Qual Quant (2010) 44:817–832
DOI 10.1007/s11135-007-9081-7

ORIGINAL PAPER

Applying jump-diffusion processes to liquidate and convert venture capital

Tyrone T. Lin · Chuan-Chuan Ko · Chia-Wen Chang

Published online: 15 February 2007

Abstract This study attempts to apply real options and expand the model designed by Lin and Huang [Lin, T.T., Huang, Y.T.: J. Technol. Manage. **8**(3), 59–78 (2003)], which helps venture capital (VC) companies to optimize project exit decisions. The expected discounted factor and a jump-diffusion process combine to assess the value of a start-up company, and determine the threshold of the exit timing of liquidation or convertibility for establishing the optimal disinvestment evaluation model for VC companies. When the project value is below V_L^*, the VC company carries out liquidation, but when the project value exceeds V_C^*, the VC company performs convertibility. The project value is ranging between $\left(V_L^*, V_C^*\right)$, and the best choice is holding the decision and waiting to carry out the rights of liquidation and convertibility next time. Besides, this work attempts to identify the expected discounted time in terms of the investment time for VC companies.

Keywords Venture capital · Jump-diffusion process · Discounted factor · Liquidation · Convertibility

1 Introduction

According to the Taiwan Venture Capital Yearbook (Year 2003), the global stock market entered a bear market after 2000 and the capital market deteriorated at the same time. The global economy faced a recession during this period, and began to turn around after 2002.

T. T. Lin (✉)
Department of International Business, National Dong Hwa University,
No. 1, Sec. 2, Da Hsueh Rd., Shou-Feng, Hualien, 974 Taiwan
e-mail: tjlin@mail.ndhu.edu.tw

C.-C. Ko
Institute of Business & Management, National Chao Tung University, Taipei, Taiwan

C.-W. Chang
Graduate School of Management, Ming Chuan University, Taipei, Taiwan

However, the initial public offering (IPO) market in Taiwan remained strong, despite the drop of almost 20% in the Taiwan stock index. 203 IPOs occurred during 2002, representing a 33.6% increase compared to 152 IPOs during 2001. This phenomenon contrasted with major global capital markets, including the New York Stock Exchange, the NASDAQ, and the London Stock Exchange, which all showed a decrease in IPOs during this period.

Generally, Taiwanese Venture Capital (VC) companies are highly active in IPOs and play important roles in creating new entrepreneurs. One third of new IPOs involve VC companies, which means that Taiwanese VC companies have much more chances to succeed and exit channels in the investments of start-up companies. According to the Taiwan Venture Capital Yearbook (Year 2003), 27.62% of VC was invested during the entrepreneur stage, 47.2% during the growth stage, and 20.33% during the mature stage. In addition to capital investments, VC companies will assist companies at the seed and early stages in business management and development by leveraging their professional experiences, and help companies to launch IPOs as early as possible. Pandey and Angela (1996) also noted that Taiwanese investment companies only put 10% of their capitals into VC Funds. According to Andreas and Uwe (2001), a couple of exit channels of start-up companies exist for VC companies, including (1) Liquidation, (2) Buy-Back: Invested companies buying back outstanding shares held by VC companies, (3) Secondary Purchase: Selling shares to institutional investors, (4) Trade Sale: Selling the shareholdings of the invested companies to other companies, and (5) IPO: the most valuable method among those exit channels of start-up companies. However, according to Mason and Harrison (2002), Trade Sale of start-up companies is the most widely used channel by business angels for realizing capital gains. According to Andres and Uwe (2001), a correlation exists between exit decision of start-up companies and contract design for VC companies. VC companies must give up the (exit) shareholdings of the invested enterprises during a certain time frame. Sometimes, VC companies and entrepreneurs have different opinions regarding exit plans (e.g. IPO or Trade Sale). Outside investors agree to the optimal exit plan of issuing convertible warranties. Thus, VC companies have frequently used convertible warranties in their investments. During the expansion period, VC companies appear to exit the current investment projects to attract new investors and increase total profits.

By applying the real options approach (ROA), reconsidering the discounted factor and following the jump-diffusion process in cash flows of the company evaluation model, this assay extends the model of Lin and Huang (2003) to establish the exit strategy and model for VC companies in start-up investment projects. This assay is also used to determine the optimal exit point (liquidation or conversion) and establish the evaluation model for the optimal exit plan of start-up companies. Furthermore, Lin et al. (2006) explored exit decision of financial institutions in duopolistic loan market with game options approach and analyzed how uncertainty influences loan decisions for the financial institution.

The remainder of this paper is organized as follows: Section 2 deals with model establishment, including probing the optimal liquidation and conversion model based on the continuous basic model and jump-diffusion process. Section 3 then conducts the relative value analysis and sensitivity analysis. Finally, Section 4 presents conclusions.

2 Proposed model

This assay assesses the optimal VC exit plan using ROA. By applying the new discounted factor concept and considering Poisson's jump-diffusion process, this study also discusses how VC companies obtain the maximum investment value by deciding to liquidate or convert to preferred stocks upon expiration of the investment contract.

2.1 Continuous model

Assuming project value of VC investment in a start-up company is defined as $V(t)$, the value variation growth over time is described by the Geometric Brownian Motion (GBM) as follows:

$$dV(t) = \alpha V(t)dt + \sigma V(t)dW(t). \tag{1}$$

Among those factors, α: drift over time, σ: volatility over time, $dW(t)$ is the increment of standard Winner process $W(t)$ of zero mean and unit standard deviation $\sqrt{dt}$.

