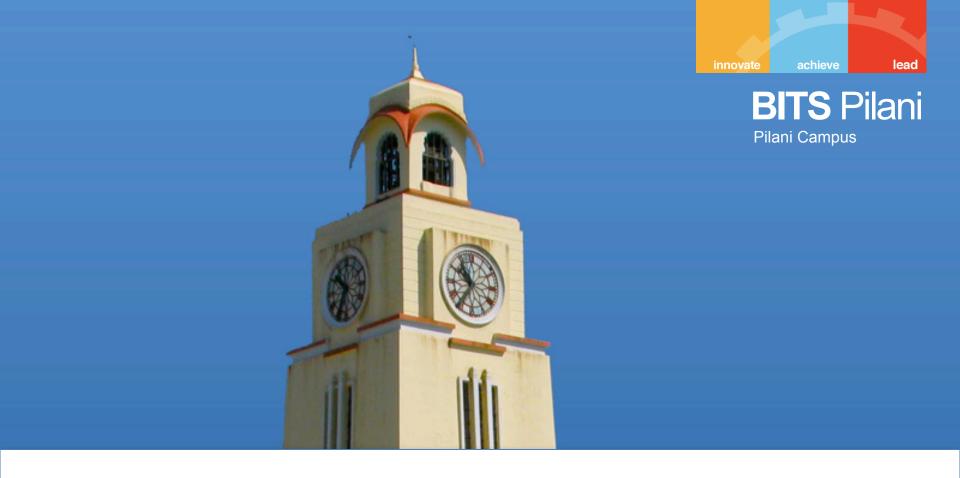




Mathematical Foundations for Machine Learning

MFML Team



S2-22_AIMLCZC416, MFML

Webinar 2

Agenda

Discussion on

- **□** Solutions of Homework Problems
- ☐ Gram-Schmidt process with example
- **☐** Example of SVD

Qus-1: Let $B=(b_1,b_2,\ldots,b_{r-1},b_r,b_{r+1},\ldots,b_n)$ be a non-singular matrix. If column b_r is replaced by a and that the resulting matrix is called B_a along with $a=\sum_{i=1}^n y_ib_i$, then state the necessary and sufficient conditions for B_a to be non-singular.

Sol: Given

$$B = \begin{bmatrix} * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$$
 Linearly independent columns

is non-singular.

$$B = \begin{bmatrix} * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$$
 Linearly independent columns

- This means no column of B can be written as a linear combination of other columns.
- lacksquare Replacing column b_r by a, we get

$$B_{a} = \begin{bmatrix} * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

Looking at,
$$a = \sum_{i=1}^{n} y_i b_i$$
, we see that

$$b_1$$
 b_2 ... b_{r-1} a b_{r+1} ... b_n

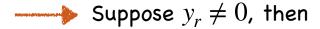
$$\blacktriangleright$$
 a is a linear combination of b_1,b_2,\ldots,b_n

Suppose
$$y_r = 0$$
, i.e.

$$B_a$$

$$a = y_1b_1 + y_2b_2 + \dots + y_{r-1}b_{r-1} + y_{r+1}b_{r+1} + \dots + y_nb_n.$$

This means a is a linear combination of other columns of B_a . Hence B_a is singular.



$$a = y_r b_r + y_1 b_1 + y_2 b_2 + \dots + y_{r-1} b_{r-1} + y_{r+1} b_{r+1} + \dots + y_n b_n.$$

Not a linear combination of columns $b_1, \ldots, b_{r-1}, b_{r+1}, \ldots, b_n$ of B_a

$$b_1 \ b_2 \ \dots \ b_{r-1} \ a \ b_{r+1} \ \dots \ b_n$$

$$\begin{bmatrix} * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ * & * & \dots & * & * & * & \dots & * \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

Qus-2: Let V be a finite dimensional vector space over \mathbb{R} . If S is a set of elements in V such that $\mathrm{span}(S)=V$, what is the relationship between S and the basis of V?

