Birla Institute of Technology and Science, Pilani

Work Integrated Learning Programmes Division

Cluster Programme - M.Tech. in Data Science and Engg.

I Semester 2022-23

Course Number	DSECL ZC416
Course Name	Mathematical Foundations for Data Science
Nature of Exam	Open Book
Weightage for grading	30%
Duration	120 minutes
Date of Exam	08/01/2023 (14:00 - 16:00)

# Pages	2
# Questions	8

Instructions

1. All questions are compulsory.
2. All parts of a question should be answered consecutively. Each answer should start from a fresh page.

(1) Find the Taylor's series expansion to three terms of the function e^{5x} around the point $x = 1$ and $x = 0$ respectively. Which approximation would give less error with respect to the original function at $x = 2$?

[3 Marks]

Solution: The Taylor's series expansion for the function $f(x)$ around the point $x = a$ to three terms is $T(x) = f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2$. Applying this formula, the Taylor's series expansion of e^{5x} around the point $x = 1$ is $T_{x=1} = e^{5*1} + \frac{5e^{5*1}}{1!}(x-1) + \frac{25e^{5*1}}{2!}(x-1)^2 = e^5 + 5e^5(x-1) + \frac{25e^5}{2}(x-1)^2$.

Around the point $x = 0$, the Taylor's formula is $T_{x=0} = e^{5*0}+\frac{5e^{5*0}}{1!}(x-0) + \frac{25e^{5*0}}{2!}(x-0)^2 = 1 + 5x + \frac{25}{2}x^2$.

The Taylor's approximation around a given point becomes a poorer and poorer approximation of the original function the further we move away from the point of expansion. Since the point $x = 1$ is closer to the point at which approximation is sought, i.e $x = 2$, the Taylor's series approximation $T_{x=1}$ is a better approximation to the function at $x = 2$ than $T_{x=0}$.

Marking Scheme: 1 mark for each Taylor's series expansion, 1 Mark for which Taylor's expansion gives a better approximation at $x = 2$.

(2) Let the vector $\mathbf{f} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ \frac{y}{\sqrt{x^2+y^2}} \end{bmatrix}$. Calculate the Jacobian matrix and find its rank, when $x \neq 0, y \neq 0$.

[4 Marks]

Solution: The Jacobian matrix is calculated as follows: $J = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$. The partial derivatives appearing in the matrix can be calculated as follows: $\frac{\partial f_1}{\partial x} = \frac{\sqrt{x^2+y^2} - x*x(x^2+y^2)^{(-1/2)}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{(3/2)}}$. $\frac{\partial f_1}{\partial y} = \frac{-xy}{(x^2+y^2)^{(3/2)}}$, $\frac{\partial f_2}{\partial x} = \frac{-xy}{(x^2+y^2)^{(3/2)}}$, $\frac{\partial f_2}{\partial y} = \frac{x^2}{(x^2+y^2)^{(3/2)}}$.

The matrix $J = \begin{bmatrix} \frac{y^2}{(x^2+y^2)^{(3/2)}} & \frac{-xy}{(x^2+y^2)^{(3/2)}} \\ \frac{-xy}{(x^2+y^2)^{(3/2)}} & \frac{x^2}{(x^2+y^2)^{(3/2)}} \end{bmatrix}$. We are given that $x, y \neq 0$, so multiplying the first row by $\frac{x}{y}$ and adding it to the second row, we get the zero row. Further the first row is non-zero which leads us to assert that the rank of J is 1.

Marking scheme: 3 Marks for Jacobian, 1 Mark for rank.

(3) Consider a 2×2 real matrix given below

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d > 0$. We claim that A has an eigenvector

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \text{ with } x, y > 0$$

Is this claim true or false? Provide a suitable justification.

[5 Marks]

Solution: The claim is true. Consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a, b, c, d > 0$. Then the characteristic polynomial is given by

$$|\begin{matrix} a-\lambda & b \\ c & d-\lambda \end{matrix}| = \lambda^2 - (a+d)\lambda + ad - bc$$

Therefore, the roots of characteristic polynomial are

$$\begin{aligned} \lambda &= \frac{(a+d) \pm \sqrt{(a+d)^2 - 4ad + 4bc}}{2} \\ &= \frac{(a+d) \pm \sqrt{(a-d)^2 + 4bc}}{2} \\ &= \frac{1}{2}(a+d) \pm \frac{1}{2}\sqrt{D}, \text{where } D = (a-d)^2 + 4bc \end{aligned}$$

Clearly $D = (a-d)^2 + 4bc > 0 \Rightarrow$ eigenvalues are real. Let

$$\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

be an eigenvector of A corresponding to $\lambda = \frac{1}{2}(a+d) + \frac{1}{2}\sqrt{D}$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating the first row of the matrix equation we get

$$
\begin{aligned}
ax + by &= (\frac{1}{2}(a+d) + \frac{1}{2}\sqrt{D})x \\
\Rightarrow 2by &= [(d-a) + \sqrt{D}]x \qquad (1) \\
\text{Now} \quad & (d-a) + \sqrt{D} \\
&= \sqrt{(a-d)^2 + 4bc} - (a-d) > 0, \text{as } b, c > 0 \\
\Rightarrow sign(x) &= sign(y), \text{from } (1) \\
\text{and} \quad & x \neq 0 \neq y \text{ as eigenvector is nonzero}
\end{aligned}
$$

