1. (1) Construct the full analytic equations for the normalized wave functions ψ_2 and ψ_3 of harmonic oscillators. (ψ_0 and ψ_1 are done in example 2.4 in the text book*)

$$\psi_0=\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}exp\left(-\frac{m\omega}{2\hbar}x^2\right)$$
$$a_+=\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}+m\omega x\right)$$

$$\begin{aligned}\psi_1=a_+\psi_0&=\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}+m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left[-\hbar\left(-\frac{m\omega}{2\hbar}\right)2x+m\omega x\right]exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}2m\omega x exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\sqrt{\frac{2m\omega}{\hbar}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}x exp\left(-\frac{m\omega}{2\hbar}x^2\right)\end{aligned}$$

在推导过程中给出a_+，ψ_0(或者ψ_1)的正确形式给 10 分

$$\begin{aligned}\psi_2=\frac{1}{\sqrt2}(a_+)^2\psi_0&=\frac{1}{\sqrt2}\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}+m\omega x\right)\frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}2m\omega x exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\frac{1}{\sqrt2}\frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar\frac{d}{dx}+m\omega x\right)x exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\frac{1}{\sqrt2}\frac{1}{\hbar}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(-\hbar\left(1-x\frac{m\omega}{2\hbar}2x\right)+m\omega x^2\right)exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\frac{1}{\sqrt2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2-1\right)exp\left(-\frac{m\omega}{2\hbar}x^2\right)\end{aligned}$$

写出a_2的推导和正确结果给 10 分，只给出推导或结果给 5 分

$$\begin{aligned}\psi_3=\frac{1}{\sqrt6}(a_+)^3\psi_0&=\frac{1}{\sqrt6}\frac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx}+m\omega x\right)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}x^2-1\right)exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\frac{1}{\sqrt6}\frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{2m\omega}{\hbar}\left(-\hbar\frac{d}{dx}+m\omega x\right)x^2 exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&-\frac{1}{\sqrt6}\frac{1}{\sqrt{2\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac14}\left(-\hbar\frac{d}{dx}+m\omega x\right)exp\left(-\frac{m\omega}{2\hbar}x^2\right)=\\&=\frac{1}{\sqrt3}\frac{1}{\sqrt{\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac14}\frac{m\omega}{\hbar}(-2x\hbar+2m\omega x^3)exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&-\frac{1}{\sqrt3}\frac{1}{\sqrt{\hbar m\omega}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac14}(m\omega x)exp\left(-\frac{m\omega}{2\hbar}x^2\right)\\&=\frac{1}{\sqrt3}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac14}\left(\frac{m\omega}{\hbar}\right)^{1/2}\left(-3x+\frac{2m\omega x^3}{\hbar}\right)exp\left(-\frac{m\omega}{2\hbar}x^2\right)\end{aligned}$$

写出a_3的推导和正确结果给 10 分，只给出推导或结果给 5 分

(2) Prove the orthonormality of the stationary states of the harmonic oscillators (textbook* page 64).

$\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \delta_{mn}$

$$\int_{-\infty}^{\infty} \psi_m^* (a_+ a_-) \psi_n dx = n \int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \int_{-\infty}^{\infty} (a_- \psi_m)^* (a_- \psi_n) dx$$

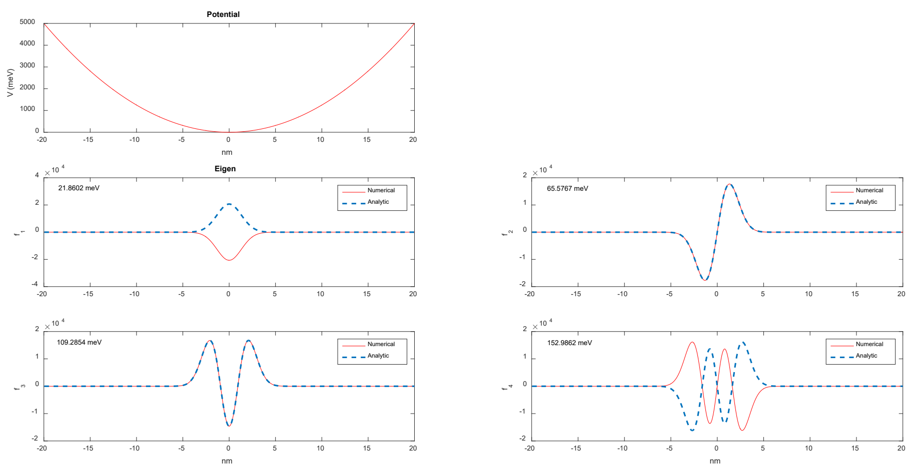
$$= \int_{-\infty}^{\infty} (a_+ a_- \psi_m)^* \psi_n dx = m \int_{-\infty}^{\infty} \psi_m \psi_n dx$$

