)=0$

$(x,y)\neq(0,0)$ 时

$\frac{\partial f}{\partial x}=\frac{y^3(x^2+y^2)-xy^3(2x)}{(x^2+y^2)^2}=\frac{y^3(y^2-x^2)}{(x^2+y^2)^2}$

$\therefore \frac{\partial f}{\partial x}=\begin{cases}\frac{y^3(y^2-x^2)}{(x^2+y^2)^2}, & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$

又 $\frac{\partial f}{\partial x}(0,y)=y$ $\therefore \frac{\partial^2 f}{\partial x\partial y}(0,0)=1$