Workhorse Model

DSGE Model

KEY: - Derivation, narrative - Comment to reader - Used for labeling

① Households

Households' lifetime utility is:

$$E_t\left[\sum_{j=0}^{\infty}\beta^j\left\{\ln(C_{t+j}-hC_{t+j-1})-\frac{\chi L_{t+j}^{1+\eta}}{1+\eta}\right\}\right]$$

Internal Habit

with budget constraint:

Value shock

Wage from being in a labor union

Nominal Dividends

Lump-sum tax

Nominal interest rate

$$P_tC_t+v_tB_t\le MRS_tL_t+DIV_t-P_tT_t+(1+i_{t-1})B_{t-1}$$

Price of goods

Bonds

Dividing by price to make this real:

$$C_t+v_t\frac{B_t}{P_t}\le mrs_tL_t+div_t-T_t+\frac{1+i_{t-1}}{P_t}B_{t-1}$$

$$C_t+v_tb_t\le mrs_tL_t+div_t-T_t+\left(\frac{P_{t-1}}{P_t}\right)(1+i_{t-1})b_{t-1}$$

Putting this in a Lagrangian:

$$\mathcal{L}=E_t\left[\sum_{j=0}^{\infty}\beta^j\left\{\ln(C_{t+j}-hC_{t+j-1})-\frac{\chi L_{t+j}^{1+\eta}}{1+\eta}\right\}+\mu_t\left(mrs_tL_t+div_t-T_t+\Pi_t^{-1}(1+i_{t-1})b_{t-1}-C_t-b_tv_t\right)\right.$$

FOCs:

$$\frac{\partial\mathcal{L}}{\partial C_t}=\frac{1}{C_t-hC_{t-1}}-\beta E_t\left[\frac{h}{C_{t+1}-hC_t}\right]-\mu_t=0$$

$$\rightarrow \mu_t=\frac{1}{C_t-hC_{t-1}}-\beta E_t\left[\frac{h}{C_{t+1}-hC_t}\right] \qquad ①$$

$$\frac{\partial\mathcal{L}}{\partial L_t}=-\chi L_t^{\eta}+mrs_t\mu_t=0$$

$$\rightarrow \chi L_t^{\eta}=mrs_t\mu_t \qquad ②$$

$$\frac{\partial\mathcal{L}}{\partial b_t}=-\mu_tv_t+\beta E_t\left[\mu_{t+1}\Pi_{t+1}^{-1}(1+i_{t+1})\right]=0$$

$$\rightarrow v_t\mu_t=E_t\left[\beta\mu_{t+1}\Pi_{t+1}^{-1}(1+i_t)\right] \qquad ③$$

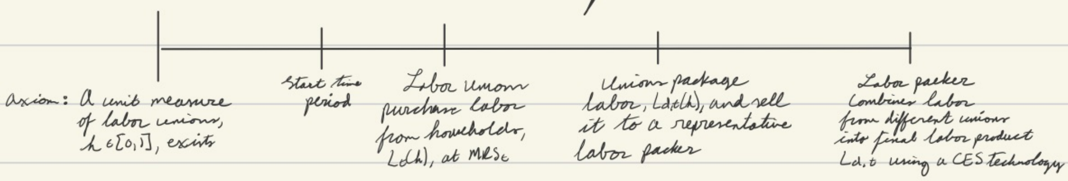
$$1=E_t\left[\Lambda_{t,t+1}\Pi_{t+1}^{-1}(1+i_t)\right]$$

$$\frac{\Pi_t}{1+i_t}=\Lambda_{t,t+1}$$

I define the stochastic discount factor as: $\Lambda_{t,t+1}=\dfrac{\beta E_t[\mu_{t+1}]}{\mu_t}$ ④

② Labor Markets

The labor market works as follows:



So overall demand for labor is:

$$L_{d,t} = \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \quad \text{where } \varepsilon_w > 1 \text{ is the elasticity of substitution}$$

Define aggregate wage expenditure on labor:

$$\int_0^1 w_t(h) L_{d,t}(h)\, dh \equiv Z_t$$

Then optimal demand for labor from each union is:

$$\max_{\{L_{d,t}(h)\}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}} \quad s.t. \int_0^1 w_t(h) L_{d,t}(h)\, dh \equiv Z_t$$

$$\langle L_{d,t}(h) \rangle \quad \cancel{\frac{\varepsilon_w}{\varepsilon_w - 1}} \cdot \cancel{\frac{\varepsilon_w - 1}{\varepsilon_w}} L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{1}{\varepsilon_w - 1}} - \lambda w_t(h) = 0$$

$$(*) \qquad \lambda w_t(h) = L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{1}{\varepsilon_w - 1}}$$

$$\lambda w_t(h) L_{d,t}(h) = L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{1}{\varepsilon_w - 1}}$$

Integrate both sides:

$$\lambda \int_0^1 w_t(h) L_{d,t}(h)\, dh = \int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{1}{\varepsilon_w - 1}}$$

$$\lambda Z_t = \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{\varepsilon_w}{\varepsilon_w - 1}}$$

Define W_t as the aggregate wage. Then $W_t L_{d,t} = Z_t$. So:

$$\lambda W_t L_{d,t} = L_{d,t}$$

$$\lambda = \frac{1}{W_t}$$

Sub λ into $(*)$:

$$\frac{w_t(h)}{W_t} = L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} \left[\int_0^1 L_{d,t}(h)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} dh \right]^{\frac{1}{\varepsilon_w - 1}}$$

$$\frac{w_t(h)}{W_t} = L_{d,t}(h)^{\frac{-1}{\varepsilon_w}} L_{d,t}^{\frac{1}{\varepsilon_w}}$$

$$L_{d,t}(h) = \left(\frac{w_t(h)}{W_t} \right)^{-\varepsilon_w} L_{d,t} \qquad (*)$$

Now take the total expenditure on labor:

$$W_t L_{d,t} = \int_0^1 W_t(h) L_{d,t}(h)\, dh \equiv Z_t \qquad \text{Sub in } L_{d,t}(h):$$

$$W_t L_{d,t} = \int_0^1 W_t(h) \left(\frac{W_t(h)}{W_t}\right)^{-\varepsilon_w} L_{d,t}\, dh$$

$$W_t = \int_0^1 W_t(h)^{1-\varepsilon_w} dh\, W_t^{\varepsilon_w}$$

$$W_t^{1-\varepsilon_w} = \int_0^1 W_t(h)^{1-\varepsilon_w} dh \qquad (**)$$

Labor Unions have dividends of:

$$DIV_{L,t}(h) = W_t(h) L_{d,t}(h) - MRS_t L_t(h)$$

We impose $L_t(h) = L_{d,t}(h)$ (labor supply = labor demand)

$$DIV_{L,t}(h) = (W_t(h) - MRS_t) L_{d,t}(h)$$

Now substituting (*):

$$DIV_{L,t}(h) = (W_t(h) - MRS_t)\left(\left(\frac{W_t(h)}{W_t}\right)^{-\varepsilon_w} L_{d,t}\right)$$

$$= W_t(h)^{1-\varepsilon_w} W_t^{\varepsilon_w} L_{d,t} - MRS_t W_t(h)^{-\varepsilon_w} W_t^{\varepsilon_w} L_{d,t}$$