The project value of VC investment in the start-up company $F_I(V)$ after dt, the increment of project value $dF_I(V)$ is calculated according to Itô's Lemma(Itô 1951),

$$dF_I(V) = F_{IV}(V)dV + \frac{1}{2}F_{IVV}(V)dV^2. \tag{2}$$

Assuming VC companies do not have any risk preference, following incremental time dt, they will expect the project value to equal the risk-free interest r_f multiplied by $F_I(V)dt$, which is

$$r_f \times F_I(V)dt = E\left[dF_I(V)\right] = \left\{\alpha V F_{IV}(V) + \frac{1}{2}\sigma^2 V^2 F_{IVV}(V)\right\} dt. \tag{3}$$

By slightly manipulating the above equation, we obtain:

$$\frac{1}{2}\sigma^2 V^2 F_{IVV}(V) + \alpha V F_{IV}(V) - r_f \times F_I(V) = 0. \tag{4}$$

Equation 4 is a second order homogeneous differential function, and its general solution form has the format $F_I(V) = AV^u$. Applying the solution to the above equation to obtain the following quadratic equation:

$$\frac{1}{2}\sigma^2 u^2 + \left(\alpha - \frac{1}{2}\sigma^2\right) u - r_f = 0. \tag{5}$$

The two roots of the above equation are:

$$u_1 = \frac{\left(\frac{1}{2}\sigma^2 - \alpha\right) + \sqrt{\left(\frac{1}{2}\sigma^2 - \alpha\right)^2 + 2\sigma^2 r_f}}{\sigma^2} > 1; \tag{6}$$

$$u_2 = \frac{\left(\frac{1}{2}\sigma^2 - \alpha\right) - \sqrt{\left(\frac{1}{2}\sigma^2 - \alpha\right)^2 + 2\sigma^2 r_f}}{\sigma^2} < 0. \tag{7}$$

The project value upon liquidation is:

$$F_{I1}(V) = a_1 V^{u_1} + b_1 V^{u_2}. \tag{8}$$

The liquidation value is zero when the project value is zero, VC companies do not need to pay an additional liquidation fee, that is, $F_{I1}(0) = 0$. b_1 in Eq. 8 must be zero. Thus, the contract value upon liquidation is $F_{I1}(V) = a_1 V^{u_1}$.

With the same inference as the liquidation value, the contract value upon conversion is $F_{I2}(V) = a_2 V^{u_1} + b_2 V^{u_2}$. Among those exit plans, if VC companies choose to liquidate, they will incur a liquidation cost L_0; if VC companies choose to give up the right to liquidate and wait for the right of conversion, they will incur two costs, the conversion cost C_0 and the liquidation cost upon conversion $L_0 \times (V_C^*/V_L^*)^{\beta_{11}}$, where $(V_L^*/V_C^*)^{\beta_{11}}$ is the present value factor. See Appendix 1 for more detailed inferences.

From the above inference, the contract value is conducted upon both liquidation $F_I(V_L^*)$ and conversion $F_I(V_C^*)$ to calculate the optimal exit threshold for VC companies to select liquidation or conversion. Refer to Dixit and Pindyck (1994) for finding the threshold and project value using value-matching and smooth-pasting conditions as follows:

According to the value matching condition (VMC), assuming the optimal liquidation threshold is V_L^*, the liquidation threshold upon liquidation V_L^* minus the liquidation cost L_0 should equal the project value upon liquidation $F_{I1}(V_L^*)$. According to the smooth pasting condition (SPC), the incremental project value should match the first order of the differential function when the project value at the threshold point produces the same marginal profit for exit plan. The above two conditions are shown as follows:

$$\begin{cases} VMC\colon V_L^* - L_0 = F_{I1}(V_L^*); \\ SPC\colon \frac{d[V_L^* - L_0]}{dV_L^*} = \frac{dF_{I1}(V_L^*)}{dV_L^*}. \end{cases} \tag{9}$$

Based on Eq. 9 and after some manipulation, the optimal liquidation threshold V_L^* and the project value coefficient a_1 upon liquidation are conducted as follows:

$$V_L^* = \frac{u_1}{u_1 - 1} L_0; \tag{10}$$

$$a_1 = \frac{1}{u_1} (V_L^*)^{1-u_1}. \tag{11}$$

Assuming the optimal conversion threshold is V_C^*, the project value is evaluated by V_C^* upon conversion minus the liquidation cost C_0 and $L_0 \times (V_C^*/V_L^*)^{\beta_{11}}$ should equal the project value upon conversion $F_{I2}(V_C^*)$. The above result is the VMC.

Moreover, according to SPC, the incremental project value should match the first order differential function. The two conditions are shown as below:

$$\begin{cases} VMC\colon V_C^* - \left[C_0 + L_0 \left(\frac{V_C^*}{V_L^*} \right)^{\beta_{11}} \right] = F_{I2}(V_C^*); \\ SPC : \frac{d\left[V_C^* - \left(C_0 + L_0 \left(\frac{V_C^*}{V_L^*} \right)^{\beta_{11}} \right) \right]}{dV_C^*} = \frac{F_{I2}(V_C^*)}{dV_C^*}. \end{cases} \tag{12}$$

Furthermore, upon conversion, the project value minus the liquidation cost and the present value of the conversion cost is assumed to equal the liquidation contract value $F_{I2}(V_L^*) = a_2 \left(V_L^*\right)^{u_1} + b_2 \left(V_L^*\right)^{u_2}$; restated,

$$V_L^* - \left[L_0 + C_0 \left(\frac{V_L^*}{Vc^*} \right)^{\beta_{11}} \right] = F_{I2}(V_L^*). \tag{13}$$

Rearrange the equations upon liquidation and conversion as follows:

$$\begin{cases} V_L^* - a_1 \left(V_L^*\right)^{u_1} - L_0 = 0; \\ 1 - a_1 u_1 \left(V_L^*\right)^{u_1 - 1} = 0, \end{cases} \tag{14}$$

and

$$\begin{cases} V_C^* - \left[C_0 + L_0\left(\frac{V_C^*}{V_L^*}\right)^{\beta_{11}}\right] - \left[a_2\left(V_C^*\right)^{u_1} + b_2\left(V_C^*\right)^{u_2}\right] = 0; \\ 1 - L_0\beta_{11}\left[\frac{(V_C^*)^{\beta_{11}-1}}{(V_L^*)^{\beta_{11}}}\right] - \left[a_2u_1\left(V_C^*\right)^{u_1-1} + b_2u_2\left(V_C^*\right)^{u_2-1}\right] = 0; \\ V_L^* - \left[L_0 + C_0\left(\frac{V_L^*}{V_C^*}\right)^{\beta_{11}}\right] - \left[a_2\left(V_L^*\right)^{u_1} + b_2\left(V_L^*\right)^{u_2}\right] = 0. \end{cases} \tag{15}$$

Since no closed form solution exists for the optimal conversion threshold V_C^* and the conversion project value coefficient a_2, b_2 upon conversion, the solution is inferred via the value analysis method. When the project value is below V_L^*, VC companies will choose to liquidate the investment in the start-up company; when the project value exceeds V_C^*, the VC companies will choose to convert and exercise the option. However, when the project value is between V_L^* and V_C^*, the optimal strategy is to maintain the current situation and continue evaluating the investment in the start-up company to determine the optimal solution.