Now if v is an eigenvector corresponding to an eigenvalue λ of matrix A then

$$
\begin{aligned}
Av &= \lambda v \\
\Rightarrow -Av &= -\lambda v \\
\Rightarrow A(-v) &= \lambda(-v)
\end{aligned}
$$

$\Rightarrow -v$ is also an eigenvector of A Therefore, WLOG, we can assume $x, y > 0$

Marking Scheme: 2 marks for showing eigenvalues are real, 3 Marks for the rest of the argument.

(4) Data analysis led to a matrix $A = (a_{i,j}), i, j = 1, \cdots, n$ where n is positive integer such that

$$
\begin{aligned}
a_{i,i} &= 4, \forall i = 1, \cdots, n \\
a_{i,i+1} &= -2, \forall i = 1, \cdots, n-1 \\
a_{i,i-1} &= -2, , \forall i = 2, \cdots, n \\
a_{i,j} &= 0, \text{otherwise}
\end{aligned}
$$

We claim that every eigenvalue of A is positive real number. Is this claim true or false? Give a mathematical justification for your answer.

[3 Marks]

Solution: The claim is true. Now,

$$
\begin{aligned}
a_{ii} &= 4, \forall i = 1, \cdots, n \\
a_{ii+1} &= -2, \forall i = 1, \cdots, n-1 \\
a_{ii-1} &= -2, \forall i = 2, \cdots, n \\
a_{ij} &= 0, \text{otherwise}
\end{aligned}
$$

$$\Rightarrow A = \begin{pmatrix} 4 & -2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -2 & 4 & -2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -2 & 4 & -2 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & -2 & 4 & -2 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -2 & 4 \end{pmatrix}$$

To prove: Every eigenvalue of A is a positive real number.
Clearly

$$\begin{aligned} A^T(j, j+1) &= A(j+1, j), \ \forall j = 1, \cdots, n-1 \\ &= A(i, i-1) = a_{ii-1} = -2, \ \forall i = 2, \cdots, n \\ &= a_{j,j+1} = A(j, j+1), \ \forall j = 1, \cdots, n-1 \\ A^T(j, j-1) &= A(j-1, j), \ \forall j = 2, \cdots, n \\ &= A(i, i+1) = a_{ii+1} = -2, \ \forall i = 1, \cdots, n-1 \\ &= a_{j,j-1} = A(j, j-1), \ \forall j = 2, \cdots, n \\ A^T(i,j) &= 0 = A(j,i) = A(i,j), \ \text{o.w.} \end{aligned}$$

$\Rightarrow A$ is symmetric.
We know that all eigenvalues of symmetric positive definite matrices are positive, real.
Since A is symmetric, it is enough to prove that A is positive definite.
So, consider,

$$\begin{aligned} & (Ax)^T x \\ = & \begin{pmatrix} 4x_1 - 2x_2 \\ -2x_1 + 4x_2 - 2x_3 \\ -2x_2 + 4x_3 - 2x_4 \\ \vdots \\ -2x_{n-2} + 4x_{n-1} - 2x_n \\ -2x_{n-1} + 4x_n \end{pmatrix}^T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \\ = & \ 4x_1^2 - 2x_1x_2 - 2x_1x_2 + 4x_2^2 - 2x_2x_3 - 2x_2x_3 + 4x_3^2 \cdots - 2x_{n-1}x_n + 4x_n^2 \\ = & \ 2[x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + \cdots + (x_{n-1} - x_n)^2 + x_n^2] \\ > & \ 0 \text{ since all } x_i \neq 0 \end{aligned}$$

This implies A is positive definite and hence all eigenvalues are positive- and real.
Marking Scheme: 1 Mark to establish symmetry, 2 marks for positive-definiteness.

(5) An engineer named H working on a machine learning problem from Oil industry encountered a square matrix $\mathbf{C} \in \mathbb{R}^{n \times n}$. He made an important observation that the matrix $\mathbf{C}$ satisfies an interesting property i.e $\mathbf{CC^T} = \mathbf{C^TC}$. Then his manager asked him to study the properties of eigenvalues and eigenvectors of this matrix with help from other team members. ($\bar{\alpha}$ represents the complex conjugate of α).

(a) A colleague named A1 then claimed that if $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{C}$ then $(\lambda, \mathbf{x})$ must always be an eigenpair of $\mathbf{C^T}$.
(b) A colleague named A2 instead claimed that if $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{C}$ then $(\bar{\lambda}, \mathbf{x})$ must always be an eigenpair of $\mathbf{C^T}$.
(c) The manager claimed that if $(\lambda, \mathbf{x})$ is an eigenpair of $\mathbf{C}$ then $\mathbf{x}$ can never be an eigenvector of $\mathbf{C^T}$.