Unless $m = n$, $\int_{-\infty}^{\infty} \psi_m^* \psi_n dx$ must be zero. Due to normalization condition

$$\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \delta_{mn}$$

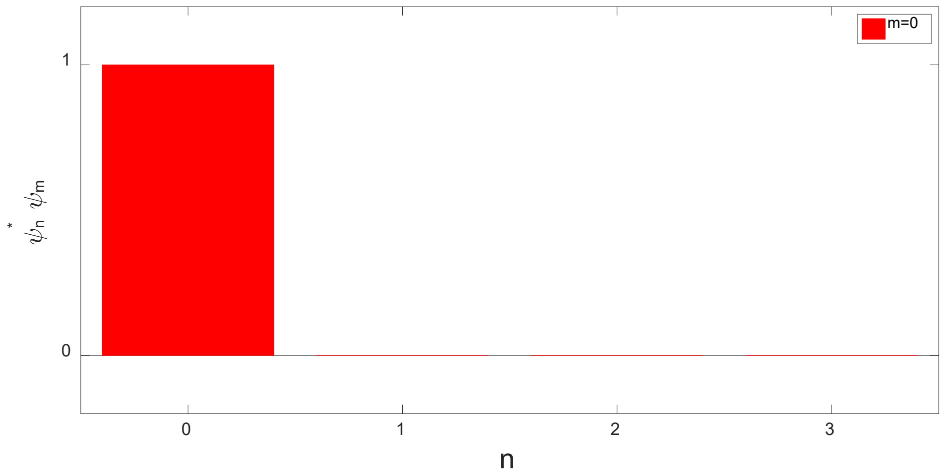
写出推导和分析给 10 分

(3) Use the Matlab code matrix_QM_annotation.m to compare the analytic and numerical results for wavefunctions $\psi_0, \psi_1,\ \psi_2$, and ψ_3 and plot out the results. Hint: First determine the parameter ω based on the coefficient given in line 21 (20^18/625) and then calculate the analytic solutions; *Since the Matlab function eig() does an automatic normalization in the obtained wave function phi(:, n), you need to compare phi(:, 1)/del_x^0.5, phi(:, 2)/del_x^0.5, phi(:, 3)/del_x^0.5, phi(:, 4)/del_x^0.5 to* $\psi_0, \psi_1,\ \psi_2$, *and* ψ_3 *respectively. And there will be a phase difference in the result.*