2.2 Jump-diffusion model

Assuming project value of VC investment in the start-up company is defined as $V(t)$, which is the same definition as the continuous process model mentioned above. The value variation growth over time is determined by GBM for the continuous process and by Poisson process for the discreet process based on the jump-diffusion model as follows:

$$dV(t) = \alpha V(t)dt + \sigma V(t)dW(t) - \theta V(t)dq(t). \tag{16}$$

Among those factors of the project value,α: drift over time, σ: volatility over time, and $dW(t)$: the increment of standard Winner process $W(t)$ of zero mean and unit standard deviation $\sqrt{dt}$. The jump-process follows Poisson process with an arrival rate of λ_1:

$$dq\left(t\right) = \begin{cases} 1, & \text{with prob. } \lambda_1 dt; \\ 0, & \text{with prob. } 1 - \lambda_1 dt, \end{cases}$$

θ_1: the magnitude of influence for the jump size in the jump-process.

Given the project value of the investment $F_I(V)$ after dt, the increment of the project value is calculated according to Itôs Lemma (Itô 1951)

$$dF_I(V) = F_{IV}(V)dV + \frac{1}{2}F_{IVV}(V)dV^2 - \lambda_1\left[F_I(V) - F_I(\theta_1 V)\right]dt. \tag{17}$$

Assuming VC companies have no risk preference, following incremental time dt, they expect the project value to equal the risk-free interest r_f multiplied by $F_I(V)dt$, namely:

$$\begin{aligned} r_f \times F_I(V)dt &= E\left[dF_I(V)\right] \\ &= \left\{\alpha V F_{IV}(V) + \frac{1}{2}\sigma^2 V^2 F_{IVV}(V) - \lambda_1 F_I(V) + \lambda_1 F_I(\theta_1 V)\right\} dt. \end{aligned} \tag{18}$$

With some manipulation, we obtain:

$$\frac{1}{2}\sigma^2 V^2 F_{IVV}(V) + \alpha V F_{IV}(V) - (\lambda_1 + r_f)F_I(V) + \lambda_1 F_I(\theta_1 V) = 0. \tag{19}$$

The general solution to Eq. 19 can be expressed as $F_I(V) = AV^u$. Applying the general solution form to the above equation with some manipulation, we obtain the following quadratic equation:

$$\frac{1}{2}\sigma^2u^2 + \left(\alpha - \frac{1}{2}\sigma^2\right)u - (\lambda_1 + r_f) + \lambda_1\theta_1^u = 0. \tag{20}$$

The project value upon liquidation is:

$$F_{I1}(V) = a_1V^{u_1} + b_1V^{u_2}. \tag{21}$$

The liquidation value is zero when the project value is zero, and VC companies do not need to pay an additional fee for liquidation (as seen in Section 2.1). To ensure the consistency with the contract value being converted into zero, b_1 in Eq. 21 must be zero. Thus, the contract value upon liquidation is $F_{I1}(V) = a_1V^{u_1}$. With the same inference of the liquidation value, the contract value upon conversion is $F_{I2}(V) = a_2V^{u_1} + b_2V^{u_2}$. Among those exit plans, if VC companies choose to liquidate, a liquidation cost of L_0 will occur; if VC companies choose to give up the right to liquidate and wait for the conversion right, two costs will occur, the conversion cost C_0 and the liquidation cost upon conversion $L_0 \times (V_C^*/V_L^*)^{\beta_{11}}$. This study conducts the contract value upon both liquidation $F_I(V_L^*)$ and conversion $F_I(V_C^*)$ to determine the optimal exit threshold for VC companies to select liquidation or conversion.

According to VMC, assuming the optimal liquidation threshold is V_L^*, the liquidation threshold upon liquidation V_L^* minus the liquidation cost L_0 should equal the project value upon liquidation $F_{I1}(V_L^*)$. According to SPC, the incremental project value should be equal to the first order of differential function, which matches the same marginal profit in threshold point. The above two conditions are demonstrated as follows:

$$\begin{cases} VMC\colon V_L^* - L_0 = F_{I1}(V_L^*), \\ SPC\colon \frac{d[V_L^*-L_0]}{dV_L^*} = \frac{dF_{I1}(V_L^*)}{dV_L^*}. \end{cases} \tag{22}$$

Based on the above description, this study conducts the optimal liquidation threshold V_L^* and the project value coefficient a_1 upon liquidation as follows:

$$V_L^* = \frac{u_1}{u_1 - 1}L_0; \tag{23}$$

and

$$a_1 = \frac{1}{u_1}(V_L^*)^{1-u_1}. \tag{24}$$

Based on the same inference as previously mentioned, assuming the optimal conversion threshold is V_C^*, the project value V_C^* upon conversion minus the liquidation cost C_0 and $L_0 \times (V_C^*/V_L^*)^{\beta_{11}}$ should equal the project value upon conversion $F_{I2}(V_C^*)$. The incremental project value should be the same as the first order of the differential function, which can match the marginal profit at decision point. The analytical result is as below:

$$\begin{cases} VMC: V_C^* - \left[C_0 + L_0\left(\frac{V_C^*}{V_L^*}\right)^{\beta_{11}}\right] = F_{I2}(V_C^*), \\ SPC: \frac{d\left[V_C^* - \left(C_0 + L_0\left(\frac{V_C^*}{V_L^*}\right)^{\beta_{11}}\right)\right]}{dV_C^*} = \frac{F_{I2}(V_C^*)}{dV_C^*}. \end{cases} \tag{25}$$

Furthermore, upon conversion, assuming the project value minus the liquidation cost and the present value of the conversion cost equals the liquidation contract value $F_{I2}(V_L^*) = a_2\left(V_L^*\right)^{u_1} + b_2\left(V_L^*\right)^{u_2}$, it will become:

$$V_L^* - \left[L_0 + C_0\left(\frac{V_L^*}{V_C^*}\right)^{\beta_{11}}\right] = F_{I2}(V_L^*). \tag{26}$$

