An Introduction to the Hitchin Moduli Space as an Integrable System

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April 30, 2021

1 Introduction

The classification of objects using moduli spaces is a recurring theme in algebraic geometry, representation theory, and gauge theory. At the heart of all three lies the moduli space of stable holomorphic vector bundles over a Riemann surface [3, 6].

In 1986, Nigel Hitchin introduced a new type of object, a holomorphic vector bundle with a structure called a Higgs field [2]. In his paper, Hitchin showed that the moduli space of these "Higgs bundles" on a compact Riemann surface is the same as the gauge-theoretic moduli space of solutions to a certain set of differential equations arising from a dimensional reduction of the Yang-Mills equations. He also showed a relationship with the moduli space of holomorphic bundles. Simpson [8] later demonstrated a relationship between Higgs bundles and representation theory, further cementing Higgs bundles in a central place in modern mathematics.

The moduli space of Higgs bundles has a rich geometry. It's a complex algebraic variety endowed with a special type of Kahler metric called a hyperKahler metric, which allows it to be studied from the perspectives of algebraic, complex, symplectic, and Riemannian geometry. It is also equipped with a natural fibration, called the Hitchin fibration, which is a key technical tool in the study of the moduli space. It also gives it the structure of an algebraically integrable system.

The study of integrable systems has its origins in a 1918 paper of Emmy Noether [9]. There she proved, roughly speaking, that in a physical system, conservation laws (such as the conservation of energy and momentum) and in 1-1 correspondence with symmetries of the system (here, 1-parameter groups of diffeomorphisms). In the century since, this theorem has revolutionized physics, and also spawned a field of mathematics.

An integrable system is one with the maximal number of possible symmetries, and therefore the maximal number of independent conserved quantities. Such systems can in some cases be solved explicitly, and in all cases have many nice properties.

Integrable systems can be studied in many contexts, but one of the most natural is that of a symplectic manifold. In this paper, we will introduce some of the ideas from symplectic geometry which will allow us to formulate Noether's theorem and integrable systems. We will then introduce Higgs bundles, and work through (with some technical details omitted) the construction of a special case of the moduli space of Higgs bundles, those where the underlying bundle is stable.

This convenience will allow us to demonstrate explicitly the connection to the moduli space of stable vector bundles and prove that the moduli space forms an integrable system using an infinite-dimensional version of the technique known as symplectic reduction.

We assume familiarity with the basic notions of smooth manifolds, Lie groups, calculus on manifolds, and some comfort with vector bundles.

2 Symplectic Geometry and Integrable Systems

Unless otherwise specified, the reference for this section is [5]. The majority of proofs in this section are short computations. To improve readability and keep technical prerequisites as low as possible, we mostly omit them.

We begin with a definition:

Definition 2.1. A *symplectic manifold* is a pair (M, ω) where M is a smooth manifold and ω is a closed, nondegenerate 2-form.

By nondegenerate we mean that at each $p \in M$, ω defines an isomorphism $T_pM \to T_p^*M$ by $v \mapsto \omega(v, \cdot)$. While the definition is straightforward enough, it's not clear why we would want to consider this particular object. However, both properties of ω are required to formulate integrable systems. Nondegeneracy implies via linear algebra that a symplectic manifold must be even-dimensional.

Example 2.2.

1. Let $M = T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ with coordinates $\{x_1, ..., x_n, y_1, ..., y_n\}$. Then the form
$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$$
makes M into a symplectic manifold.

2. Let M be any smooth manifold. The cotangent bundle T^*M has a natural symplectic structure defined as follows. Let $\pi : T^*M \to M$ be the projection map. Then define a 1-form θ pointwise by
$$\theta_p = \pi^*\xi,$$
where $p = (x, \xi) \in T_x^*M$. One can check that in standard cotangent bundle coordinates $\{x_1, ..., x_n, y_1, ..., y_n\}$, the form θ has the form
$$\theta = \sum y_i dx^i,$$

and its differential

$$\omega = d\theta = \sum dy^i \wedge dx^i$$

is a symplectic form.

In physics, the coordinates correspond to n position coordinates x and n momentum coordinates y, where one considers a system on M as living in the "phase space" T^*M spanned by the position and momentum coordinates.