We also impose "sticky wages", where unions have a probability of adjusting wages of $(1-\phi_w)$ with $\phi_w \in [0,1]$. For unions that can't update wages in a given period, wages can be written as: $W_t(h)\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{\gamma_w} = W_{t+j}(h)$, with $\gamma_w \in [0,1]$ being an index on inflation

Putting all this together, the union's maximization problem is:

$$\max_{\{W_t(h)\}} E_t\left[\sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j}\left\{W_{t+j}(h)^{1-\varepsilon_w} W_{t+j}^{\varepsilon_w} L_{d,t+j} - MRS_{t+j} W_{t+j}(h)^{-\varepsilon_w} W_{t+j}^{\varepsilon_w} L_{d,t+j}\right\}\right]$$

Probability of keeping last period's wage

Discounting by household stochastic discount factor

Then sub in inflation indexing and make terms real:

$$\max_{\{W_t(h)\}} E_t\left[\sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j}\left\{W_t(h)^{1-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} W_{t+j}^{\varepsilon_w} L_{d,t+j} - MRS_{t+j} W_t(h)^{-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} W_{t+j}^{\varepsilon_w} L_{d,t+j}\right\}\right]$$

$$\max_{\{W_t(h)\}} E_t\Bigg[\sum_{j=0}^{\infty} \phi_w^j \Lambda_{t,t+j}\Bigg\{W_t(h)^{1-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} \frac{W_{t+j}^{\varepsilon_w}}{P_{t+j}^{\varepsilon_w}} L_{d,t+j} P_{t+j}^{\varepsilon_w - 1}$$
$$- P_{t+j}^{\varepsilon_w}\frac{MRS_{t+j}}{P_{t+j}} W_t(h)^{-\varepsilon_w}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w}\frac{W_{t+j}^{\varepsilon_w}}{P_{t+j}^{\varepsilon_w}} L_{d,t+j}\Bigg\}\Bigg]$$

$$\max_{\{W_t(h)\}} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\left\{\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} W_t(h)^{1-\varepsilon_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d,t+j} - mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} W_t(h)^{-\varepsilon_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right\}\right]$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial W_t(h)} = (1-\varepsilon_w) W_t(h)^{-\varepsilon_w} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right]$$
$$+\varepsilon_w W_t(h)^{-\varepsilon_w-1} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j} mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right] = 0$$

Since all unions are the same, the reset price is the same across all unions, so $W_t(h) = W_t^*$.

$$(\varepsilon_w - 1) W_t^{*\,-\varepsilon_w} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right] =$$
$$\varepsilon_w W_t^{*\,-\varepsilon_w-1} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j} mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right]$$
$$W_t^* = \frac{\varepsilon_w}{\varepsilon_w-1} E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j} mrs_{t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_w\gamma_w} P_{t+j}^{\varepsilon_w} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right]$$
$$E_t\left[\sum_{j=0}^{\infty}\phi_w^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_w)\gamma_w} P_{t+j}^{\varepsilon_w-1} w_{t+j}^{\varepsilon_w} L_{d,t+j}\right]^{-1}$$
$$\underline{W}_t^* = \frac{\varepsilon_w}{\varepsilon_w-1}\frac{F_{1,t}}{F_{2,t}}$$

where $F_{1,t} = mrs_t P_t^{\varepsilon_w} w_t^{\varepsilon_w} L_{d,t} + \phi_w \Lambda_{t,t+1} \Pi_t^{-\varepsilon_w\gamma_w} F_{1,t+1}$

$F_{2,t} = P_t^{\varepsilon_w-1} w_t^{\varepsilon_w} L_{d,t} + \phi_w \Lambda_{t,t+1} \Pi_t^{(1-\varepsilon_w)\gamma_w} F_{2,t+1}$

We can divide through by P_t to obtain real wage:

$$w_t^* = \frac{\varepsilon_w}{\varepsilon_w-1}\frac{f_{1,t}}{f_{2,t}}$$, where $f_{i,t}$ denotes the real value of the $F_{i,t}$ term: (5)

$$f_{1,t} = mrs_t w_t^{\varepsilon_w} L_{d,t} + \phi_w E_t\left[\Lambda_{t,t+1}\Pi_{t+1}^{\varepsilon_w}\Pi_t^{-\varepsilon_w\gamma_w} f_{1,t+1}\right] \quad (6)$$

$$f_{2,t} = w_t^{\varepsilon_w} L_{d,t} + \phi_w E_t\left[\Lambda_{t,t+1}\Pi_{t+1}^{\varepsilon_w-1}\Pi_t^{\gamma_w(1-\varepsilon_w)} f_{2,t+1}\right] \quad (7)$$

- Aggregating:

We can now aggregate, integrating (*) over all unions:

$$\int_0^1 L_{d,t}(h)\,dh = \int_0^1\left(\frac{w_t(h)}{w_t}\right)^{-\varepsilon_w} L_{d,t}\,dh$$
$$L_t = L_{d,t}\int_0^1\left(\frac{w_t(h)}{w_t}\right)^{-\varepsilon_w} dh$$
$$L_t = L_{d,t} V_t^w \qquad \text{where } V_t^w = \int_0^1\left(\frac{w_t(h)}{w_t}\right)^{-\varepsilon_w} dh \quad (8)$$

Looking closer at V_t^w, we can impose wage stickiness:

$$V_t^w = \int_0^{1-\phi_w} \left(\frac{W_t(h)}{W_t}\right)^{-\varepsilon_w} dh + \int_{1-\phi_w}^1 \left(\frac{\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w} W_{t-1}(h)}{W_t}\right)^{-\varepsilon_w} dh$$ Using the same reset price:

$$= (1-\phi_w)\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w} + \Pi_{t-1}^{-\varepsilon_w\gamma_w} W_t^{\varepsilon_w} \int_{1-\phi_w}^1 W_{t-1}(h)^{-\varepsilon_w} dh.$$ Multiply right term by $\left(\frac{W_{t-1}}{W_{t-1}}\right)^{-\varepsilon_w}$

$$= (1-\phi_w)\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w} + \Pi_{t-1}^{-\varepsilon_w\gamma_w} W_t^{\varepsilon_w} W_{t-1}^{-\varepsilon_w} \int_{1-\phi_w}^1 \left(\frac{W_{t-1}(h)}{W_{t-1}}\right)^{-\varepsilon_w} dh$$

$$= (1-\phi_w)\left(\frac{W_t^*}{W_t}\right)^{-\varepsilon_w} + \phi_w \Pi_{t-1}^{-\varepsilon_w\gamma_w} W_t^{\varepsilon_w} W_{t-1}^{-\varepsilon_w} V_{t-1}^w$$

We need the right term in real values:

The first term is already real, as we just multiply & divide by P_t initially

$$\longrightarrow = (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \phi_w \Pi_{t-1}^{-\varepsilon_w\gamma_w} \frac{W_t^{\varepsilon_w}}{P_t^{\varepsilon_w}} \frac{W_{t-1}^{-\varepsilon_w}}{P_{t-1}^{-\varepsilon_w}} V_{t-1}^w \left(\frac{P_t}{P_{t-1}}\right)^{\varepsilon_w}$$

$$V_t^w = (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \phi_w \left(\frac{\Pi_t}{\Pi_{t-1}^{\gamma_w}}\right)^{\varepsilon_w} \left(\frac{w_t}{w_{t-1}}\right)^{\varepsilon_w} V_{t-1}^w \qquad (9)$$