Rearrange the liquidation and conversion equations as follows:

$$\begin{cases} V_L^* - a_1\left(V_L^*\right)^{u_1} - L_0 = 0, \\ 1 - a_1 u_1\left(V_L^*\right)^{u_1 - 1} = 0, \end{cases} \tag{27}$$

and

$$\begin{cases} V_C^* - \left[C_0 + L_0\left(\frac{V_C^*}{V_L^*}\right)^{\beta_{11}}\right] - \left[a_2\left(V_C^*\right)^{u_1} + b_2\left(V_C^*\right)^{u_2}\right] = 0, \\ 1 - L_0\beta_{11}\left[\frac{(V_C^*)^{\beta_{11}-1}}{(V_L^*)^{\beta_{11}}}\right] - \left[a_2 u_1\left(V_C^*\right)^{u_1-1} + b_2 u_2\left(V_C^*\right)^{u_2-1}\right] = 0, \\ V_L^* - \left[L_0 + C_0\left(\frac{V_L^*}{V_C^*}\right)^{\beta_{11}}\right] - \left[a_2\left(V_L^*\right)^{u_1} + b_2\left(V_L^*\right)^{u_2}\right] = 0. \end{cases} \tag{28}$$

Since no closed form solution exists for the optimal conversion threshold V_C^* and the conversion project value coefficient a_2, b_2 upon conversion, the solution is inferred using the value analysis method. When the project value is below V_L^*, VC companies choose liquidation; when the project value exceeds V_C^*, they choose conversion and exercise the option. However, when the project value is between V_L^* and V_C^*, the optimal strategy is to maintain the current status and keep evaluating the project value to determine the optimal solution.

2.3 Expected discount time

First, a discounted time $E\left[T_1^*\right]$ interval is assumed to exist in $\left[V_L^*, V_C^*\right]$. Referring to Appendix 1, the expected discounted factor is defined as $E[\mathrm{e}^{-rT_1^*}] = (V_L^*/V_C^*)^{\beta_{11}}$. The change in the logarithm using the rule of Jensen's inequality ($E(-rT_1^*) \geq \log E(\mathrm{e}^{-rT_1^*})$) is derived as follows: $-r \times E\left[T_1^*\right] \geq \beta_{11} \times \log\left(V_L^*/V_C^*\right)$. After rearranging the equation, the expected discounted time can be derived as:

$$E\left[T_1^*\right] \geq \frac{\beta_{11}}{r}\log\left(\frac{V_C^*}{V_L^*}\right) \tag{29}$$

Secondly, an expected discounted time $E\left[T_2^*\right]$ is assumed to exist in interval $\left[V_0, V_L^*\right]$. From Appendix 2, the expected discounted factor is calculated by $E[\mathrm{e}^{-rT_2^*}] = (V_0/V_L^*)^{\beta_{21}}$. Rearranging the equation based on the same inference, the expected discounted time $E\left[T_2^*\right]$ is greater than or equals:

$$\frac{\beta_{21}}{r}\log\left(\frac{V_L^*}{V_0}\right) \tag{30}$$

2.4 Model comparison

Lin and Huang (2003) discuss the exit plans for VC companies using ROA based on Cossin et al. (2002). This proposed model compares exercising liquidation with exercising conversion to obtain the optimal exit evaluation model by establishing two exit thresholds. This assay continues the research of Lin and Huang (2003), based on mutual preference of risk by revising the discounted factor and applying the Jump process to identify the threshold

Table 1 Model comparison

	Andreas and Uwe (2001)	Lin and Huang (2003)	Proposed model
Approach	Utility function	ROA	ROA
Optimal threshold	None	V_R^*: Optimal redemption threshold	V_L^*: Optimal liquidation and convertibility threshold
Discounted Factor	None	$(\frac{V_C^*}{V_L^*})^{\frac{r}{\alpha}}$: discounted factor $\frac{r}{\alpha}$	$(\frac{V_C^*}{V_L^*})^{\beta_{11}}$: discounted factor β_{11}
Project of value path		According to geometric Brownian motion $dV_t = \alpha V_t dt + \sigma V_t dW(t)$	1. Continuous model-according to geometric Brownian motion $Dv_{(t)} = \alpha V_{(t)}dt + \sigma V_{(t)}dW_{(t)}$ 2. Jump-diffusion model-according to geometric Brownian motion (GBM) and Poisson process $dV_{(t)} = \alpha V_{(t)}dt + \sigma V_{(t)}dW_{(t)} - \theta V_{(t)}dq_{(t)}$

upon liquidation and conversion. In addition, the expected time required to reach the decision threshold is derived by applying the expected discounted factor to the expected discount point. The model comparison is listed as Table 1.

3 Numerical and sensitivity analysis

Since it is difficult to obtain data from single case for value analysis, this study applies the relative variance and coefficients from the Taiwan Venture Capital Yearbook, published by the Taiwan Venture Capital Association (Year 2003). The numerical solution is also obtained using the non-liner function solution from the software polyamth5.1.

3.1 Continuity model

Now we apply continuity model to turn to numerical solutions and sensitivity analysis to verify these intuition.

3.1.1 Numerical analysis

First, the parameters β_{11} and u_1, u_2 of the discounted factor $u_1 > 1, u_2 < 0$ are calculated. The simulated parameter data are as follows: volatility $\sigma = 0.790$; drift rate $\alpha = 3.150\text{E-}04$; risk-free interest rate $r_f = 0.020$; discount interest rate $r = 0.220$; project value $V_0 = 21$ (million NT dollars). Meanwhile, the simulation results are $\beta_{11} = 1.476$, $u_1 = 1.059$, and $u_2 = -0.060$. Other variables by hypothesis and parameter date are liquidation cost $L_0 = 3.25$ million NT dollars and conversion cost $C_0 = 4.2$ million NT dollars. The results are obtained as: $V_L^* = 57.9$ million NT dollars; $V_C^* = 133.9$ million NT dollars; $a_1 = 0.975$; $a_2 = 0.816$; $b_2 = 0.074$. Thus, if the invested project value is lower than the optimal liquidation threshold $V_L^* = 57.9$ million NT dollars, the optimum exit plan for VC companies is to liquidate the project; if the invested project value exceeds the optimal conversion threshold $V_C^* = 133.9$ million NT dollars, the optimum exit plan for VC companies is to exercise the

