Topics in Topology

Prof: Andy Putman

Math 80430: 3-manifolds, fall 2021

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Topics: This course will cover the topology of 3-manifolds, interpreted broadly. No background beyond the standard first-year graduate material will be assumed. The focus will be on techniques that are useful elsewhere (especially in group theory), as well as examples. One major goal will be to carefully explain the motivation for and statement of Thurston's Geometrization Conjecture, as proved by Perelman. Other topics will depend on the interests of the audience.

Text: I won't follow a textbook, but I'll post references to the course website as we go.

Grading: To get an A in the class, you should show up to the lectures and also act as the "scribe" for 2-3 of them (depending on enrollment). The scribe should prepare good lecture notes for that lecture that I can post on the course website. There is a sign-up sheet on the website for scribe duties.

Color Palette:

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Lecture 01: August 23, 2021 First year notes

Reminder: The following is a complete list of compact, connected orientable, n-manifolds,

dim 0: a point (pt)

dim 1: S^1 (also $\mathbb{R}$ if not compact)

dim 2: Σ_g = genus g surfaces (without orientable, also need connected sum of $\mathbb{RP}^2$; without compact, also terror)
$S^2 \setminus$ cantor

$\Sigma_0 = S^2$ $\Sigma_1 = S^1 \times S^1$ $\Sigma_2 = T^2 \# T^2$

dim 3: Geometrization Theorem (conjectured by Thurston, proved by Perelem) is sort of a classification

dim 4: no hope for classification

main goal of course: learn the statement and context for dimension 3

Technical Remarks:

differentiable everywhere — Piecewise linear i.e. has atlases

- In low dimensions, there is no difference between top, smooth, and PL-manifolds (Radó in dim 2, Moise in dim 3)
  1920s? late 50s?
  I will ignore these point-set differences
- This will not be a fundamental course in smooth or PL-topology
  Trust me for technical details (will post notes on PL topology on website)

Dimension 3 Examples

easy ex. S^3, $\mathbb{RP}^3$, $T^3 = S^1 \times S^1 \times S^1$

lie group ex.

- $SO(3) = \{M \in GL_3(\mathbb{R}) \mid M \text{ preserves dot product}\} \cong \mathbb{RP}^3$ fun fact
- $SU(2) = \left\{\begin{bmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix} \in GL_2(\mathbb{C}) \mid |\alpha|^2 + |\beta|^2 = 1\right\} \cong \{(\alpha,\beta) \in \mathbb{C}^2 \mid |\alpha|^2 + |\beta|^2 = 1\} = S^3$
- $\mathcal{H}_3(\mathbb{R})$ = Heisenberg group $= \left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \middle| a,b,c \in \mathbb{R}\right\}$
  not compact, but
  has cocompact lattice $\mathcal{H}_3(\mathbb{Z})$ so $\mathcal{H}_3(\mathbb{Z}) \backslash \mathcal{H}_3(\mathbb{R})$ is a interesting compact 3-manifold

important ex. — X is top. sp. G is gp. action X/G is compact

- $\mathbb{H}^3 = SL_2(\mathbb{C})/SU(2)$, $\dim(\mathbb{H}) = \dim(SL_2(\mathbb{C})) - \dim(SU(2)) = 6 - 3 = 3$ not compact
  ↳ But, if $\Gamma < SL_2(\mathbb{C})$ is discrete, torsion free subgroup then $\Gamma \cap SU(2)$ = discrete, compact, torsion free = 1 (1 is only element of finite order)
  $\Rightarrow \Gamma \curvearrowright \mathbb{H}^3$ (acts on) freely on the left. Additionally, if Γ is cocompact then $\Gamma \backslash \mathbb{H}^3 = \Gamma \backslash SL_2(\mathbb{C})/SU(2)$ is a compact 3-manifold
  Geometrization roughly says most 3-manifolds can be uniquely written of this form (none of ours listed can)

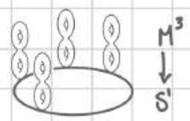
$\Sigma_g \times S^1$

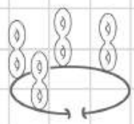
locally product space — part of product

the projection $f: \Sigma_g \times S^1 \to S^1$ makes it into a fiber bundle with fiber Σ_g

basic top. fact: A fiber bundle over a contractible space is trivial (i.e. a product)
can shrink to a point

Consider a fiber bundle $f: M^3 \to S^1$ with fiber Σ_g. What does this look like?



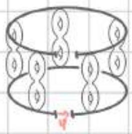


remove small $U = (-\epsilon, \epsilon)$

$[-\epsilon,\epsilon]$ and $S^1 \setminus (-\epsilon,\epsilon)$ are both contractible so $A = f^{-1}([-\epsilon,\epsilon])$ and $B = f^{-1}(S^1 \setminus (-\epsilon,\epsilon))$ are trivial fiber bundles $A = \Sigma_g \times [-\epsilon,\epsilon]$ and $B = \Sigma_g \times (S^1 \setminus (-\epsilon,\epsilon))$ which are glued together at $\Sigma_g \times -\epsilon$ and $\Sigma_g \times \epsilon$.

isomorphism of smooth manifolds

$\Rightarrow$ There must exist a diffeomorphism $\varphi: \Sigma_g \to \Sigma_g$
s.t. $M^3 = \Sigma_g \times [0,1] / (p,1) \sim (\varphi(p), 0)$ notation for this is $M^3 = M_\varphi$



Remarks on mapping tori

- If φ is isotopic to φ' then $M_\varphi \cong M_{\varphi'}$
  homotopy but every point is a diffeomorphism
- M_φ is orientable $\Leftrightarrow$ φ is orientation preserving

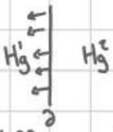
def. The mapping class group of Σ_g is $\mathrm{Mod}(\Sigma_g) = \mathrm{Diff}^+(\Sigma_g)/\text{isotopy}$

so M_φ only depends on image of φ in $\mathrm{Mod}(\Sigma_g)$

Theorem (Agol). If M^3 a compact hyperbolic 3-manifold then M^3 has finite cover that is M_φ for some $\varphi \in \mathrm{Mod}(\Sigma_g)$
most of them by geometrization

ex. A genus g handlebody H_g is $H_g =$  so $\partial H_g \cong \Sigma_g$

Take two genus g handlebodies H_g^1 and H_g^2
Goal: Glue together to get oriented 3-manifold
orient ∂H_g^1 s.t. H_g^1 inside
∂H_g^2 s.t. H_g^2 outside



Choose orientation-preserving diffeomorphism $f_i: \partial H_g^i \to \Sigma_g$
Pick diffeo. $\varphi: \Sigma_g \to \Sigma_g$ and define $\mathcal{H}_\varphi = H_g^1 \amalg H_g^2 / {\text{for } p \in \partial H_g^1 \atop p \sim f_2^{-1} \circ \varphi \circ f(p)}$ is a Heegaard splitting of 3-manifold

ex. $S^3 =$ two 3-balls glued along boundary
$U = \{(a,b,c,d) \in S^3 \mid d \geq 0\}$ and $V = \{(a,b,c,d) \in S^3 \mid d \leq 0\}$
Heegaard splitting along $id: S^2 \to S^2$.

Construction

Given genus g H-g can stabilize to get genus $(g+1)$

drill out of H_g^2

drill in also gives a new handle

Lecture 02: August 25, 2021

Last time

Defined Heegaard splitting of M^3 a decomposition into two handlebodies w/ ∂ (boundary) glued together.

Alt. def. Givem M^3, a Heegaard surface in M^3 is embedded closed surface $\Sigma \hookrightarrow M^3$ s.t. if you cut M^3 open along Σ, you get two handlebodies

ex. usual $S^1 \hookrightarrow S^3$

ex. usual torus in S^3 is Heegaard surface

it is "clear" why "inside" a genus-1 handlebody $\mathbb{D}^2 \times S^1$
but why "outside"?

What is a handlebody

H_g = handlebody of genus g
= 3-manifold bounded by Σ_g
$\partial H_g = \Sigma_g$

$H_2 =$ 

Alt. def. Pick a finite graph, $G \hookrightarrow \mathbb{R}^3$
Let $H = \epsilon$-thick of G.
H is a handlebody of genus $g = b_1(G)$.

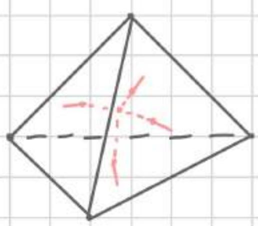
Formula-Based Argument (for examples of handlebody)

$S^3 = \{(z,w) \mid |z|^2 + |w|^2 = 1\} \subseteq \mathbb{C}^2$

$S^1 \times S^1 = T = \{(z,w) \mid |z|^2 = 1/2,\ |w|^2 = 1/2\} \subseteq \mathbb{C}^2$
"inside T" $= \{(z,w) \mid |z|^2 \leq 1/2,\ |z|^2 + |w|^2 = 1\} \cong \mathbb{D}^2 \times S^1$
↳ z-coord disc, for every z w/ $|z|^2 \leq 1/2$
w that work are a circle, $|w|^2 = 1 - |z|^2$
"outside T" $= \{(z,w) \mid |z|^2 + |w|^2 = 1,\ |w|^2 \leq 1/2\} \cong S^1 \times \mathbb{D}^2$

Theorem. Let M^3 be compact, oriented 3-manifold without boundary. Then M^3 has a Heegaard splitting.

proof. Fix a triangulation of M.
Let G = 1-skeleton $M^{(1)}$, a graph in M^3.
Let G_2 = dual graph to triangulation.
vertices = on in middle of 3-simplex Δ_i,
edges = connect $v_1 \in \Delta_1$ to $v_2 \in \Delta_2$ if $\Delta_1 \cap \Delta_2$ = triangle



G_1

G_2

Let U_i = slightly thicken up G_i (a handle body). As you thicken up G_i, U_i start out disjoint. Imagine thickening up more and more until they touch at boudary. Then $M^3 = U_1 \cup U_2$ w/ $U_1 \cap U_2 = \partial U_1 = \partial U_2$. Hence $\partial U_1 = \partial U_2$, a Heegaard surface #

Recall: For compact oriented closed surface Σ, the mapping class group $\mathrm{Mod}(\Sigma)$ is $\mathrm{Diff}^+(\Sigma)/\text{isotopy}$

Theorem (Epstein). If $f,g: \Sigma \to \Sigma$ are diffeomorphisms (Σ as above) that are homotopic, then f is isotopic to g.
$\Rightarrow \mathrm{Mod}(\Sigma) = \mathrm{Diff}^+(\Sigma)/\text{homotopy}$

Theorem. $\mathrm{Mod}(S^2) = 1$.

proof. Given orientation preserving diffeo $f: S^2 \to S^2$. We know f is a degree-1 map. Hence f and $id: S^2 \to S^2$ are the same element of $\pi_2(S^2) = \mathbb{Z}$ ($=$ is degree), so f is homotopic to id. Hence f is isotopic to id. #

Theorem. $\mathrm{Mod}(\underset{S^1\times S^1}{\Sigma_1}) \cong SL_2\mathbb{Z}$

proof. We have $\psi: \mathrm{Mod}(\Sigma_1) \to \mathrm{Aut}(H_1(\Sigma_1)) = \mathrm{Aut}(\mathbb{Z}^2) = GL_2(\mathbb{Z})$.
Poincaré duality ex: some elements of $\mathrm{Mod}(\Sigma_1)$ are orientation preserving, $\mathrm{im}(\psi) \subseteq SL_2(\mathbb{Z})$.
The map $\mathrm{Diffeo}(T^2) \to GL_2\mathbb{Z} \xrightarrow{\det} \pm 1$ is the same map as $\mathrm{Diffeo}(T^2) \to \mathrm{Aut}(H_1(\Sigma_1)) = \mathrm{Aut}(\mathbb{Z}^2) = \pm 1$.
(Claim 1) $\psi: \mathrm{Mod}(\Sigma_1) \to SL_2\mathbb{Z}$ is surjective.
proof. Given $M \in SL_2\mathbb{Z}$, we have a linear map $L_M: \mathbb{R}^2 \to \mathbb{R}^2$ that preserves $\mathbb{Z}^2$.
L_M descends to $f_M: \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ and $\psi(f_M) = M$. //
(claim 2) $\psi: \mathrm{Mod}(\Sigma_1) \to \mathrm{Aut}(H_1(\Sigma_1))$ is injective.
proof. consider $f: \Sigma_1 \to \Sigma_1$ s.t. f acts trivially on $H_1(\Sigma_1)$. We want to prove f is homotopic to id.
Let $* \in \Sigma_1$ be inverse of 0 under universal cover $\pi: \mathbb{R}^2 \to \Sigma_1 = \mathbb{R}^2/\mathbb{Z}^2$. Homotopy f, can assign $f(*) = *$.
Lift f to $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ with $\tilde{f}(0) = 0$. The map $\tilde{f}$ takes $\mathbb{Z}^2$ to $\mathbb{Z}^2$. Moreover, the map $\tilde{f}|_{\mathbb{Z}^2}: \mathbb{Z}^2 \to \mathbb{Z}^2$ is an action on $\pi_1(\Sigma_1) = H_1(\Sigma_1) = \mathbb{Z}^2$. So $\tilde{f}|_{\mathbb{Z}^2} = id$. Let $\tilde{f}_t: \mathbb{R}^2 \to \mathbb{R}^2$ be $\tilde{f}_t(p) = (1-t)\tilde{f}(p) + tp$. Since $\tilde{f}$ commutes w/ deck group $\mathbb{Z}^2$ and fixes $\mathbb{Z}^2$, we know $\tilde{f}_t$ descends to homotopy $f_t: \Sigma_1 \to \Sigma_1$ w/ $f_0 = f$ and $f_1 = id$. //
Thus ψ is an isomorphism. #

def. An element $v \in H_1(\Sigma_g)$ is primitive if we cannot write $v = kw$ for $k \geq 2$ and $w \in H_1(\Sigma_g)$.

no crossing

Theorem. A nonzero $v \in H_1(\Sigma_g) = \mathbb{Z}^{2g}$ can be written as $v = [\gamma]$ for γ an oriented simple closed curve (scc) iff v is primitive.

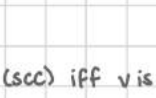
proof. $\Rightarrow$ Assume $v = [\gamma]$, $v \neq 0 \Rightarrow \gamma$ nonintersecting. Can find oriented scc δ that intersects γ once. Assume $v = kw$ with $k \in \mathbb{Z}$, $w \in H_1(\Sigma_g)$. Then let $i_{alg}: H_1(\Sigma_g) \times H_1(\Sigma_g) \to \mathbb{Z}$ be the algebraic intersection pairing. Then $\pm 1 = i_{alg}([\gamma],[\delta]) = i_{alg}(kw,[\delta]) = k \cdot i_{alg}(w,[\delta])$ so $k = \pm 1$, i.e. primitive.

$\Leftarrow$ Assume v is primitive.
(step 1) Find disjoint oriented scc $\gamma_1, \dots, \gamma_n$ with $v = \sum [\gamma_i]$.
proof. can realize $v = [\gamma']$ w/ γ' not simple. Can assume all self-intersection of γ' are tranverse, and resolve them. This turns γ' into union of disjoint scc without changing homology class.

(step 2) Choose $v = \sum_{i=1}^{n} [\gamma_i]$ with minimal n. Then $n = 1$.
proof. Let $S \subseteq \Sigma_g$ be a component of Σ_g cut open along γ_i. Hence $\partial S =$ a bunch of γ_i

(i) ∂S has ≥ 2 components. otherwise, ∂S is null-homologous γ_i and discard γ_i.
(ii) For any two components γ_i and γ_j of ∂S, S lies on different sides.
  other wise, $[\delta] = [\gamma_i] + [\gamma_j]$ and we can decrease n by replacing $\gamma' \,\&\, \gamma^2$ by δ

So, ∂S must have exactly 2 components and γ_1 and γ_2 lay on different sides of them, i.e. $[\gamma_i] = [\gamma_j]$. Hence in $v = [\gamma_1] + \dots + [\gamma_n]$ with all terms the same. Let $u = [\gamma_i]$, we have $v = n \cdot u$, but $n = 1$ by primitivity. #



Lecture 03: August 30th 2021

Last Time:

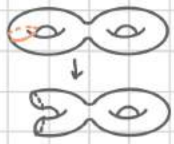
a. $\mathrm{Mod}(\Sigma_1) = \mathrm{Diff}^+(\Sigma)/\text{isotopy} = SL_2\mathbb{Z}$

b. For $v \in H_1(\Sigma_g)$ nonzero, we can write $v = [\gamma]$ with γ a nonseparating oriented simple closed curve iff v is primitive, i.e. cannot write $v = k\omega$ for $k \geq 2$

examples of simple closed curves

There are <u>many</u> nonseparating simple closed curves.