The last derivation in this section starts from (**):

$$W_t^{1-\varepsilon_w} = \left[\int_0^1 W_t(h)^{1-\varepsilon_w} dh\right]$$ Use sticky wages:

$$W_t^{1-\varepsilon_w} = \int_0^{1-\phi_w} W_t(h)^{1-\varepsilon_w} dh + \int_{1-\phi_w}^1 \left(\left(\frac{P_{t-1}}{P_{t-2}}\right)^{\gamma_w} W_{t-1}(h)\right)^{1-\varepsilon_w} dh$$ Impose reset wage w^*:

$$W_t^{1-\varepsilon_w} = (1-\phi_w)(W_t^*)^{1-\varepsilon_w} + \int_{1-\phi_w}^1 \left(\Pi_{t-1}^{\gamma_w} W_{t-1}(h)\right)^{1-\varepsilon_w} dh.$$

If we consider W_{t-1} as the average wage (also as the aggregate wage) in $t-1$, then as $h \to \infty$, by the weak law of large numbers:

$$W_t^{1-\varepsilon_w} = (1-\phi_w)(W_t^*)^{1-\varepsilon_w} + \phi_w\left(\Pi_{t-1}^{\gamma_w} W_{t-1}\right)^{1-\varepsilon_w}$$

Set this into real terms:

$$\frac{W_t^{1-\varepsilon_w}}{P_t^{1-\varepsilon_w}} = (1-\phi_w)\left(\frac{W_t^*}{P_t}\right)^{1-\varepsilon_w} + \phi_w\left(\Pi_{t-1}^{\gamma_w} \frac{P_{t-1}}{P_t} \frac{W_{t-1}}{P_{t-1}}\right)^{1-\varepsilon_w}$$

$$w_t^{1-\varepsilon_w} = (1-\phi_w) w_t^{*\,1-\varepsilon_w} + \phi_w \left(\frac{\Pi_{t-1}^{\gamma_w}}{\Pi_t} w_{t-1}\right)^{1-\varepsilon_w} \qquad (10)$$

③ Production

There are 4 firms:

i. Representative whole-saler – combines capital and labor to make $Y_{m,t}$.

ii. Competitive Capital Producer – Creates new physical capital $\hat{I}_t$.

iii. Retail Firms: Repackage wholesale output using $Y_t(f) = Y_{m,t}(f)$.

iv. Competitive Final Goods Firm: Aggregate $Y_t(f)$ into Y_t using CES aggregation.

Then retailers face demand:

$$Y_t = \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} df\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}}$$

Smets-Wouters parameter

$$\frac{1}{1+\lambda} = \frac{\varepsilon_p - 1}{\varepsilon_p}$$

$$\frac{\varepsilon_p}{\varepsilon_p - 1} = 1+\lambda \rightarrow \frac{\varepsilon_p}{\varepsilon_p - 1} - 1 = \lambda \qquad \frac{\varepsilon_p - \varepsilon_p + 1}{\varepsilon_p - 1} = \lambda \rightarrow \frac{1}{\varepsilon_p - 1} = \lambda$$

Redefine Z_t as total expenditure on final goods: $Z_t = \int_0^1 P_t(f) Y_t(f)\, df$

Final goods firm face the following problem:

$$\max_{\{Y_t(f)\}} \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} df\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \quad \text{s.t.} \quad Z_t = \int_0^1 P_t(f) Y_t(f)\, df$$

$$\langle Y_t(f) \rangle \quad \frac{\cancel{\varepsilon_p}}{\cancel{\varepsilon_p - 1}} \cdot \frac{\varepsilon_p - 1}{\cancel{\varepsilon_p}} Y_t(f)^{\frac{-1}{\varepsilon_p}} \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} df\right]^{\frac{1}{\varepsilon_p - 1}} - \lambda P_t(f) = 0 \qquad (**)$$

$$Y_t(f)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} \left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} df\right]^{\frac{1}{\varepsilon_p - 1}} = \lambda P_t(f) Y_t(f)$$

$$\left[\int_0^1 Y_t(f)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} df\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} = \lambda Z_t$$

$$Y_t = \lambda Z_t \longrightarrow Y_t = \lambda P_t Y_t \longrightarrow \frac{1}{P_t} = \lambda$$

Sub into $(**)$:

$$Y_t(f)^{\frac{-1}{\varepsilon_p}} Y_t^{\frac{1}{\varepsilon_p}} = \frac{P_t(f)}{P_t}$$

$$Y_t(f)^{\frac{1}{\varepsilon_p}} = Y_t^{\frac{1}{\varepsilon_p}} \left(\frac{P_t(f)}{P_t}\right)^{-1}$$

$$Y_t(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t$$

The aggregate price dynamics are:

$$P_t Y_t = \int_0^1 P_t(f) Y_t(f)\, df = Z_t$$

$$P_t Y_t = \int_0^1 P_t(f) \left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t\, df$$

$$P_t^{1-\varepsilon_p} = \int_0^1 P_t(f)^{1-\varepsilon_p} df$$

I. Retail Firms

First, we define nominal dividends:

$$DIV_{R,t}(f) = P_t(f) Y_t(f) - P_{m,t} Y_{m,t}(f)$$

$$P_t Y_t - P_{m,t} Y_{m,t}$$

Set $Y_t(f) = Y_{m,t}(f)$, imposing demand = supply:

$$DIV_{R,t}(f) = P_t(f)\, Y_t(f) - P_{m,t} Y_t(f) \quad \text{Sub in for } Y_t(f)$$
$$= P_t(f)\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t - P_{m,t}\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t$$
$$= P_t(f)^{1-\varepsilon_p} P_t^{\varepsilon_p} Y_t - P_{m,t} P_t(f)^{-\varepsilon_p} P_t^{\varepsilon_p} Y_t$$

Like with wages, we impose sticky prices, where $(1-\phi_p)$ is the probability of adjusting price each period. Also like with wages, we can index to inflation w/ γ_p: $P_{t+j}(f) = P_t(f)\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{\gamma_p}$ for a firm that can't update prices. The retailers problem is thus:

$$\max_{\{P_t(f)\}} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left\{P_{t+j}(f)^{1-\varepsilon_p} P_{t+j}^{\varepsilon_p} Y_{t+j} - P_{m,t+j} P_{t+j}(f)^{-\varepsilon_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right\}\right]$$
$$= \max_{\{P_t(f)\}} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left\{P_t(f)^{1-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j} - P_{m,t+j} P_t(f)^{-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right\}\right]$$

Convert to real values:

$$= \max_{\{P_t(f)\}} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left\{P_t(f)^{1-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p-1} Y_{t+j} - p_{m,t+j} P_t(f)^{-\varepsilon_p}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right\}\right]$$

$$\langle P_t(f)\rangle \quad (1-\varepsilon_p) P_t(f)^{-\varepsilon_p} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p-1} Y_{t+j}\right]$$
$$+ \varepsilon_p P_t(f)^{-\varepsilon_p-1} E_t\left[\sum_{j=0}^{\infty} \phi_p^j \Lambda_{t,t+j}\, p_{m,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}\right] = 0$$

Let $X_{1,t} = \sum_{j=0}^{\infty} (\phi_p\beta)^j \frac{u_{t+j}}{u_t}\, p_{m,t+j}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{-\varepsilon_p\gamma_p} P_{t+j}^{\varepsilon_p} Y_{t+j}$

& $X_{2,t} = \sum_{j=0}^{\infty} (\phi_p\beta)^j \frac{u_{t+j}}{u_t}\left(\frac{P_{t+j-1}}{P_{t-1}}\right)^{(1-\varepsilon_p)\gamma_p} P_{t+j}^{\varepsilon_p-1} Y_{t+j}$.