Our main object of study will be some given smooth function $H \in C^\infty(M)$, called the Hamiltonian function. A triplet (M, ω, H) is called a *Hamiltonian system*. In classical mechanics, this represents the total energy at a fixed position and velocity. The isomorphism $T_pM \to T_p^*M$ induced by ω allows us to translate back and forth between sections of TM and T^*M. Thus there is a unique vector field X_H such that

$$dH = \omega(X_H, \cdot).$$

This is the *Hamiltonian vector field* of H. Recall that given a vector field X, one can integrate it to a unique flow ρ_t^X, a time varying diffeomorphism such that for any $p \in M$,

$$\frac{d}{dt}\, \rho_t^X(p)\big|_{t=0} = X(p).$$

The action of a flow on a point draws out a curve on M, called an *integral curve* of X. A flow is the same thing as an $\mathbb{R}$-action provided the flow exists for all t. This will always be true if M is compact, or X compactly supported. Given an $\mathbb{R}$-action on M, one can obtain a vector field by differentiating. So we have, roughly speaking, a bijection between vector fields and $\mathbb{R}$-actions.

One can show that the equations of motion in Lagrangian or Newtonian classical mechanics are equivalent to the flow equation for X_H, where H is the total energy. We can see this in the following example:

Example 2.3. Let $M = T^*\mathbb{R}^3$, spanned by $\{x_1, x_2, x_3, y_1, y_2, y_3\}$, and $\omega = \sum_{i=1}^3 dx_i \wedge dy_i$. Let $a, b > 0$, and let $H = ax_3 + \frac{1}{2}b|y|^2$. If x is the position and y is the momentum, H is the total energy for an object moving in a constant gravitational field. In this case,

$$dH = adx_3 + \sum_{i=1}^3 by_i dy_i,$$

and

$$X_H = -a\frac{\partial}{\partial y_3} + \sum_{i=1}^3 by_i \frac{\partial}{\partial x_i}.$$

Thus the flow equation is

$$\begin{cases} \frac{dx_i}{dt} & = by_i \quad i = 1, 2, 3 \\ \frac{dy_3}{dt} & = -a \end{cases}, \tag{1}$$

or, combining the equations,

$$\frac{d^2x}{dt^2} = -ab,$$

which for appropriate values of a and b is precisely Newton's second law.

Given this, the conserved quantities we are looking for are functions $f \in C^\infty(M)$ which are constant along the integral curves of X_H. The symmetries we're looking for are vector fields such that H is constant along the flow. This will guarantee that after acting by a symmetry, X_H and thus the equations of motion remain the same.

The final piece we need is a way to connect these. Given two smooth functions f, g on M define their *Poisson bracket* as the new function given by the formula

$$\{f, g\} = \omega(X_f, X_g).$$

The Poisson bracket is bilinear, antisymmetric, and turns $C^\infty(M)$ into a Lie algebra. It also satisfies the property

$$X_{\{f,g\}} = -[X_f, X_g].$$

Its use is exemplified by the following theorem:

Theorem 2.4. *The Poisson bracket $\{f, H\} = 0$ if and only if f is constant along integral curves of X_H.*

In physics speak: $\{f, H\} = 0$ if and only if f is a conserved quantity! This immediately implies Noether's theorem:

Corollary 2.5 (Noether's Theorem)**.** *The function f is a conserved quantity if and only if the flow generated by X_f is a symmetry.*

Proof. If f is a conserved quantity, then by the theorem $\{f, H\} = 0$. This implies $\{H, f\} = 0$, so by the theorem again H is constant along the integral curves of X_f, i.e. X_f generates a symmetry. $\square$

Remark 2.6. We've actually only tackled most of Noether's theorem. The last question that remains is: given a vector field X which generates symmetry, is there some function f such that $X = X_f$? The answer is "yes" locally, and "no" globally. The extent to which this fails is measured by the first de Rham cohomology $H^1(M)$.