Prove/Disprove the claims made by A1, A2 and the manager. (Note : answers without proper reasoning will not be awarded marks).

[4 Marks]

Solution: It is given that $C^TC = CC^T$

Let $G = C - \lambda I$

It can be proved easily that $G^*G = GG^*$ (1 marks)

It can be proved that $< Gx, Gx > = < G^*x, G^*x >$

This is same as$||Gx||_2 = ||G^*x||_2$ (1 marks)

Now let $Ax = \lambda x$

This is possible if and only if $||(C - \lambda I)x||_2 = 0$. By previous result , this is equivalent to $||(C^T - \bar{\lambda})x||_2 = 0$ (1 marks)

In conclusion $C^T x = \bar{\lambda}x$. (1 marks)

Hence A1 and managers claim is incorrect. But A2 claim is correct.

Instruction : Marks only if above argument is made. No marks for simply stating who is correct or incorrect .

(6) Consider the last 2 digits of your BITS email id. For example, if your email is **2022da098**$\beta_1\beta_2$**@wilp.bits-pilani.ac.in**, then look at the last 2 digits before the @ symbol represented by β_1 and β_2 here.
(a) Write down your BITS email id
(b) Write down the β_1 and β_2 values as extracted from your BITS email id.
(c) Construct a matrix $\mathbf{B}$ using the extracted values of β_1 and β_2 as follows :

$$\mathbf{B} = \begin{bmatrix} \beta_1 & \beta_2 \\ 0 & 0 \end{bmatrix}$$

(d) Derive the largest singular value σ_1 of this matrix $\mathbf{B}$.
(e) Calculate the value of $\alpha = \sigma_1^2$.
(f) Derive left singular vector $\mathbf{u_1}$ corresponding to σ_1.
(g) Derive right singular vector $\mathbf{v_1}$ corresponding to σ_1.
(h) Find a matrix $\mathbf{C} = \mathbf{u_1 v_1^T}$ and Calculate $\|\mathbf{B} - \mathbf{C}\|_\mathbf{F}$ where $\|\mathbf{Q}\|_\mathbf{F}$ denotes the square-root of the sum of the squares of the entries of the matrix $\mathbf{Q}$.
(i) Find a matrix $\mathbf{E} = \sigma_1 \mathbf{u_1 v_1^T}$ and Calculate $\|\mathbf{B} - \mathbf{E}\|_\mathbf{F}$.

[4 Marks]

Solution: The solution is to found out numerically by doing standard SVD steps. Every student will get a different answer as their email is different.

(d) and (e) 0.5 marks

(f) and (g) 0.5 marks

(h) 1.5 marks if answer is $||B - C||_F = |\sigma_1 - 1|$ where $\sigma_1 = \sqrt{\beta_1^2 + \beta_2^2}$ is

from (d)
(i) 1.5 marks if answer is $||B - E||_F = 0$

(7) Find two mutually orthogonal vectors each of which is orthogonal to the vector $(4, -1, 3)$ of $\mathbb{R}^3$ w.r.t the standard inner product.

[4 Marks]

Solution: To find the first vector orthogonal to $(4, -1, 3)$, assume the vector to be (a, b, c) and solve $4a - b + 3c = 0$. One solution to this equation is $(a, b, c) = (1/4, 1, 0)$. To find the other orthogonal vector (p, q, r), we set up two equations:

$$\begin{aligned} 4p - q + 3r &= 0 \\ 1/4p + q &= 0 \end{aligned}$$

a solution to which is $(p, q, r) = (1, -1/4, -17/12)$. Therefore one possible solution is $(1/4, 1, 0), (1, -1/4, -17/12)$.

Another way of doing this is through Gram-Schmidt orthogonalization.

Marking Scheme: 2 Marks for each orthogonal vector.

(8) Using row elementary operations, transform the basis $\{(1, 0, 1), (1, 0, -1), (0, 3, 5)\}$ of $\mathbb{R}^3$ to obtain an orthonormal basis.

[3 Marks]

Solution: Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & -1 & 5 \end{bmatrix}$. The augmented system $[A^T A | A^T]$ becomes $\begin{bmatrix} 2 & 0 & 5 & | & 1 & 0 & 1 \\ 0 & 2 & -5 & | & 1 & 0 & -1 \\ 5 & -5 & 34 & | & 0 & 3 & 5 \end{bmatrix}$.

Performing Gaussian elimination of this system gives us $\begin{bmatrix} 2 & 0 & 5 & | & 1 & 0 & 1 \\ 0 & 2 & -5 & | & 1 & 0 & -1 \\ 0 & 0 & 9 & | & 0 & 3 & 0 \end{bmatrix}$.

Thus the orthonormal basis for the system is $(1/\sqrt{2}, 0, 1/\sqrt{2}), (1/\sqrt{2}, 0, -1/\sqrt{2}), (0, 1,$

Marking Scheme: 2 Marks for Setting up the augmented matrix and Gaussian elimination, 1 Mark for final normalized basis.