Recursively: $X_{1,t} = p_{m,t} P_t^{\varepsilon_p} Y_t + \phi_p \Lambda_{t,t+1} \Pi_t^{-\varepsilon_p\gamma_p} X_{1,t+1}$

$X_{2,t} = P_t^{\varepsilon_p-1} Y_t + \phi_p \Lambda_{t,t+1} \Pi_t^{(1-\varepsilon_p)\gamma_p} X_{2,t+1}$

Because each retailer has the same pricing problem, all reset prices are the same, P_t^*. Rearranging the first-order condition gives:

$$(\varepsilon_p - 1) P_t^{*\,-\varepsilon_p} X_{2,t} = \varepsilon_p P_t^{*\,-\varepsilon_p-1} X_{1,t}$$
$$P_t^* = \frac{\varepsilon_p}{\varepsilon_p-1}\cdot\frac{X_{1,t}}{X_{2,t}} \quad \text{Get in real terms:}$$
$$\frac{P_t^*}{P_t} = \frac{\varepsilon_p}{\varepsilon_p-1}\cdot\frac{X_{1,t}}{X_{2,t}}\cdot\frac{P_t^{\varepsilon_p-1}}{P_t^{\varepsilon_p}}$$
$$p_t^* = \frac{\varepsilon_p}{\varepsilon_p-1}\cdot\frac{x_{1,t}}{x_{2,t}}, \text{ where} \quad (11)$$
$$\frac{X_{1,t}}{P_t^{\varepsilon_p}} = x_{1,t} = p_{m,t} Y_t + \phi_p E_t\left[\Lambda_{t,t+1} \Pi_t^{-\varepsilon_p\gamma_p}\cdot\frac{P_{t+1}^{\varepsilon_p}}{P_t^{\varepsilon_p}}\, x_{1,t+1}\right] \quad (12)$$

$$\frac{x_{2,t}}{P_t^{\varepsilon_p-1}} = x_{2,t} = Y_t + \phi_p E_t\left[\Lambda_{t,t+1}\,\Pi_t^{(1-\varepsilon_p)\gamma_p}\,\frac{P_{t+1}^{\varepsilon_p-1}}{P_t^{\varepsilon_p-1}}\,x_{2,t+1}\right] \quad (13)$$

- Aggregating:

Integrate $Y_t(f)$ over all f:

$$\int_0^1 Y_t(f)\,df = \int_0^1\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p} Y_t\,df \qquad \text{Imposing eq'm:}$$

$$\int_0^1 Y_{m,t}(f)\,df = Y_t\int_0^1\left(\frac{P_t(f)}{P_t}\right)^{-\varepsilon_p}df$$

Price dispersion

$$Y_{m,t} = Y_t\,v_t^p \quad (14)$$

Looking at just v_t^p and introducing Calvo pricing:

$$v_t^p = \int_0^{1-\phi_p} P_t^{*\,-\varepsilon_p}\,df + \int_{1-\phi_p}^1\left(\frac{\Pi_{t-1}^{\gamma_p}P_{t-1}(f)}{P_t}\right)^{-\varepsilon_p}df$$

$$= (1-\phi_p)P_t^{*\,-\varepsilon_p} + \Pi_{t-1}^{-\gamma_p\varepsilon_p}P_t^{\varepsilon_p}P_{t-1}^{-\varepsilon_p}\int_{1-\phi_p}^1\left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{-\varepsilon_p}df$$

$$\xrightarrow[f\to\infty]{P} \quad v_t^p = (1-\phi_p)P_t^{*\,-\varepsilon_p} + \phi_p\left(\frac{\Pi_t}{\Pi_{t-1}^{\gamma_p}}\right)^{\varepsilon_p}v_{t-1}^p \quad (15)$$

Now rewriting A.21 w/ Calvo pricing:

$$P_t^{1-\varepsilon_p} = \int_0^{1-\phi_p}P_t^{*\,1-\varepsilon_p}\,df + \int_{1-\phi_p}^1\Pi_{t-1}^{\gamma_p(1-\varepsilon_p)}P_{t-1}(f)^{1-\varepsilon_p}\,df$$

$$\xrightarrow[f\to\infty]{P}(1-\phi_p)P_t^{*\,1-\varepsilon_p} + \phi_p\Pi_{t-1}^{\gamma_p(1-\varepsilon_p)}P_{t-1}^{1-\varepsilon_p} \qquad \text{Divide by } P_t^{1-\varepsilon_p}:$$

$$1 = (1-\phi_p)p_t^{*\,1-\varepsilon_p} + \phi_p\Pi_{t-1}^{\gamma_p(1-\varepsilon_p)}\Pi_t^{\varepsilon_p-1} \quad (16)$$

II. Wholesale Firms

Production technology: $Y_{m,t} = A_t(\upsilon_t K_t)^\alpha L_{d,t}^{1-\alpha}$ (17)

Capital utilization (apparently a choice variable)

Law of Motion: $K_{t+1} = \hat{I}_t + (1-\delta(\upsilon_t))K_t$ (18)

where $\delta(\upsilon_t) = \delta_0 + \delta_1(\upsilon_t - 1) + \delta_2(\upsilon_t-1)^2$

Dividends: $DIV_{m,t} = P_{m,t}A_t(\upsilon_t K_t)^\alpha L_{d,t}^{1-\alpha} - W_t L_{d,t} - P_t^k\hat{I}_t$

Write dividends in real terms:

$$\frac{DIV_{m,t}}{P_t} = \frac{P_{m,t}}{P_t}A_t(\upsilon_t K_t)^\alpha L_{d,t}^{1-\alpha} - \frac{W_t}{P_t}L_{d,t} - \frac{P_t^k}{P_t}\hat{I}_t$$

$$div_{m,t} = p_{m,t}A_t(\upsilon_t K_t)^\alpha L_{d,t}^{1-\alpha} - w_t L_{d,t} - p_t^k\hat{I}_t$$

Write in Lagrangian:

$$\mathcal{L} = E_t\left[\sum_{j=0}^\infty \Lambda_{t,t+j}\left\{p_{m,t+j}A_{t+j}(\upsilon_{t+j}K_{t+j})^\alpha L_{d,t+j}^{1-\alpha} - w_{t+j}L_{d,t+j} - p_{t+j}^k\hat{I}_{t+j} + \lambda_{1,t+j}\left(\hat{I}_{t+j} + (1-\delta(\upsilon_{t+j}))K_{t+j} - K_{t+j+1}\right)\right.\right]$$

Take FOCs wrt $L_{d,t}$; $\hat{I}_t$; U_t; K_{t+1}

$$\frac{\partial \mathcal{L}}{\partial L_{d,t}} = (1-\alpha) p_{m,t} A_t (U_t K_t)^{\alpha} L_{d,t}^{-\alpha} - w_t = 0 \qquad \text{I}$$

$$\frac{\partial \mathcal{L}}{\partial \hat{I}_t} = -p_t^k + \lambda_{1,t} = 0 \longrightarrow \lambda_{1,t} = p_t^k \qquad \text{II}$$

$$\frac{\partial \mathcal{L}}{\partial U_t} = p_{m,t} A_t \alpha K_t^{\alpha} U_t^{\alpha-1} L_{d,t}^{1-\alpha} - \lambda_{1,t} \delta'(U_t) K_t = 0 \qquad \text{III}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_{1,t} + E_t\left[\alpha \Lambda_{t,t+1} p_{m,t+1} A_{t+1} U_{t+1}^{\alpha} K_{t+1}^{\alpha-1} L_{d,t+1}^{1-\alpha} + \lambda_{1,t+1} \Lambda_{t,t+1} (1-\delta(U_{t+1}))\right] = 0 \qquad \text{IV}$$