Theorem. Given any oriented nonseparating simple closed curves γ_1, γ_2 on Σ_g, there exists a $f \in \mathrm{Diff}^+(\Sigma_g)$ with $f(\gamma_1) = \gamma_2$.



proof. Let $S_i = \Sigma_g$ cut along γ_i. S_i are connected 2 boundary components, $\chi(S_i) = \chi(\Sigma_g)$. classification of surfaces $\Rightarrow S_i \cong$ genus $(g-1)$ surface with 2 boundary components. we can find diffeo $\bar{f}: S_1 \to S_2$. Glue ∂-components together, get f with $f(\gamma_1) = \gamma_2$. #

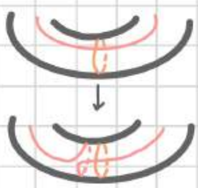
maximal connected subset (of the boundary)

remark: For Σ_1 we can prove theorem b from a.
given primitive $v = (a,b) \in H_1(\Sigma_1) = \mathbb{Z}^2$. we know $\gcd(a,b) = 1$ so there exists $c, d \in \mathbb{Z}$ with $ac + bd = 1$.
set $f = \begin{pmatrix} a & -d \\ b & c \end{pmatrix} \in \underbrace{SL_2\mathbb{Z}} = \mathrm{Mod}(\Sigma_1)$. Then letting $\gamma =$  $= (1,0) \in H_1(\Sigma_1)$, $f(\gamma) =$ a curve with boundary class (a,b).

$SL_2\mathbb{Z} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z},\ ad - bc = 1 \}$
= special linear group of degree 2 over the field $\mathbb{Z}$

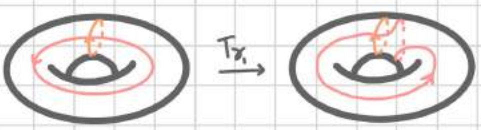
notation. Σ_g^n = genus g surface with n boundary components, $\mathrm{Mod}(\Sigma_g^n) = \mathrm{Diff}^+(\Sigma_g^n, \partial\Sigma_g^n)/\text{isotopy}$

fix ∂ pointwise



def. Given simple closed curve on S, the Dehn twist $T_\gamma \in \mathrm{Mod}(S)$ is "cut along γ, give 360° twist, reglue"

example. the matrices $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in $\mathrm{Mod}(\Sigma_1) = SL_2\mathbb{Z}$ are Dehn twist (or their inverses are).



$T_{\gamma_1}(\gamma_2) = (-1,1)$
$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$SL_2\mathbb{Z}$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$
$\Rightarrow \mathrm{Mod}(\Sigma_1)$ is generated by Dehn twist

Theorem. (Dehn, Lickorish). $\mathrm{Mod}(\Sigma_g^n)$ is generated by Dehn twists
True for $\mathrm{Mod}(\Sigma_1) = SL_2\mathbb{Z}$, $\mathrm{Mod}(\Sigma_0) = 1$, $\mathrm{Mod}(\Sigma_0^1) = 1$

proof done by induction on genus g and number of boundary components n.
step 1: If $\mathrm{Mod}(\Sigma_g)$ generated by Dehn twists, so is $\mathrm{Mod}(\Sigma_g^n)$ for all $n \geq 0$
step 2: Given any nonseparating oriented simple closed curve γ_1 and γ_2 on Σ_g exists product f of Dehn twists s.t. $f(\gamma_1) = \gamma_2$
claim: step 2 implies that if $\mathrm{Mod}(\Sigma_{g-1}^2)$ generated by Dehn twists then so is $\mathrm{Mod}(\Sigma_g)$. (will prove $g \geq 2$)

proof of claim. Consider $f \in \mathrm{Mod}(\Sigma_g)$. Let $\gamma =$ . By step 2, can multiply by Dehn twists s.t. $f(\gamma) = \gamma$.
Gluing ∂-components together gives homomorphism $\pi: \mathrm{Mod}(\Sigma_{g-1}^2) \to \mathrm{Mod}(\Sigma_g)$, $\mathrm{Im}(\pi)$ = mapping classes fix γ.

So we can write $f = \pi(f')$, write f' = product of Dehn twists, $\pi(\text{D.T.}) = \text{D.T.}$ so f is a product of Dehn twists. #

Black Box Theorem. ($g \geq 2$) Given nonseparating simple closed curve γ, γ' there exists a sequence $\gamma = \gamma_1, \gamma_2, \dots, \gamma_k = \gamma'$ of nonseparating simple closed curves s.t.

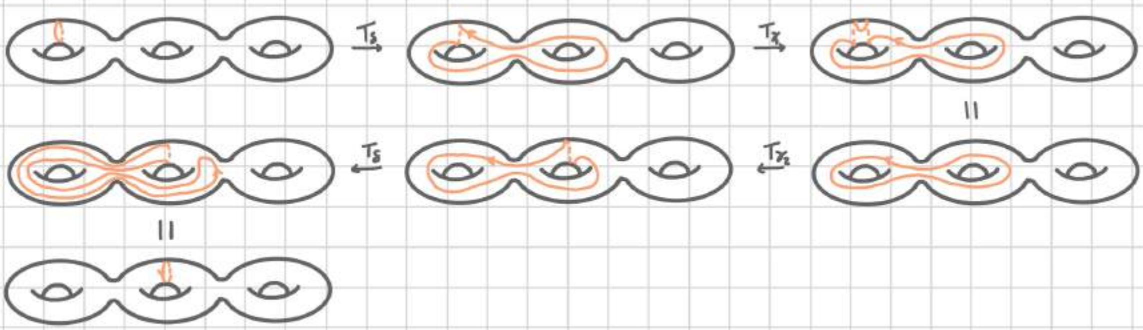
- γ_i & γ_{i+1} are disjoint
- $\gamma_i \cup \gamma_{i+1}$ does not separate

example. 

$\gamma = \gamma_1, \gamma_2, \gamma_3 = \gamma'$

proof of black box is combinatorial. start cutting curve open, splicing pieces together to decurve # intersecting points (will post link to proof on course website)

proof of step 2. Consider nonseparating simple closed curve γ_1 and γ_2. Want to find a product f of Dehn twists s.t. $f(\gamma_1) = \gamma_2$. By Black Box, we can assume $\gamma_1 \cap \gamma_2 = \emptyset$ and $\gamma_1 \cup \gamma_2$ is nonseparating. Using classification of surfaces, we can assume γ_1 and γ_2 are . Use $f = T_\delta T_{\gamma_2} T_{\gamma_1} T_\delta$. We want $f(\gamma_1) = \gamma_2$.



Construction. Let ∂_0 be a boundary component of Σ_g^n

Regard $\Sigma_g^{n-1} = \Sigma_g^n$ with disk D glued to ∂_0

Restricting elements of $Diff^+(\Sigma_g^{n-1})$ to D gives map $\varphi : Diff^+(\Sigma_g^{n-1}, \partial) \to Emb(D, \Sigma_g^{n-1})$.

Delete interior of D

Basic fact: φ is a fiber bundle and if $i : D \hookrightarrow \Sigma_g^{n-1}$ inclusion, $\varphi^{-1}(i)$ = Diffeo fixing ∂ and $D \cong Diff^+(\Sigma_g^n, \partial)$

So I have fiber bundle $Diff^+(\Sigma_g^n, \partial) \hookrightarrow Diff^+(\Sigma_g^{n-1}, \partial) \xrightarrow{\varphi} Emb^+(D, \Sigma_g^{n-1})$

Look at long exact sequence in π_*

$$\pi_1(Emb^+(D,\Sigma_g^{n-1})) \to \underset{Mod(\Sigma_g^n)}{\underset{\parallel}{\pi_0(Diff^+(\Sigma_g^n,\partial))}} \to \underset{Mod(\Sigma_g^{n-1})}{\underset{\parallel}{\pi_0(Diff^+(\Sigma_g^{n-1},\partial))}} \to \underset{0}{\underset{\parallel}{\pi_0(Emb^+(D,\Sigma_g^{n-1}))}} \to 0$$

$Emb^+(D, \Sigma_g^{n-1})$ = choice of point $p \in \Sigma_g^{n-1}$ and oriented tubular neighborhood of $p \simeq$ oriented $\underbrace{\text{frame bundle}}$ of $\Sigma_g^{n-1} \simeq UT\Sigma_g^{n-1}$

principal fiber bundle $F(E)$ of a vector bundle E, which consist of all ordered bases of the vector space attached to each point.

a fiber bundle $\pi : P \to X$ w/a continuous right action $P \times G \to P$ s.t. G preserves the fibers of P

Summary, we have exact sequence $\pi_1(UT\Sigma_g^{n-1}) \to Mod(\Sigma_g^n) \to Mod(\Sigma_g^{n-1}) \to 1$

Lecture 04: September 1st, 2021

Recall Proving:

Theorem. $\mathrm{Mod}(\Sigma_g^n)$ given by Dehn twists T_γ

proven: • $\mathrm{Mod}(\Sigma_1)$, $\mathrm{Mod}(\Sigma_0)$, $\mathrm{Mod}(\Sigma_{g-1}^2)$ is generated by T_α's

• For $g \geq 2$, if $\mathrm{Mod}(\Sigma_{g-1}^2)$ is generated by Dehn twists so is $\mathrm{Mod}(\Sigma_g)$

↖ Σ_g cut along nonseparating curve

Remains to Show:

Step 1: If $\mathrm{Mod}(\Sigma_g)$ generated by Dehn twists, so is $\mathrm{Mod}(\Sigma_g^n)$

key tool from last time:

(Birmen Exact Sequence (weak version)) $\pi_1(\mathrm{Emb}^+(D^2, \Sigma_g^{n-1})) \xrightarrow{\beta} \mathrm{Mod}(\Sigma_g^n) \xrightarrow{\psi} \mathrm{Mod}(\Sigma_g^{n-1}) \to 1$

Define $\mathrm{Mod}_{DT}(\Sigma_g^n)$ = subgroup (of $\mathrm{Mod}(\Sigma_g^n)$) generated by Dehn twists

By induction, we know $\mathrm{Mod}_{DT}(\Sigma_g^{n-1}) = \mathrm{Mod}(\Sigma_g^{n-1})$.

Also clear that $\psi(DT) = DT$ so $\psi(\mathrm{Mod}_{DT}(\Sigma_g^n)) = \mathrm{Mod}_{DT}(\Sigma_g^{n-1}) = \mathrm{Mod}(\Sigma_g^{n-1})$.

Hence, we have enough to prove $\mathrm{Im}(\beta) \subseteq \mathrm{Mod}_{DT}(\Sigma_g^n)$.

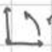
$\mathrm{Im}(\beta)$ = "Disc-pushing subgroup"

$\mathrm{Emb}^+(D^2, \Sigma_g^{n-1})$ = pairs (p, tubular neighborhood of p) with $p \in \Sigma_g^{n-1}$

$\simeq \mathrm{Frames}^+(T\Sigma_g^{n-1})$

Two kinds of loops in $\mathrm{Frame}^+(T\Sigma_g^{n-1})$ gen π_1:

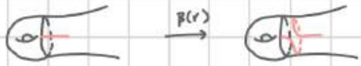
• rotate base frame without moving



rotate by 2π

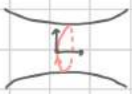
$r \in \pi_1(\mathrm{Frame}^+(T\Sigma_g^{n-1})) = \pi_1(\mathrm{Emb}^+(D^2, \Sigma_g^{n-1}))$

$\beta(r)$ = rotate disc D glued to $\partial \Sigma_g^{n-1}$



i.e. $\beta(r)$ = Dehn twist about component of $\partial \Sigma_g^n$

• parallel translation of frame about a scc on the surface



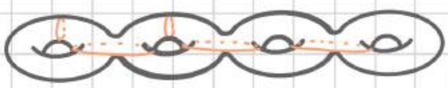
In $\mathrm{Emb}^+(D^2, \Sigma_g^{n-1})$ this drags D around a loop

$\beta(\text{loop}) = T_{\gamma_1} T_{\gamma_2}^{-1} \in \mathrm{Mod}_{DT}(\Sigma_g^n)$

Addendum: In fact,

Theorem. $\mathrm{Mod}(\Sigma_g^n)$ is generated by finitely many Dehn twists

$n=0$ | can use the following (Humphries):



Knots

def. A knot in M^3 is a smooth embedding $S^1 \hookrightarrow M^3$

examples.

unknot   trefoil   figure 8

Consider knots K_1 & K_2 equivalent if we can get from one to the other by:
- reverse direction
- ambient isotopy ($f_t : M \to M$ family of diffeo w/ $f_0 = id$)

Q. How to tell knots apart?

neighborhood of K (interior)

Knot complement $M \setminus K$ is very interesting 3-manifold, even for $M = S^3$ invariants of $M \setminus K$ (or $M \setminus \mathring{nb}(K)$) $\mathbb{D}^2 \times S^1$
($\pi_n, H_n, \ldots$ of $M \setminus K$ give invariants of K)

Theorem (Gordon-Lueck '89). For knots $K_1, K_2 \hookrightarrow S^3$, K_1 equivalent to K_2 $\iff$ $\exists$ orientation preserving diffeo $f : S^3 \setminus K_1 \to S^3 \setminus K_2$

example. $S^3 \setminus$ neighborhood(unknot) $= \mathbb{D}^2 \times S^1$ so $\pi_1(S^3 \setminus \text{unknot}) = \mathbb{Z}$

Theorem (Papakyriakopoulos). Fix knot $K \subseteq S^3$. $K =$ unot $\iff \pi_1(S^3 \setminus K) = \mathbb{Z}$.

H_1 is not so useful:

Lemma. For knot $K \subseteq S^3$, $\underset{\pi_1(S^3\setminus K)^{ab}}{\underset{\parallel}{H_1(S^3 \setminus K)}} = \mathbb{Z}$

proof. Either
- Alexander duality (If you have a polyhedron is S^n then the homology is determined by the polyhedron) the circle in this case
- $S^3 = (S^3 \setminus K) \cup (\mathring{nb}(K))$ with intersection $\simeq T^2$, use Mayer-Vietoris

def. For $p, q \in \mathbb{Z}$ relatively prime, the (p,q)-torus knot $T_{p,q}$ is a knot in S^3 that equals the scc on usual $T^2 \hookrightarrow S^3$ corresponding to $(p,q) \in H_1(T^2) = \pi_1(T^2) = \mathbb{Z}^2$

example. trefoil $= T_{2,3}$

Theorem. For $p, q \in \mathbb{Z}$ coprime $\pi_1(S^3 \setminus T_{p,q}) = \langle x, y \mid x^p = y^q \rangle$

proof. Let $U =$ inside of $T^2 \cong D^2 \times S^1$, $V =$ outside of $T^2 \cong D^2 \times S^1$, $U \cap V = T^2$. $U' = U \setminus T_{p,q} \overset{he}{\simeq} S^1$, $V' = V \setminus T_{p,q} \overset{he}{\simeq} S^1$. (he: homotopy equivalent)
Seifert van Kampen Diagram

$$\begin{array}{ccc} \pi_1(U' \cap V') & \longrightarrow & \pi_1(V') \\ \downarrow & & \downarrow \\ \pi_1(U') & \longrightarrow & \pi_1(S^3 \setminus T_{p,q}) \end{array} \quad = \quad \begin{array}{ccc} \mathbb{Z} & \xrightarrow{p} & \mathbb{Z} = \langle x \rangle \\ {\scriptstyle q}\downarrow & & \downarrow \\ \langle y \rangle = \mathbb{Z} & \longrightarrow & \pi_1(S^3 \setminus T_{p,q}) \end{array}$$

this pushout diagram preserves $\langle x, y \mid x^p = y^p \rangle$

Now Bass-Serre theory
$Z(C_p * C_q) =$ trivial $\Rightarrow \langle x^p \rangle = Z(G) \Rightarrow G / Z(G) = C_p * C_q$ #

Construction: Given knot $K \hookrightarrow M^3$, Dehn surgery along K is
- remove $\mathring{nb}(K) = D^2 \times S^1$
- glue back in some other way

regluing data = self-map of $\partial nb(K) = T^2$
$= SL_2 \mathbb{Z}$
(need much less info) next time!

Theorem. Every component oriented 3-manifold can be obtained as surgery on some link in S^3
knot w/ multiple parts

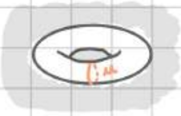
Lecture 05: September 6th, 2021

Recall: If M is a 3-manifold, $K \hookrightarrow M$ a knot, then surgery on K is a 2-step process:

- (drilling) Cut out $Nb(K) = \mathbb{D}^2 \times S^1$ to get a 3-manifold M' w/ new boundary components $\partial \cong S^1 \times S^1$
- (filling) Given 3-manifold M' and boundary components $\partial \cong S^1 \times S^1$, <u>choose</u> homeomorphism $f: \partial(\mathbb{D}^2 \times S^1)$ w/ ∂, glue $\mathbb{D}^2 \times S^1$ to M' to get 3-manifold $\hat{M}$

process: Drill $\mathbb{D}^2 \times S^1$ out of ambient space with boundary component $S^1 \times S^1$, Fill in $S^1 \times \mathbb{D}^2$ by homeomorphism

Q. How to enumerate choice of homeomorphism?

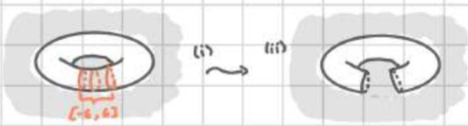


new solid torus from drilling out unknot
$\mu = \partial \mathbb{D}^2 \times \{pt\}$ bounds a disk in $\mathbb{D}^2 \times S^1$

To do a Dehn filling:

(i) Attach $\mathbb{D}^2 \times [-\epsilon, \epsilon]$ = tubular neighborhood of $\mathbb{D}^2$ bounded by μ, along $f(\mathbb{D}^2 \times [\epsilon, \epsilon])$ creating a 3-manifold w/ S^2-boundary component

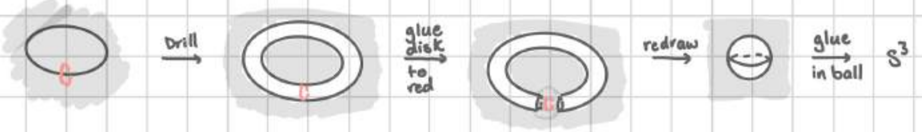
(ii) Glue $\mathbb{D}^3$ to S^2-boundary component
(only one way to do this for orientation)

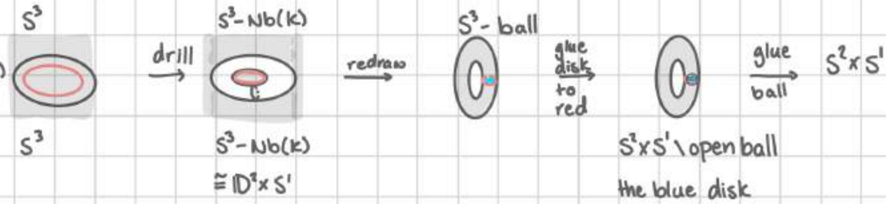


Upshot: To specify surgery, it is enough to specify $f(\mu) \subseteq \partial = S^1 \times S^1$. In other words, for surgery on a knot K, the piece of data needed to specify surgery is a non-nullhomotopic scc on $\partial Nb(K)$.