Sol: Given

$$span(S) = V$$

- lacksquare S is a generating set of V.
- Since basis is a minimal generating set, we have

Number of elements in a basis of $V \leq$ Number of elements in S

Qus-3:

- a) Let P be a real square matrix satisfying $P = P^T$ and $P^2 = P$.
 - 1. Can the matrix P have complex eigenvalues? If so, construct an example, else, justify your answer.
 - 2. What are the eigenvalues of P?

Sol: 1. $P = P^T \implies P$ is symmetric.

<u>Spectral theorem</u>: If $A \in \mathbb{R}^{n \times n}$ is symmetric there exists an orthonormal basis of the corresponding vector space V consisting of the eigenvectors of A, and each eigenvalue is real.

lacktriangleright Therefore, by the Spectral theorem, P cannot have complex eigenvalues.

Sol: 2. Suppose λ is an eigenvalue of P and let v be the corresponding eigenvector. Then

$$Pv = \lambda v$$

$$\Rightarrow P^{2}v = \lambda Pv = \lambda^{2}v$$

$$\Rightarrow Pv = \lambda^{2}v$$

$$\Rightarrow \lambda v = \lambda^{2}v \qquad (\because P^{2} = P)$$

$$\Rightarrow (\lambda^{2} - \lambda)v = \mathbf{0}$$

Since $v \neq 0$, we have

$$\lambda^2 - \lambda = 0 \implies \lambda(\lambda - 1) = 0 \implies \lambda = 0 \text{ or } \lambda = 1$$

Hence, the eigenvalues of P are 0 and 1.

b) Given the following matrix

$$A = \begin{bmatrix} 1 & 2 & r \\ c & 1 & 7 \\ c & 1 & 7 \end{bmatrix}$$

where c and r are arbitrary real numbers and $5.5 < r \le 6.5$, and the fact that λ_1 = 3 is one of the eigenvalues, is it possible to determine the other two eigenvalues? If so, compute them and give reasons for your answer.

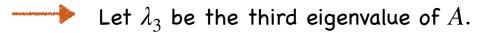
Sol: (ii) Since A has two identical rows

$$det(A) = 0$$

i.e. 0 is an eigenvalue of A.

Next, we recall that

Trace of a matrix = sum of its eigenvalues



$$A = \begin{bmatrix} 1 & 2 & r \\ c & 1 & 7 \\ c & 1 & 7 \end{bmatrix}$$

Since trace
$$(A) = 9$$
, we have

$$3 + 0 + \lambda_3 = 9 \implies \lambda_3 = 6.$$

Hence, the other two eigenvalues of
$$A$$
 are 0 and 6 .

Qus-4: The Fibonacci sequence is defined by

$$V_n = V_{n-1} + V_{n-2}$$
, for $n \ge 2$

with starting values $V_0=0$ and $V_1=1$. Observe that the calculation of V_k requires the calculation of V_2,V_3,\ldots,V_{k-1} . To avoid this, could this problem be written as an eigenvalue problem and solved for V_n directly? If so, find the explicit formula for V_n .

Sol: We can write

$$V_{n+1} = V_n + V_{n-1}$$

$$V_n = 1.V_n$$
or
$$\begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix}$$

Define

$$F_n = \begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix}$$

and

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Then, we have

$$F_n = A \begin{bmatrix} V_n \\ V_{n-1} \end{bmatrix} = AF_{n-1}$$

$$F_1 = AF_0$$

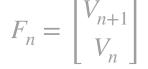
$$F_2 = AF_1 = A^2 F_0$$

$$F_3 = AF_2 = A^3F_0, \dots$$

Continuing like this, we get

$$F_n = A^n F_0$$

where
$$F_0 = \begin{bmatrix} V_1 \\ V_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
.



- If we find A^n , then from F_n , we can find the value of V_n .
- lacksquare Since A is a symmetric matrix, it is diagonalisable, i.e.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = PDP^{-1}$$

where columns of P consists of eigenvectors of A and D is the diagonal matrix with eigenvalues of A as its diagonal entries.

