Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

M.Tech. in Artificial Intelligence and Machine Learning.

I Semester 2022-23

Course Number	AIMLCZ C416
Course Name	Mathematical Foundations for Machine Learning
Nature of Exam	Open Book
Weightage for grading	30%
Duration	120 minutes
Date of Exam	08/01/2023 (14:00 - 16:00)

# Pages	2
# Questions	8

Instructions

1. All questions are compulsory.
2. All parts of a question should be answered consecutively. Each answer should start from a fresh page.

(1) Find the Taylor's series expansion to three terms of the function e^{5x} around the point $x = 1$ and $x = 0$ respectively. Which approximation would give less error with respect to the original function at $x = 2$?

[3 Marks]

(2) Let the vector $\mathbf{f} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$. Calculate the Jacobian matrix and find its rank, when $x \neq 0, y \neq 0$.

[4 Marks]

(3) Consider a 2×2 real matrix given below

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d > 0$. We claim that A has an eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \text{ with } x, y > 0$$

Is this claim true or false? Provide a suitable justification.

[5 Marks]

(4) Data analysis led to a matrix $A = (a_{i,j}), i, j = 1, \cdots, n$ where n is positive integer such that

$$\begin{aligned} a_{i,i} &= 4, \forall i = 1, \cdots, n \\ a_{i,i+1} &= -2, \forall i = 1, \cdots, n-1 \\ a_{i,i-1} &= -2, , \forall i = 2, \cdots, n \\ a_{i,j} &= 0, \text{otherwise} \end{aligned}$$

We claim that every eigenvalue of A is positive real number. Is this claim true or false? Give a mathematical justification for your answer.

[3 Marks]

(5) An engineer named H working on a machine learning problem from Oil industry encountered a square matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$. He made an important observation that the matrix $\mathbf{C}$ satisfies an interesting property i.e $\mathbf{CC^T} = \mathbf{C^TC}$. Then his manager asked him to study the properties of eigenvalues and eigenvectors of this matrix with help from other team members. ($\bar{\alpha}$ represents the complex conjugate of α).

(a) A colleague named A1 then claimed that if $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{C}$ then $(\lambda, \mathbf{x})$ must always be an eigenpair of $\mathbf{C^T}$.

(b) A colleague named A2 instead claimed that if $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{C}$ then $(\bar{\lambda}, \mathbf{x})$ must always be an eigenpair of $\mathbf{C^T}$.

(c) The manager claimed that if $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{C}$ then $\mathbf{x}$ can never be an eigenvector of $\mathbf{C^T}$.

Prove/Disprove the claims made by A1, A2 and the manager. (Note : answers without proper reasoning will not be awarded marks).

[4 Marks]

(6) Consider the last 2 digits of your BITS email id. For example, if your email is **2022da098**$\beta_1\beta_2$**@wilp.bits-pilani.ac.in**, then look at the last 2 digits before the @ symbol represented by β_1 and β_2 here.

(a) Write down your BITS email id

(b) Write down the β_1 and β_2 values as extracted from your BITS email id.

(c) Construct a matrix $\mathbf{B}$ using the extracted values of β_1 and β_2 as follows :

$$\mathbf{B} = \begin{bmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{bmatrix}$$

(d) Derive the largest singular value σ_1 of this matrix $\mathbf{B}$.

(e) Calculate the value of $\alpha = \sigma_1^2$.

(f) Derive left singular vector $\mathbf{u_1}$ corresponding to σ_1.

(g) Derive right singular vector $\mathbf{v_1}$ corresponding to σ_1.

(h) Find a matrix $\mathbf{C} = \mathbf{u_1 v_1^T}$ and Calculate $\|\mathbf{B} - \mathbf{C}\|_\mathbf{F}$ where $\|\mathbf{Q}\|_\mathbf{F}$ denotes the sum of the squares of the entries of the matrix $\mathbf{Q}$.

(i) Find a matrix $\mathbf{E} = \sigma_\mathbf{1}\mathbf{u_1 v_1^T}$ and Calculate $\|\mathbf{B} - \mathbf{E}\|_\mathbf{F}$.

[4 Marks]

(7) Find two mutually orthogonal vectors each of which is orthogonal to the vector $(4, -1, 3)$ of $\mathbb{R}^3$ w.r.t the standard inner product.

[4 Marks]

(8) Using row elementary operations, transform the basis $\{(1, 0, 1), (1, 0, -1), (0, 3, 5)\}$ of $\mathbb{R}^3$ to obtain an orthonormal basis.

[3 Marks]