

Exercises set I

PhD course on Sequential Monte Carlo methods 2017

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This document contains exercises to make you familiar with the content of the course. *The exercises in this document are not mandatory, and you do not need to hand in your solutions.* The mandatory assignment is found in a separate document named "Hand-in". We strongly recommend that you carefully work through these exercises before starting with the mandatory assignments.

I.1 Importance sampling.

Let us for this problem assume that you can only generate random numbers with a standard normal distribution, $q(x) = \mathcal{N}(x; 0, 1)$, but are interested in (possibly weighted) samples from $\pi(x) = \mathcal{U}(x; 0, 4)$. \mathcal{N} denotes the normal (Gaussian) distribution, and \mathcal{U} the uniform distribution.

- (a) Consider an importance sampler with proposal $q(x) = \mathcal{N}(x; 0, 1)$ and target $\pi(x)$. Is this a valid importance sampler?
- (b) Implement the suggested importance sampler with $N = 10\,000$. Plot the result as, for example, a kernel density estimate or a weighted histogram. What problems do you experience with your sampler, and how can you improve it?
- (c) For importance sampling it holds that an estimate of any test function ϕ is unbiased. Use the weighted samples to estimate the mean of the target ($\phi(x)$ is the identity function), and make a simulation study to support that this estimate is indeed unbiased (regardless of choice of proposal q). *Note that this claim holds for any finite number of samples N !* Confirm this by using only a small number of samples, e.g., $N = 10$, in your simulations.
- (d) We have so far assumed that we can evaluate $\pi(x)$ exactly. However, sometimes we can only evaluate the target $\pi(x)$ up to proportionality, as $\pi(x) = \frac{\tilde{\pi}(x)}{Z}$, where we can evaluate $\tilde{\pi}(x)$ but the normalizing constant Z is unknown to us. Actually, sometimes Z is the quantity of interest. (If, e.g., the target $\pi(x)$ is a posterior, then $\tilde{\pi}(x)$ is the likelihood times the prior, and Z is the marginal likelihood which can be useful for, e.g., model selection.)

Give an informal derivation to the estimator

$$\hat{Z} = \frac{1}{N} \sum_{i=1}^N \tilde{W}^i, \text{ where } \tilde{W}^i = \frac{\tilde{\pi}(X_i)}{q(X_i)}.$$

Hint: Start with $Z = \int \tilde{\pi}(x) dx$.

- (e) Implement the estimator to estimate Z if $\tilde{\pi}$ is the indicator function for the interval $[0, 4]$. Make a simulation study supporting the theoretical claim that \hat{Z} is an unbiased estimate of Z (for some finite N). Also explore how the variance of \hat{Z} changes with more or less 'good' choices of proposals q .
- (f) The estimator \hat{Z} is indeed unbiased, but we still have to take care when using it. A typical case is when we can only evaluate π up to proportionality (i.e., $\tilde{\pi}$), but are still interested in functionals of π . (Note that Z is actually a functional of $\tilde{\pi}$ rather than π .) Let us say that we are once again (cf. (c)) interested in estimating the mean of the target, but we can only access $\tilde{\pi}$, which here is the indicator function for the interval $[0, 4]$. Derive your

estimator for the mean of π for this case, and confirm with a simulation study that it is *not* unbiased. (Note: your proposal q has to be asymmetric around the true mean in order to see the effect. Use, e.g. $q(x) = \mathcal{N}(x; 0, 3^2)$ and a small number of samples, e.g., $N = 10$.) Also confirm that the bias vanishes as $N \rightarrow \infty$.

- (g) Show that the solution to the previous problem essentially corresponds to simply normalizing the sample weights w_i such that $\sum_i w_i = 1$ before reporting the mean estimate.

I.2 Importance sampling in higher dimensions.

Let us consider importance sampling in a D -dimensional space. Assume that you have access to D -dimensional random vectors from the standard multivariate normal distribution $\mathcal{N}(\vec{0}, I_D)$ (I_D is here the D -dimensional identity matrix), and use importance sampling to generate weighted samples from the D -dimensional multivariate uniform distribution $\pi(x) = \mathcal{U}(x; [-.5, .5]^D)$, i.e., the unit cube centered around the origin. (In this problem, you can evaluate the target $\pi(x)$ exactly.) Repeat the experiment for different values of D , and plot the proportion of samples with non-zero weights as a function of D . Conclude at what rate this fraction decreases (constant, linear, polynomial, exponential, etc.) based on both your simulations as well as a theoretical argument.

I.3 An important numerical aspect

Consider again the high-dimensional problem, but this time with focus on the numerical aspects in $D = 1000$ dimensions:

- (a) Consider importance sampling in \mathbb{R}^D , with target $\pi(x) = \mathcal{N}(x; 0, I_D)$ (i.e., x is a D -dimensional vector). Use the proposal $q(x) = \mathcal{N}(x; 0, 2^2 \cdot I_D)$. Generate $N = 10$ samples x^i and compute, for each sample, first $\pi(x^i)$ and then $q(x^i)$. Note that since the covariance matrices are diagonal, the densities factorize $\mathcal{N}(x; 0, I_D) = \prod_{k=1}^D \mathcal{N}(x_k; 0, 1)$ etc, where x_k is the k th component of x . Next, compute the weights $\tilde{w}^i = \frac{\pi(x^i)}{q(x^i)}$ and the normalized version of it $w^i = \frac{\tilde{w}^i}{\sum_i \tilde{w}^i}$. What normalized weights w^i do you obtain? Why?
- (b) The perhaps most useful remedy to this problem is to consider the logarithm of the weights instead. Hence, use that $\log \mathcal{N}(x; 0, I_D) = \sum_{k=1}^D \log \mathcal{N}(x_k; 0, 1)$, and compute $\log w^i = \log \pi(x^i) - \log q(x^i)$ instead. Do you experience the same problem?
- (c) If $\log \tilde{w}^i$ is too small, $\exp(\log \tilde{w}^i)$ may still be smaller than what the system can represent. In order to obtain a normalized version of the weights, explore the ‘trick’ of computing $\underline{w}^i = \exp(\log w^i - \max_j \{\log w^j\})$ instead, and use \underline{w}^i to obtain the normalized weights instead. Why is this a valid approach?

The aspects explored in this problem are highly relevant also for state-space filtering problems when the state dimension is small, but T large. Our recommendation is to always implement the logarithm of the weights when working with Monte Carlo!

I.4 Bootstrap particle filter for the stochastic volatility model

Consider the so-called stochastic volatility model

$$x_t | x_{t-1} \sim \mathcal{N}(x_t; \phi x_{t-1}, \sigma^2), \quad (1a)$$

$$y_t | x_t \sim \mathcal{N}(y_t; 0, \beta^2 \exp(x_t)), \quad (1b)$$

where the parameter vector is given by $\theta = \{\phi, \sigma, \beta\}$. Here, x_t denotes the underlying latent volatility (the variations in the asset price) and y_t denotes the observed scaled log-returns from some financial asset. The $T = 500$ observations that we consider in this task are log-returns from the NASDAQ OMX Stockholm 30 Index during a two year period between January 2, 2012 and January 2, 2014. We have calculated the log-returns by $y_t = 100[\log(s_t) - \log(s_{t-1})]$, where s_t denotes the closing price of the index at day t . The data is found in `seOMXlogreturns2012to2014.csv`. For more details on stochastic volatility models, see e.g. [1, 2].

Assume that the parameter vector is given by $\theta = \{0.98, 0.16, 0.70\}$. Estimate the marginal filtering distribution at each time index $t = 1, \dots, T$ using the bootstrap particle filter with $N = 500$ particles. Make a reasonable assumption about the initial state x_0 . Plot the mean of the filtering distribution $p(x_t | y_{1:t-1})$ at each time step and compare with the observations. Is the estimated volatility reasonable?

I.5 Bootstrap particle filter central limit theorem

Recall the bootstrap particle filter central limit theorem,

$$\sqrt{N} \left(\sum_{i=1}^N W_t^i \varphi(X_t^i) - I_t(\varphi) \right) \xrightarrow{d} \mathcal{N}(0, V_t(\varphi))$$

with

$$V_t(\varphi) = \sum_{k=0}^t \int \frac{p(x_k | y_{1:t})^2}{p(x_k | y_{1:k-1})} (I_{k,t}(\varphi | x_k) - I_t(\varphi))^2 dx_k, \quad (2)$$

$$I_t(\varphi) = \mathbb{E}[\varphi(X_t) | y_{1:t}],$$

$$I_{k,t}(\varphi | x_k) = \mathbb{E}[\varphi(X_t) | y_{k+1:t}, x_k] = \begin{cases} \varphi(x_t), & \text{if } k = t, \\ \int \varphi(x_t) p(x_t | x_k, y_{k+1:t}) dx_t, & \text{if } k < t. \end{cases}$$

Use the recursive expressions for the asymptotic variance (see lecture 5)

$$\tilde{V}_{t-1}(\varphi) = V_{t-1}(\varphi) + \text{Var}[\varphi(X_{t-1}) | y_{1:t-1}], \quad (\text{resampling})$$

$$\tilde{V}_t(\varphi) = \tilde{V}_{t-1}(\mathbb{E}[\varphi(X_t) | x_{t-1}]) + \mathbb{E}[\text{Var}[\varphi(X_t) | X_{t-1}] | y_{1:t-1}], \quad (\text{propagation})$$

$$V_t(\varphi) = \tilde{V}_t \left(\frac{p(y_t | x_t)}{p(y_t | y_{1:t-1})} \cdot \{\varphi(x_t) - \mathbb{E}[\varphi(X_t) | y_{1:t}]\} \right), \quad (\text{weighting})$$

to verify the additive expression for the variance (2).

Hint: Start by verifying that

$$V_t(\varphi_t) = V_{t-1}(\varphi_{t-1}) + \text{Var}[\zeta_t(X_t) | y_{1:t-1}]$$

for some functions $\varphi_{t-1}(x_{t-1})$ and $\zeta_t(x_t)$ which are expressed in terms of $\varphi_t(x_t) = \varphi(x_t)$. Then compute the two terms of the above expression explicitly using an inductive argument.

References

- [1] Marc Chesney and Louis Scottt. “Pricing European currency options: A comparison of the modified Black-Scholes model and a random variance model”. In: *Journal of financial and quantitative analysis* 24.3 (1989), pp. 267–284.
- [2] Angelo Melino and Stuart M. Turnbull. “Pricing foreign currency options with stochastic volatility”. In: *Journal of Econometrics* 45.1–2 (1990), pp. 239–265.