Example by picture (knot = gray, surgery curve = pink):

(a) trivial surgery





(c)

$\rightsquigarrow$?

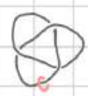
Fact: For knots in S^3, we can parametrize pink (surgery) curves by elements of $\mathbb{Q} \cup \{\infty\}$ by "slopes of surgery"
nontrivial scc on $S^1 \times S^1 \longleftrightarrow$ elements of $H_1(S^1 \times S^1) = \mathbb{Z}^2$. So we need to specify basis for $H_1(\partial(S^3 \setminus Nb(K)))$

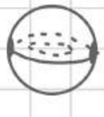
two curves (meridian and longitude):

- meridian curve: choose pt. $p \in K$, take ϵ-ball B around p

$B^3 \setminus Nb(K)$ = solid torus

eg.





the <u>meridian</u> is unique (up to homotopy) loop on $\partial(B \setminus Nb(K)) = S^1 \times S^1$ that does not bound a disk in $B \setminus Nb(K)$

canonical, varying p just moves pink curve around knot.

• Given a knot $K \hookrightarrow M^3$, where the meridian $m \subseteq \partial Nb(K)$ makes sense. A longitude is any curve in $\partial Nb(K)$ that intersects m exactly once

ex.  or 

The pair (meridian, longitude) is the basis for $H_1(\partial(M \setminus Nb(K))) = \mathbb{Z}^2$.

In S^3, there is a way to choose canonical longitude.

def. Given disjoint oriented knots K_1, K_2 in S^3, their linking number is defined as follows:
we know $H_1(S^3 \setminus K_1) \cong \mathbb{Z}$, then $lk(K_1, K_2)$ = elements of $H_1(S^3 \setminus K_1) \cong \mathbb{Z}$ given by K_2
↑orientation on K_1 specify isometry

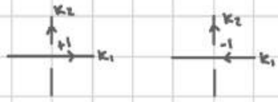
examples:

(a) 
$lk(K_1,K_2)=0$

(b) 
$lk(K_1,K_2)=1$

(c) 
$lk(K_1,K_2)=-1$

Recipe for computing $lk(K_1,K_2)$:
Draw link diagram, look at all places where K_2 crosses under K_1, each crossing gets a sign, $lk(K_1,K_2)$ = sum of these signs



def. The canonical longitude of $K \hookrightarrow S^3$ is the unique $\ell \subseteq \partial Nb(K)$ that intersects the meridian once and $lk(\ell, K)=0$.

examples:


$lk=0$


nonexample
$lk(\ell,K)=-3$


$lk=0$

Naming surgeries:

 $\infty \rightsquigarrow S^3$

 $0 \rightsquigarrow S^2 \times S^1$

def. Given $K \hookrightarrow M^3$, surgery with surgery curve a longitude is called an integral surgery (in S^3, slope $\in \mathbb{Z}$).

Thm. If $\hat{M}$ is obtained by integral surgery on a knot in closed oriented 3-manifold M, then M and $\hat{M}$ are cobordant, i.e. there exists compact oriented 4-manifold W with $\partial W = -\hat{M} \sqcup M$
↑with opposite orientation

pf. Given longitude ℓ for surgery on K and meridian m, we can find $f: \mathbb{D}^2 \times S^1 \to M$ with $f(\cdot \times S^1) = K$, $f(S^1 \times \{*\}) = m$, and $f(\{*\} \times S^1) = \ell$.

Let $W = (M \times [0,1]) \sqcup (\mathbb{D}^2 \times \mathbb{D}^2)/\sim$ where $\sim$ glues $\mathbb{D}^2 \times S^1 \subseteq \mathbb{D}^2 \times \mathbb{D}^2$ to $M \times 1$ via f.

$\partial W = M \cup (M \setminus Im(f)$, plus rest of ∂ of $\mathbb{D}^2 \times \mathbb{D}^2$, i.e. $S^1 \times \mathbb{D}^2)$. ← $-\hat{M}$ $\#$

Lecture 06: September 8^{th}, 2021

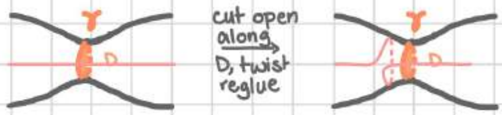
Given M, a 3-manifold, and Σ, a surface in M that cuts M into sub-3-manifolds X and Y such that $M = X \sqcup Y/\text{glue}: \Sigma \hookrightarrow \partial X$ to $\Sigma' \hookrightarrow \partial Y$ via homeomorphism $\psi: \Sigma \to \Sigma'$. Consider a homeomorphism $f: \Sigma \to \Sigma$, and let $M' = X \sqcup Y/\text{glue}$ via $\psi \circ f: \underset{\partial X}{\overset{\cap\!|}{\Sigma}} \to \underset{\partial Y}{\overset{\cap\!|}{\Sigma'}}$. When is $M \cong M'$?

Lemma. If f extends to a homeomorphism $F: X \to X$, then $M \cong M'$.

pf. Define $\lambda: M \to M'$ by $\lambda|_X = F$ and $\lambda|_Y = \text{id}$. Then λ is a homeomorphism. #

example. Consider the Heegaard splitting $M = \mathcal{H}_g \sqcup \mathcal{H}_g' /\sim$ with gluing map $\psi: \underset{\Sigma_g}{\overset{}{\partial \mathcal{H}_g}} \to \underset{\Sigma_g}{\partial \mathcal{H}_g'}$.

Let γ = scc on $\partial \mathcal{H}_g$ that bounds a disk D in $\mathcal{H}_g$. Then T_γ extends to a "disk twist" in $\mathcal{H}_g$.



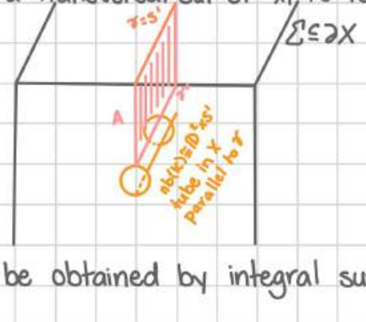
Lemma. Given the notation from the example, $M = \mathcal{H}_g \sqcup \mathcal{H}_g' /\text{glue}$ via $\psi \circ T_\gamma$.

We could also take δ, δ' = disjoint curves on $\partial \mathcal{H}_g$ that bounds an annulus (i.e. $I \times S'$) in $\mathcal{H}_g$. Then $T_\delta T_{\delta'}^{-1}$ extends to an "annulus twist" in H_g by cutting along the annulus and twisting.

Theorem. Let M, a 3-manifold, be of the form $M = X \sqcup Y/\sim$ where $X, Y \subseteq M$ are sub-3-manifolds and $\sim$ glues components $\Sigma \subseteq \partial X$ to $\Sigma' \subseteq \partial Y$ via homeomorphism $\psi: \Sigma \to \Sigma'$. Let $\gamma \subseteq \Sigma$ be a scc and set $M' = X \sqcup Y/\text{glue}$ via $\psi \circ T_\gamma: \Sigma \to \Sigma'$. Then M' is obtained from M by integral surgery on a knot $K \hookrightarrow \mathring{X}$. ← interior
In fact, K = push γ into interior of X.

pf. T_γ might not extend to X, so M' and M might not be the same. But, let $X_1 = X$ with neighborhood of K drilled out. Then define $M_1 = X_1 \sqcup Y_1 /\text{glue}$ via ψ and $M_1' = X_1 \sqcup Y/\text{glue}$ via $\psi \circ T_\gamma$.
claim: $M_1 \cong M_1'$. In fact, T_γ extends to X_1 (hence M' is obtained by surgery on K).
A = annulus ($I \times S'$) bounded by γ and the curve γ' on $nb(K)$ parallel to γ. Then $T_\gamma: \Sigma \to \Sigma$ extends to an annulus twist in X_1 along $T_\gamma T_{\gamma'}^{-1}$ which is an integral surgery. #

Take a transversal cut of X_1 to recieve cross section:



corollary. Every closed oriented 3-manifold M can be obtained by integral surgery on link in S^3.

pf. M has Heegaard splitting, $M = \mathcal{H}_g \sqcup \mathcal{H}_g' /\sim$ with $\sim$ glues $\Sigma = \partial \mathcal{H}_g$ to $\Sigma' \cong \partial \mathcal{H}_g'$ via homeomorphism $\psi: \Sigma \to \Sigma'$.
Write $\psi = T_{\gamma_1} \circ T_{\gamma_2} \circ \dots \circ T_{\gamma_n}$ as a product of Dehn twists. Define $M_k = \mathcal{H}_g \sqcup \mathcal{H}_g' /\text{glue}$ via $T_{\gamma_1} \circ \dots \circ T_{\gamma_k}$. Then Then $S^3 = M_0, M_1, \dots, M_n = M$ and M_{k+1} obtained from M_k by integral surgery along a knot (push γ_{k+1} into one piece). #

Corollary. Every compact oriented 3-manifold is a boundary component of oriented 4-manifold.

Q. How can we organize all these examples?

1st step: Cut into simple pieces

Connected Sum

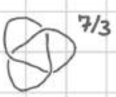
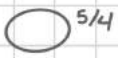
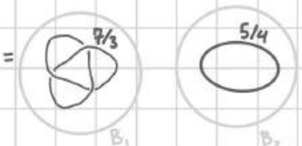
def. Let M_1 and M_2 be connected orientable n-manifolds. Their connected sum $M_1 \# M_2$ is:

- remove ball B_i from M_i to get M_i',
- glue ∂B_1 to ∂B_2 via (orientation-reversing) diffeomorphism that extends over the disk.

(a pain to see this is well-defined, but it is)

Ex. $T^2 \# T^2$

Example. If M_1 = surgery on some link $L_1 \subseteq S^1$ and M_2 = surgery some link $L_2 \subseteq S^3$. Move L_1 and L_2 s.t. they lie in disjoint balls. Then $M_1 \# M_2$ = surgery on $L_1 \sqcup L_2$.

$M_1 =$  $M_2 =$  $\Rightarrow M_1 \# M_2 =$ 

Example. $M_1 = \mathcal{H}_{g_1} \sqcup \mathcal{H}_{g_1}$/glued via $\varphi_1 : \Sigma_{g_1} \to \Sigma_{g_1}$ and $M_2 = \mathcal{H}_{g_2} \sqcup \mathcal{H}_{g_2}$/glued via $\varphi_2 : \Sigma_{g_2} \to \Sigma_{g_2}$ where isotopy φ_i fixes disk $D_i \subseteq \Sigma_{g_i}$. Cut out D_i and glue $F : \Sigma_{g_1+g_2} \to \Sigma_{g_1+g_2}$. Then $M_1 \# M_2 = \mathcal{H}_{g_1+g_2} \sqcup \mathcal{H}_{g_1+g_2}$/glue by F.

Example. $S^n \# M^n = M^n$

def. We say that M^n is prime if $M^n \not\cong S^n$ and M^n cannot be written as as a nontrivial connected sum

no S^n- factors

Theorem (Kneser, Milnor). Let M be a compact oriented 3-manifold other than S^3. Then we can write, $M = M_1 \# \dots \# M_k$ where M_i are prime 3-manifolds and this is unique up to reordering.

Remarks:

Existence is true in all dimensions (with irritating subtle issues in dim 4)

Uniqueness is false without orientability and in all cases where dim ≥ 4.

example (non-orientable). $\mathbb{P}^1 \# \mathbb{P}^1 \# \mathbb{P}^1 \cong T^2 \# P^1$

example (higher dim) In dim ≥ 7, the set of exotic spheres form a finite group under connect sum (no uniqueness).

example (Hirzebrach). $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2} \cong (S^2 \times S^2) \# \overline{\mathbb{C}P}^2$

orientable

Before we can prove Kneser-Milnor, we need to prove:

Theorem (Alexander) Every smooth, embedded $S^2 \hookrightarrow S^3$ bound a smooth disk on both sides.

We call S^3 irreducible which is stronger than prime.

Lecture 07: September 13th, 2021

Definition from Last Time

Recall: A 3-manifold M is
- prime if we can't write $M = M_1 \# M_2$ with $M_i \cong S^3$
- irreducible if all $S^2 \hookrightarrow M$ bounds balls

irreducible $\Rightarrow$ prime, but converse is false

Alexander's Theorem

Theorem (Alexander). Every smoothly embedded $S^2 \hookrightarrow \mathbb{R}^3$ bounds a smooth ball.

Remarks:
- powerful tool for showing M^3's are prime: If $\tilde{M} = \mathbb{R}^3$ then any $S^2 \hookrightarrow M^3$ lifts to $S^2 \hookrightarrow \tilde{M} = \mathbb{R}^3$ which bounds a ball B. Since $S^2 \hookrightarrow M^3$, the deck group (identifying pts. back to same pt.) of cover can't contain nontrivial $\varphi : \tilde{M} \to \tilde{M}$ with $\varphi(B) \cap B = \emptyset$, such a φ would have to have $\varphi(S^2) \cap \hat{S} = \emptyset$. So $S^2 \hookrightarrow M$ bounds a ball (ex. T^3 is prime). Similarly, Alexander shows if $\tilde{M} = S^3$ then $\tilde{M}$ is prime ($S^3 = \mathbb{R}^3 \sqcup \infty$).

Theorem. $S^2 \times S^1$ is the only orientable 3-manifold that is prime but not irreducible.

pf. (prime) $S^2 \times S^1$ is prime, otherwise we could find $S^2 \hookrightarrow S^2 \times S^1$ that separates into 2 non-ball pieces.

$\underbrace{S^2 \times S^1}_{\text{universal cover}} = \mathbb{R}^3 \setminus 0$, $S^2 \to S^2 \times S^1$

So $S^2 \hookrightarrow \mathbb{R}^3 \setminus 0$ (later: all $S^2 \times S^1$ in $\mathbb{R}^3$ bound balls in $\mathbb{R}^3$)
$\Rightarrow S^2$ bounds ball in $\mathbb{R}^3$, does not in $\mathbb{R}^3 \setminus 0$
$\Rightarrow S^2 \hookrightarrow S^2 \times S^1$ is a generator for $\pi_2(S^2 \times S^1) = \pi_2(S^2) = \mathbb{Z}$. which does not separate.

(not irreducible) $S^2 = S^2 \times \{pt.\} \hookrightarrow S^2 \times S^1$.

(only) Let M orientable be prime but not irreducible, then there exists nonseparating $f : S^2 \hookrightarrow M$. We can find $g : S^1 \to M$ s.t. $g(S^1) \cap f(S^2) = \{pt.\}$. connects one side to other

Set $X = nb(f(S^2 \cup g(S^1))$.

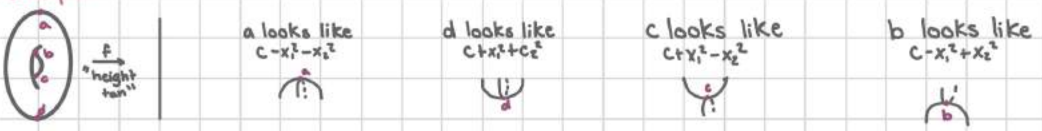
Claim: $X \cong (S^2 \times S^2) \setminus$ open ball. Let $X =$ . inside of each $S^2 \times [0,1]$ glued to each other

So $\partial X = S^2$ and if glued $D^2 \times [0,1]$ to tube would get $S^2 \times S^1$.
$\Rightarrow M^3 = (S^2 \times S^1) \# N$ since M is prime, $N \cong S^3$, so $M = S^2 \times S^1$. #

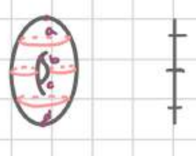
Morse Theory for Proof of Alexander

def. A Morse function on compact M^n is $M^n \to \mathbb{R}$ s.t. all critical points ($p \in M$ s.t. $df_p = 0$) are nondegenerate, i.e. there exists coordinates around p with $p = 0$ s.t. in those coords $f(x_1, \dots, x_n) = \overset{f(p)}{C} + x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots - x_n^2$ (all x_n show up, no zero coefficient).

example.