From I : $w_t = (1-\alpha) p_{m,t} A_t (U_t K_t)^{\alpha} L_{d,t}^{-\alpha}$ (19)

From III : $p_{m,t} A_t \alpha K_t^{\alpha} U_t^{\alpha-1} L_{d,t}^{1-\alpha} = \lambda_{1,t} \delta'(U_t) K_t$

$$p_{m,t} A_t \alpha K_t^{\alpha} U_t^{\alpha-1} L_{d,t}^{1-\alpha} = p_t^k \delta'(U_t) K_t$$

$$p_{m,t} A_t \alpha K_t^{\alpha-1} U_t^{\alpha-1} L_{d,t}^{1-\alpha} = p_t^k \delta'(U_t) \qquad (20)$$

From IV : $E_t\left[\alpha \Lambda_{t,t+1} p_{m,t+1} A_{t+1} U_{t+1}^{\alpha} K_{t+1}^{\alpha-1} L_{d,t+1}^{1-\alpha} + \lambda_{1,t+1} (1-\delta(U_{t+1})) \Lambda_{t,t+1}\right] = \lambda_{1,t}$

$$E_t\left[\alpha \Lambda_{t,t+1} p_{m,t+1} A_{t+1} U_{t+1}^{\alpha} K_{t+1}^{\alpha-1} L_{d,t+1}^{1-\alpha} + p_{t+1}^k (1-\delta(U_{t+1})) \Lambda_{t,t+1}\right] = p_t^k \qquad (21)$$

III. Capital Producer

The capital producer has the following technology:

$$\hat{I}_t = \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t \qquad (22)$$

where $I_t \equiv$ unconsumed final output, $S(\cdot) \equiv$ adjustment cost function. The dividend function is:

$DIV_{k,t} = P_t^k \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t - P_t I_t$ Make into real terms:

$div_{k,t} = p_t^k \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t - I_t$

The maximization problem is:

$$\max_{\{I_t\}} E_t\left[\sum_{j=0}^{\infty} \Lambda_{t,t+j} \left\{p_{t+j}^k \left(1 - S\left(\frac{I_{t+j}}{I_{t+j-1}}\right)\right) I_{t+j} - I_{t+j}\right\}\right]$$

$$\langle I_t \rangle \quad 1 = p_t^k \left(1 - S\left(\frac{I_t}{I_{t-1}}\right)\right) - p_t^k \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right) + E_t\left[\Lambda_{t,t+1} p_{t+1}^k \left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)\right] \qquad (23)$$

where $S\left(\frac{I_t}{I_{t-1}}\right) = \frac{\kappa_I}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2$

5 Fiscal Authority

The government produces an exogenously stochastic portion of Y_t. It is financed by lump-sum taxes on the household and by bonds, B_t.

$$P_t G_t + B_{t-1}(1+i_{t-1}) = P_t T_t + B_t$$

$$\rightarrow \quad G_t + \frac{B_{t-1}}{P_t}(1+i_{t-1})\frac{P_{t-1}}{P_{t-1}} = T_t + b_t$$

$$G_t + b_{t-1}(1+i_{t-1})\Pi_t^{-1} = T_t + b_t \qquad (24)$$

6 Central Bank

The central bank sets interest rates according to:

$$\ln(1+i_t) = (1-\rho_r)\ln(1+i_{ss}) + \rho_r \ln(1+i_{t-1}) + (1-\rho_r)\left[\phi_\pi(\ln(\Pi_t)-\ln(\Pi^*)) + \phi_y(\ln(Y_t)-\ln(Y_{t-1}))\right] + \sigma_r \varepsilon_{r,t} \qquad (25)$$

I test the model with an NGDP rule:

$$\ln(1+i_t) = (1-\rho_r)\ln(1+i_{ss}) + \rho_r \ln(1+i_{t-1}) + (1-\rho_r)\phi_n\left[\ln(\Pi_t) + \ln(Y_t)-\ln(Y_{t-1})\right] + \sigma_r \varepsilon_{r,t}$$

At the ZLB, $(1+\tilde{i}_t) = \max\{1, 1+i_t\}$ (26)

7 Exogenous Processes:

$$\ln(A_t) = \rho_a \ln(A_{t-1}) + \sigma_a \varepsilon_{a,t} \qquad (27)$$

$$\ln(G_t) = (1-\rho_g)\ln(G_{ss}) + \rho_g \ln(G_{t-1}) + \sigma_g \varepsilon_{g,t} \qquad (28)$$

$$cp_t = (1-\rho_{cp})cps + \rho_{cp} cp_{t-1} + \sigma_{cp}\varepsilon_{cp,t}, \text{ where } cp = \frac{1}{\varepsilon_p - 1} \qquad (29)$$

8 Aggregate Resource Constraint

Start with the household budget constraint:

$$C_t + b_t = mrs_t L_t + div_t - T_t + \Pi_{t-1}^{-1}(1+i_{t-1})b_{t-1} \qquad \text{Plug-in for } T_t:$$

$$C_t \cancel{+ b_t} = mrs_t L_t + div_t - G_t \cancel{- b_{t-1}(1+i_{t-1})\Pi_t^{-1}} \cancel{+ b_t} \cancel{+ \Pi_{t-1}^{-1}(1+i_{t-1})b_{t-1}}$$

$$C_t = mrs_t L_t + div_t - G_t$$

$$C_t + G_t = \cancel{mrs_t L_t} + p_{m,t}A_t(u_t K_t)^\alpha L_{d,t}^{1-\alpha} \cancel{- w_t L_{d,t}} \cancel{- p_t^k \hat{I}_t} + \cancel{p_t^k\left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right]I_t} - I_t$$

$$+ P_t Y_t - P_{m,t} Y_{m,t} + \cancelto{0}{div_{F,t}} \cancel{+ w_t L_{d,t}} \cancel{- mrs_t L_t}$$

$$C_t + I_t + G_t = p_{m,t} A_t (u_t K_t)^\alpha L_{d,t}^{1-\alpha} + Y_t - p_{m,t} Y_{m,t} + \cancel{div_{F,t}} \rightarrow 0$$

$$C_t + I_t + G_t = \cancel{p_{m,t} Y_{m,t}} + Y_t - \cancel{p_{m,t} Y_{m,t}}$$

$$C_t + I_t + G_t = Y_t$$

30

⑨ *Equilibrium Conditions*

1) $$u_t = \frac{1}{C_t - hC_{t-1}} - \beta E_t\left[\frac{h}{C_{t+1} - hC_t}\right]$$

2) $$\chi L_t^{\eta} = mrs_t u_t$$

3) $1 = \Lambda \Pi^{-1}(1+i)$ $$v_t u_t = E_t\left[\beta u_{t+1} \Pi_{t+1}^{-1} (1+i_t)\right]$$ ✓ °