Fig. 1 Drift rate α to the optimal threshold

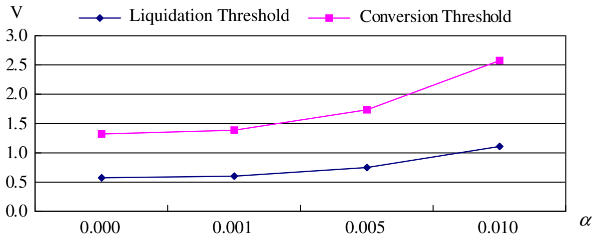
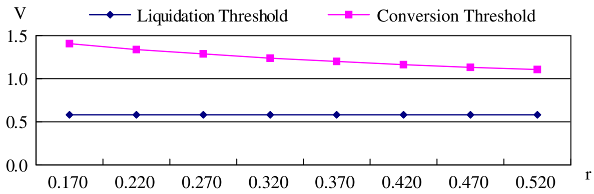


Fig. 2 Discounted interest rates r to the optimal threshold



conversion option. Moreover, the contract value upon the liquidation threshold is:

$$F_1(V_L^*) = a_1 \times \left(V_L^*\right)^{u_1} = 0.975 \times (0.579)^{1.059} = 0.547 \text{ (million NT dollars)}.$$

The contract value upon the conversion threshold is:

$$\begin{aligned} F_2(V_C^*) &= a_2 \left(V_C^*\right)^{u_1} + b_2 \left(V_C^*\right)^{u_2} = 0.816 \times (1.339)^{1.059} + 0.074 \times (1.339)^{-0.060} \\ &= 1.184 \text{ (million NT dollars)}. \end{aligned}$$

3.1.2 Sensitivity analysis

This section performs a sensitivity analysis of the simulated variances in this assay, including: drift rate α; discounted interest rate r; liquidation cost L_0; conversion cost C_0; parameters u_1, u_2, and β_{11}. Under the hypothesis that the other parameter's variances are consistent, the variable is varied to identify the changes in optimal thresholds V_L^* and V_C^* for both magnitude and direction.

From Fig. 1, both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* increase with increasing α, and the incremental also increases with increasing α.

Figure 2 shows that the optimal liquidation threshold V_L^* is not influenced by r when the optimal conversion threshold V_C^* decreases with increasing r.

Figure 3 displays that both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* increase with increasing L_0.

Figure 4 illustrates that the optimal liquidation threshold V_L^* is not influenced by C_0 since the optimal conversion threshold V_C^* increases with increasing C_0.

Figure 5 demonstrates that both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* decrease with increasing u_1 (but in less magnitude than u_1).

Figure 6 shows that the optimal liquidation threshold V_L^* is not influenced by u_2 since the optimal conversion threshold V_C^* increases with increasing u_2. However, the increase is minimal.

Figure 7 illustrates that the optimal liquidation threshold V_L^* is not influenced by β_{11} when the optimal conversion threshold V_C^* decreases with increasing β_{11} (but in less magnitude than β_{11}).

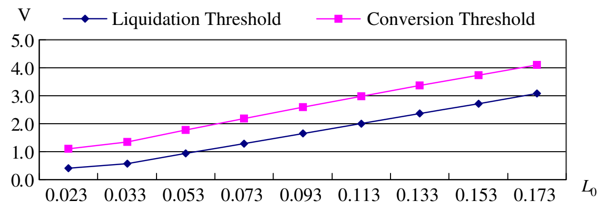


Fig. 3 Liquidation cost L_0 to the optimal threshold

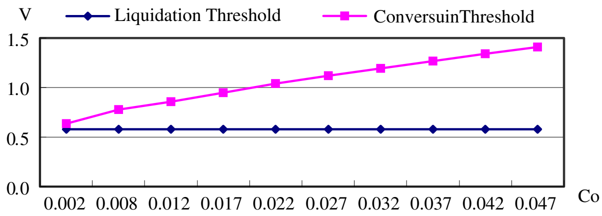


Fig. 4 Conversion cost C_0 to the optimal threshold

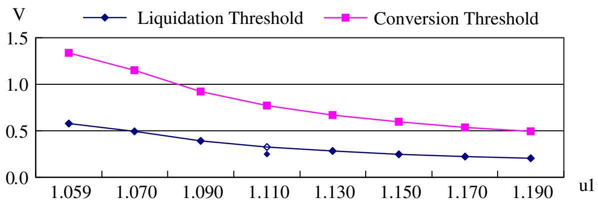


Fig. 5 Parameter u_1 to the optimal threshold

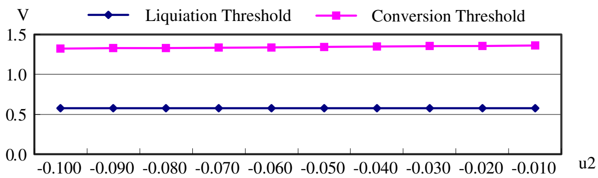


Fig. 6 Parameter u_2 to the optimal threshold

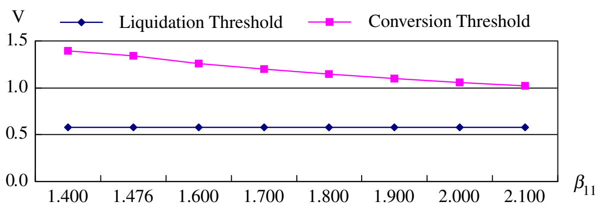


Fig. 7 Parameter β_{11} to the optimal threshold

3.2 Jump-diffusion model

Now we apply jump-diffusion model to turn to numerical solutions and sensitivity analysis to verify these intuition.