Example 2.7. Let (M, ω, H) as in Example 2.3. Let $f_1(x, y) = y_1$, $f_2(x, y) = y_2$. We can see from the equations 1 that these are conserved quantities. Now note

$$X_{f_1} = -\frac{\partial}{\partial x_1}$$
$$X_{f_2} = -\frac{\partial}{\partial x_2}.$$

The flows for these are given by translation of x_1 and x_3. So the translation symmetry for our system implies by Noether's theorem the conservation of momenta f_1 and f_2. Indeed, for $i = 1, 2$,

$$\begin{aligned}\{f_i, H\} &= \omega(X_{f_i}, X_H) \\ &= \left(\sum_{i=1}^{3} dx_i \wedge dy_i\right)\left(-\frac{\partial}{\partial x_i}, -a\frac{\partial}{\partial y_3} + \sum_{i=1}^{3} by_i \frac{\partial}{\partial x_i}\right) \\ &= 0,\end{aligned}$$

so these are commuting functions.

We can now define integrable systems. A set of functions $f_1, ..., f_k$ is said to be *independent* if $df_1, ..., df_k$ is linearly independent at each point. Some linear algebra shows that if $\dim M = 2n$, the maximum number of independent functions such that $\{f_i, f_j\} = 0$ for all i, j is $k = n$.

A Hamiltonian system (M, ω, H) for which there exist n independent functions $f_1 = H, f_2, ..., f_n$ which Poisson commute is called an *integrable system*.

Example 2.8. Let (M, ω, H) as above and set $a = 0$, so

$$H = \frac{1}{2}b|y|^2.$$

Let $f_i = y_i$, $i = 1, 2, 3$. Then it is easy to see that $\{df_i = dy_i\}_{i=1}^3$ is linearly independent at each point. Observe that it is generally true for symplectic manifolds of the form T^*N that if functions only depend on vertical coordinates, and not horizontal coordinates, then they commute. So our functions commute and $\{M, \omega, H)$ is an integrable system.

Integrable systems have many nice properties. We discuss one important one. Let $f : M \to \mathbb{R}^n$ be given by $f(x) = (f_1(x), ..., f_n(x))$.

Lemma 2.9. *Suppose $c \in \mathbb{R}^n$ is a regular value of f, and $X_{f_1}, ..., X_{f_n}$ are complete on $f^{-1}(c)$, then the connected components of $f^{-1}(c)$ are of the form*

$$\mathbb{R}^{n-k} \times T^k,$$

where T^k is the k-torus. In particular, if f is a proper map (i.e. preimages of compact sets are compact), then f gives a (possibly singular) fibration of M by tori.

Proof. Since the vector fields in question commute, their flows commute; thus, one may simply follow the flow lines. $\square$

The preimage $f^{-1}(c)$ also always has the property that it is a *Lagrangian* submanifold, i.e. its dimension is $\frac{1}{2}\dim M$ and $\iota^*\omega = 0$, where ι is the inclusion map. Such submanifolds are important in much of symplectic geometry and have many nice properties.

In actuality, the integrable system we construct will be a complex integrable system, with complex functions and a complex-valued symplectic form. However, what we have said in this section still holds with very little difference.

3 Higgs Bundles

We now turn our attention to Higgs Bundles. Unless otherwise specified, for the remainder of this paper we will be drawing on [3] and [2].

Definition 3.1 ([4])**.** Let Σ be a compact Riemann surface. A *Higgs bundle* on Σ is a pair (V, Φ), where V is a holomorphic vector bundle and

$$\Phi : E \to E \otimes K$$

is a holomorphic bundle map, called the *Higgs field*, where K is the canonical bundle $K = T^*\Sigma$.

Another way of thinking about Φ is as an "endomorphism-valued 1-form," i.e. a section of $\mathrm{End}(E) \otimes K$. In coordinates, this will look like a matrix of holomorphic 1-forms, such as

$$\Phi = \begin{pmatrix} f_{11}dz & f_{12}dz \\ f_{21}dz & f_{22}dz \end{pmatrix}.$$

We wish to construct a moduli space of Higgs bundle structures over a given topological vector bundle V. However, the set of all isomorphism classes of Higgs bundles will not patch together into a nice topological space; to get a smooth manifold, we need to restrict to a subset of Higgs bundles. First, we must restrict the genus of our Riemann surface to be greater than 1. Second, we must only consider bundles which are *stable*.