4) $$\Lambda_{t,t+1} = \beta \frac{E_t[u_{t+1}]}{u_t}$$ ✓ °

5) $$w_t^* = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{f_{1,t}}{f_{2,t}}$$

6) $$f_{1,t} = mrs_t w_t^{\varepsilon_w} L_{d,t} + \phi_w E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{\varepsilon_w} \Pi_t^{-\varepsilon_w \gamma_w} f_{1,t+1}\right]$$ ✓ °

7) $$f_{2,t} = w_t^{\varepsilon_w} L_{d,t} + \phi_w E_t\left[\Lambda_{t,t+1} \Pi_{t+1}^{\varepsilon_w - 1} \Pi_t^{\gamma_w(1-\varepsilon_w)} f_{2,t+1}\right]$$ ✓ °

8) $$L_t = L_{d,t} V_t^w$$

9) $$V_t^w = (1-\phi_w)\left(\frac{w_t^*}{w_t}\right)^{-\varepsilon_w} + \phi_w \left(\frac{\Pi_t}{\Pi_{t-1}^{\gamma_w}}\right)^{\varepsilon_w} \left(\frac{w_t}{w_{t-1}}\right)^{\varepsilon_w} V_{t-1}^w$$

10) $$w_t^{1-\varepsilon_w} = (1-\phi_w) w_t^{*\,1-\varepsilon_w} + \phi_w \left(\frac{\Pi_{t-1}^{\gamma_w}}{\Pi_t} w_{t-1}\right)^{1-\varepsilon_w}$$

11) $$p_t^* = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{X_{1,t}}{X_{2,t}}$$

12) $$X_{1,t} = p_{m,t} Y_t + \phi_p E_t\left[\Lambda_{t,t+1} \Pi_t^{-\varepsilon_p \gamma_p} \Pi_{t+1}^{\varepsilon_p} X_{1,t+1}\right]$$ ✓ °

13) $$X_{2,t} = Y_t + \phi_p E_t\left[\Lambda_{t,t+1} \Pi_t^{(1-\varepsilon_p)\gamma_p} \Pi_{t+1}^{\varepsilon_p - 1} X_{2,t+1}\right]$$ ✓ °

14) $$Y_{m,t} = Y_t v_t^p$$

15) $$v_t^p = (1-\phi_p) p_t^{*\,-\varepsilon_p} + \phi_p \Pi_t^{\varepsilon_p} \Pi_{t-1}^{-\varepsilon_p \gamma_p} v_{t-1}^p$$

16) $$1 = (1-\phi_p) p_t^{*\,1-\varepsilon_p} + \phi_p \Pi_{t-1}^{\gamma_p(1-\varepsilon_p)} \Pi_t^{\varepsilon_p - 1}$$

17) $$Y_{m,t} = A_t (u_t K_t)^\alpha L_{d,t}^{1-\alpha}$$

18) ° $$K_{t+1} = \hat{I}_t + (1-\delta(u)) K_t$$

19) $$w_t = (1-\alpha) p_{m,t} A_t (u_t K_t)^\alpha L_{d,t}^{-\alpha}$$

20) $$p_{m,t} A_t \alpha K_t^{\alpha-1} u_t^{\alpha-1} L_{d,t}^{1-\alpha} = p_t^k \delta'(u_t)$$

21) 6 $$E_t\left[\alpha \Lambda_{t,t+1} p_{m,t+1} A_{t+1} u_{t+1}^{\alpha} K_{t+1}^{\alpha-1} L_{d,t+1}^{1-\alpha} + p_{t+1}^k (1-\delta(u_{t+1})) \Lambda_{t,t+1}\right] = p_t^k$$ ✓ ·

22) 6 $$\hat{I}_t = \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_t$$

23) $1 = \rho_t^k\left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] - \rho_t^k \frac{I_t}{I_{t-1}} S'\left(\frac{I_t}{I_{t-1}}\right) + E_t\left[\Lambda_{t,t+1}\rho_{t+1}^k\left(\frac{I_{t+1}}{I_t}\right)^2 S'\left(\frac{I_{t+1}}{I_t}\right)\right]$ ✓

24) $\ln(1+i_t) = (1-\rho_r)\ln(1+i_{ss}) + \rho_r\ln(1+i_{t-1}) + (1-\rho_r)\left[\phi_\pi(\ln(\Pi_t) - \ln(\Pi^*)) + \phi_y(\ln(Y_t) - \ln(Y_{t-1}))\right] + \sigma_r\varepsilon_{r,t}$

25) $(1+\tilde{i}_t) = \max\{1, 1+i_t\}$

26) $\ln(A_t) = \rho_a\ln(A_{t-1}) + \sigma_a\varepsilon_{a,t}$

27) $\ln(G_t) = (1-\rho_g)\ln(G_{ss}) + \rho_g\ln(G_{t-1}) + \sigma_g\varepsilon_{g,t}$

28) $cp_t = (1-\rho_{cp})cps + \rho_{cp}cp_{t-1} + \sigma_{cp}\varepsilon_{cp,t}$, where $cp = \frac{1}{\varepsilon_p - 1}$

29) $C_t + I_t + G_t = Y_t$

30) $welf_t = \ln(C_t - hC_{t-1}) - \frac{\chi L_{t+j}^{1+\eta}}{1+\eta} + \beta E_t\left[welf_{t+1}\right]$

⑩ Steady State

Impose $\Pi_{ss} = 1$ and $A = 1$ and $\upsilon = 1$ $\quad v = 1$

Starting with 4) $\Lambda = \beta\frac{E_t[u]}{u} \longrightarrow \Lambda_{ss} = \beta$ $\quad\quad \Lambda = \frac{\Pi}{(1+i)}$

Go to 3) $u = \beta u \Pi^{-1}(1+i)] \rightarrow 1 = \beta(1+i) \rightarrow (1+i_{ss}) = \frac{1}{\beta}$ $\quad$?

$\chi L^\eta = mrs\, u \longrightarrow u = \frac{\chi L^\eta}{mrs}$

$u = \frac{1}{C - hC} - \beta\frac{h}{C - hC} \longrightarrow \frac{\chi L^\eta}{mrs} = \frac{1}{C-hC}(1-\beta h)$

$f_1 = mrs\, w^{\varepsilon_w} L_d + \phi_w \Lambda \Pi^{\varepsilon_w}\Pi^{-\varepsilon_w\gamma_w} f_1 \rightarrow f_1 = (1-\phi_w\beta)^{-1}(mrs \cdot w^{\varepsilon_w} \cdot L_d)$

$f_2 = w^{\varepsilon_w} L_d + \phi_w \Lambda_{t,t+1}\Pi_{t+1}^{\varepsilon_w - 1}\Pi^{\gamma_w(1-\varepsilon_w)} f_2 \rightarrow f_2 = (1-\phi_w\beta)^{-1}(w^{\varepsilon_w} \cdot L_d)$

$w^* = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{f_1}{f_2}$

$\longrightarrow w^* = \frac{\varepsilon_w}{\varepsilon_w - 1} mrs$

$w^{1-\varepsilon_w} = (1-\phi_w)w^{*1-\varepsilon_w} + \phi_w w^{1-\varepsilon_w} \rightarrow w^{1-\varepsilon_w} = w^{*1-\varepsilon_w} \rightarrow w = w^*$