3.2.1 Numerical analysis

First, this study calculates the parameters β_{11} and u_1, u_2 of the discounted factor with $u_1 > 1, u_2 < 0$. The simulated parameter data are as follows: volatility $\sigma = 0.790$; drift rate $\alpha = 3.150\text{E-}04$; risk-free interest rate $r_f = 0.020$; discounted interest rate $r = 0.220$;

magnitude $\theta_1 = 1.200\text{E-}03$; arrival rate $\lambda_1 = 0.006$; project value $V_0 = 21$ (million NT dollars). Moreover, the simulation results are $\beta_{11} = 1.486$, $u_1 = 1.076$, and $u_2 = -0.053$. Other variables by hypothesis and parameter date include liquidation cost $L_0 = 3.25$ million NT dollars and conversion cost $C_0 = 4.2$ million NT dollars. The numerical results are obtained as: $V_L^* = 45.8$ million NT dollars; $V_C^* = 106.3$ million NT dollars; $a_1 = 0.986$; $a_2 = 0.781$; $b_2 = 0.073$. Thus, if the invested project value is below the optimal liquidation threshold $V_L^* = 45.8$ million NT dollars, the optimum exit plan for VC companies is liquidation; if the invested project value exceeds the optimal conversion threshold $V_C^* = 106.3$ million NT dollars, the best exit plan for VC companies is conversion. In addition, the contract value upon the liquidation threshold is:

$$F_1(V_L^*) = a_1 \times \left(V_L^*\right)^{u_1} = 0.986 \times (0.458)^{1.076} = 0.426 \text{ (Million NT dollars)}.$$

The contract value upon the conversion threshold is:

$$\begin{aligned} F_2(V_C^*) &= a_2 \left(V_C^*\right)^{u_1} + b_2 \left(V_C^*\right)^{u_2} = 0.781 \times (1.063)^{1.076} + 0.073 \times (1.063)^{-0.053} \\ &= 0.907 \text{ (Million NT dollars)}. \end{aligned}$$

3.2.2 Sensitivity analysis

This subsection performs a sensitivity analysis of the simulated variances in this assay, including: drift rate α; discounted interest rate r; project value effect intensity θ_1; arrival rate λ_1; liquidation cost L_0; conversion cost C_0; parameters u_1, u_2 and β_{11}. Assuming the other variances are consistent, the variable is varied to identify the changes in the optimal thresholds V_L^* and V_C^* for both magnitude and direction.

Figure 8 shows that both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* increase with increasing α, but the magnitude of the increase is minimal.

Figure 9 reveals that the optimal liquidation threshold V_L^* is not influenced by r as the optimal conversion threshold V_C^* decreases with increasing r (but in less magnitude than r).

Figure 10 indicates that both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* decrease with increasing λ_1 (but in less magnitude than λ_1).

Figure 11 illustrates that both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* increase with increasing θ_1, but the magnitude of the increase is very minimal.

Figure 12 illustrates that both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* increase with increasing L_0.

Figure 13 illustrates that the optimal liquidation threshold V_L^* is not influenced by C_0 as the optimal conversion threshold V_C^* increases with increasing C_0.

Figure 14 illustrates that both the optimal liquidation threshold V_L^* and the optimal conversion threshold V_C^* decrease with increasing u_1 (but in less magnitude than u_1).

Figure 15 reveals that the optimal liquidation threshold V_L^* is not influenced by u_2 as the optimal conversion threshold V_C^* increases with increasing u_2, but the magnitude of increase is minimal.

Figure 16 shows that the optimal liquidation threshold V_L^* is not influenced by β_{11} since the optimal conversion threshold V_C^* decreases with increasing β_{11} (but in less magnitude than β_{11}).

Fig. 8 Drift rate α to the optimal threshold

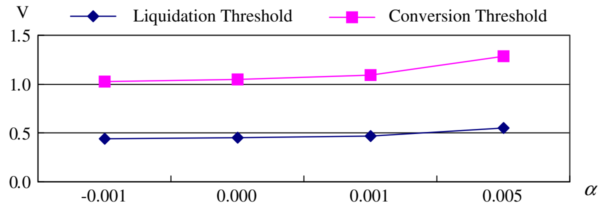


Fig. 9 Discounted interest rate r to the optimal threshold

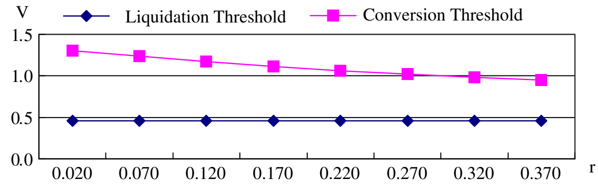


Fig. 10 Average arrival rate λ_1 to the optimal threshold

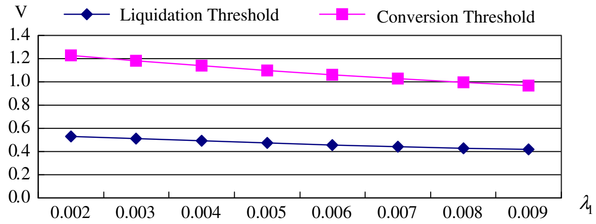


Fig. 11 Magnitude θ_1 to the optimal threshold

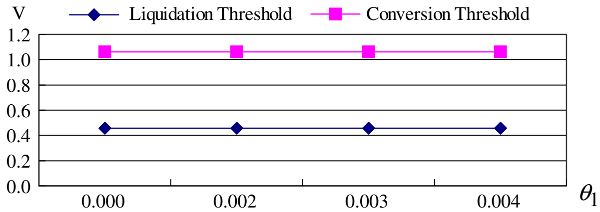


Fig. 12 Liquidation cost L_0 to the optimal threshold

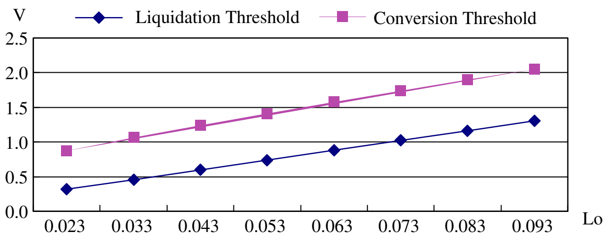
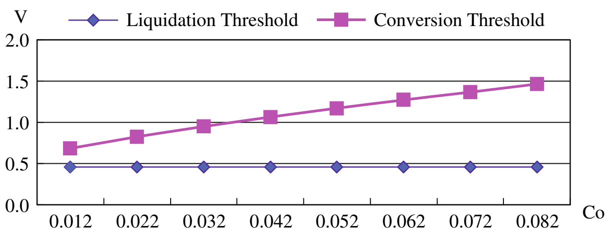