On a 2-manifold M, for a Morse function $f : M \to \mathbb{R}$, we have
- for every non critical value, $z \in \mathbb{R}$, $f^{-1}(z) =$ union of circles



- at critical points, 3 behaviors

Alexander in Other Dimension

Given $f : S^{n-1} \hookrightarrow \mathbb{R}^n$. It follows from Alexander duality that $f(S^{n-1})$ separates into two components, one of which is bounded. ($\tilde{H}_0(\mathbb{R}^n \setminus f(S^{n-1})) = \mathbb{Z}$). Hard part is that the bounded component is smooth.

remark. Go to one-point compactification.

$S^n \hookrightarrow S^{n+1}$ separates S^{n+1} into two components $S^{n+1} = X \cup Y$, $X \cap Y \cong S^{n-1}$.

Seifert-van Kampen, $\pi_1(S^{n+1}) = \pi_1(X) * \pi_1(Y) \Rightarrow \pi_1(X) = \pi_1(Y) = 1$

Similarly, using Meyer-Vietoris, $\tilde{H}_k(X) = \tilde{H}_k(Y) = 0$ for all k.

$\Rightarrow X$ and Y are contractible $n+1$ manifolds w/ boundary S^n.

So if we had a tool to prove Poincaré conjecture then we could prove they are disk.

(n=1) $S^0 \hookrightarrow \mathbb{R}$, trivial

(n=2) bounded component is a surface, $S \hookrightarrow \mathbb{R}^2$ w/ $\partial S = S^1$, $\chi(S) = 1$ Euler characteristic
(in $S^2 = \mathbb{R}^2 \sqcup \infty$, bounded thing on both sides
$\Rightarrow S^2 = S \cup_{S^1} S'$, $2 = \chi(S^2) = \chi(S) + \chi(S')$
$\Rightarrow \chi(S) = \chi(S') = 1 \Rightarrow$ C.S. $S = \mathbb{D}$) classification of surfaces

(n=4) Known that $S^3 \hookrightarrow \mathbb{R}^4$ bounds topological disk (smooth is open)

(n≥5) smooth, Schoenflier's conjecture
($S^n \hookrightarrow \mathbb{R}^n$ bounds smooth ball, uses h-cobordism thm)

Caution. for Alexander theorem in $\mathbb{R}^3$ might think $S^2 \hookrightarrow \mathbb{R}^3$ looks like

but next time, we will prove Alexander must somehow overcome "local knotting"

Strategy: Start with $f: S^2 \hookrightarrow \mathbb{R}^3$.

basic fact. Morse functions are generic, so we can perturb f s.t. function $S^2 \xhookrightarrow{f} \mathbb{R}^3 \xrightarrow{z\text{-coord}} \mathbb{R}$ is a Morse function.

=> Except for finitely many critical values $c_1 < c_2 < \dots < c_n$, $h^{-1}(d) = (\mathbb{R}^2 \times \{d\}) \cap f(S) =$ union of circles in $\mathbb{R}^2$

So the intersection of $f(S^2)$ with the plane looks like:

We will cut open sphere into piece, show pieces bound disks (each has ≤ 1 critical pt) and then reassemble.

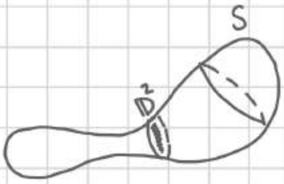
Lecture 08: September 15th, 2021

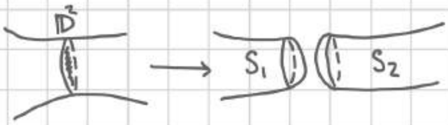
Proof of Alexander's Theorem

Theorem. Every smooth $S^2 \hookrightarrow \mathbb{R}^3$ bounds a ball.

reduction. Consider a 2-sphere $S \hookrightarrow \mathbb{R}^3$. A surgery disc for S is $\mathbb{D}^2 \hookrightarrow \mathbb{R}^3$ s.t. $\mathbb{D}^2 \cap S = \partial \mathbb{D}^2$.

surgery on this disc: cup and cap



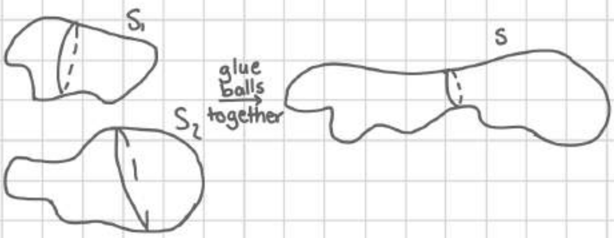


Cuts S into disjoint 2-spheres S_1 and S_2

claim: If S_1 and S_2 bound balls, so does S.

Let B_i be balls with $\partial B_i = S_i$.

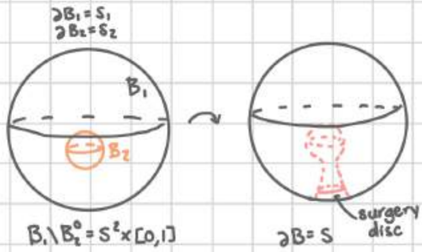
case 1: $B_1 \cap B_2 = \emptyset$



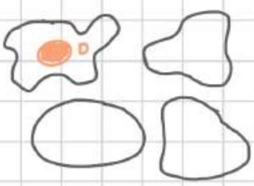
case 2: $B_1 \cap B_2 \neq \emptyset$

Since $S_1 \cap S_2 = \emptyset$, we must have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$.

W.L.O.G., $B_2 \subseteq B_1$. Find ball w/ $\partial = S$ by "scooping B_2 out of B_1"

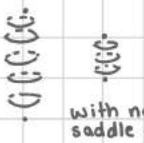


proof of Alexander. Consider smoothly embedded sphere $S \hookrightarrow \mathbb{R}^3$. Perturb slightly so $\pi: \mathbb{R}^3 \to \mathbb{R}$, $\pi(x,y,z) = z$ s.t. $\pi|_S := \varphi$ is a Morse function. So there exists finitely many critical values, and each is nondegenerate therefore we can find $h_1 < h_2 < \ldots < h_n$ s.t. each $h_i \in \mathbb{R}$ is regular value of $\varphi: S \to \mathbb{R}$ (i.e. $\varphi^{-1}(h_i)$ = union of circles) and there exists at most one critical value in (h_i, h_{i+1}) for all i. For each i, look at $\varphi^{-1}(h_i) \subseteq \mathbb{R}^2 \times h_i$, a union of circles in $\mathbb{R}^2$. Each circle C bounds a disk in $\mathbb{R}^2$. An innermost such disc D is a surgery disk. Do surgery on this



Do this again and again. This causes $\varphi^{-1}(h_i)$ to be empty, while cutting S into many components each of which has either a new minimum or new maximum (but no new saddles). Do this to each h_i. Let T be one of the components of result. It is enough to show T bounds a ball. key property of T: For height function $\varphi: T \to \mathbb{R}$, there is at most one saddle point (all other will be critical points). In fact, $T \subseteq \mathbb{R}^2 \times (h_i, h_{i+1})$ for some i.

case 1: T has no saddles: so we have some upwards, only births and deaths. specifically only one of each since T is connected

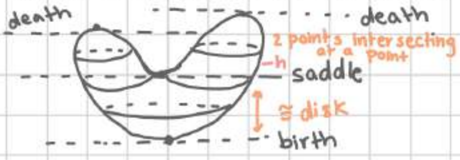


with no saddle cpts births never merge

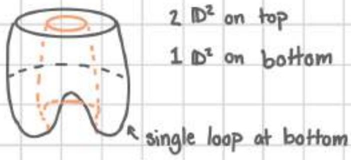
Let birth be at height b and death be at height d. So $\varphi^{-1}(b) = pt$, $\varphi^{-1}(d) = pt$ and $\varphi^{-1}(c)$ = circle for $b < c < d$. $\varphi^{-1}(c)$ bounds a disc in $\mathbb{R}^2 \times c$. These assemble to a ball bounded by T.

case2: T has one saddle
Up to reflection in z-coordinate, T has one birth and two deaths.
Let h be a height w/ $\varphi^{-1}(h)$ = two circles.

case 2a: Circles in $\varphi^{-1}(h)$ are not nesteds



case 2b: circles in $\varphi^{-1}(h)$ are nested
minus the caps and cues, looks like:



this also bounds a B^3
(link to notes of Hatcher w/ more geo. details)

Next Goal:

Theorem (Kneser). All closed orientable 3-manifolds M can be written as $M = M_1 \# M_2 \# \ldots \# M_n$ with M_k prime.

def. For a group G, the rank of G, $rk(G)$ = min. siz of generating set. (very hard to calculate)

example. $rk(SL_n\mathbb{Z}) \le \underbrace{n(n-1)}_{\text{estimate}}$ but $rk(SL_n\mathbb{Z}) = \underbrace{2}_{\text{actual}}$

elements matrices
e_{ij} = id w/ 1 at position (i,j) ($1 \le i,j \le n$ distinct)

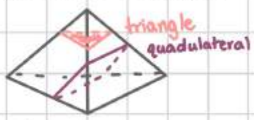
Theorem (Grushko). For groups G_1 and G_2, $rk(G_1 * G_2) = rk(G_1) + rk(G_2)$. (lower bound is hard)

observe. For 3-manifolds $M_1, \ldots, M_n$, $\pi_1(M_1 \# \ldots \# M_n) = \pi_1(M_1) * \ldots * \pi_1(M_n)$. So by Grusko, there can be at most $rk(\pi_1(M))$ connect summands in $M = M_1 \# \ldots \# M_n$ which have nontrivial π_1. If we have Poincaré conjecture (M closed 3-manifold w/ $\pi_1(M) = 1$ $\Rightarrow M \cong S^3$) then would know existence of prime de comp.

So we need a mechanism (other than π_1) to force process to stop

Kneser's idea: Normal Surface Theory

def. Let M be a 3-manifold with fixed triangulation. A normal surface in M is a closed surface $S \subseteq M$ (not necessarily connected) s.t. for each 3-complex Δ of M, each component $S \cap \Delta$ is one of two forms



two steps:

① Given collection of 2-spheres $S_1 \sqcup \ldots \sqcup S_m$, cutting up as in connect sum decomposition, we can find a different collection $S_1' \sqcup \ldots \sqcup S_m'$ that is a normal surface

② In a normal surface $S \subseteq M$ with no parallel components, there is a universal bounded on # components of S.

Lecture 09: September 20th 2021

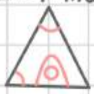
Definitions

M = compact manifold with triangulation.

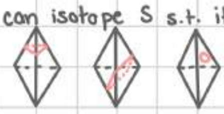
def. A surface $S \hookrightarrow M$ is transverse to a triangulation if for all k-simplices Δ^k, $S \cap \Delta^k$ is a properly embedded submanifold of dimension $k-\Delta$.

- $S \cap \Delta^0 = \emptyset$
- $S \cap \Delta^1$ = finitely many points

- $S \cap \Delta^2$ = 1-manifold

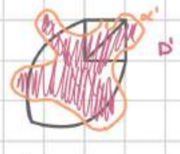


lemma. Consider surface $S \hookrightarrow M$, we can isotope S s.t. it is transverse to triangulation and for all 3-simplicies $\Delta = \Delta^3$ all components of $S \cap \partial\Delta$ of form:



pf. For S transverse to triangulation, weight$(S) = \#(S \cap M^{(1)})$. Isotope S s.t. weight(S) is minimalized. We will prove $S \cap \partial\Delta$ of 3 forms all 3-simplices Δ. Consider 3-simplices Δ and component T of $S \cap \partial\Delta$, what could go wrong is that T could intersect on edge in ≥ 2 pts. Assume this happens. We can find segment β of an edge of $\partial\Delta$ and arc α' of T and disk $D' \subseteq \partial\Delta$ s.t. $\alpha' \cap \beta$ = endpoints (so $\alpha' \cup \beta$ = circle) and $D' \cap S = \alpha'$.

Picture $\partial\Delta$ =

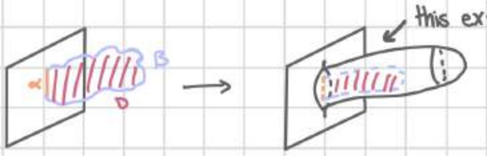


$D' \subseteq \partial\Delta$

"Tilt" D' into interior of Δ to get disk $D \subseteq \Delta$ with:

- $D \cap \partial\Delta = \beta$
- $D \cap S$ = single arc α

Push S along D:

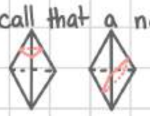


enveloped D by this extension

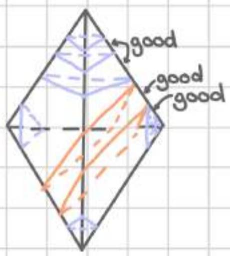
We have now eliminated two intersection points x, y of $S \cap M^{(1)}$ without introducing more, which decreasing the weight contradicting minimal weight. #

Normal Surfaces

Recall that a normal surface $S \subseteq M$ intersects each 3-simplex Δ in union of disks with boundaries either:



so intersections look like:



A componenent of Δ's is good if it lies between two parallel triangles or two parallel quad., all others are bad.

(nb) There are at most 6 bad components

A component X of $M \setminus S$ is good if $X \cap \Delta$ union of good components for all Δ, otherwise bad.

(nb) If there are t 3-simplices then $\leq 6t$ bad components of $M \setminus S$.

observation. The local (polygon) $\times I$ structures on good components glue up to show each good component is an I-bundle over a closed surface.

def. An untwisted good component is one where this I-bundle structure is trivial: $X \cong \Sigma \times I$. Otherwise, it is twisted.

(nb) ∂X submanifold of S, X untwisted implies ∂X = two components, twisted implies ∂X = one components.

lemma. If $X \subseteq M \setminus S$ (S normal) is a good components, then $H_1(X, \partial X; \mathbb{F}_2) = \mathbb{F}_2$

pf. $H_1(X, \partial X) \cong H^2(X) \cong H_2(X) = \mathbb{F}_2$ since X deformation retracts to closed surface. #

lemma $S \subseteq M$ a normal surface, $r = \#$ of components of $M \setminus S$ that are not twisted good implies $r \geq \#(\text{cpts. of } S) - \dim(H_1(M;\mathbb{F}_2)) + 1$

pf. Let the components of $S = S_1 \sqcup \dots \sqcup S_n$. Let $X_1, \dots, X_k$ = twisted good components of $M \setminus S$ indexed s.t. $\partial X_i = S_i$. For $k+1 \leq i \leq n$, let X_i = regular neighborhood of S_i choose s.t. X_i are all disjoint. Let $X = X_1 \sqcup \dots \sqcup X_n \Rightarrow M \setminus X$ = components of $M \setminus S$ not twisted good. So $H_0(M \setminus X; \mathbb{F}_2) = \mathbb{F}_2^n$. Look at LES of pair $(M, M \setminus X)$ with $\mathbb{F}_2$-coefficient.

$$\begin{tikzcd}
H_1(M) \arrow[r, "\dim(\mathrm{im}) \leq m"'] \arrow[d, equal] & H_1(M, M\setminus X) \arrow[r, "\mathrm{im} = \mathbb{F}_2^{r-1}"'] \arrow[d, equal, "\text{excision}"] & H_0(M \setminus X) \arrow[r] \arrow[d, equal] & H_0(M) \arrow[r] \arrow[d, equal] & H_0(M, M \setminus X) \arrow[r] \arrow[d, equal] & 0 \\
\mathbb{F}_2^m & \bigoplus_{i=1}^{n} H_1(X_i, \partial X_i) \arrow[d, equal] & \mathbb{F}_2^r & \mathbb{F}_2 & 0 & \\
& \mathbb{F}_2^n & & & &
\end{tikzcd}$$

Thus $n \leq m + r - 1$ so $r \geq n - m + 1$. #

corollary. Let $S \subseteq M$ be a normal surface. Letting $T = \#$ of tetrahedron in triangulation. Assume that $M \setminus S$ has at least $6t + \dim(H_1(M, \mathbb{F}_2)) + 1$ components. Then there exists two components of S that are parallel, i.e. bound $\Sigma \times I$ where Σ = surface.

pf. Assume the conclusion is false, then all good components of $M \setminus S$ are twisted good. We know there are $\leq 6t$ bad components. These are all components that are not twisted good. The lemma gives $6t \geq \#(\text{cpts. of } S) - \dim(H_1(M, \mathbb{F}_2)) + 1$, a contradiction. #

Lecture 10: September 22^{nd}, 2021

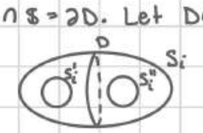
Prime Decomposition

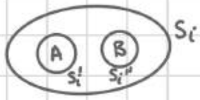
Theorem. M is a closed 3-manifold, then we can write $M = M_1 \# \dots \# M_n$ with all M_i prime.

pf. Since $H_1(N \# (\# S^2 \times S^1)) = \mathbb{Z}^k \oplus H_1(N)$, we can have at most $\dim(H_1(M;\mathbb{Q}))$ $(S^2 \times S^1)$-factors. Splitting those off reduces to where all 2-spheres in M are separating. We need a bound on K s.t. there exists K disjoint 2-spheres $S_1 \sqcup \dots \sqcup S_K \hookrightarrow M$ whose complement has no component that is a punctured 3-sphere. Call such an $S_1 \sqcup \dots \sqcup S_K$ a good sphere system. Since $(\# \text{cpts of } M \setminus \bigsqcup_{i=1}^{k} S_i) = k+1$, there is a bound on good sphere systems that are normal surfaces (too many and we must have $1 \le i < j \le K$ with S_i and S_j parallel, so there exists a component of complement of form $S^2 \times [0,1] \cong S^3 \setminus \{2 \text{ balls}\}$). So, for existence, it is enough to prove:

claim: If there exists a good sphere system $S_1 \sqcup \dots \sqcup S_K$, then there exists good normal sphere system $S_1' \sqcup \dots \sqcup S_K'$.