→ Note that

$$A^{n} = (PDP^{-1})(PDP^{-1})...(PDP^{-1})$$
 (*n* times)
 $= (PD^{2}P^{-1})(PDP^{-1})...(PDP^{-1})$ (*n* - 1 times)
 \vdots
 $= PD^{n}P^{-1}$

Therefore, to find A^n , it is enough to find P and D.

To find P and D, we need to find the eigenvalues and eigenvectors of A.

Suppose λ is an eigenvalue of A. Then

$$\det(A - \lambda I) = 0 \qquad \Longrightarrow \begin{vmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0$$

$$\Longrightarrow -\lambda(1-\lambda)-1=0$$

$$\implies \lambda^2 - \lambda - 1 = 0$$

$$\implies \lambda = \frac{1 \pm \sqrt{5}}{2}$$

Thus, the eigenvalues of
$$A$$
 are $\frac{1\pm\sqrt{5}}{2}$ and hence $D=\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0\\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$.

Suppose $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ be an eigenvector of A corresponding to the eigenvalue $\frac{1+\sqrt{5}}{2}$. Then

$$Av = \frac{1+\sqrt{5}}{2}v \qquad \Longrightarrow \begin{bmatrix} 1 & 1\\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_1\\ v_2 \end{bmatrix} = \frac{1+\sqrt{5}}{2} \begin{bmatrix} v_1\\ v_2 \end{bmatrix}$$

$$\implies v_1 + v_2 = \frac{1 + \sqrt{5}}{2} v_1 \quad \text{and } v_1 = \frac{1 + \sqrt{5}}{2} v_2$$

- Choosing $v_2 = 1$, we get $v = \begin{bmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{bmatrix}$

Similarly, we can see
$$\left[\frac{1-\sqrt{5}}{2}\right]$$
 is an eigenvector of A corresponding to the eigenvalue $\frac{1-\sqrt{5}}{2}$.

Hence,
$$P = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix}$$
.

It is easy to compute that,
$$P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$
.



Using $A^n = PD^nP^{-1}$, we compute

$$A^{n} = \frac{1}{\sqrt{5}} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}^{n} \begin{bmatrix} 1 & \frac{-1+\sqrt{5}}{2} \\ -1 & \frac{1+\sqrt{5}}{2} \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} \quad \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} \right] \left[1 \quad \frac{-1+\sqrt{5}}{2} \\ * \quad * \right]$$

$$= \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{n+1} * \right].$$



$$F_n = A^n F_0,$$

$$F_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$F_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad F_n = \begin{bmatrix} V_{n+1} \\ V_n \end{bmatrix}$$

we get

$$V_{n+1} = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left(\frac{1 - \sqrt{5}}{2} \right)^{n+1} \right]$$

or

$$V_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right].$$

Qus-5: Prove that if A is a square diagonalisable matrix of size $n \times n$, then $A^k \to 0$ as $k \to \infty$ if and only if $|\lambda_i| < 1, \forall i$.

Sol:



$$A = PDP^{-1}$$

where D is the diagonal matrix with eigenvalues of A as its diagonal entries.

 \blacktriangleright Multiplying A by P^{-1} on the left and by P on the right, we get

$$P^{-1}AP = P^{-1}PDP^{-1}P$$
$$D = P^{-1}AP$$

$$\implies D^k = P^{-1}A^kP$$

Suppose $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. Then

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix} \implies D^k = \begin{bmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda_n^k \end{bmatrix}$$

$$A^k \to 0 \text{ as } k \to \infty$$

$$\iff D^k \to 0 \text{ as } k \to \infty$$

$$\iff (\lambda_i)^k \to 0 \text{ as } k \to \infty, \ \forall i$$

$$\iff |\lambda_i| < 1, \forall i$$

Qus-6: Construct examples of matrices for which the defect is positive, negative and zero wherever possible.