Fig. 13 Conversion cost C_0 to the optimal threshold



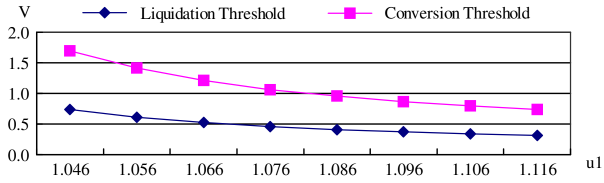


Fig. 14 Parameter u_1 to the optimal threshold

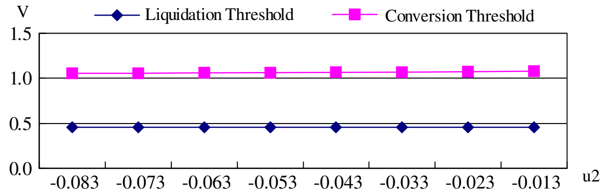


Fig. 15 Parameter u_2 to the optimal threshold

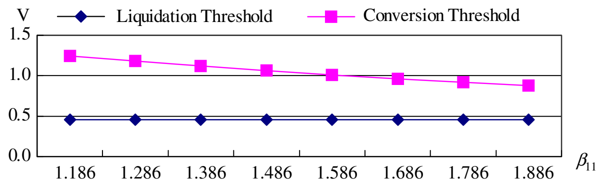


Fig. 16 Parameter β_{11} to the optimal threshold

Table 2 The value of the relative variables

Variable and parameter	Continuity model	Jump-diffusion model
r	0.22	0.22
β	$\beta_{11} = \beta_{21} = 1.476$	$\beta_{11} = 1.486$ $\beta_{21} = 1.540$
V_0	0.21	0.21
V_L^*	0.579	0.458
V_C^*	1.339	1.063

3.3 The expected discount time

Now we turn to numerical solutions and sensitivity analysis to verify these intuitions.

3.3.1 Numerical analysis

The simulated parameter data include the following: $\lambda_1 = 0.006$, $\theta_1 = 1.200\text{E-}03$, $\lambda_2 = 0.04$, and $\theta_2 = 0.008$. According to the numerical analysis from the previous model, the variables are found to have the following values (Shown as Table 2).

This study also computes the expected discounted time on both the original model and the Jump-diffusion model (Shown as Table 3)

3.3.2 Sensitivity analysis

This section conducts a sensitivity analysis of the simulated variances in this assay, including: r, β_{11}, β_{21}, V_0, V_L^*, and V_C^*. Assuming that the other variances are consistent, the variable is adjusted to identify the changes of the expected discounted time $E\left[T_1^*\right]$ and $E\left[T_2^*\right]$.

Table 3 Expected discounted time

	Continuity model	Jump-diffusion model
Interval $\left[V_0, V_L^*\right]$	$E\left[T_2^*\right] \geq \frac{\beta_{21}}{r}\log\left(\frac{V_L^*}{V_0}\right) =$ 2.955 (years)	$E\left[T_2^*\right] \geq \frac{\beta_{21}}{r}\log\left(\frac{V_L^*}{V_0}\right) =$ 2.371 (years)
Interval $\left[V_L^*, V_C^*\right]$	$E\left[T_1^*\right] \geq \frac{\beta_{11}}{r}\log\left(\frac{V_C^*}{V_L^*}\right) =$ 2.443 (years)	$E\left[T_1^*\right] \geq \frac{\beta_{11}}{r}\log\left(\frac{V_C^*}{V_L^*}\right) =$ 2.470 (years)

Table 4 Summary of sensitivity analysis for expected discount time

Parameters and variables	$E\left[T_1^*\right]$	$E\left[T_2^*\right]$
r	−	−
β_{11}	+	×
β_{21}	×	+
V_0	×	−
V_L^*	−	+
V_C^*	+	×

+: Positive correlation;
−: Negative correlation;
×: Zero correlation

First, the discounted point in interval $\left[V_L^*, V_C^*\right]$: $E\left[T_1^*\right] \geq (\beta_{11}/r)\log\left(V_C^*/V_L^*\right)$. Separately taking the first order of differential to β_{11}, r, V_L^*, and V_C^* yields the following: (1) $\partial E\left[T_1^*\right]/\partial r \geq -(\beta_{11}/r^2)\left(V_C^*/V_L^*\right)$; (2) $\partial E\left[T_1^*\right]/\partial\beta_{11} \geq (1/r)\log\left(V_C^*/V_L^*\right) > 0$; (3) $\partial E\left[T_1^*\right]/\partial V_L^* \geq -(\beta_{11}/r)\left(1/V_L^*\right)$; (4) $\partial E\left[T_1^*\right]/\partial V_C^* \geq (\beta_{11}/r)\left(1/V_C^*\right) > 0$.

Secondly, the discounted point in interval $\left[V_0, V_L^*\right]$ $E\left[T_2^*\right] \geq (\beta_{21}/r)\log\left(V_L^*/V_0\right)$. Separately taking the first order of differential to β_{21}, r, V_0, and V_L^* yields the following: (1) $\partial E\left[T_2^*\right]/\partial r \geq -(\beta_{21}/r^2)\left(V_L^*/V_0\right)$; (2) $\partial E\left[T_2^*\right]/\partial\beta_{21} \geq (1/r)\log\left(V_L^*/V_0\right) > 0$; (3) $\partial E\left[T_2^*\right]/\partial V_0^* \geq -(\beta_{21}/r)\left(1/V_0^*\right)$; (4) $\partial E\left[T_2^*\right]/\partial V_L^* \geq (\beta_{21}/r)\left(1/V_L^*\right) > 0$. The sensitivity analysis for the expected discounted point to all variables in both intervals is summarized as below (Table 4).

4 Conclusion

By applying ROA, reconsidering the present value interest factor, and applying the jump-diffusion process company evaluation model, the proposed assay extends the research of the model of Lin and Huang (2003) to establish an exit strategy and model for VC companies who invest in the start-up companies. This assay can also determine the optimal exit threshold (liquidation or conversion) and build up an evaluation model for the optimal exit plan. When the project value is below the liquidation threshold V_L^*, VC companies should decide to liquidate the start-up companies; meanwhile, when the project value exceeds the conversion threshold V_C^*, they choose to convert by converting their shares in an open market. However, when the project value is between V_L^* and V_C^*, the best strategy is to maintain the current status and continue assessing the optimal solution.