Observation. If $\mathcal{S} = S_1 \sqcup \dots \sqcup S_K$ is a good sphere system and $D \subseteq M$ is a surgery disk for $\mathcal{S}$, i.e. $D \cong \mathbb{D}^2$ and $D \cap \mathcal{S} = \partial D$. Let $D \cap \mathcal{S} \subseteq S_i$. Let $S_i' \sqcup S_i''$ = result of doing surgery:





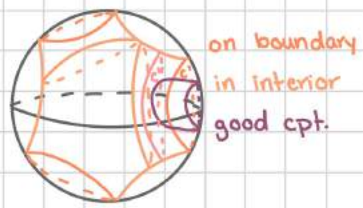
then either $S_1 \sqcup \dots \sqcup S_i' \sqcup \dots \sqcup S_K$ or $S_1 \sqcup \dots \sqcup S_i'' \sqcup \dots \sqcup S_K$ is a good sphere system.
One of A or B must not be a punctured S^3, say A. Then $B \sqcup (\substack{\text{nonpunctured} \\ S^3 \text{ on outside}}) \neq$ punctured S^3 so can replace S_i with S_i'.

pf of claim. Last time we proved that after isotoping $\mathcal{S}$, we can assume that for all 3-simplicies Δ, all cpts of $\mathcal{S} \cap \partial \Delta$ are of the form:

Step 1: Eliminate all circle components of $\mathcal{S} \cap \partial \Delta$
If there are circle components on some $\partial \Delta$, an innermost one can be used as a surgery disk. Surgery as in the observation eliminates it. Repeat this until no circle components.

Step 2: Make it normal

issue: in normal surface, all components of $\mathcal{S} \cap \partial \Delta$ are triangles and quads and all components of $\mathcal{S} \cap \Delta$ are disks. Assume there is a non-disk component. Since each component of $\mathcal{S}$ is a 2-sphere, all components of $\mathcal{S} \cap \Delta$ have genus 0. View Δ as $\mathbb{D}^3$, this bad cpt. looks like:



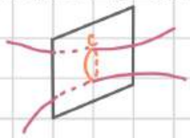
Choose innermost bad component, there could be a good component inside. Letting C be some intersect of bad cpt with $\partial \Delta$, push C inside cpt. until bounds disk disjoint from all good cpts and hence all cpts. Call this disk D and its boundary C'. We can do surgery on D. Doing this over and over gets us down to just good components //

Uniqueness of connected sum: Let M be a closed 3-manifold and let $M = M_1 \# \dots \# M_m \# (\#_k S^2 \times S^1)$, call this $\downarrow$, and $M = N_1 \# \dots \# N_n \# (\#_\ell S^2 \times S^1)$, call this $\downarrow\downarrow$, be two prime decomposition. Then $k = \ell$ and $m = n$ and the M_i & N_i are just reorderings of each other.

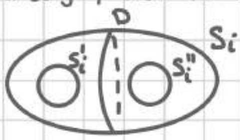
pf. A superbad sphere system for $(\downarrow)$ is disjoint 2-spheres $\mathcal{S} = S_1 \sqcup \dots \sqcup S_r$ s.t. all components of $M \setminus \mathcal{S}$ are either (multiple) punctured S^3's or (multiple) punctured M_i's. Similarly for super bad sphere systems for $(\downarrow\downarrow)$. These exist.

Key observation. If $\mathcal{S}$ is superbad for $(\downarrow)$ and $\mathcal{S}'$ sphere systems with $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{S}'$ also superbad for $(\downarrow)$.
(we can't cut M_i nontrivialy since prime)

claim. Exists sphere system $\mathbb{U}$ which is superbad for $(\downarrow)$ and $(\downarrow\downarrow)$.
pf. Let $\mathcal{S}$ be superbad for $(\downarrow)$ and $\mathcal{T}$ be superbad for $(\downarrow\downarrow)$. If we can make $\mathcal{S}$ and $\mathcal{T}$ disjoint then key observation $\mathbb{U} = \mathcal{S} \sqcup \mathcal{T}$ will be superbad for $(\downarrow)$ and $(\downarrow\downarrow)$. Assume they are not disjoint. We can assume they intersect transversely so $\mathcal{S} \cap \mathcal{T}$ = union of circles. Let T_a = cpt of $\mathcal{T}$ containing a circle and let $C \subseteq \mathcal{S} \cap T_a$ be innermost on $T_a \cong S^2$. Thus C bounds a disk $D \subseteq T_a$ whose interior is disjoint from $\mathcal{S}$. Let S_b be a component of $\mathcal{S}$ with $C \subseteq T_a \cap S_b$.



D is a surgery disk for S_b. Let S_b' and S_b'' be two components of surgery:

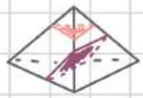


It is enough to show result, $\mathcal{S}'$, of replacing S_b with $S_b' \sqcup S_b''$ is still superbad for $(\downarrow)$ since the region bounded by $S_b \sqcup S_b' \sqcup S_b''$ is 3-punctured S^3. Do this over and over again to make them disjoint. $\mathbb{U}$ is superbad for $(\downarrow)$ and $(\downarrow\downarrow)$. //
We know that:
- the M_i are obtained from $M \setminus \mathbb{U}$ by gluing disks to all ∂-components and throwing away the S^3's.
- Similarly for N_j's

therefore $n=m$ and the sets $\{M_1, \dots, M_m\}$ and $\{N_1, \dots, N_n\}$ are the same. To see $k=\ell$, i.e. # of $S^2 \times S^1$ are the same, let $A = M_1 \# \dots \# M_m = N_1 \# \dots \# N_n$ so $M = A \# (\#_k S^2 \times S^1) = A \# (\#_\ell S^2 \times S^1)$ thus $H_1(M;\mathbb{Q}) = H_1(A;\mathbb{Q}) \oplus \mathbb{Q}^k = H_1(A;\mathbb{Q}) \oplus \mathbb{Q}^\ell$ so $k=\ell$. //

Proof Break Down

main theorem. If M is a closed (connected) 3-manifold the M has a prime decomposition.



Existence — connecting space between prime submanifolds — every S bounds a disk and looks like

Since there is a bound on good sphere systems that are normal, we reduce the theorem to the following claim:
Claim: If there exists a good sphere system, then there exists a normal sphere system.
Step 1: Eliminate all circles
↳ Key observation: If $\mathcal{S}$ is a good sphere system and D is a surgery disk for $\mathcal{S}$, then surgery on D replaces circle components with two parts, one of which is not a punctures S^3 and goes in place of the circle component
Step 2: Make it normal
↳ Assume non-disk bounding component. Choose innermost non disk-bounding component. There could be a disk bounding component contained inside so we can't just remove it. Let C be some intersection of the non-disk bounding component and the boundary, push C in until it bounds a disk that is disjoint from all other components then complete surgery on this disk to release the disk bounding component.
This completes the claim and thus the existence of a prime decomposition.

Uniqueness — punctured sphere system

Let $\downarrow = M_1 \# \dots \# M_m \# (\#_k S^2 \times S^1)$ and $\downarrow\downarrow = N_1 \# \dots \# N_n \# (\#_\ell S^2 \times S^1)$ be two prime decompositions of M. (WTS $m=n$, $k=\ell$, $\{M_i\} = \{N_i\}$)
We use the notation of superbad sphere system in order to break down M and N into equal parts.
↳ Key observation: If $\mathcal{S}$ is superbad for $\downarrow$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{S}'$ is superbad for $\downarrow\downarrow$.
claim: There exists $\mathbb{U}$ that is superbad for $\downarrow$ and $\downarrow\downarrow$. Particularly, if $\mathcal{S}, \mathcal{T}$ are superbad for $\downarrow, \downarrow\downarrow$ respectively & $\mathcal{S} \cap \mathcal{T} = \emptyset$, then $\mathbb{U} = \mathcal{S} \sqcup \mathcal{T}$.
↳ Assume not disjoint. Then they intersect transverly, so $\mathcal{S} \cap \mathcal{T}$ = union of circles. Complete surgery multiple times to make disjoint, this does not affect superbadness.
Them M_i, N_j are obtained from $M \setminus \mathbb{U}$, $N \setminus \mathbb{U}$ respectively, so $n=m$ & $\{M_i\} = \{N_j\}$. Also $M = M_1 \# \dots M_n \# (\#_k S^2 \times S^1) = N_1 \# \dots \# N_n \# (\#_\ell S^2 \times S^1)$ so $k=\ell$. //

Lecture 11: September 27th, 2021

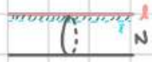
Dehn's, Loop, Sphere Lemmas

lemma. (Dehn) Let M^3 be a 3-manifold, let $f:(\mathbb{D}^2, S^1) \to (M^3, \partial M^3)$ be a continuous map s.t. there exists a neighborhood $U \subseteq \mathbb{D}^2$ of S^1 with $f|_U$ an embedding. Then there exists an embedding $g:(\mathbb{D}^2, S^1) \to (M^3, \partial M^3)$ s.t. $g|_{S^1} = f|_{S^1}$. (nb: can weaken to $f|_{S^1}$ embedding)

remark: Dehn stated this with a wrong proof (pointed out by Kneser) and proven correctly by Papakyriakopolous.

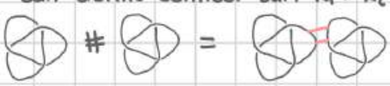
corollary. (Dehn's motivation) Let $K \hookrightarrow S^3$ be a knot with $\pi_1(S^3 \setminus K) = \mathbb{Z}$ then K is the unknot.

pf. Let $N = nb(K) \cong \mathbb{D}^2 \times S^1$ and let $\ell \subseteq \partial(N)$ be the canonical longitude so $\text{link}(\ell, K) = 0$. Thus ℓ represents 0 in $H_1(S^3 \setminus K) = \mathbb{Z} \cong \pi_1(S^3 \setminus K)$. Thus ℓ is nullhomotopic in $S^3 \setminus \mathring{N}$ so there exists continuous map $f':(\mathbb{D}^2, S^1) \to (S^3 \setminus \mathring{N}, \ell)$ with $f|_{S^1}$ an embedding of ℓ. Push ℓ into N a little bit to get $\hat{\ell}$ with embedded annulus in N bounding ℓ and $\hat{\ell}$. Letting $\hat{N}$ be the smaller tubular neighborhood with $\hat{\ell} \subseteq \partial \hat{N}$. we can find map $f:(\mathbb{D}^2, S^1) \to (S^3 \setminus \mathring{\hat{N}}, \hat{\ell})$ satisfying Dehn lemma hypothesis. Applying Dehn's lemma and then connecting disk from Dehn with annulus to K, finally find embedding $h:(\mathbb{D}^2, S^1) \to (S^3, K)$. All embeddings of $\mathbb{D}^2 \hookrightarrow S^3$ are (up to possibly changing orientations) isotopic to standard one (can extend to embedding $\mathbb{D}^3 \hookrightarrow S^3$, all of these are the same.)



Application. Given knots $K_1, K_2 \hookrightarrow S^3$ can define connect sum $K_1 \# K_2$

- Put them in disjoint balls
- band them together

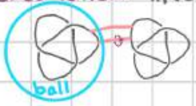


Is it well defined?

must be the same

observation. $\pi_1(S^3 \setminus (K_1 \# K_2)) = \pi_1(S^3 \setminus K_1) *_{\mathbb{Z}} \pi_1(S^3 \setminus K_2)$ using Serfeit van Kampen

$\mathbb{Z}$ generated by meridian



Theorem. Every knot in S^3 can be written as $K_1 \# \ldots \# K_n$ with each $K_i \neq$ unknot and each K_i prime.
(nb. this is also unique by a different proof.)

pf. $\text{rank}(G)$ = min. size of generating set. $\text{rank}(G) \leq 1$ iff G is cyclic hence for knots L, L = unknot iff $\text{rank}(\pi_1(S^3 \setminus L)) = 1$. Now you prove that if $\mathbb{Z} \hookrightarrow A, B$ then $\text{rank}(A *_{\mathbb{Z}} B) = \text{rank}(A) + \text{rank}(B) - 1$ (like Grushko) from this, if $K_1, \ldots, K_n$ not unknot then $\text{rank}(\pi_1(S^3 \setminus (K_1 \# \ldots \# K_n))) \geq n$. Hence if you keep refining connected sum decomposition of knot, you have to stop at some point and you have a decomposition. ✗

Theorem. (Loop). Let M^3 be a 3-manifold, let $T \subseteq \partial M^3$ be a compact 3-manifold and let $N \trianglelefteq \ker(\pi_1(T) \to \pi_1(M^3))$ be a proper subgroup of kernal Then we can find embedding $f:(\mathbb{D}^2, S^1) \to (M^3, T)$ s.t. $f(S^1) \subseteq T$ is freely homotopic to element of $\pi_1(T) \setminus N$.

special case. $N = 1$. Then since N must be proper, we must have $\ker(\pi_1(T) \to \pi_1(M)) \neq 1$ and get $f(S^1)$ not nullhomotopic in T.
This implies Dehn's Lemma, sketch:
Assume $\varphi:(\mathbb{D}^2, S^1) \to (M, \partial M)$ that is an embedding on neighborhood of S^1.
Let $T \subseteq \partial M$ be a small neighborhood of $\varphi(S^1)$.

$\pi_1(T) = \mathbb{Z}$, generated by $\varphi(S^1)$ so we know $\pi_1(T) \to \pi_1(M)$ is trivial map so loop theorem gives embedding $f:(\mathbb{D}^2, S^1) \to (M, T)$ s.t. $f(S^1)$ is an embedded loop in T not nullhomotopic. Thus $f(S^1)$ is isotopic to $\underbrace{\text{core curve}}_{S^1 \times \frac{1}{2}}$ $\varphi(S^1)$ of $T = S^1 \times [0,1]$. Can then use $\varphi|_{nb(S^1)}$ embedding to isotope f s.t. agrees with φ on S^1. ✗

Corollary. Let M be a 3-manifold and Σ_g = genus g surface with $\varphi: \Sigma_g \hookrightarrow \mathring{M}$ be an embedding s.t. $\varphi_*: \pi_1(\Sigma_g) \to \pi_1(M)$ is not inj. Then we can find embedding $f: \mathbb{D}^2 \to M$ s.t. $f(\mathbb{D}^2) \cap \varphi(\Sigma_g) = f(\partial \mathbb{D}^2)$ is non-null homotopic loop in $\varphi(\Sigma_g) \cong \Sigma_g$.

pf. Let $\hat{M} = M$ cut open along $\varphi(\Sigma_g)$. Let $T_1, T_2 \cong \Sigma_g$ be two boundary components of $\hat{M}$ coorespanding to $\varphi(\Sigma_g)$. To apply loop Theorem, it is enough to show either $\pi_1(T_1)$ or $\pi_1(T_2)$ does not inject into $\pi_1(\hat{M})$. Assume they both inject. We will show this implies $\varphi_*: \pi_1(\Sigma_g) \to \pi_1(M)$ an injection, contradiction.

(case 1) $\varphi(\Sigma_g)$ separates M into M_1 and M_2 with $T_i \subseteq \partial M_i$.

$\Rightarrow \pi_1(M) = \pi_1(M_1) *_{\pi_1(T_i)} \pi_1(M_2)$. since amalgation subgroup $\pi_1(T_i)$ injects into 2 factors injects into $\pi_1(M)$

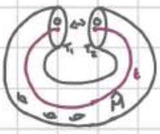
(case 2) $\varphi(\Sigma_g)$ does not separate

so $\hat{M}$ connected and M = glue 2 boundary components T_i of $\hat{M}$ together

$\Rightarrow \pi_1(M)$ = HNN extension of $\pi_1(\hat{M})$ (take $\pi_1(\hat{M})$, add new generator t conjugate $\pi_1(T_1) \leq \pi_1(\hat{M})$ to $\pi_1(T_2)$)

Same proof as case 1 works

picture of HNN



Nb. This proves following famous conjecture if φ is embedding:

Simple Loop Conjecture: If M is a 3-manifold and $\varphi: \Sigma_g \to M$ a continuous map s.t. $\varphi_*: \pi_1(\Sigma_g) \to \pi_1(M)$ not injective then there exists simple closed curve $\gamma \in \ker(\varphi_*) \setminus 1$.

corollary. Let $T^2 \hookrightarrow S^3$ be an embedded 2-torus. Then T^2 is boundary of tubular neighborhood of a knot K.