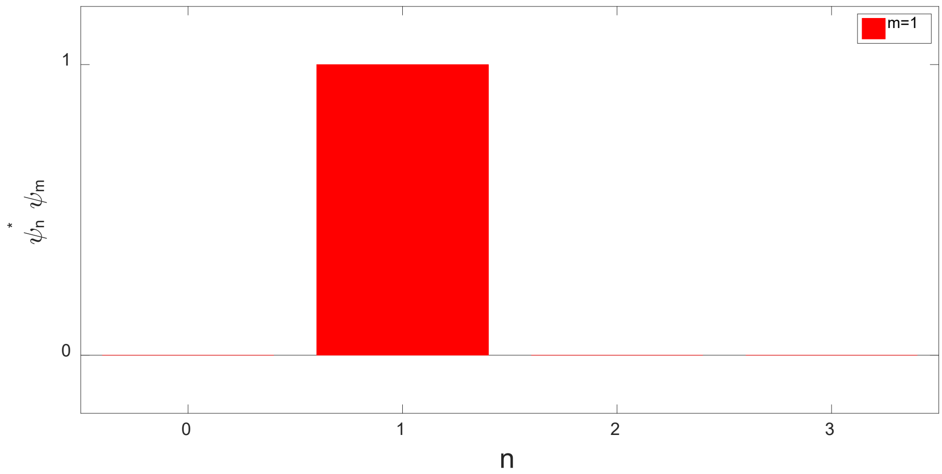


给出结果的比较并符合(仅有相位差距)的给 20 分。强度有差距曲线形状一致的扣 5 分，每一个形状不符合的曲线扣 5 分。

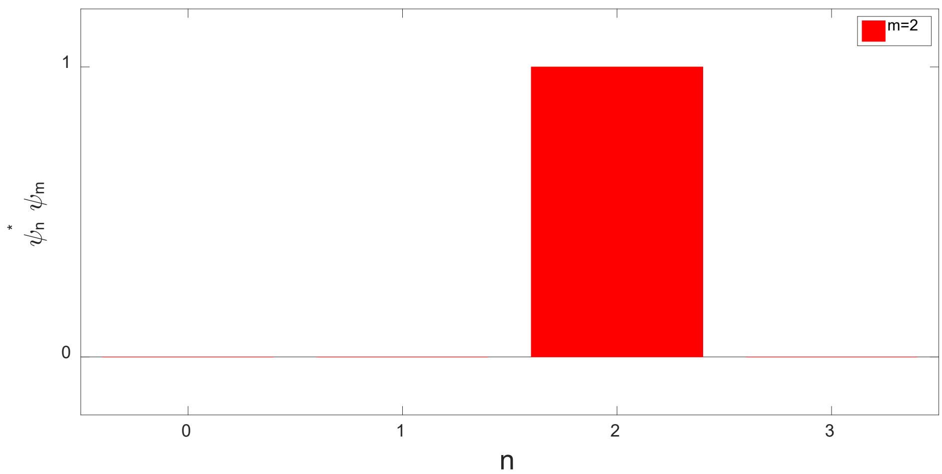
(4) Use the Matlab code eigenfunction.m to test the orthonormality condition between $\psi_0, \psi_1,\ \psi_2$, and ψ_3 and plot out results.