In the example involving the numerical analysis of the threshold and coefficients, the liquidation model coefficient a_1 in the continuity-model is lower than the liquidation model coefficient a_1 in the jump-diffusion model. Other values including liquidation threshold V_L^*, conversion threshold V_C^*, and project value coefficients, a_2 and b_2, are all higher in the continuity-model than the jump-diffusion model. In the numerical analysis example for the expected discounted point, in interval $\left[V_0, V_L^*\right]$, the expected time in the continuity-model exceeds that in the jump-diffusion model, while in interval $\left[V_L^*, V_C^*\right]$, the expected time in the jump-diffusion model exceeds that in the continuity-model. The above explains that the decisions are more strongly influenced by a sudden change in the period between liquidation and conversion than before liquidation; in the sensitivity analysis, the expected discounted point is positively correlated to all the parameters in the denominator and negatively correlated to all the parameters in the numerator.

Acknowledgements The first author would like to thank the National Science Council of the Republic of China for financially supporting this research under Contract No. NSC95-2416-H-259 -015-.

Appendix 1

Assuming the expected discounted factor in the continuous period $\left[V_L^*, V_C^*\right]$ is $E[\mathrm{e}^{-rT_1^*}]$, the function $f_1(v)$ is defined as:

$$f_1(v) = E[\mathrm{e}^{-rT_1^*}]. \tag{A1}$$

Equation A1 can be also shown as:

$$f_1(v) = \mathrm{e}^{-rdt_1} E[f_1(v+dv) \mid v] = \mathrm{e}^{-rdt_1}\left\{f_1(v) + E[df_1(v)]\right\}, \tag{A2}$$

with v based on GBM, the distribution is shown as below:

$$dv = \alpha v dt_1 + \sigma v dw - \theta_1 v dq_1, \tag{A3}$$

where

$$dq_1 = \begin{cases} 1, & \text{with prob. } \lambda_1 \mathrm{d}_{\mathrm{t}_1} \\ 0, & \text{with prob. } 1 - \lambda_1 \mathrm{d}_{\mathrm{t}_1} \end{cases}.$$

Following period dt_1, the incremental value of function $f_1(v)$ is:

$$df_1(v) = \left\{\alpha v f_{1v}(v) + \frac{1}{2}\sigma^2 v^2 f_{1vv}(v) - \lambda_1[f_1(v) - f_1(\theta_1 v)]\right\} dt_1. \tag{A4}$$

As already known, $\mathrm{e}^{-rdt_1} \cong 1 - rdt_1$, applying it to Eq. A4, we obtain:

$$\frac{1}{2}\sigma^2 v^2 f_{1vv}(v) + \alpha v f_{1v}(v) - (\lambda_1 + r) f_1(v) + \lambda_1 f_1(\theta_1 v) = 0 \tag{A5}$$

Rearranging Eq. A5 yields:

$$\frac{1}{2}\sigma^2 \beta_{11}^2 + (\alpha - \frac{1}{2}\sigma^2)\beta_{11} - (\lambda_1 + r) + \lambda_1 \theta_1^{\beta_{11}} = 0 \tag{A6}$$

Assuming the general solutions are:

$$f_1(v) = A_{11} v^{\beta_{11}} + A_{12} v^{\beta_{12}} \tag{A7}$$

Since $f_1(v)$ equals zero upon $v = 0$, the coefficient A_{12} in Eq. A7 must be zero. Thus, the value function is $f_1(v) = A_{11} v^{\beta_{11}}$. T_1^* is very minimal when $V_L^* \to V_C^*$ and the discounted factor $f_1(v) \approx 1$; thus $f_1(V_L^*) = 1$. In other words,

$$A_{11} V_L^{*\beta_{11}} = 1 \Rightarrow A_{11} = \left(\frac{1}{V_L^*}\right)^{\beta_{11}} \tag{A8}$$

The value function of $f_1(v) = (v/V_C^*)^{\beta_{11}}$ can be obtained.

Upon the point of V_C^* turning back to the point of V_L^*, the value function is $f_1(V_0) = (V_L^*/V_C^*)^{\beta_{11}}$. Thus the discounted factor is $E[\mathrm{e}^{-rT_1^*}] = (V_L^*/V_C^*)^{\beta_{11}}$.

Appendix 2

Assuming the expected discounted factor in the continuous period $\left[V_0, V_L^*\right]$ is $E[\mathrm{e}^{-rT_2^*}]$ with the same inference as Appendix 1, based on Eq. A6, the equation of parameter β_2 can be derived as:

$$\frac{1}{2}\sigma^2\beta_{21}^2 + \left(\alpha - \frac{1}{2}\sigma^2\right)\beta_{21} - (\lambda_2 + r) + \lambda_2\theta_2^{\beta_{21}} = 0, \tag{A9}$$

and the value function is $f_2(v) = (v/V_L^*)^{\beta_{21}}$.

Upon the point of V_L^* turning back to the point of V_0, the value function is $f_2(V_0) = (V_0/V_L^*)^{\beta_{21}}$. Thus the discounted factor is $E[\mathrm{e}^{-rT_2^*}] = (V_0/V_L^*)^{\beta_{21}}$.

References

1. Andreas, B, Uwe, W.: Convertible securities and optimal exit decisions in venture capital finance. J. Corp. Finan. **7**(3), 285–306 (2001)
2. Taiwan Venture Capital Association (2003). Taiwan Venture Capital Yearbook (Year 2003). CVCBA Pub., Taipei
3. Cossin, D., Leleuxand, B., Saliasi, E.: Understanding the economic value of legal covenants in investment contracts: a real-options approach to venture equity contracts. Working Paper, University of Lausanne (2002)
4. Dixit, A.K., Pindyck, R.S.: Investment under Uncertainty. Princeton University Press, New Jersey (1994)
5. Itô, K.: On stochastic differential equation. Mem. Am. Math. Soc. **4**, 1–51 (1951)
6. Lin, T.T., Huang, Y.T. : The optimal disinvestment evaluation model for venture capital: applying real options in redemption and convertibility. J. Technol. Manage. **8**(3), 59–78 (in Chinese) (2003)
7. Lin, T.T., Ko, C.C., Liao, C.Y.: Applying game options in a loan exit model. Qual. Quan. **40**, 561–575 (2006)
8. Mason, C.M., Harrison, R.T.: Is it worth it? The rates of return from informal venture capital investments. J. Bus. Venturing **17**(3), 211–236 (2002)
9. Pandey, I.M., Angela, J.: Venture capital for financing technology in Taiwan. Technovation **16**(9), 499–514 (1996)