Sol: Suppose λ is an eigenvalue of a matrix A. Then

Algebraic multiplicity of λ ——— Multiplicity of λ as a root of the characteristic polynomial of A

Geometric multiplicity of λ — \blacktriangleright the maximum number of linearly independent eigenvectors associated with λ

Defect of λ = Algebraic multiplicity — Geometric multiplicity



Algebraic multiplicity \geq Geometric multiplicity

The defect cannot be negative.

Example of matrix with zero defect.

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

 $\cal A$ is a diagonal matrix with eigenvalues 2 and 3. Since all the eigenvalues are distinct, the defect of each eigenvalue is $\cal O$.

It is also easy to compute the eigenvectors corresponding to each eigenvalue and check algebraic and geometric multiplicity is same for each eigenvalue.



Example of matrix with positive defect.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

The characteristic polynomial of A is

$$\lambda^2 - 2\lambda + 1 = 0$$

Therefore, 1 is the only eigenvalue of A and has multiplicity 2.

 \implies Algebraic multiplicity of 1 = 2.

Suppose
$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$
 be an eigenvector of A corresponding to the eigenvalue 1. Then

$$Av = v \qquad \Longrightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\Longrightarrow v_1 + v_2 = v_1$$

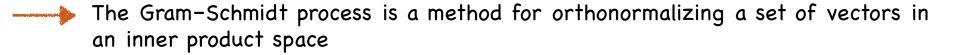
$$\Longrightarrow v_2 = 0 \quad \text{and } v_1 \text{ is a free variable}$$

$$\Longrightarrow v = v_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

 \implies Geometric multiplicity of 1 = 1.

Hence, Defect of eigenvalue 1 = 2 - 1 = 1 > 0.

Gram Schmidt process



The Gram-Schmidt process takes a finite, linearly independent set of vectors

$$S = \{v_1, \dots, v_k\}$$
 for $k \le n$

and generates an orthogonal set

$$S' = \{u_1, ..., u_k\}$$

The Gram-Schmidt process works as follows:

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle v_2, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1$$

$$u_3 = v_3 - \frac{\langle v_3, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 - \frac{\langle v_3, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

Gram Schmidt process (contd.)

$$u_{4} = v_{4} - \frac{\langle v_{4}, u_{1} \rangle}{\langle u_{1}, u_{1} \rangle} u_{1} - \frac{\langle v_{4}, u_{2} \rangle}{\langle u_{2}, u_{2} \rangle} u_{2} - \frac{\langle v_{4}, u_{3} \rangle}{\langle u_{3}, u_{3} \rangle} u_{3}$$

$$\vdots$$

$$\vdots$$

$$u_{k} = v_{k} - \sum_{j=1}^{k-1} \frac{\langle v_{k}, u_{j} \rangle}{\langle u_{j}, u_{j} \rangle} u_{j}$$



The normalized vectors

$$e_1 = \frac{u_1}{||u_1||}, e_2 = \frac{u_2}{||u_2||}, \dots, e_k = \frac{u_k}{||u_k||}$$

form an orthonormal set.

Qus: Obtain an orthogonal basis for the subspace of \mathbb{R}^4 spanned by

$$x_1 = (1,0,1,0), x_2 = (1,1,1,1), x_3 = (-1,2,0,1)$$

Sol: Following the Gram-Schmidt process, we set

$$v_1 = x_1 = (1,0,1,0)$$
.

Next, we have

$$v_2 = x_2 - \frac{\langle x_2, v_1 \rangle}{||v_1||^2} v_1 = (1, 1, 1, 1) - \frac{2}{2} (1, 0, 1, 0) = (0, 1, 0, 1)$$

and

innovate

Example of Gram Schmidt process (contd.)

$$v_3 = x_3 - \frac{\langle x_3, v_1 \rangle}{||v_1||^2} v_1 - \frac{\langle x_3, v_2 \rangle}{||v_2||^2} v_2$$

$$= (-1, 2, 0, 1) + \frac{1}{2} (1, 0, 1, 0) - \frac{3}{2} (0, 1, 0, 1)$$

$$= (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

The orthogonal basis so obtained is

$$\{(1,0,1,0), (0,1,0,1), (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})\}$$

We can normalise the vectors to get

$$\{\frac{1}{\sqrt{2}}(1,0,1,0), \frac{1}{\sqrt{2}}(0,1,0,1), (-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2})\}$$

Example of SVD

Qus: Find Singular value decomposition of the matrix

$$A = \begin{bmatrix} -3 & 1\\ 6 & -2\\ 6 & -2 \end{bmatrix}$$

Sol: Step 1: Compute A^TA

$$A^{T}A = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

Step 2: Find the eigenvalues of A^TA .

