

《量子信息基础》2020.4.2 随堂作业:

1. (Text book* Problem 11.11)

You could derive the spontaneous emission rate (Equation 11.63) without the detour through Einstein's A and B coefficients if you knew the ground state energy density of the electromagnetic field, $\rho_0(\omega)$, for then it would simply be a case of stimulated emission (Equation 11.54). To do this honestly would require quantum electrodynamics, but if you are prepared to believe that the ground state consists of one photon in each classical mode, then the derivation is fairly simple:

- (a) To obtain the classical modes, consider an empty cubical box, of side l , with one corner at the origin. Electromagnetic fields (in vacuum) satisfy the classical wave equation

$$\left(\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2\right) f(x, y, z, t) = 0$$

where f stands for any component of \mathbf{E} or of \mathbf{B} . Show that separation of variables, and the imposition of the boundary condition $f = 0$ on all six surfaces yields the standing wave patterns

$$f_{n_x, n_y, n_z} = A \cos(\omega t) \sin\left(\frac{n_x \pi}{l} x\right) \sin\left(\frac{n_y \pi}{l} y\right) \sin\left(\frac{n_z \pi}{l} z\right)$$

with

$$\omega = \frac{\pi c}{l} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

There are two modes for each triplet of positive integers $(n_x, n_y, n_z = 1, 2, 3, \dots)$, corresponding to the two polarization states.

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Consider the wave function f can be separated into four different variables, e.g.

$$f(x, y, z, t) = X(x)Y(y)Z(z)T(t)$$

Then

$$XYZ \frac{1}{c^2} \frac{\partial^2 T}{\partial t^2} = YZT \frac{\partial^2 X}{\partial x^2} + XZT \frac{\partial^2 Y}{\partial y^2} + XYT \frac{\partial^2 Z}{\partial z^2} \quad (1)$$

Assuming

$$\begin{aligned} X(x) &= A_x \sin(k_x x) + B_x \cos(k_x x) \\ Y(y) &= A_y \sin(k_y y) + B_y \cos(k_y y) \\ Z(z) &= A_z \sin(k_z z) + B_z \cos(k_z z) \\ T(t) &= A_t \sin(\omega t) + B_t \cos(\omega t) \end{aligned}$$

Since $X(0) = Y(0) = Z(0) = 0$

So

$$B_x = B_y = B_z = 0$$

Since $X(l) = Y(l) = Z(l) = 0$

$$\begin{aligned} k_x &= \frac{n_x \pi}{l} \\ k_y &= \frac{n_y \pi}{l} \end{aligned}$$

$$k_z = \frac{n_z \pi}{l}$$

$$(n_x, n_y, n_z = 1, 2, 3, \dots)$$

The equation (1) evolves to be

$$\begin{aligned} A_x A_y A_z A_t \frac{\omega^2}{c^2} \sin(k_x x) \sin(k_y y) \sin(k_z z) \sin(\omega t) \\ + A_x A_y A_z B_t \frac{\omega^2}{c^2} \sin(k_x x) \sin(k_y y) \sin(k_z z) \cos(\omega t) \\ = A_x A_y A_z A_t k_x^2 \sin(k_x x) \sin(k_y y) \sin(k_z z) \sin(\omega t) \\ + A_x A_y A_z A_t k_y^2 \sin(k_x x) \sin(k_y y) \sin(k_z z) \sin(\omega t) \\ + A_x A_y A_z A_t k_z^2 \sin(k_x x) \sin(k_y y) \sin(k_z z) \sin(\omega t) \\ + A_x A_y A_z B_t k_x^2 \sin(k_x x) \sin(k_y y) \sin(k_z z) \cos(\omega t) \\ + A_x A_y A_z B_t k_y^2 \sin(k_x x) \sin(k_y y) \sin(k_z z) \cos(\omega t) \\ + A_x A_y A_z B_t k_z^2 \sin(k_x x) \sin(k_y y) \sin(k_z z) \cos(\omega t) \end{aligned}$$

and

$$\omega^2 = c^2(k_x^2 + k_y^2 + k_z^2) = \frac{\pi^2 c^2}{l^2}(n_x^2 + n_y^2 + n_z^2)$$

$$\omega = \frac{\pi c}{l} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

If the normalization coefficient is A, then

$$f_{n_x, n_y, n_z} = A \cos(\omega t) \sin\left(\frac{n_x \pi}{l} x\right) \sin\left(\frac{n_y \pi}{l} y\right) \sin\left(\frac{n_z \pi}{l} z\right)$$

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(b) The energy of a photon $E = h\nu = \hbar\omega$ is (Equation 4.92), so the energy in the mode (n_x, n_y, n_z) is

$$E_{n_x, n_y, n_z} = 2 \frac{\pi \hbar c}{l} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

What, then, is the total energy per unit volume in the frequency range $d\omega$, if each mode gets one photon? Express your answer in the form.

$$\frac{1}{l^3} dE = \rho_0(\omega) d\omega$$

and read off $\rho_0(\omega)$. Hint: refer to Figure 5.3.

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Considering there are two different polarization in **E** or **B**

$$E_{n_x, n_y, n_z} = 2 \hbar \omega = 2 \frac{\pi \hbar c}{l} \sqrt{n_x^2 + n_y^2 + n_z^2}$$

In the ω -space, each mode has a volume of

$$\frac{\pi^3 c^3}{l^3}$$

and

$$dN = 2 \cdot \frac{1}{8} \cdot 4\pi \omega^2 d\omega \cdot \frac{l^3}{\pi^3 c^3} = \frac{1}{2} \frac{\omega^2 l^3}{\pi^2 c^3} d\omega$$

$$\frac{1}{l^3} dE = \frac{1}{l^3} 2\hbar\omega dN = \frac{\hbar\omega^3}{\pi^2 c^3} d\omega$$

$$\rho_0(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3}$$

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- (c) Use your result, together with Equation 11.54, to obtain the spontaneous emission rate. Compare Equation 11.63.

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$$R_{b \rightarrow a}(t) \cong \frac{\pi}{3\varepsilon_0 \hbar^2} |\mathcal{P}|^2 \rho_0(\omega) = \frac{\pi}{3\varepsilon_0 \hbar^2} |\mathcal{P}|^2 \frac{\hbar\omega^3}{\pi^2 c^3} = \frac{\omega^3}{3\pi\varepsilon_0 \hbar c^3} |\mathcal{P}|^2$$

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2. (Text book* Problem 11.12)

The half-life $t_{1/2}$ of an excited state is the time it would take for half the atoms in a large sample to make a transition. Find the relation between $t_{1/2}$ and τ (the “lifetime” of the state).

$$N(t) = N(0)e^{-t/\tau} = N(0)/2$$

$$e^{-t/\tau} = 1/2$$

$$t = \tau \ln 2$$

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* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).