High-dimensional SMC

Christian A. Naesseth

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Outline

Aim: Understand some of the problems with SMC in high dimensions, and ways of mitigating them.

1. High-dimensional SSM
2. Examples
3. Weight degeneracy
4. Mitigating the problem:
   - Improving the proposal
   - Localization
   - Ensemble transform filters

Several talks, and a session, during the workshop!

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High-dimensional SSM

High-dimensional state-space model:

$$x_t|x_{t-1} \sim p(x_t|x_{t-1}),$$
$$y_t|x_t \sim p(y_t|x_t),$$

with

$$x_t = \left(x_{t,1}, \ldots, x_{t,d}, \ldots, x_{t,D_x}\right),$$
$$y_t = \left(y_{t,1}, \ldots, y_{t,d}, \ldots, y_{t,D_y}\right),$$

where D_x and/or D_y are “large”.

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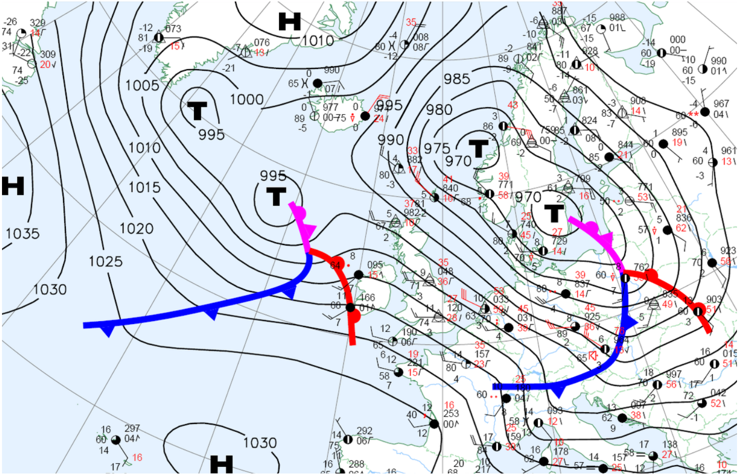
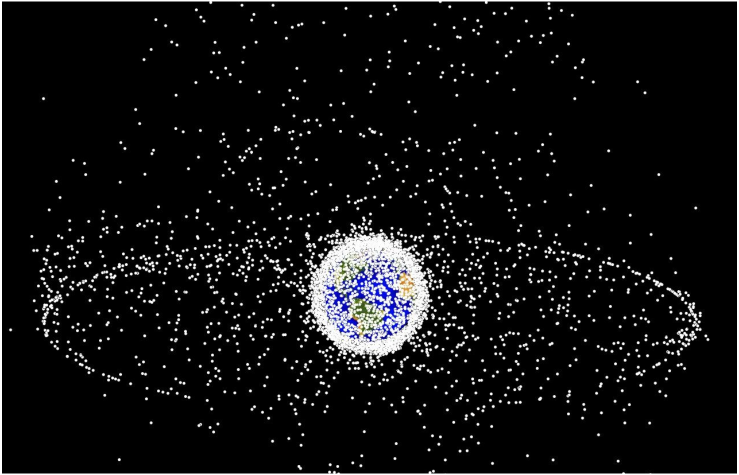
Example: Spatio-temporal statistics



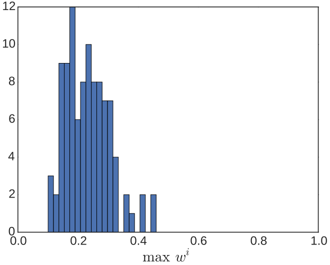
Example: Target tracking



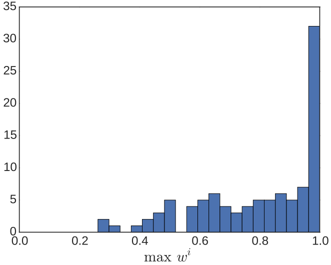
Weight degeneracy: $D_x = D_y$

$$x \sim \mathcal{N}(0, I),$$
$$y|x \sim \mathcal{N}(x, I)$$

Sampling from the prior, weighting by the likelihood we get:



$D_x = D_y = 5$



$D_x = D_y = 50$

Weight degeneracy: Theory

$$\tau^2 = \mathrm{Var}\left[-\log \widetilde{w}_t^i\right]$$

$$\mathbb{E}\left[\frac{1}{\max_i w_t^i}\right] \sim 1 + \frac{\sqrt{2\log N}}{\tau}$$

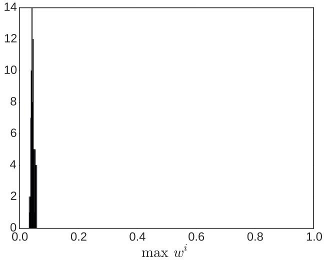
Obstacles to High-Dimensional Particle Filtering, Snyder et. al., 2008