$v^w = (1-\phi_w)\left(\frac{w^*}{w}\right)^{-\varepsilon_w} + \phi_w v_w \rightarrow v^w = \left(\frac{w^*}{w}\right)^{-\varepsilon_w} \rightarrow v^w = 1$

$L = L_d v^w \rightarrow L = L_d$

$x_1 = p_m Y + \phi_p \Lambda \Pi^{-\varepsilon_p\gamma_p}\Pi^{\varepsilon_p} x_1 \longrightarrow x_1 = p_m Y + \phi_p\beta x_1 \rightarrow x_1 = (1-\phi_p\beta)^{-1}p_m Y$

$x_2 = Y + \phi_p\Lambda\Pi^{(1-\varepsilon_p)\gamma_p}\Pi^{\varepsilon_p - 1}x_2 \longrightarrow x_2 = Y + \phi_p\beta x_2 \longrightarrow x_2 = (1-\phi_p\beta)^{-1}Y$

$p^* = \frac{\varepsilon_p}{\varepsilon_p - 1}\frac{x_1}{x_2} \longrightarrow p^* = \frac{\varepsilon_p}{\varepsilon_p - 1}p_m \longrightarrow p_m = \frac{\varepsilon_p - 1}{\varepsilon_p}$

$1 = (1-\phi_p)p^{*1-\varepsilon_p} + \phi_p\Pi^{\gamma_p(1-\varepsilon_p)}\Pi^{\varepsilon_p - 1} \rightarrow 1 = (1-\phi_p)p^{*1-\varepsilon_p} + \phi_p \rightarrow (1-\phi_p) = (1-\phi_p)p^{*1-\varepsilon_p} \rightarrow 1 = p^{*1-\varepsilon_p} \rightarrow p^* = 1$

$v^p = (1-\phi_p)p^{*-\varepsilon_p} + \phi_p\Pi^{\varepsilon_p}\Pi^{-\varepsilon_p\gamma_p}v^p \rightarrow v^p = (1-\phi_p) + \phi_p v^p \rightarrow v^p = 1$

$$Y_m = Y_V{}^{\rho} \longrightarrow Y_m = Y$$

$$Y_m = A(UK)^{\alpha} L_d^{1-\alpha} \longrightarrow Y_m = K^{\alpha} L_d^{1-\alpha}$$

$$K = \hat{I} + (1-\delta(U))K \longrightarrow K(1-1+\delta(U)) = \hat{I} \longrightarrow K = \delta(U)^{-1}\hat{I}$$

$$w = (1-\alpha)p_m A(UK)^{\alpha} L_d^{-\alpha} \longrightarrow w = (1-\alpha)p_m K^{\alpha} L_d^{-\alpha}$$

$$p_m A \alpha K^{\alpha-1} U^{\alpha-1} L_d^{1-\alpha} = p^k \delta'(U) \longrightarrow p_m \alpha K^{\alpha-1} L_d^{1-\alpha} = p^k \delta'(U)$$

$$\alpha \Lambda p_m A U^{\alpha} K^{\alpha-1} L_d^{1-\alpha} + \Lambda p^k(1-\delta(U)) = p^k \longrightarrow \alpha\beta p_m K^{\alpha-1} L_d^{1-\alpha} + p^k(1-\delta(U))\beta = p^k$$

$$\hat{I} = \left[1 - S\left(\tfrac{I}{I}\right)\right] I \longrightarrow \hat{I} = I$$

$$1 = p^k\left[1 - S\left(\tfrac{I}{I}\right)\right] - p_t^k \tfrac{I}{I} S'\left(\tfrac{I}{I}\right) + \Lambda p^k \left(\tfrac{I}{I}\right)^2 S'\left(\tfrac{I}{I}\right) \longrightarrow 1 = p^k$$

$$p_m \alpha K^{\alpha-1} L_d^{1-\alpha} = p^k \delta'(U) \longrightarrow \frac{\varepsilon_p - 1}{\varepsilon_p} \alpha K^{\alpha-1} L^{1-\alpha} = \delta'(U) \longrightarrow \frac{\varepsilon_p - 1}{\varepsilon_p} \alpha \left(\frac{K}{L}\right)^{\alpha-1} = \delta_1$$

$$\alpha\beta p_m K^{\alpha-1} L_d^{1-\alpha} + p^k(1-\delta(U))\beta = p^k \longrightarrow \alpha\beta p_m \left(\frac{K}{L}\right)^{\alpha-1} + \beta(1-\delta(U)) = 1 \longrightarrow \alpha p_m K^{\alpha-1} L_d^{1-\alpha} + 1 - \delta_0 = \frac{1}{\beta}$$

$$\alpha p_m K^{\alpha-1} L_d^{1-\alpha} = \frac{1}{\beta} - 1 + \delta_0$$

$$K^{\alpha-1} = \frac{1}{\alpha p_m}\left(\frac{1}{\beta} - 1 + \delta_0\right) L_d^{\alpha-1}$$

$$K = \left[\frac{1}{\alpha p_m}\left(\frac{1}{\beta} - 1 + \delta_0\right)\right]^{\frac{1}{\alpha-1}} L_d$$

$$\alpha \Lambda p_m A U^{\alpha} K^{\alpha-1} L_d^{1-\alpha} + \Lambda p^k (1-\delta(U)) = p^k \qquad (p^k = 1)$$

$$\alpha \Lambda p_m A U^{\alpha} K^{\alpha-1} L_d^{1-\alpha} + \Lambda(1-\delta(U)) = 1$$

$$\alpha p_m A U^{\alpha} K^{\alpha-1} L_d^{1-\alpha} + (1-\delta(U)) = \frac{1}{\Lambda}$$

$$\alpha p_m A K^{\alpha-1} L_d^{1-\alpha} + 1 - \delta_0 = \frac{1}{\Lambda}$$

$$\alpha p_m K^{\alpha-1} L_d^{1-\alpha} = \frac{1}{\Lambda} + \delta_0 - 1$$

$$K^{\alpha-1} = \frac{1}{\alpha p_m}\left(\frac{1}{\Lambda} + \delta_0 - 1\right) L_d^{\alpha-1}$$

$$K = L_d\left[\frac{1}{\alpha p_m}\left(\frac{1}{\Lambda} + \delta_0 - 1\right)\right]^{\frac{1}{\alpha-1}}$$

$$\cancel{\frac{\varepsilon_p - 1}{\varepsilon_p}} \frac{1}{\cancel{p_m}}\left(\frac{1}{\Lambda} + \delta_0 - 1\right) = \delta_1$$

$$\frac{1}{\Lambda} + \delta_0 - 1 = \delta_1$$

$$\frac{K}{L} = \left[\frac{1}{\alpha p_m}\left(\frac{1}{\Lambda} + \delta_0 - 1\right)\right]^{\frac{1}{\alpha-1}}$$

$$w = (1-\alpha)\frac{\varepsilon_p - 1}{\varepsilon_p}\left(\frac{K}{L}\right)^{\alpha}$$

$$\frac{\varepsilon_p - 1}{\varepsilon_p} \alpha \left(\frac{K}{L_d}\right)^{\alpha-1} = \delta_1 \qquad \cancel{\frac{\varepsilon_p - 1}{\varepsilon_p} \alpha} \frac{1}{\cancel{\alpha p_m}}\left(\frac{1}{\beta} - 1 + \delta_0\right) = \delta_1$$

