0.1 Limits of sequences

0.1.1 Basic ideas

Sequences: *Example 1:* We consider the sequences of real numbers (a_n) with

$$a_n := \frac{1}{n}, \qquad n = 1, 2, ...$$

As n grows in magnitude, the value of a_n approaches 0 (see Table 1.1). To describe this behaviour we write

$$\boxed{\lim_{n\to\infty} a_n = 0}$$

and say that the *limit* of the sequence (a_n) is 0.

Table 1: 1

n	1	2	10	100	1000	10000	$\cdots$
an	1	0.5	0.1	0.01	0.001	0.0001	$\cdots$

Example 2: The sequences (b_n) with $b_n :=\frac{n}{n+1}$ approaches for large n the value 1. We again write $\lim_{n\to\infty} b_n = 1$.
Functions: In many applications of mathematics in science, the technology and economics, the netion of limits plays a particularity important role. The notion of the $limitofafunction$ in reduced to the notion of the limit of a sequence as above.
Example 3: Consider the function

$$f(x) = \begin{cases} x^2 & \text{for att real numbers } x \neq 0, \\ 1 & \text{for } x = 0 \end{cases}$$

We write

$$\boxed{\lim_{x\to a} f(x) = b,}$$

if and only if for every sequence (a_n) with $a_n \neq a$ for all n, we have the following:

$$\boxed{\text{From } \lim_{n\to\infty} a_n = 0 \text{ it follows that } \lim_{n\to\infty} f(a_n) = b.}$$

For the function $f(x)$ in this example one has

$$\lim_{x\to 0} f(x) = 0, \tag{1}$$

since from $a_n \neq 0$ for all n and $\lim\limits_{n\to\infty} a_n = 0$ it follows that $\lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} {a_n}^2 = 0$.
The relation (1) corresponds to our intuitive impression: if the point x approaches from the right (of the left) the point 0, then the corresponding values of the function approach 0. The value of f at 0 is irrelevant to these considerations.
Since the limit of a sequence of rational numbers can be irrational, one needs a rigorous development of the theory of limits, arising from a rigorous introduction of the real numbers, wich we describe in the following section.