Performance Bounds for Particle Filters Using the Optimal Proposal, Snyder et. al., 2015

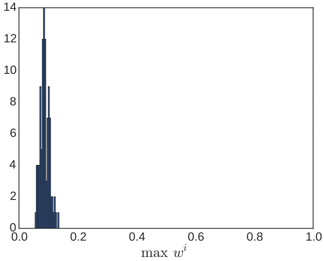
Weight degeneracy: $D_x >> D_y$

$$x \sim \mathcal{N}(0, I),$$
$$y|x \sim \mathcal{N}\left((x_1, x_2)^\top, I\right)$$

Sampling from the prior, weighting by the likelihood we get:



$D_x = 5,\ D_y = 2$

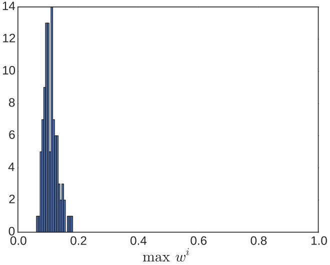


$D_x = 50,\ D_y = 2$

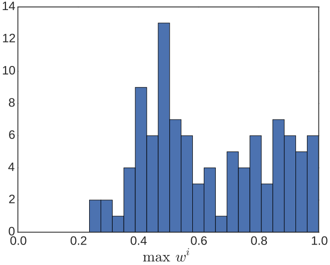
Weight degeneracy: $D_x >> D_y$

$$x \sim \mathcal{N}(0, I),$$
$$y|x \sim \mathcal{N}(Cx, I),$$

Sampling from the prior, weighting by the likelihood we get:



$D_x = 5,\ D_y = 2$

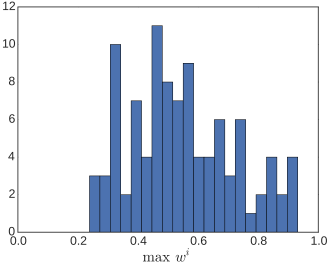


$D_x = 50,\ D_y = 2$

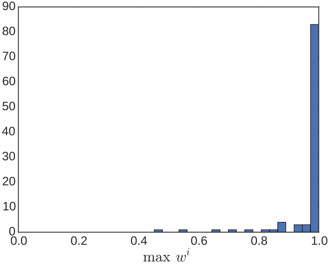
Weight degeneracy: $D_x << D_y$

$$x \sim \mathcal{N}(0, I),$$
$$y|x \sim \mathcal{N}(Cx, I),$$

Sampling from the prior, weighting by the likelihood we get:



$D_x = 5,\ D_y = 5$



$D_x = 5,\ D_y = 50$

Weight degeneracy: Summary

- To prevent *weight degeneracy* we need

$$N = e^{\frac{\alpha}{2}\tau^2}, \text{ where } \tau^2 = \mathrm{Var}\big[-\log \widetilde{w}_t^i\big]$$

- τ^2 depends on the interaction between x_t and y_t
  Ex. $p(x) = \mathcal{N}(x|0, I)$

$$\begin{aligned} p(y|x) &= \mathcal{N}(y|x, I) && \Longrightarrow \tau^2 = \mathcal{O}(D_y) = \mathcal{O}(D_x) \\ p(y|x) &= \mathcal{N}(y|(x_1, x_2)^\top, I) && \Longrightarrow \tau^2 = \mathcal{O}(D_y) = \mathcal{O}(2) \end{aligned}$$

- Good effective sample size is necessary, *not* sufficient

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Improving the proposal

The **fully adapted particle filter**,

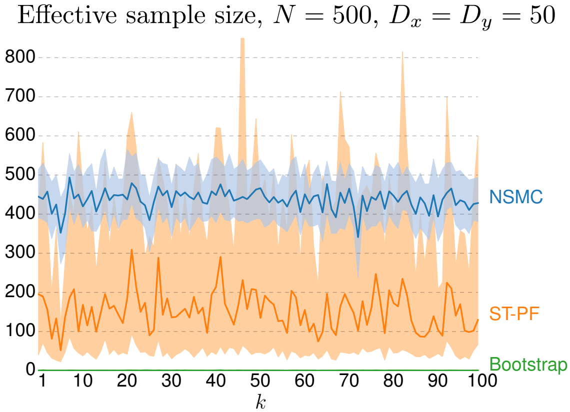
$$q(x_t|x_{t-1}, y_t) = p(x_t|x_{t-1}, y_t),$$

can improve $\tau^2 (= \mathrm{Var}\big[-\log p(y_t|x^i_{t-1})\big])$ by a factor of $\sim 10 - 100$.