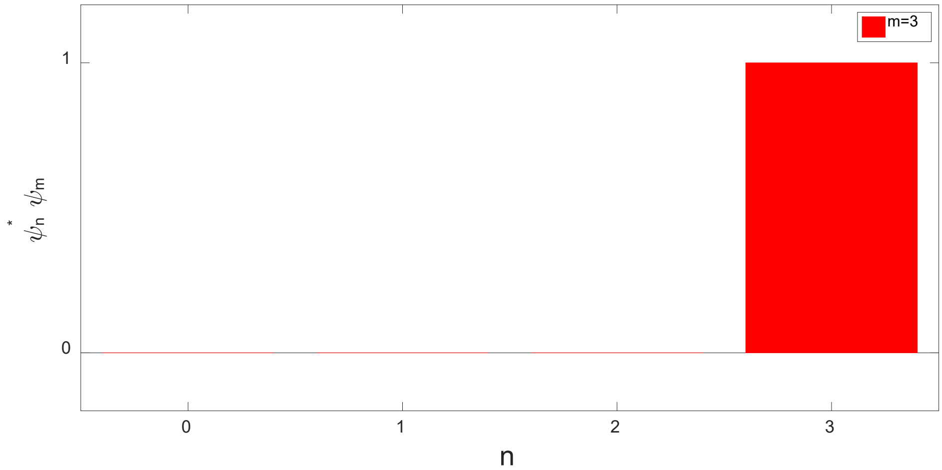
m=0
1
0
0
1
2
3
n

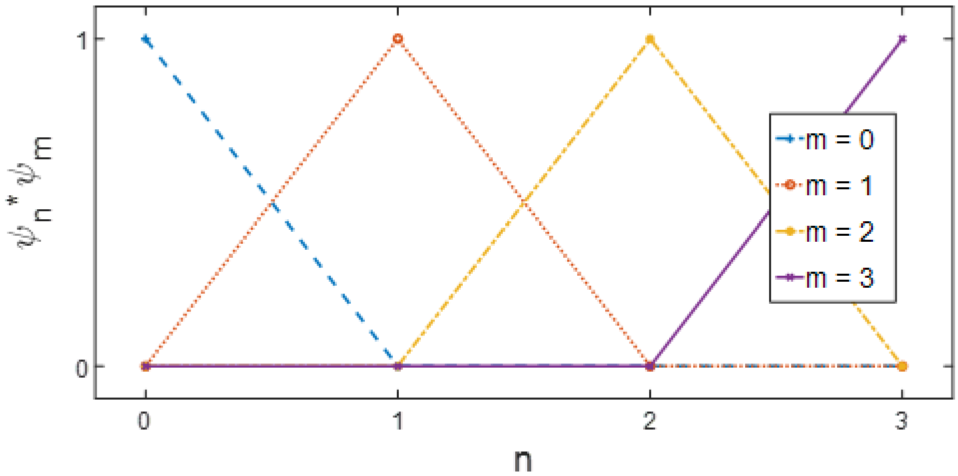


m=1
1
0
0
1
2
3
n



m=2
1
0
0
1
2
3
n





给出比较结果的给 10 分

2. <即教材*问题 2.12 和 Example 2.5>
Starting from equation 2.69, find $\langle x\rangle$, $\langle p\rangle$, $\langle x^2\rangle$, $\langle p^2\rangle$, and $\langle T\rangle$ for the n-th stationary state of the harmonic oscillator. Check the uncertainty principle between $\langle x\rangle$ and $\langle p\rangle$ is satisfied.

$$x=\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-),\qquad p=i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-)$$

$$a_+\psi_n=\sqrt{n+1}\psi_{n+1},\qquad a_-\psi_n=\sqrt{n}\psi_{n-1}$$

$$\langle x\rangle=\sqrt{\frac{\hbar}{2m\omega}}\int \psi_n^*(a_+ + a_-)\psi_n dx$$

$$=\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}dx+\sqrt{n}\int \psi_n^*\psi_{n-1}dx\right]=0$$

同时给出推导和答案给 5 分，只给出其中一个给 3 分

$$\langle p\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\int \psi_n^*(a_+ - a_-)\psi_n dx$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\left[\sqrt{n+1}\int \psi_n^*\psi_{n+1}dx - \sqrt{n}\int \psi_n^*\psi_{n-1}dx\right] = 0$$

同时给出推导和答案给 5 分，只给出其中一个给 3 分

$$x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2 = \frac{\hbar}{2m\omega}(a_+^2 + a_+a_- + a_-a_+ + a_-^2)^2$$

$$\begin{cases} a_+^2\psi_n = a_+\left(\sqrt{n+1}\psi_{n+1}\right) = \sqrt{n+1}\sqrt{n+2}\psi_{n+2} \\ a_+a_-\psi_n = a_+\left(\sqrt{n}\psi_{n-1}\right) = n\psi_n \\ a_-a_+\psi_n = a_-\left(\sqrt{n+1}\psi_{n+1}\right) = (n+1)\psi_n \\ a_-^2\psi_n = a_-\left(\sqrt{n}\psi_{n-1}\right) = \sqrt{n}\sqrt{n-1}\psi_{n-2} \end{cases}$$

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\left[0 + n\int |\psi_n|^2dx + (n+1)\int |\psi_n|^2dx + 0\right] = \frac{\hbar}{2m\omega}(2n+1)$$

$$= \left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}$$

同时给出推导和答案给 5 分，只给出其中一个给 3 分

$$p^2 = -\frac{\hbar m\omega}{2}(a_+ - a_-)^2 = -\frac{\hbar m\omega}{2}(a_+^2 - a_+a_- - a_-a_+ + a_-^2)^2$$

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\left[0 - n\int |\psi_n|^2dx - (n+1)\int |\psi_n|^2dx + 0\right] = \frac{\hbar m\omega}{2}(2n+1)$$

$$= \left(n+\frac{1}{2}\right)\hbar m\omega$$

同时给出推导和答案给 5 分，只给出其中一个给 3 分

$$\langle T\rangle = \langle\frac{p^2}{2m}\rangle = \left(n+\frac{1}{2}\right)\frac{\hbar\omega}{2}$$

同时给出推导和答案给 5 分，只给出其中一个给 3 分

$$\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{\frac{\hbar}{m\omega}},\quad \sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{n+\frac{1}{2}}\sqrt{\hbar m\omega}$$

$$\sigma_x\sigma_p = \left(n+\frac{1}{2}\right)\hbar \geq \frac{\hbar}{2}$$

同时给出推导和答案给 5 分，只给出其中一个给 3 分

* David J. Griffiths, and Darrell F. Schroeter, Introduction to Quantum Mechanics (3rd Edition), Cambridge University Press (2018).