The characteristic polynomial of A^TA is:

$$\lambda^{2} - trace(A^{T}A)\lambda + det(A^{T}A) = 0$$

$$\implies \lambda^{2} - 90\lambda = 0 \qquad (\because det(A^{T}A) = 0)$$

Example of SVD (contd.)

$$\implies \lambda(\lambda - 90) = 0$$

 $\implies \lambda_1 = 90$ and $\lambda_2 = 0$ are the two eigenvalues of A^TA .

Step 3: Find the singular values of A.

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{90}$$
 and $\sigma_2 = \sqrt{\lambda_2} = 0$.

Step 4: Construct the diagonal matrix of same size as A with singular values as its diagonal entries in decreasing order.

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example of SVD (contd.)

<u>Step 5</u>: Find the eigenvectors (called right singular vectors) corresponding to each eigenvalue of A^TA .

Suppose
$$z=\begin{bmatrix} z_1\\z_2 \end{bmatrix}$$
 be an eigenvector of A corresponding to the eigenvalue λ_2 . Then

$$A^{T}Az = \lambda_{2}z \implies \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix} \begin{bmatrix} z_{1} \\ z_{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\implies 81z_{1} - 27z_{2} = 0 \text{ and } -27z_{1} + 9z_{2} = 0$$

$$\implies z_{2} = 3z_{1}$$

$$\implies z = z_{1} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Choose
$$v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$
 as an eigenvector corresponding to λ_2 .

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Example of SVD (contd.)

Similarly, we can find
$$v_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$
 as an eigenvector corresponding to λ_1 .

Step 6: Write the orthogonal matrix consisting of the normalised eigenvectors of A^TA .

$$V = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 & 1\\ 1 & 3 \end{bmatrix}.$$

Step 7: Find the left eigenvectors (called left singular vectors) \hat{u}_i using

$$\hat{u}_i = \frac{Av_i}{\sigma_i}.$$

For i = 1, we have.

$$\hat{u}_1 = \frac{Av_1}{\sigma_1} = \frac{1}{\sqrt{90}} \begin{bmatrix} -3 & 1\\ 6 & -2\\ 6 & -2 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}} \end{bmatrix} = \frac{1}{\sqrt{900}} \begin{bmatrix} 10\\ -20\\ -20 \end{bmatrix}.$$





Since $\sigma_2=0$, to find the other vector we need to use the orthogonality condition

Let
$$w = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
 be the next vector. Then

$$w \cdot u_1 = 0 \implies 10x - 20y - 20z = 0 \implies x = 2y + 2z$$

$$\implies w = y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Let

$$w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$
 And

$$w_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Example of SVD (contd.)

Hence the vectors orthogonal to u_1 are

$$u_2 = w_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \implies \hat{u_2} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$
 and

$$u_3 = w_2 - \frac{\langle w_2, u_2 \rangle}{||u_2||^2} u_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{4}{5} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} \implies \hat{u_3} = \frac{1}{\sqrt{45}} \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}$$

Hence,
$$U=$$

$$\begin{vmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{vmatrix}.$$

Example of SVD (contd.)

Step 8: The singular value decomposition of A is

$$A = U\Sigma V^{T} = \begin{bmatrix} \frac{10}{\sqrt{900}} & \frac{2}{\sqrt{5}} & \frac{2}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & \frac{1}{\sqrt{5}} & -\frac{4}{\sqrt{45}} \\ \frac{-20}{\sqrt{900}} & 0 & \frac{5}{\sqrt{45}} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix}$$

Thank You