Nested sequential Monte Carlo allows for an exact approximation (cf. PMCMC).

Nested Sequential Monte Carlo Methods, Naesseth et. al., 2015
High-dimensional Filtering using NSMC, Naesseth et. al., 2016

Improving the proposal



Nested Sequential Monte Carlo Methods, Naesseth et. al., 2015

A stable PF for a class of high-dimensional SSM, Beskos et. al., 2017

Localization

Idea: Estimate the marginals $p(x_{t,d}|y_{1:t})$, $d = 1, \dots, D_x$.

- Typical approximation

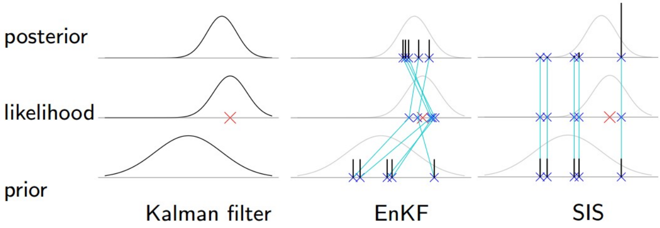
$$p(x_{t-1,\neg d}|y_{1:t-1}) \approx \prod_{k \neq d} p(x_{t-1,k}|y_{1:t-1})$$

- Run a separate particle filter for each dimension d
- Exchange information to propagate and weight particles

Particle Filtering for High-Dimensional Systems, Djurić & Bugallo, 2013

Ensemble transform filters

Idea: Sample and *move* an ensemble of points.



Understanding the Ensemble Kalman Filter, Katzfuss et. al., 2016

Ensemble Kalman filter

Kalman filter for LG-SSM

- Time update:

$$\widehat{x}_{t|t-1} = A\widehat{x}_{t-1|t-1},$$
$$P_{t|t-1} = AP_{t-1|t-1}A^{\top} + Q$$

- Measurement update:

$$\widehat{x}_{t|t} = \widehat{x}_{t|t-1} + K_t \left(y_t - C\widehat{x}_{t|t-1}\right),$$
$$P_{t|t} = (I - K_t C)\, P_{t|t-1},$$

where $K_t = P_{t|t-1}C^{\top}\left(CP_{t|t-1}C^{\top} + R\right)^{-1}$

Ensemble Kalman filter

Ensemble Kalman filter for LG-SSM

- Time update:

$$\widehat{x}^i_{t|t-1} = A\widehat{x}^i_{t-1|t-1} + v^i_t,$$
$$\widehat{P}_{t|t-1} = \mathrm{Cov}\left[\widehat{x}^{1:N}_{t|t-1}\right]$$

- Measurement update:

$$\widehat{x}^i_{t|t} = \widehat{x}^i_{t|t-1} + \widehat{K}_t\left(y_t - e^i_t - C\widehat{x}^i_{t|t-1}\right),$$

where $\widehat{K}_t = \widehat{P}_{t|t-1}C^\top\left(C\widehat{P}_{t|t-1}C^\top + R\right)^{-1}$

Ensemble Kalman filter

Ensemble Kalman filter for general SSM

- Time update:

$$\widehat{x}^i_{t|t-1} = f\left(\widehat{x}^i_{t-1|t-1}, v^i_t\right),$$

$$\widehat{P}_{t|t-1} = \mathrm{Cov}\left[\widehat{x}^{1:N}_{t|t-1}\right]$$

- Measurement update:

$$\widehat{x}^i_{t|t} = \widehat{x}^i_{t|t-1} + \widehat{K}_t\left(y_t - g\left(\widehat{x}^i_{t|t-1}, e^i_t\right)\right),$$

$$\widehat{K}_t = \mathrm{Cov}\left[\widehat{x}^{1:N}_{t|t-1}, g\left(\widehat{x}^{1:N}_{t|t-1}, e^{1:N}_t\right)\right]\mathrm{Cov}\left[g\left(\widehat{x}^{1:N}_{t|t-1}, e^{1:N}_t\right)\right]^{-1}$$

Take-home messages

- Weight degeneracy depends on interaction between x_t and y_t
- N needs to scale exponentially with $\text{Var}\big[-\log \widetilde{w}_t^i\big]$
- Ways of mitigating:
  - Improved proposal
  - Localization
  - Ensemble transform filters
  - ...

Contact: **christian.a.naesseth@liu.se**

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References

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High-dimensional Filtering using Nested Sequential Monte Carlo, Naesseth et. al., 2016

A stable particle filter for a class of high-dimensional state-space models, Beskos et. al., 2017

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