$$\frac{1}{\beta} - 1 + \delta_0 = \delta_1$$

This implies that as long as $\delta_1 = \frac{1}{\beta} - 1 + \delta_0$, L can be any value. I choose $L = 1$:

$$L = 1 \qquad L_p = 1$$

$$w = (1-\alpha)p_m K^{\alpha} L_d^{-\alpha} \longrightarrow w = (1-\alpha)\frac{\varepsilon_p - 1}{\varepsilon_p} K^{\alpha} L^{-\alpha} \qquad w = w^*$$

$$w^* = \frac{\varepsilon_w}{\varepsilon_w - 1} mrs \longrightarrow \frac{\varepsilon_w - 1}{\varepsilon_w} w = mrs$$

$$\frac{\chi L^{\eta}}{mrs} = \frac{1}{c - hc}(1-\beta h) \longrightarrow \frac{\chi L^{\eta}}{mrs}(1-\beta h)^{-1} = (c - hc)^{-1} \longrightarrow c(1-h) = \frac{mrs(1-\beta h)}{\chi L^{\eta}}$$

$$\longrightarrow c = \frac{mrs(1-\beta h)}{(1-h)\chi L^{\eta}}$$

$$K = \delta(U)^{-1}\hat{I} \longrightarrow K = \delta_0^{-1} I \longrightarrow I = \delta_0 K$$

$$gY = Y - C - I \longrightarrow g = \frac{Y - C - I}{Y}$$

$$f_1 = (1 - \phi_w \beta)^{-1}(mrs \cdot w^{\varepsilon_w} \cdot L_d) \qquad f_1 = (1 - \phi_w \Lambda)^{-1}(mrs\, w^{\varepsilon_w} L_d)$$

$$f_2 = (1 - \phi_w \beta)^{-1}(w^{\varepsilon_w} \cdot L_d) \qquad f_2 = (1 - \phi_w \Lambda)^{-1}(w^{\varepsilon_w} \cdot L_d)$$

$$x_1 = (1 - \phi_p \beta)^{-1} p_m Y \qquad x_1 = (1 - \phi_p \Lambda)^{-1}(p_m Y)$$

$$x_2 = (1 - \phi_p \beta)^{-1} Y \qquad x_2 = (1 - \phi_p \beta)^{-1} Y$$

Order of variables in Dynare (for ZLB extraction)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
u	C	L	mrs	Π	i	Δ	w^*	f_1	f_2	L_d	v^w	w	p^*	x_1	x_2	p_m	Y	Y_m	v^p

A	υ	K	$\hat{I}$	p^k	I	G	ε_p	CP	V	welf	logA	logC	logY	logI	logPi	logL
21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37

38	39
rate	logw

$$\tilde{i}_t = \sigma(E_t[\tilde{y}_{t+1}] - \tilde{y}_t) + E_t[\tilde{\pi}_{t+1}]$$

Instead of observing $\tilde{y}$, say we observe $y = \tilde{y} - y^A$. Then:

$$\tilde{i}_t = \sigma(E_t[\tilde{y}_{t+1} - y^A_{t+1}] - \tilde{y}_t + y^A_t) + E_t[\tilde{\pi}_{t+1}]$$

$$= \sigma(y^A_t - E_t[y^A_{t+1}]) + \sigma(E_t[\tilde{y}_{t+1}] - \tilde{y}_t) + E_t[\tilde{\pi}_{t+1}]$$ Set $\sigma = 1$. Then:

$$= y^A_t - E_t[y^A_{t+1}] + \phi_N^{-1} E_t[\tilde{i}_{t+1}]$$

$$E_t[\tilde{i}_{t+1}] = \phi_N \tilde{i}_t \underbrace{- \phi_N(y^A_t - E_t[y^A_{t+1}])}_{\text{Adjustment cost wedge}}$$

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(\tilde{i}_t - E_t[\tilde{\pi}_{t+1}])$$
$$\frac{1}{\sigma}\tilde{i}_t = E_t[\tilde{y}_{t+1}] - \tilde{y}_t + \frac{1}{\sigma}E_t[\tilde{\pi}_{t+1}]$$
$$\tilde{i}_t = \sigma(E_t[\tilde{y}_{t+1}] - \tilde{y}_t) + E_t[\tilde{\pi}_{t+1}]$$

$$\tilde{i}_t = \phi_N(\tilde{\pi}_t + \tilde{y}_t - \tilde{y}_{t-1})$$
$$\tilde{i}_t = \phi_N\tilde{\pi}_t + \phi_N(\tilde{y}_t - \tilde{y}_{t-1})$$

$$\tilde{i}_t = \sigma(E_t[\tilde{y}_{t+1}] - \tilde{y}_t) + E_t[\tilde{\pi}_{t+1}]$$
$$\tilde{i}_t = (\sigma-1)(E_t[\tilde{y}_{t+1}] - \tilde{y}_t) + E_t[\tilde{\pi}_{t+1}] + E_t[\tilde{y}_{t+1}] - \tilde{y}_t$$
$$\tilde{i}_t = \phi_N^{-1}E_t[\tilde{i}_{t+1}] + (\sigma-1)(E_t[\tilde{y}_{t+1}] - \tilde{y}_t)$$
$$E_t[\tilde{i}_{t+1}] = \phi_N\tilde{i}_t + \phi_N(1-\sigma)(E_t[\tilde{y}_{t+1}] - \tilde{y}_t)$$

Shadow cost of consumption

$V_t \; u_t = E_t[\beta u_{t+1}\Pi_{t+1}^{-1}(1+i_t)]$

$u_t = \frac{1}{V_t}A$

$\frac{\partial u_t}{\partial V_t} = -A/V_t^2$ (A is >0, as $i_t > 0$, Π_{t+1} is almost-surely >0, and $u_{t+1} > 0$)

$V_t \uparrow \; \rightarrow u_t \downarrow$

If $u \downarrow \; \rightarrow \frac{1}{C_t - hC_{t-1}} - \beta\frac{h}{C_{t+1} - hC_t} \downarrow$ $\quad \frac{-1}{(C_t - hC_{t-1})^2} - \frac{\beta h^2}{(C_{t+1} - hC_t)^2}$ as $C_t \uparrow$ $\rightarrow \downarrow - (\downarrow)$

For this to decrease, $C_t \uparrow$ $\quad \rightarrow \downarrow - \downarrow$

Turn habit off for a moment:

$u_t = \frac{1}{C_t}$. If $u_t \downarrow \; \Rightarrow C_t \uparrow$. This makes intuitive economic sense. If the shadow cost of consumption falls, you'll consume more today.

$mrs_t L_t + div_t - T_t + \Pi_t^{-1}(1+i_{t-1})a_{t-1} - c_t - a_t$

Using Fisher: $mrs_t L_t + div_t - T_t + (1+r_t)a_{t-1} - c_t - a_t$

Plug-in div_t: $\cancel{mrs_t L_t} + Y_t - I_t - \cancel{mrs_t L_t} - T_t + (1+r_t)a_{t-1} - c_t - a_t$

$Y_t - I_t - T_t + (1+r_t)a_{t-1} - c_t - a_t$

Income $= Y_t$ $\quad coh = Y_t + (1+r_t)a_{t-1}$

$$\frac{(1+i_{t-1})}{(1+\Pi_t)}$$

$(1+i_t) - (1+\Pi_t)$

$r_{t-1}^{ante} = i_{t-1} - \Pi_t$

$r_t = r_{t-1}^{ante}$