Recall that the *degree* $\deg(L)$ of a complex line bundle L is defined as the number of zeroes of a nonzero meromorphic section minus the number of poles, counted with multiplicity. This number is a topological invariant, and equal to the first Chern class of the bundle. The degree of a complex vector bundle V of arbitrary rank is defined as the degree of its determinant line bundle, i.e. the line bundle whose transition functions are given by the determinant of the transition functions of V.

A holomorphic vector bundle bundle V is *stable* if for any holomorphic subbundle L of V,

$$\frac{\deg(L)}{\mathrm{rk}(L)} < \frac{\deg(V)}{\mathrm{rk}(V)}.$$

Remark 3.2.

1. There is a notion of stability for Higgs bundles used in [2,4] which considers only those subbundles satisfying $\Phi(L) \subseteq L \otimes K$, but we will not use it here.
2. We restrict to $g > 1$ because the for $g = 0, 1$ there are much fewer stable bundles. More details can be found in [2,3]

It is a nonobvious fact that stable bundles do not admit any nontrivial holomorphic automorphisms (i.e. except multiples of the identity). If we consider only stable Higgs bundles up to isomorphism on a Riemann surface of genus

$g > 1$, the set of such bundles will form a smooth manifold. To describe precisely how this manifold is constructed, we formulate a space of holomorphic structures on V.

By a theorem of differential geometry, a holomorphic structure on a complex vector bundle V is equivalent to a choice of differential operator $d''_A : \Omega^0(\Sigma, V) \to \Omega^{0,1}(\Sigma, V)$, which satisfies

$$d''_A(fs) = \bar{\partial} f \otimes s + f d''_A s$$

for all sections s and smooth functions f. In this case a section is defined to be holomorphic if $d''_A s = 0$. We denote by $H^0(\Sigma, V)$ the space of holomorphic sections of V. If d''_A and d''_B are two complex structures, their difference

$$d''_A - d''_B \in \Omega^{0,1}(\Sigma, \operatorname{End} V),$$

and so the space of holomorphic structures $\mathcal{A}$ is an affine space modeled on $\Omega^{0,1}(\Sigma, \operatorname{End} V)$. Denote by $\mathcal{A}^s$ the set of stable complex structures.

However, we wish to consider complex structures up to isomorphism. If $\mathcal{G}$ is the group of automorphisms of V (also known as the gauge group), then $g \in \mathcal{G}$ acts on the space of holomorphic structures by

$$d''_A \mapsto g^{-1} d''_A g.$$

Since stability of the bundle implies that there are no nontrivial holomorphic automorphisms of V (i.e. automorphisms which preserve the holomorphic structure), $\mathcal{G}$ acts freely on $\mathcal{A}^s$.

We now incorporate the Higgs field into our consideration. Fix a complex vector bundle V of rank m. Let $\Omega = \Omega^0(\Sigma, \operatorname{End} V \otimes K)$ be the set of smooth (not necessarily holomorphic) endomorphism-valued 1-forms. Note that the set $\mathcal{H} \subseteq \Omega \times \mathcal{A}^s$ of stable Higgs bundles is a sort of "infinite dimensional vector bundle" over $\mathcal{A}^s$. Let

$$\mathcal{M} = \mathcal{H}/\mathcal{G}$$

denote the *moduli space of stable Higgs bundles of rank m*. Note that this moduli space is dependent on both a choice of V and m.

Using either techniques either of geometric invariant theory or of gauge theory, one can show that since the action is free, this is a complex manifold of dimension $2m^2(g-1)+2$.

We will denote by $\mathcal{N}$ the space $\mathcal{A}^s/\mathcal{G}$. This is the moduli space of stable holomorphic vector bundles, fixing an underlying bundle V of rank m. We will see in the next section that $\mathcal{M} \cong T^*\mathcal{N}$.

4 The Integrable System and Relationship with Stable Bundles

We are now ready to construct the integrable system. This requires two parts: functions and a symplectic form. The symplectic form will be the canonical symplectic form from the isomorphism $\mathcal{M} \cong T^*\mathcal{N}$. Our functions will come from the Hitchin fibration.

The strategy will be to compute a symplectic form and commuting functions on the infinite dimensional manifold $\Omega^0(\Sigma, \operatorname{End} V \otimes K) \times \mathcal{A}^s$ and then argue that their properties are preserved under the quotient.