We know map on π_1 is not injective. So we can find disk $\mathbb{D}^2 \subseteq S^3$ s.t. $\mathbb{D} \cap T^2 = S^1$ nontrivial loop in T^2.

T^2 divides S^3 into 2 components. Let A = component containing D. Will prove $A \cong$ solid torus. Put points at ∞ outside A, so $A \hookrightarrow \mathbb{R}^3$. Let B = surgery on A along D.

$\partial B \cong S^2$ so by Alexander's theorem $B \cong \mathbb{B}^3$. #

Lecture 12: September 29th, 2021

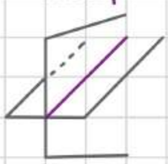
Proof of Dehn's Lemma

lemma. (Dehn) Let M^3 be a 3-manifold, let $f:(\mathbb{D}^2, S^1) \to (M^3, \partial M^3)$ be a continuous map s.t. there exists a neighborhood $U \subseteq \mathbb{D}^2$ of S^1 with $f|_U$ an embedding. Then there exists an embedding $g:(\mathbb{D}^2, S^1) \to (M^3, \partial M^3)$ s.t. $g|_{S^1} = f|_{S^1}$.

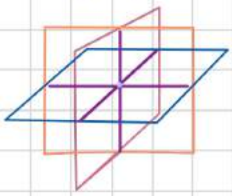
Homotoping f, we can assume the self-intersections of $\mathrm{Im}(f)$ are transverse. One consequence: we can triangulate $\mathbb{D}^2$ and M such that f is a simplicial map.

What does $\underbrace{\mathrm{Sing}(f)}_{\text{singularity}} = \{p \in \mathbb{D}^2 \mid |f^{-1}(f(p))| \geq 2\}$ look like? Image of sing(f)

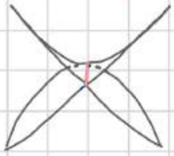
- double pts



- triple pts



- branch pt



In $\mathbb{D}^2$, Sing(f) is a finite graph:

Dehn's wrong pf: tried to cut and paste to "resolve" all singularities.

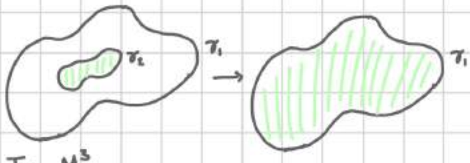
One case where you can do this:

claim. pf is easy if sing(f) only has double pts !!

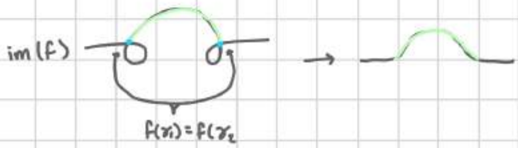
$\Rightarrow$ sing(f) = disjoint union of circles. Circles come in pairs, mapped by f to same loops in M. Will show how to modify f to get rid of pair of identified circles. Repeating this will yield an embedding. Let γ_1 and γ_2 be circles in sing(f) with $f(\gamma_1) = f(\gamma_2)$.

case 1: nested, say γ_2 inside γ_1

change f on $\mathrm{Int}(\gamma_1)$ to do what it does on $\mathrm{Int}(\gamma_2)$ eliminating singularities

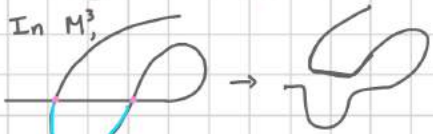


In M^3,

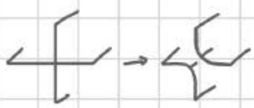


case 2: γ_1 and γ_2 not nested

In M^3,



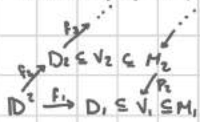
local picture



pf of Dehn's lemma. observation. If $H_1(\mathrm{Im}(f); \mathbb{F}_2) = 0$, then proof is easy. Let V = regular neighborhood of $\mathrm{Im}(f)$ = 3-manifold deformation retracting to $\mathrm{Im}(f)$. $\mathbb{F}_2$-coefficients: $0 = H^1(V) = H_2(V, \partial V) \to H_1(\partial V) \to H_1(V) = 0$. Thus $H_1(\partial V; \mathbb{F}_2) = 0$, ∂V = union of component surfaces without boundary, so ∂V = union of S^2. Let U = components of ∂V intersecting ∂M so $f(\partial \mathbb{D}^2) \subseteq U$. $f(\partial \mathbb{D}^2) \subseteq U$ bounds a disk $D' \subseteq U$. //

Image of g, an embedded disk w/ boundary $f(\partial \mathbb{D}^2)$ in green

If $H_1(\mathrm{Im}(f); \mathbb{F}_2) \neq 0$, then $\mathrm{Im}(f)$ has a double cover. Take double covers and iterate to get:



In general: we have a double cover $M_{k+1} \xrightarrow{p_{k+1}} M_k$

lift $f_k: \mathbb{D}^2 \to D_k \subseteq V_k$ to a map $f_{k+1}: \mathbb{D}^2 \to M_{k+1}$,

let $D_{k+1} = \mathrm{Im}(f_{k+1})$, V_{k+1} = reg. nghd. of D_{k+1}

Claim. This process must stop.

Recall everything is triangulated. Since p_2 is a nontrivial double cover, the # of simplicies of D_{k+1} must be greater than D_k. But # of simplices of D_k is $\leq$ # of simplicies of $\mathbb{D}^2$. So this has to stop with $\mathbb{D}^2 \xrightarrow{f_n} D_n \subseteq V_n$ and $H_1(V_n;\mathbb{F}_2) = 0$. So we can find an embedding $g_n: \mathbb{D}^2 \to V_n$ with $g_n|_{nbhd(\partial)} = f_n|_{nbhd(\partial)}$. We now have to descend down the tower of double covers.

Given an embedding $g_k: \mathbb{D}^2 \to M_k$ and a double cover $p_k: M_k \to V_{k-1}$. We want to find an embedding $g_{k-1}: \mathbb{D}^2 \to V_{k-1}$ s.t. $g_k|_{nbhd(\partial)} = p_k \circ g_k|_{nbhd(\partial \mathbb{D}^2)}$. Letting $t \in C_2$ be the generator, C_2 is the deck group of cover. The singular set of $p_k \circ g_k$ = pts of $g_k(\mathbb{D}^2)$ intersect $t \cdot g_k(\mathbb{D}^2)$ only double pts so by beginning of lecture we can always solve this. //

Lecture 13: October 4^{th}, 2021

Sphere Theorem

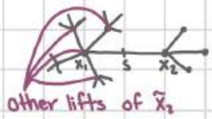
Theorem (Sphere). Let M be a compact 3-manifold. Assume that $\pi_2(M) \neq 0$. Then there exists embedding $S^2 \hookrightarrow M$ that is nontrivial in $\pi_2(M)$.

Corollary. If M^3 is a compact irreducible 3-manifold s.t. either $\partial M^3 \neq \emptyset$ or $\pi_1(M^3)$ infinite then the universal cover of M is contractible so e.g. for all knots $K \hookrightarrow S^3$, $S^3 \setminus K$ has countable universal cover.

pf. Let $\tilde{M}$ = universal cover. Then $\pi_1(\tilde{M}) = 1$, $H_2(\tilde{M}) = \pi_2(M) \overset{\text{by sphere thm \& irreducibility}}{=} 0$. Since $\partial M^3 \neq \emptyset$ or $\pi_1(M^3)$ is infinite, $\tilde{M}$ is not a closed 3-manifold. Thus $H_3(\tilde{M}) = 0$. Since $H_n(\tilde{M}) = 0$ for $n \geq 4$, Whitehead implies $\tilde{M}$ is contractible. # (compactness necessary for sphere thm)

Q. How to tell if, a 2-sphere, $S \hookrightarrow M$ is nontrivial in $\pi(M)$?

1. S is nonseparating then $[S] \neq 0$ in $H_2(M)$ since we can find loop $\gamma \subseteq M$ with $\gamma \cap S = \{pt\}$ then algebra intersections with γ gives $i_\gamma : H_2(M) \to \mathbb{Z}$ with $i_\gamma([\gamma]) = \pm 1$.
2. S separates into X_1 and X_2 s.t. for each $i = \{1,2\}$, either X_i has ≥ 2 boundary components or $\pi_1(X) \neq 1$. Lift S to universal cover $\tilde{S} \hookrightarrow \tilde{M}$. I claim $[\tilde{S}] \neq 0$ in $H_2(\tilde{M}) = \pi_2(M)$. We can find embedded curve γ in $\tilde{M}$ s.t. a) $\gamma \cap \tilde{S} = \{pt\}$ & b) on either side of $\tilde{S}$, either γ ends at a boundary component or goes to ∞ (i.e. leaving every component set.)
   ($\tilde{M}$ is built from copies of $\tilde{X}_1$ and $\tilde{X}_2$, glued together in tree-like fashion if $\partial X_i = S$ for $i = 1,2$ then $\tilde{X}_i$ has ≥ 2 lifts of S since $\pi_1(X_i) \neq 1$ and graph with edges lifts of S, vertices lifts of X_i)



Then algebraic intersections with γ gives map $i_\gamma : H_2(\tilde{M}) \to \mathbb{Z}$ with $i_\gamma([\gamma]) = \pm 1$.

pf of sphere thm.

Reductions:

① We can assume all components $T \subseteq \partial M$ are incompressible, i.e. injective in π_1. If $T \subseteq \partial M$ is not incompressible, the loop theorem implies there exists an embedding $\psi : (D^2, \partial D^2) \to (M, T)$ s.t. $\psi(\partial D^2) \cong S'$ is nontrivial in $\pi_1(T)$.

Let $M' = M \setminus \mathrm{Int}(\mathrm{nbhd}(\psi(D^2)))$, $M' \subseteq M$ a 3-manifold and since T is connected M' is connected and $M \overset{\text{hmtpy equiv.}}{\simeq} M' \vee S'$. Looking at univeral cover, if $\pi_2(M) \neq 0$ then $\pi_2(M') \neq 0$. universal cover:

Thus it is enough to prove theorem for M', repeating this over and over again, we get a 3-manifold with incompressible boundary.

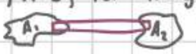
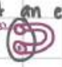
② We can assume none of the components of ∂M are 2-spheres. If a 2-sphere component $T \subseteq \partial M$ is trivial in $\pi_2(M)$, then M is contractible.

Group Theory Black Box:

Assume that M is a compact 3-manifold, $\pi_2(M) \neq 0$, s.t. each component of ∂M is incompressible and not a 2-sphere. Then either:

- 1 $\pi_1(M) = A_1 *_B A_2$ where $A_1, A_2 \neq 1$ and B finite, or
- 2 $\pi_1(M)$ = an HNN-extension with finite amalgamating subgroup, or
- 3 we have a short exact sequence $1 \to B \to \pi_1(M) \to \mathbb{Z} \to 1$ with B finite.

def. The HNN, $A *_B$, for A = group, $B < A$ a subgroup, $\psi : B \to A$ an embedding, is defined as $A *_B = \langle A, t \mid tbt^{-1} = \psi(b) \text{ for } b \in B \rangle$

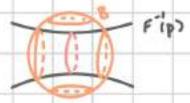
$A_1 *_B A_2$ =  (free amalgamation) $A *_B$ = 

Back to Sphere Theorem:

Consider M^3, a compact 3-manifold, with each component of ∂M incompressible and not a 2-sphere. Assume sphere theorem false. Then all embedded 2-spheres $S \hookrightarrow M$ must be separating and one component of $M \setminus S$ must have no other boundary components and $\pi_1 = 1$. Then S bounds a homotopy 3-ball. We can apply group theory black box. One of three things is true:

case 1: We have a short exact sequence $1 \to B \to \pi_1(M) \to \mathbb{Z} \to 1$ with B finite.

we can find $F: M \to S^1$ s.t. $F_*: \pi_1(M) \to \pi_1(S^1) = \mathbb{Z}$ is f. Choose regular value $p \in S^1$, so $F^{-1}(p) \subseteq M$ a surface.



claim. By homotoping F, we can assume each component of $F^{-1}(p)$ is incompressible, i.e. π_1-injective.

pf. assume not, loop thm then gives a compressing disk $\psi: (\mathbb{D}^2, \partial \mathbb{D}^2) \to (M, F^{-1}(0))$ with $\psi(\mathrm{int}(\mathbb{D}^2)) \subseteq M \setminus F^{-1}(0)$ and $\psi(\partial \mathbb{D}^2) \subseteq F^{-1}(p)$ not null homotopic. Choose B = ball that is a nbhd $(\psi(\mathbb{D}^2))$ s.t. $B \cap F^{-1}(p)$ = two circles, C_1, C_2, parallel to $\psi(\partial \mathbb{D}^2)$ with $C_1, C_2 \subseteq \partial B$. Define a new map, $G: M \to S^1$ via: $G|_{M \setminus B} = F$ and $G|_{\text{disk bounding } C_i \text{ in } B} = p$ and on each of the three 3-balls, X_1, X_2, X_3, making up $B \setminus$ disk bound C_i, define G: we have $G|_{\partial X_i}$ already defined. Since $\pi_2(S^1) = 0$, we extend in any way s.t. $G(\mathrm{int}(X_i)) \subseteq S^1$ does not contain p. Then $\bar{G}$ is homotopic to F. Repeating this over and over again, we can assume $F^{-1}(p)$ is incompressible. //

For a component $T \subseteq F^{-1}(p)$, we have that $F(T) = p$, hence $f = F_*: \pi_1(M) \to \mathbb{Z}$ must have $\pi_1(T)$ in its kernal, i.e. $\pi_1(T)$ is finite. Thus $T \cong S^2$. So by assumption bounds a homotopy ball, and by homotoping F, we can get rid of that component. Doing this, we can assume $F^{-1}(p) = \emptyset$, so $F_*: \pi_1(M) \to \mathbb{Z}$ is zero map, a contradiction. //

Lecture 14: October 6th, 2021

General Version of Last Time

def. A subspace Y of a topological space X is 2-sided codimensional-1 if there exists a neighborhood $\mathcal{U}$ of Y with $\mathcal{U} \cong Y \times (-1,1)$ with $Y \subseteq \mathcal{U}$ equal to $Y \times 0$.

example. X is a n-manifold, $Y \subseteq X$ is 2-sided $(n-1)$-manifold (properly embedded)

last time. $X = S^1$, $Y = pt$ 

Given such $Y \subseteq X$, we can define $\lambda: X \to S^1 = [-1,1]/-1\sim 1$, $\lambda(p) = \begin{cases} -1 \equiv 1 & \text{if } p \notin \mathcal{U} \\ t & \text{if } p = (y,t) \in \mathcal{U} \end{cases}$ and $\lambda^{-1}(0) = Y$. λ is the indicator function.

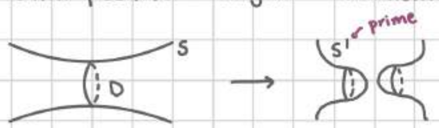
lemma. Given n-manifold M^n and $Y \subseteq X$, 2-sided codimensional-1 subspace and $f: M \to X$. We can homotope f such that $f^{-1}(Y)$ = properly embedded $(n-1)$-submanifold of M.

pf. Let $\lambda: X \to S^1$ be indicator function, then $\lambda \circ f: M \to S^1$ continuous function. We can homotope f (while fixing all points of $M \setminus (\lambda\circ f)^{-1}((-1/2, 1/2))$ such that $\lambda \circ f$ is smooth on $(\lambda\circ f)^{-1}((-1/2,1/2))$ and transverse to 0. Then $f^{-1}(Y) = (\lambda \circ f)^{-1}(0)$.

Theorem. Let $f: M^3 \to X$ be a continuous map from a 3-manifold and $Y \subseteq X$ is a 2-sided codimensional-1 subspace.
Assume:
- X, Y are connected
- $\pi_1(Y) \to \pi_1(X)$ is injective
- $\pi_3(X) = 0$
- for each component Z of $X \setminus Y$, $\pi_2(Z) = 0$

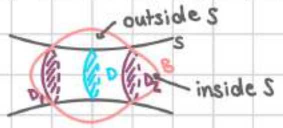
Then we can homotope f such that $f^{-1}(Y)$ is an incompressible surface.

pf. Using the lemma, we can assume f is "transverse" to Y, so $S = f^{-1}(Y)$ is a properly embedded 2-sided surface. Let $\lambda: X \to S^1$ be the indicator function. Assume S is not incompressible. The loop theorem gives that there exists a compressing disk D with $D \cap S = \partial D$ and ∂D is non-null homotopic loop in S. Near ∂D, $\lambda \circ f: M \to S^1 = [-1,1]/\sim$ is either positive or negative (vanishes on ∂D). Assume positive. Let $S' = S$ surgered along D.



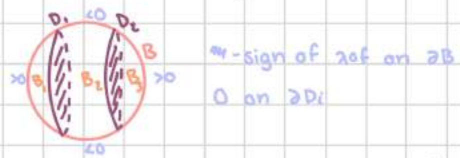
claim. We can homotope f to $g: M \to X$ s.t. $g^{-1}(Y) = S'$
(repeat over and over to make $f^{-1}(Y)$ incompressible).

Let $B \cong \mathbb{D}^3$ be a ball around D such that $\partial B \cap S$ = two loops bounding disks D_1 and D_2 parallel to D with $D_1, D_2 \subseteq B$.



