NATIONAL UNIVERSITY OF SINGAPORE

MA2216/ST2131 PROBABILITY

(Semester 1: AY2015/2016)

Time allowed : 2 hours

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FOUR (4)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. Marks for each question are indicated at the beginning of the question.
3. This is a CLOSED BOOK (with **helpsheet**) examination.
4. Candidates may bring in **ONE (1)** handwritten A4-size help sheet.
5. Candidates may use calculators. However, they should lay out systematically the various steps in the calculations.

1. [15 marks] Let the point (u, v) be chosen uniformly from the square $0 < u \leq 1$, $0 < v \leq 1$. That is, the joint density function of U and V is given by

$$f(u, v) = \begin{cases} 1, & \text{if } 0 < u, v < 1, \\ 0, & \text{otherwise.} \end{cases}$$

To the point (u, v) chosen, put X to be the random variable that takes the value $\sqrt{-2 \ln u}\, \cos(2\pi v)$, and let Y be the random variable that assigns to the point (u, v) the number $\sqrt{-2 \ln u}\, \sin(2\pi v)$.

(i) [8 marks] Find the joint density function of X and Y, and identify the distributions of X and Y

(ii) [7 marks] Evaluate the probability $\mathbb{P}\left\{X^2 + Y^2 \leq 1\right\}$.

2. [15 marks]

(A) [5 marks] Let X have a Gamma distribution with parameters $\frac{n}{2}$ and $\frac{1}{2}$. Find its mean and variance first, and then, apply the Central Limit Theorem to approximate the probability, when $n \to \infty$,

$$\mathbb{P}\left\{X > n + 2\sqrt{n}\right\}.$$

(It is known that $\mathbb{P}\left\{Z > 1.414\right\} \approx 0.0787$.)

(B) [10 marks] Suppose that A, B, C, are independent random variables, each being uniformly distributed over $(0, 1)$.

(i) [5 marks] Put $W = A\,C$. Find the probability density function of W.

(ii) [5 marks] What is the probability that both roots of the equation

$$A\,x^2 + \mathbf{2}\,B\,x + C = 0$$

are real?

3. [15 marks] Let X and Y have the joint density function

$$f(x,y) = c\,x\,(y-x)\,e^{-y}, \quad 0 < x < y < \infty,$$

of which c is the normalizing constant

(i) [2 marks] Determine the value of c.

(ii) [3 marks] Find the marginal density functions $f_X(x)$ and $f_Y(y)$, respectively.

(iii) [4 marks] Evaluate the means $\boldsymbol{E}[X]$ and $\boldsymbol{E}[Y]$, respectively.

(iv) [3 marks] Find $f_{Y|X}(y|x)$, $0 < x < y$, and then evaluate the conditional expectation $\boldsymbol{E}[Y \mid X]$.

(v) [3 marks] Evaluate the covariance $\mathrm{Cov}(X, Y)$.

4. [15 marks] The number, X, of people who enter an elevator on the ground floor is a Poisson random variable with parameter 10. There are 10 floors above the ground floor and assume that each person is equally likely to get off at any one of these 10 floors, independently of where the others get off.

For $j = 1, 2, \ldots, 10$, let $I_j = 1$ if the elevator stops at floor j, and 0, otherwise. Put $N = \sum_{j=1}^{10} I_j$, which is the number of stops that the elevator will make before discharging all of its passengers.

(i) [2 marks] Evaluate $\mathbb{P}\{I_j = 0 \mid X = k\}$ for each j.

(ii) [2 marks] Evaluate $\boldsymbol{E}[I_j \mid X = k]$ for each j.

(iii) [4 marks] Compute $\boldsymbol{E}[N]$, the expected number of stops that the elevator will make before discharging all of its passengers.

(iv) [4 marks] Evaluate $\mathbb{P}\{I_i = 1,\, I_j = 1 \mid X = k\}$, and then find the probability $\mathbb{P}\{I_i = 1,\, I_j = 1\}$, where $1 \le i < j \le 10$, and k is a non-negative integer.

(v) [3 marks] Find $\mathrm{Cov}(I_i,\, I_j)$ for $1 \le i < j \le 10$, and then evaluate $\mathrm{Var}(N)$.

[HINT: $\mathrm{Var}(N) = \sum_{j=1}^{10} \mathrm{Var}(I_j) + 2 \sum\sum_{1 \le i < j \le 10} \mathrm{Cov}(I_i,\, I_j)$.]

END OF PAPER