Let P be the ring of homogeneous complex polynomials in the entries of elements of the Lie algebra $\operatorname{Mat}(n, \mathbb{C})$ of $\operatorname{GL}(n, \mathbb{C})$ which are *Ad-invariant*, i.e. which satisfy

$$f(AT_1A^{-1}, ..., AT_kA^{-1}) = f(T_1, ..., T_k)$$

for any $T_1, ..., T_k \in \operatorname{Mat}(n, \mathbb{C})$ and $A \in \operatorname{GL}(n, \mathbb{C})$. Fix some basis $p_1, ..., p_R$ of this algebra.

The invariance of these polynomials means they patch together on transition functions to define maps

$$H^0(\Sigma, \operatorname{End} V) \to C^\infty(\Sigma, \mathbb{C})$$

on sections, for any bundle V. By homogeneity we can extend these to maps

$$H^0(\Sigma, \operatorname{End} V \otimes K) \to \bigoplus_{i=1}^{R} H^0(\Sigma, K^{\otimes d_i}),$$

where d_i is the degree of p_i. This map is called the *Hitchin fibration*. From it we get R coefficient functions $f_1, ..., f_R : H^0(\Sigma, \operatorname{End} V \otimes K) \to \mathbb{C}$. Using some algebraic geometry one can show that if the genus of Σ is greater than 1, and if V is stable,

$$\dim H^0(M, \operatorname{End} V \otimes K) = \dim \bigoplus_{i=1}^{R} H^0(\Sigma, K^{\otimes d_i}),$$

and moreover $R = m^2(g-1)+1$, which implies that this is the right number of functions to make an integrable system.

Theorem 4.1. *We have that $T^*\mathcal{N} \cong \mathcal{M}$. Moreover, the functions f_i are well-defined on $\mathcal{M}$, and they Poisson commute with respect to the natural symplectic structure on $T^*\mathcal{N}$.*

The proof uses an infinite-dimensional version of a symplectic geometry technique known as *symplectic reduction*. We will show that both spaces are isomorphic to

$$\mu^{-1}(0)/\mathcal{G},$$

for a map μ called a *moment map*, constructed from an infinite-dimensional symplectic form on $\Omega \times \mathcal{A}$. That our functions Poisson commute will then follow from the construction of μ.

We work through the finite-dimensional version of our special case (symplectic reduction works in much more general cases). We then assume everything works out nicely in the infinite dimensional case and complete the proof.

Let G be a Lie group acting freely on some smooth manifold N. Then we can lift this action to a free action on T^*N by the formula $g \cdot \xi(v) = \xi(g_*^{-1}v)$, where $g \cdot p = q$, $v \in T_qN$, $\xi \in T_p^*N$. If G is n-dimensional, the action of G generates vector fields $X_1, ..., X_n$ on N which form a basis for the Lie algebra $\mathfrak{g}$. These vector fields define functions $f_1, ..., f_n$ on on T^*N by $f_i(\xi) = \xi(X_i)$.

Using the natural symplectic structure on T^*N, we can define Hamiltonian vector fields X_{f_i}, where are canonical extensions of X_i to all of T^*N, viewing N as the 0-section.

Let $\varepsilon^1, ..., \varepsilon^n \in \Omega^1(T^*N)$ be the dual basis of X_{f_1}, ..., X_{f_n}, and define $\mu : M \to \mathfrak{g}^*$ by

$$\mu(x) = \sum f_i(x)\varepsilon^i.$$

Here μ is an example of a *moment map*, and it contains information about the action G. Note that μ is equivariant with respect to G (taking the coadjoint action of G on $\mathfrak{g}^*$), and so since G acts freely $\mu^{-1}(0)/G$ is a well-defined smooth manifold.

We will need two facts:

First, since G acts freely, N/G is a smooth manifold, and moreover

$$T^*(N/G) \cong \mu^{-1}(0)/G.$$

Second, suppose g and h are two G-invariant functions on T^*N which Poisson commute. Since g and h are G-invariant, they descend to well-defined functions $\tilde{g}$ and $\tilde{h}$ on $\mu^{-1}(0)/G$. In this case, $\{\tilde{g}, \tilde{h}\} = 0$ with respect to the natural symplectic structure on $T^*(N/G)$. (One can prove this by looking at the orbits and applying Theorem 2.4.)