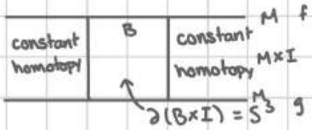
construction of $g: M \to X$
(i) $g|_{M \setminus \mathring{B}} = f|_{M \setminus \mathring{B}}$
(ii) $g(\partial D_i) \subseteq Y$ is a null-homotopic loop in X, $\pi_1(Y) \hookrightarrow \pi_1(X)$, so $g(\partial D_i)$ is null-homotopic in Y, so we can extend $g|_{\partial D_i}$ to a map $D_i \to Y$.

(i) and (ii) define g on $(M \setminus \mathring{B}) \cup D_1 \cup D_2$, the remainding part of B is three balls B_1, B_2, B_3.



We have defined g on the boundary of B_i, we must extend it to the interior s.t. $g(\mathring{B}_i)$ is disjoint from Y. Since each component has $\pi_2(Z) = 0$, no obstruction to doing this. This defines g on all of M and by construction $g^{-1}(Y) = S'$.

Since f and g only differ on $B \cong \mathbb{D}^3$ and $\pi_3(X) = 0$, we can homotope f to g.



Since $\pi_3(X) = 0$, we can extend homotopy over $B \times I$. ⌘

Back to Sphere Theorem

It remains to prove the following:

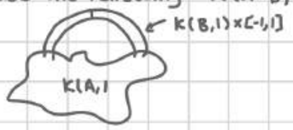
Theorem. Let M be an orientable 3-manifold such that every 2-sphere in M bounds a homotopy 3-ball, then:
(a) there does not exist $f: \pi_1(M) \to \mathbb{Z}$ with $|\ker(f)| < \infty$
(b) don't have $\pi_1(M) = A_1 *_B A_2$ with $|B| < \infty$
(c) don't have $\pi_1(M) = A *_B$ with $|B| < \infty$

pf. We have already done a. We will show b and hint at c as they are similar. Assume $\pi_1(M) = A_1 *_B A_2$ with $|B| < \infty$. Let $Z_i = $ a $K(A_i, 1)$ and $W = $ a $K(B,1)$. Define $X = Z_1 \cup (W \times [-1,1]) \cup Z_2 /\sim$ with $W \times \{-1\}$ glued to Z_1 via map $W \to Z_1$ with $B = \pi_1(W) \to \pi_1(Z) = A_1$ the inclusion, and similarly for $W \times \{1\}$ glued to Z_2.

X = 

Let $Y = W \times 0 \subseteq X$. Thus $X = $ a $K(A_1 *_B A_2, 1)$. We can find a map $f: M^3 \to X$ s.t. $f_*: \pi_1(M) \to \pi_1(X)$ an isomorphism. By the theorem, we can assume $f^{-1}(Y)$ is an incompressible surface, S. Letting T be a component of S, since $f(T) \subseteq Y$, we know $\pi_1(T) \subseteq \pi_1(M)$ is conjugate of B. Since $|B| < \infty$ and M orientable and T 2-sided, T must be orientable surface with finite π_1. Thus $T \cong S^3$ so T bounds a homotopy sphere. Same obstruction as before lets us homotope f to eliminate T. Doining this to all components of S gives a homotope of f with $f^{-1}(Y) = \emptyset$. Thus we can homotope f s.t. the image lies in Z_1 or Z_2 so $f_*: \pi_1(M) \to A_1 *_B A_2$ is <u>not</u> surjective, a contradiction. ※

The proofs of (a) and (c) are similar:
(a) realize $f: \pi_1(M) \twoheadrightarrow \mathbb{Z}$ by $F: M \to S^1$. Let $X = S^1$, $Y = pt$. Run same argument.
(c) use the following $K(A *_B, 1)$ and run the same argument:



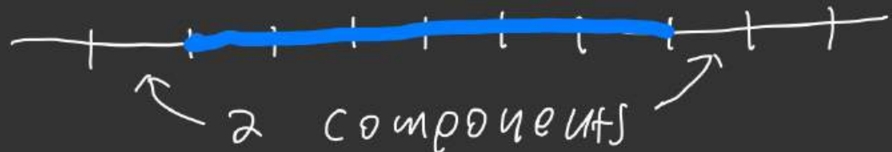
§1: Ends of spaces

Def: The # of ends of a loc. finite simp cpx X is

$$e(X) = \sup\{n \mid \exists \text{ cpt subcpx } K \subseteq X \text{ s.t. } X \setminus K \text{ has } \geq n \text{ unbounded components}\}$$

Ex: $e(X) = 0 \iff X$ cpt

Ex: $e(\mathbb{R}) = 2$



Ex: $e(\mathbb{R}^n) = 1$ for $n \geq 2$

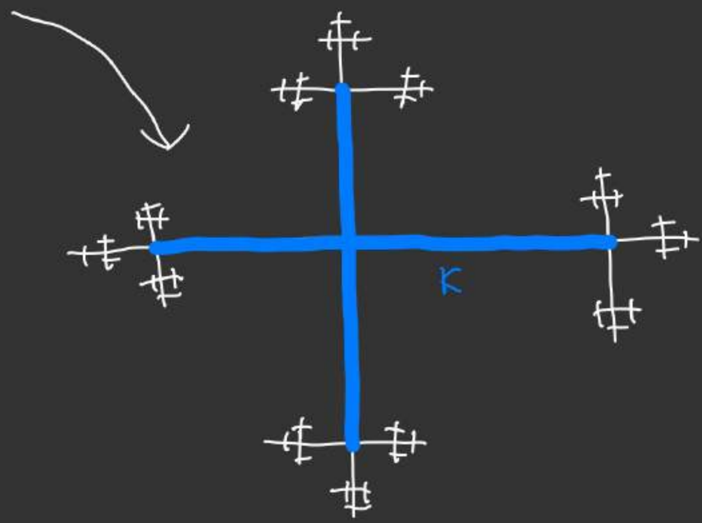
$\mathbb{R}^n \setminus K$ has 1 unbounded cpt for all cpt K:

A, B bounded
C unbounded

Ex: X = infinite 4-valent tree

$e(X) = \infty$

eg $X \setminus K$ has 12 unbounded cpts



Goal: Interpret cohomologically

$C^\bullet(X)$ = simplical cochain cpx, so (3)

$C^k(X) = \{$fcns fin set of k-simplices to $\mathbb{Z}\}$

Def: $C_c^k(X) = \{ f \in C^k(X) \mid f(\sigma)=0$ for all but fin many k-simplices $\sigma\}$

X loc finite $\Rightarrow$ $C_c^\bullet(X)$ subcochain cpx of $C^\bullet(X)$

Def: $H_c^k(X) = H^k(C_c^\bullet(X))$ is cohomology w/ cpt support.

Def: $C_e^\bullet(X) = C^\bullet(X)/C_c^\bullet(X)$, so have SES

$$0 \to C_c^\bullet(X) \to C^\bullet(X) \to C_e^\bullet(X) \to 0$$

Def: $H_e^k(X) = H^k(C_e^\bullet(X))$

For $f \in C^k(X)$, let $[f]_e$ = im in $C_e^k(X)$ (4)

Meaning of $H_e^0(X)$:

Lemma: For loc finite simp cpx X and $f \in C^0(X)$, have

$$[f]_e \in H_e^0(X) = \ker(C_e^0(X) \xrightarrow{d} C_e^1(X))$$

iff exists finite subcpx $K \subseteq X$ s.t. for all cpts A of $X \setminus K$, $f(v)=f(w)$ for all vertices of A

pf:

$[f]_e \in H_e^0(X)$ iff $d(f) \in C_c^1(X)$

ie if there exists finite subcpx K s.t. $d(f)(\sigma)=0$ for all 1-simplices σ not in K.

$\sigma = \underset{v}{\bullet} \longrightarrow \underset{w}{\bullet}$: $d(f)(\sigma) = f(w) - f(v)$

So $f(v)=f(w)$ on all 1-simplices σ not in K, ie f constant on cpt of $X \setminus K$ $\blacksquare$

Cor: For a locally finite simp cpx X, have (5)

- $H^0_e(X)$ fin gen $\iff e(X)<\infty$
- if $n=e(X)<\infty$, then $H^0_e(X)\cong\mathbb{Z}^n$

For a torsion-free abelian grp A, let $rk(A)=\infty$ if A not fin gen, and $rk(A)=n$ if $A\cong\mathbb{Z}^n$.

Then Cor means:

Cor': For a locally finite simp cpx X, have

$$e(X)=rk\, H^0_e(X)$$

Alt $H^\bullet$-interpretation of ends: (6)

Thm: For an infinite 1-connected loc finite simp cpx X have

$$e(X)=rk\, H^1_c(X)+1$$

pf:

The SES

$$0\to C^\bullet_c(X)\to C^\bullet(X)\to C^\bullet_e(X)\to 0$$

of cochain cpx's gives LES in $H^\bullet$:

$$\underset{\substack{\| \\ 0}}{H^0_c(X)}\to \underset{\substack{\| \\ \mathbb{Z}}}{H^0(X)}\to H^0_e(X)\to H^1_c(X)\to \underset{\substack{\| \\ 0}}{H^1(X)}$$

X infinite and connected

X 1-connected

Thus

$$e(X)=rk\, H^0_e(X)=rk\, H^1_c(X)+1$$

∎

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Key Example for Sphere Thm:

Thm: M cpt oriented 3-mnfld w/ incompressible bdry and no spherical bdry cpts
$\widetilde{M}$ = universal cover of M
If $\pi_2(M) \neq 0$, then $e(\widetilde{M}) \geq 2$

pf:
Since each cpt of ∂M is π_1-injective, cpts of $\partial \widetilde{M}$ = univ covers of cpts of ∂M
None spheres $\Rightarrow$ all cpts of $\partial \widetilde{M}$ contractible.
$\Longrightarrow$

$$0 \neq \pi_2(M) = H_2(\widetilde{M}) = H_2(\widetilde{M}, \partial\widetilde{M}) = H^1_c(\widetilde{M})$$

↖ Hurewicz ↖ all cpts of ∂M contractible ↖ Poincaré Duality

$$\Rightarrow e(\widetilde{M}) = \operatorname{rk} H^1_c(\widetilde{M}) + 1 \geq 1 + 1 = 2$$

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Thus to prove sphere thm, need criterion like:

Thm: X cpt simp cpx w/ universal cover $\widetilde{X}$. Then $e(\widetilde{X}) \geq 2$, iff either

- $\pi_1(X) \cong A_1 *_B A_2$ w/ $A_1, A_2 \neq 1$ and $|B| < \infty$, or
- $\pi_1(X) \cong A *_B$ w/ $|B| < \infty$

Stallings proved this. But it seems strange that $e(\widetilde{X}) \geq 2$ only depends on $\pi_1(X)$...

Next Goal: Give defn of ends of a group

§2: Ends of Groups

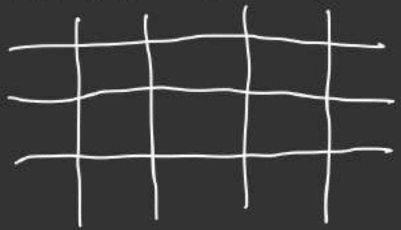
Def: For a group G w/ genset S, the Cayley graph of G is

$Cay(G,S)$ = graph w/
vertices $= G$
edges $=$ g — gs $\quad (g \in G,\ s \in S)$

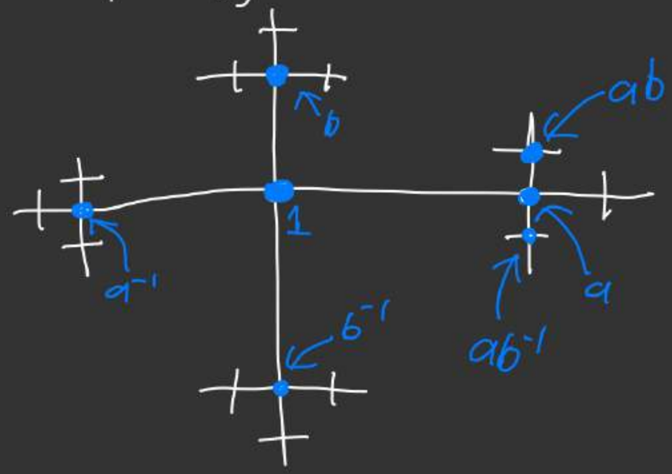
S genset $\Longrightarrow$ $Cay(G,S)$ connected

Ex: $Cay(\mathbb{Z}, \{1\}) =$

Ex: $Cay(\mathbb{Z}^2, \{(1,0),(0,1)\}) =$



Ex: $Cay(F_2, \{a,b\}) =$



Def: For a grp G w/ fin genset S, the # of ends of G is

$$e(G) = e(\,Cay(G,S))$$

(will soon prove indep of S)

Ex: G finite $\Longleftrightarrow$ $e(G)=0$

Ex: $e(\mathbb{Z}) = 2$

Ex: $e(\mathbb{Z}^2) = 1$

Ex: $e(F_2) = \infty$

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<u>Lemma</u>: For a fin gen grp G, $e(G)$ is indep of genset

pf:

S = fin genset

$$H^0_e(\mathrm{Cay}(G,S)) \subseteq C^0_e(\mathrm{Cay}(G,S)) = \frac{C^0(\mathrm{Cay}(G,S))}{C^0_c(\mathrm{Cay}(G,S))}$$

$C^0(\mathrm{Cay}(G,S))$ = fcns $f: G \to \mathbb{Z}$ (<u>not</u> known)

$C^0_c(\mathrm{Cay}(G,S))$ = fcns $f: G \to \mathbb{Z}$ w/ $f(g)=0$ for all but fin many g.

These don't depend on S, so $C^0_e(\mathrm{Cay}(G,S))$ indep of S

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$H^0_e(\mathrm{Cay}(G,S)) = V / C^0_c(\mathrm{Cay}(G,S))$,

w/

$$V = \left\{ f: G \to \mathbb{Z} \;\middle|\; \begin{array}{l} \text{For all } s \in S^{\pm}, \text{ for all} \\ \text{but fin many } g \in G \text{ we} \\ \text{have } f(g) = f(gs) \end{array} \right\}$$

Since S generates G, this is equivalent to

$$\left\{ f: G \to \mathbb{Z} \;\middle|\; \begin{array}{l} \text{For all } h \in G, \text{ for all} \\ \text{but fin many } g \in G \text{ we} \\ \text{have } f(g) = f(gh) \end{array} \right\}$$

This is indep of S, so

$e(G) = \mathrm{rk}\, H^0_e(\mathrm{Cay}(G,S))$ is too

(13)

Thm: X = cpt simplicial cpx w/ univ cover $p: \tilde{X} \to X$

$$e(\pi_1(X)) = e(\tilde{X})$$

pf:

$T \subseteq X^{(1)}$ max tree

X' = collapse T to single vertex

$(X' \overset{he}{\simeq} X)$

Univ cover $\tilde{X}'$ = collapse each cpt of $p^{-1}(T)$ to a vertex

$\tilde{X} \to \tilde{X}'$ is proper h.e, so

$$e(\tilde{X}) = rk\, H^0_c(\tilde{X}) = rk\, H^0_c(\tilde{X}') = e(\tilde{X}')$$

$\Rightarrow$ Can asm X has one vertex

(14)

Let $S \subseteq \pi_1(X)$ be genset of loops coor to 1-simplices

Then

$$\tilde{X}^{(1)} = Cay(\pi_1(X), S)$$

so

$$e(\tilde{X}) = e(\tilde{X}^{(1)}) = e(Cay(\pi_1(X), S)) = e(\pi_1(X))$$

∎

Cor: G = fin presented grp

$H < G$ finite-index subgrp

$\Rightarrow e(G) = e(H)$

pf:

X = cpt simp cpx w/ $\pi_1(X) = G$

Y = cover coor to H (cpt since H f.i.)

X, Y have same univ cover $\tilde{X}$

$\Rightarrow e(G) = e(\tilde{X}) = e(H)$ ∎

Thm (Freudenthal): G = f.g. grp 15

Then $e(G) \in \{0, 1, 2, \infty\}$

pf:

Asm $n = e(G)$ has $2 \leq n < \infty$

Z = Cayley graph wrt some genset

$K \subseteq Z$ cpt subgraph w/

$Z \setminus K$ having unbounded cpts $A_1, \ldots, A_n$

$e(G) \geq 2 \Rightarrow |G| = \infty$, so can find

$g \in G$ s.t. $g \cdot K \cap K = \emptyset$

Asm $gK \subseteq A_1$

$A_1 \setminus gK$ can only have 1

unbounded cpt since otherwise

$Z \setminus (K \cup gK)$ would have $(n+1)$

unbounded cpts

$\Longrightarrow Z \setminus gK$ has 2 unbounded 16

cpts:

- unbounded cpt of $A_1 \setminus gK$
- $A_2 \cup \cdots \cup A_n \cup K$

But

$$Z \setminus gK \cong g^{-1}(Z \setminus gK)$$
$$= g^{-1}Z \setminus K$$
$$= Z \setminus K,$$

so $n = 2$ ∎

Major Thm (Stallings): G f.g grp

Then $e(G) \geq 2$ iff either

- $G \cong A_1 *_B A_2$ w/ $A_1, A_2 \neq 1$ and $|B| < \infty$, or
- $G \cong A*_B$ w/ $|B| < \infty$

Rmk: Can have $A = 1$ in 2nd case

This implies $B = 1$, so

$$G \cong 1 *_1 \cong \mathbb{Z}$$

Lecture 14: October 25th, 2021 Theme: 3-manifolds $M \longleftrightarrow \pi_1(M)$

Guiding Theorem. Let M_1, M_2 be closed irreducible 3-manifolds. Assume that $\pi_1(M_1) \cong \pi_1(M_2)$. Then
a. if $|\pi_1(M_i)| < \infty$, then each M_i is S^3/Γ_i where $\Gamma_i < O(3)$
b. if $|\pi_1(M_i)| = \infty$, then $M_1 \cong M_2$. remark. Needs irreducible, e.g. can have $M \# M' \not\cong M \# \bar{M}'$.