We now repeat this construction in our infinite dimensional setting, and so it will suffice to show that our functions Poisson-commute on the infinite-dimensional, where we can avoid doing explicit computations on the moduli space.

If we view our gauge group $\mathcal{G}$ as an infinite-dimensional Lie group acting freely on $\mathcal{A}^s$, then we can construct a moment map μ on $T^*\mathcal{A}^s$ as above such that

$$T^*\mathcal{N} = \mu^{-1}(0)/\mathcal{G}.$$

So what is $T^*\mathcal{A}^s$? Since $\mathcal{A}^s$ is an open set in an affine space, this bundle is a trivial product of $\mathcal{A}^s$ with the dual of space $\mathcal{A} = \Omega^{0,1}(\Sigma, \operatorname{End} V)$.

Consider the bilinear form $b : \Omega^{1,0}(\Sigma, \operatorname{End} V) \times \Omega^{0,1}(\Sigma, \operatorname{End} V) \to \mathbb{C}$ defined by

$$b(A, B) = \int_\Sigma \operatorname{Tr}(AB),$$

where there is an implicit wedge product on forms. Then b gives an isomorphism

$$\mathcal{A}^* \cong \Omega^{1,0}(\Sigma, \operatorname{End} V).$$

Moreover, note that

$$\Omega^{1,0}(\Sigma, \operatorname{End} V) \cong \Omega^0(\Sigma, \operatorname{End} V \otimes K),$$

so the set $\mathcal{H}$ of Higgs fields is a subset of $\mathcal{A}^s$. Our claim is that $\mu^{-1}(0) = \mathcal{H}$, which will then imply that

$$T^*\mathcal{N} \cong \mu^{-1}(0)/\mathcal{G} \cong \mathcal{M}.$$

An element $g \in \mathcal{G}$, i.e. an automorphism of V, corresponds to an element of $\mathrm{GL}(n, V_x)$ at each $x \in \Sigma$. Therefore the "Lie algebra" of the "Lie group" $\mathcal{G}$ is

$$\mathfrak{g} = \Omega^0(\Sigma, \operatorname{End} V).$$

Our vector fields on our base $N = \mathcal{A}^s$ can be generated by left translation as objects of the form

$$\dot{A} = d''_A \psi \in \Omega^{0,1}(\Sigma, \operatorname{End} V),$$

where $\psi \in \Omega^0(M, \operatorname{End} V)$. Our functions will be of the form

$$f : T^*\mathcal{A}^s = \mathcal{A}^s \times \Omega^{1,0}(\Sigma, \operatorname{End} V) \to \mathbb{C}$$
$$f(A, \Phi) = \int_\Sigma \operatorname{Tr}(d''_A \psi \Phi).$$

Recall that we want $f(A, \Phi) = 0$ for all f in order to get $\mu(A, \Phi) = 0$, and therefore by Stokes' Theorem

$$0 = \int_\Sigma \operatorname{Tr}(d''_A(\psi)\Phi) = \int_\Sigma \operatorname{Tr}(\psi d''_A(\Phi))$$

for all $\psi \in \Omega^0(\Sigma, \operatorname{End} V)$. This condition is equivalent to $d''_A \Phi = 0$, i.e. Φ is holomorphic. Therefore $\mu(A, \Phi) = 0$ if and only if (A, Φ) defines a Higgs bundle. Thus we have established that

$$\mu^{-1}(0)/\mathcal{G} = \mathcal{M}.$$

From the discussion earlier, we now only have to prove that our functions $f_1, ..., f_R$ Poisson commute with respect to the natural cotangent symplectic structure on $T^*\mathcal{A}^s \cong \mathcal{A}^s \times \Omega^0(\Sigma, \operatorname{End} V \otimes K)$. However, since the functions only depend on the second factor, as in Example 2.8 they Poisson commute.

One can prove moreover that $f = (f_1, ..., f_R)$ is proper, and so the Hitchin fibration gives $\mathcal{M}$ a natural fibration of (possibly singular) tori. In fact, one can say even more: the Hitchin system is an *algebraically integrable system*, i.e. that the generic fibers are Jacobians of complex algebraic curves covering Σ [3]. This natural fibration is a crucial tool in understanding the geometry and topology of $\mathcal{M}$ [4].

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