Theorem. Let M be a compact orientable prime 3-manifold with $\pi_1(M)$ free, then either $M \cong S^1 \times S^2$ or $M \cong$ a handlebody.
pf. (case 1) M is reducible, i.e. there exists $S^2 \hookrightarrow M$ that is not nullhomotopic. Thus M is prime $\Rightarrow S^2 \hookrightarrow M$ is nonseparating $\Rightarrow M \cong (S^2 \times S^1) \# M'$ and M is prime $\Rightarrow M \cong S^2 \times S^1$. (case 2) M is irreducible. By the sphere theorem, $\tilde{M}$ is contractible $\Rightarrow M$ is a $K(F_n, 1)$ for some $n \geq 0$, i.e. $M \overset{h.e.}{\cong}$ graph. Thus $H_k(M) = 0$ for $k \geq 2$. Poincaré duality and orientability implies $H_3(M, \partial M) = \mathbb{Z}$ so $\partial M \neq \emptyset$. Let B be a component of ∂M. (case 2 a) $B = S^2$. Then M is irreducible gives B bounds a disk so $M \cong \mathbb{D}^3$ (case 2 b) B is a positive genus surface
thus $\pi_1(B)$ not a free group so $\pi_1(B) \to \underset{\text{free group}}{\pi_1(M)}$ not injective. The loop theorem gives that there exists an embedding $(\mathbb{D}^2, S^1) \to (M, B)$ with $S^1 \hookrightarrow B$ nonnull homotopic loop. If $\mathbb{D}^2$ separates M into pieces M_1, M_2 then $\pi_1(M_1), \pi_1(M_2)$ are free groups with $rk(\pi_1(M_i)) < rk(\pi_1(M))$. By induction the M_i's are handlebodies so M is too. If $\mathbb{D}^2$ is nonseparating, let M' be M cut along $\mathbb{D}^2$, then $\pi_1(M) = \pi_1(M') * \mathbb{Z}$, thus by induction M' is a handlebody so M is too. ※

Theorem. (Kneser's conjecture, thm of Stallings). Let M be a closed 3-manifold with $\pi_1(M) = G_1 * G_2$ with $G_i \neq 1$.
Then we can write $M = M_1 \# M_2$ with $\pi_1(M_i) = G_i$.

Theorem. (Grusko's thm from Stalling's PhD thesis). Let F be a free group and $G = G_1 * G_2$ and $\varphi: F \to G$ be a surjection. Then we can write $F = F' * F''$ with $\varphi(F') \leq G_1$ and $\varphi(F'') \leq G_2$.

corollary. $rk(G_1 * G_2) = rk(G_1) + rk(G_2)$. if $rk = \infty$ it is trivial, so assume rk is finite.
pf. clear that $rk(G_1 * G_2) \leq rk(G_1) + rk(G_2)$. For other inequality, let $n = rk(G_1 * G_2)$ and $\varphi: F_n \to G_1 * G_2$ be a surjection. Grushko implies $F_n = F_{n_1} * F_{n_2}$ with $F_{n_i} \twoheadrightarrow G_i$, so $rk(G_i) \leq n_i$. Since $rk(G) = n_1 + n_2$, we are done. ※

pf of Grusko. Let $(X_i, v_i) =$ a $K(G_i, 1)$. Let $X =$ 

. Thus $(X, v) =$ a $K(G, 1)$.

def. For a based space (K, p) and a map $f: (K, p) \to (X, v)$, say f represents φ if there is an isomorphism $\pi_1(K, p) \cong F$ s.t. the diagram commutes:

$$\begin{tikzcd} \pi_1(K,p) \arrow[r, "f_*"] \arrow[d, phantom, "\cong"] & \pi_1(X,v) \arrow[d, phantom, "\cong"] \\ F \arrow[r, "\varphi"] & G_1 * G_2 \end{tikzcd}$$

Goal: Find f representing φ s.t. $f^{-1}(v) =$ a tree T. Indeed, letting $K_i = f^{-1}(X_i \cup [v, v_i])$, we have $K = K_1 \cup_T K_2$.
so, since T is 1-connected, we have $F \cong \pi_1(K, p) = \pi_1(K_1, p) * \pi_1(K_2, p)$ and f_* takes $\pi_1(K_i, p)$ to G_i as Grusko claims.

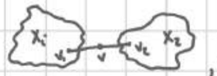
weaker statement. We can find $f: (K, p) \to (X, v)$ represting φ with $f^{-1}(v) =$ a forest.
pf. Let $F =$ free group on $\{a_1, \dots, a_n\}$. For each a_i, write $\varphi(a_i) = g_i^1 g_i^2 \cdots g_i^{k_i}$ where $g_i^j \in G_1$ or G_2, all nontrivial and alternating between G_1 & G_2. WLOG, $\varphi(a_i) \neq 1$ for all i. Let $K =$ a wedge of n circles with i^{th} circle subdivided into paths of length k_i. $p =$ wedge pt. Define f to take j^{th} edge in i^{th} circle to v-based loop in $X_{1 or 2} \cup [v, v_{1 or 2}]$ representing $g_i^j \in \pi_1(X_{1 or 2}) = G_{1 or 2}$. Then f represents φ and $f^{-1}(v) =$ the vertices, i.e. a forest of seeds. //
Pick $f: (K, p) \to (X, v)$ representing φ s.t. $f^{-1}(v)$ is a forest with fewest possible components. We want to show this is a tree. Assume $f^{-1}(v)$ has multiple components. Below, we will find a path γ in K connecting two different cpts T_1 and T_2 of $f^{-1}(v)$ s.t. $f(\gamma) \subset X_{1 or 2} \cup [v, v_{1 or 2}]$ and $f(\gamma)$ is contractible. Glue $\mathbb{D}^2$ to K s.t. it extends f over $\mathbb{D}^2$ s.t. upper edge goes to v and lower goes to $f(\gamma)$, and interior goes to $X_1 \cup (v, v_{1 or 2}]$. For this new f, $f^{-1}(v)$ has one fewer cpt. since T_1 and T_2 get connected by path in upper edge of $\mathbb{D}^2$. First, pick any γ connecting 2 distinct cpts T_1 and T_2 of $f^{-1}(v)$. Since $\varphi: \pi_1(K, v) = F \to G_1 * G_2$ is surjective, we can find loop γ_2 in K based at endpt. of γ s.t. $f(\gamma_2) \in G_1 * G_2$ represents inverse of $f(\gamma)$. Replacing γ with $\gamma \cdot \gamma_2$ we can assume $f(\gamma)$ contractible loop. Write $\gamma = \gamma_1 \cdots \gamma_r$ where each γ_i maps to $X \cup [v, v_{1 or 2}]$ and the 1 or 2 alternates and if endpts of γ_i lie in same cpts of $f^{-1}(v)$ then $f(\gamma_i) \neq 1$. Then $1 = f_*(\gamma) = f_*(\gamma_1) \cdot f_*(\gamma_2) \cdots f_*(\gamma_r)$. The $f_*(\gamma_j)$ alternate between G_1 and G_2 so must be the case that $f_*(\gamma_{i_0}) = 1$

for some i_0. By second point, $f_*(\gamma_i)$ connects two different cpts of $f^{-1}(v)$ and we are done. ※

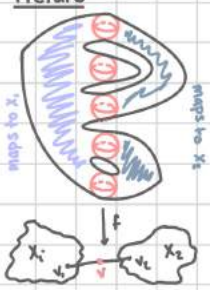
Lecture 15: October 27th, 2021

Kneser's Conjecture (thm of Stallings)

Let M be a compact 3-manifold with incompressible boundary. Assume $\pi_1(M) \cong G_1 * G_2$ with $G_i \neq 1$. Then $M = M_1 \# M_2$ with $\pi_1(M_i) = G_i$.

pf. Let $(X_i, v_i) =$ a $K(G_i, 1)$ and $X =$  , a $K(G_1 * G_2, 1)$. $K(\pi,1)$-theory implies that there exists $f: M \to X$ such that $f_*: \pi_1(M) \to \pi_1(X) = G_1 * G_2$ is an isomorphism. Using our theorem about "codimensional-1 subspaces", we can homotope f such that $f^{-1}(v)$ is an incompressible surface. For component S of $f^{-1}(v)$,
$\underbrace{\pi_1(S) \hookrightarrow \pi_1(M) \xrightarrow{\cong} \pi_1(X)}_{\text{trivial map}}$ thus $\pi_1(S) = 1$ so $S \cong S^2$.

Picture



If $f^{-1}(v)$ is one S^2 then $f^{-1}(v)$ separates M into 2 components $\bar{M}_1$ and $\bar{M}_2$. Letting $M_i = \bar{M}_i$ + disk glued to boundary component cooresponding to $f^{-1}(v)$, get $M = M_1 \# M_2$, and we are done. So pick f in its homotopy class such that $f^{-1}(v)$ has fewest number of components, we wish to show there is 1 component.

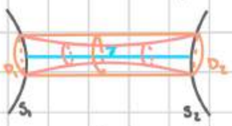
claim: $f^{-1}(v)$ is connected

Assume otherwise. Following proof of Grushko, we can find embedding arc γ s.t.

- endpoints of γ are in 2 different components of $f^{-1}(v)$
- $f(\gamma) \subseteq X_{1 \text{ or } 2}$
- $f(\gamma)$ is contractible loop in $X_{1 \text{ or } 2}$.

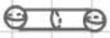
We will use γ as a guide to homotope f to decrease # of components of $f^{-1}(v)$, a contradiction.

We "tube" spheres at endpoint of γ together:



$S_1, S_2 =$ components at endpoints of γ.
$S = \mathbb{D}^2 \times (0,1)$, $\bar{S} = S \cup D_1 \cup D_2 \cong \mathbb{D}^2$.
$S' = S$ pushed in itself a little

We homotope f to $g: M \to X$, $g = f$ on $M \setminus \bar{S}$ since $\bar{S} \cong \mathbb{D}^3$ and $\pi_k(X) = 0$ for $k \geq 2$, g is homotopic to f and $g|_{\partial S'} =$ constant map to v.

We will show that we can extend this over the rest of M, i.e. to $\text{int}(S')$ and $\text{int}(S) \setminus S'$ in such a way that no other points map to v, then $g^{-1}(v) = f^{-1}(v) \cup \partial S'$. One component of this is the sphere $S_1 \cup \partial S' \cup S_2$ which looks like . So how do we extend?

Over $\text{int}(S')$:

D_1 and D_2 are disks with ∂ mapping to v and the interior mapping to the same $X_{1 \text{ or } 2}$. Since $X_{1 \text{ or } 2}$ are highly connected, there is no obstruction to extending. The obstruction lives in $\pi_2(X_{1 \text{ or } 2}) = 1$.

Over $\text{int}(S) \setminus S'$

$\text{Int}(S) \setminus S' = A \times (0,1)$ where A is an annulus, $S' \times (0,1)$. So the obstruction to extending lies in $\pi_1(X_{1 \text{ or } 2}) = G_{1 \text{ or } 2}$.
Since $f(\gamma)$ is contractible, the obstruction vanishes. //

This concludes the proof of the claim and thus the theorem. #

Q. What finitely generated abelian group can be subgroups of π_1(compact 3-manifold)?

we will answer in series of many steps.

First: Which $\mathbb{Z}^n$ can be π_1(closed 3-manifold)?

Theorem. Let M^3 be a closed 3-manifold such that $\pi_1(M^3) \cong \mathbb{Z}^n$. Then $n = 0, 1, 3$.

pf. If M is not orientable, then passing to oriented double cover changes π_1 to index-2 subgroup of π_1 so still $\mathbb{Z}^n$. So we can assume M is orientable. Next, we can assume $\pi_1(M) \neq \Delta$, so in particular, $\pi_1(M)$ is infinite. Since $\pi_1(M)$ is not nontrivial free product (its abelian), we can assume M is prime (this way we avoid Poincaré conjecture.) If M is reducible (i.e. has non-separating 2-sphere) then since M is prime and orientable, $M \cong S^1 \times S^2$ so $\pi_1(M) = \mathbb{Z}$ so we can assume M is irreducible. So by the sphere theorem, the universal cover is contractible. Since sphere theorem gives $\pi_2(M) = 0$ and $\pi_1(M)$ is infinite, $\tilde{M}$ is not compact so $H_k(\tilde{M}) = 0$ for $k \geq 3$. So $\tilde{M}$ has $\pi_1 = 0$, and $\pi_2 = H_2 = 0$ and $H_k = 0$ for $k \geq 3$. Thus $\tilde{M}$ is contractible. So M is a $K(\mathbb{Z}^n, 1)$ and hence $M \simeq (S^1)^{\times n}$, M closed orientable 3-manifold, so $\mathbb{Z} = H_3(M) = H_3((S^1)^{\times n}) = \Lambda^3 \mathbb{Z}^n$ so $n = 3$. #

def. The discrepancy of finitely presented group $G = \langle s_1, \dots, s_n \mid r_1, \dots, r_m \rangle$ is $n-m$.
The deficiency of finitely many presentable group G denoted $\mathrm{def}(G)$ is the supremum of discrepancy of a finite presentation for G.

Very hard to calculate, but will do in many cases and in particular, prove that:

- $\mathrm{def}(G) < \infty$
- for most abelian groups A, $\mathrm{def}(A) < 0$
- M closed 3-manifold $\mathrm{def}(\pi_1(M)) \geq 0$.

01/11/21

Def For a finitely presentable gp G, the deficiency of G denoted $\mathrm{def}(G)$ is the max of # generators − # relations as we range over the presentations $G = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$

Exple G finite, then $\mathrm{def}\, G \leq 0$

Reason: G^{ab} = torsion $\Rightarrow G^{ab} \otimes \mathbb{Q} = 0$

$$G = \langle x_1 \dots x_n \mid r_1 \dots r_m \rangle$$ ← generators

$\Rightarrow G^{ab} \otimes \mathbb{Q} = \mathrm{coker}(\mathbb{Q}^m \longrightarrow \mathbb{Q}^n)$ which has dim $\geq n-m$
↑ relations — so must have $n \leq m$

more gally:

Lem For a f.p gp G, $\mathrm{def}\, G \leq \dim_{\mathbb{Q}}(G^{ab} \otimes \mathbb{Q}) = b_1(G)$

Pf of au-dessus ☒

$(\Rightarrow \mathrm{def}(G) < \infty)$

Exple $\mathrm{def}(F_n) = n$ $\quad b_1(F_n) = n$

F_n = free gp on n generators $= \langle x_1 \cdots x_n \mid \rangle$

Thm (P. Hall) For a f.p. gp G $\mathrm{def}(G) \leq b_1(G) - \mathrm{rk}\, H_2(G)$ ← min size of gen. set.

Exple $G = \pi_1$ (genus g surface)

$= \langle a_1, b_1 \cdots a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle$ ← has def $2g-1$

but Thm $\Rightarrow \mathrm{def}(G) \leq b_1(G) - \mathrm{rk}\, H_2(G) = 2g-1$

$\Rightarrow \mathrm{def}(G) = 2g-1$

For pf of thm, need a tool from gp homology

Def: For $\mathbb{Z}[G]$-module M, the coinvariants are:

$$M_G = M / \langle m - gm \mid m \in M, g \in G \rangle$$

Basic fact: $H_0(G, M) = M_G$ (homology derived fctor of coinvar functor)

Given a SES of gps $1 \to K \to G \to Q \to 1$

$G \curvearrowright K$ by conj $\rightsquigarrow G \curvearrowright H_n(K)$

But (basic fact) $K \curvearrowright H_n(K)$ is trivial $\rightsquigarrow Q = G/K$ acts on $H_n(K)$

Five-Term Exact Seq. (Stallings)

Given a SES of gps $1 \to K \to G \to Q \to 1$

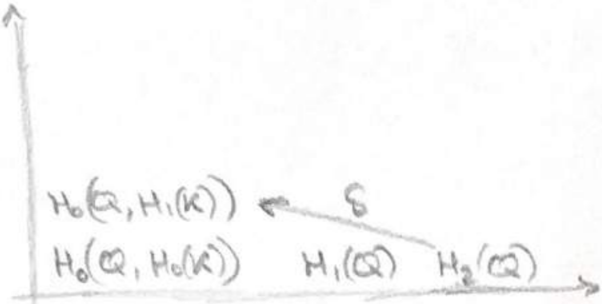
have an exact seq $H_2(G) \to H_2(Q) \to H_1(K)_Q \to H_1(G) \to H_1(Q) \to 0$

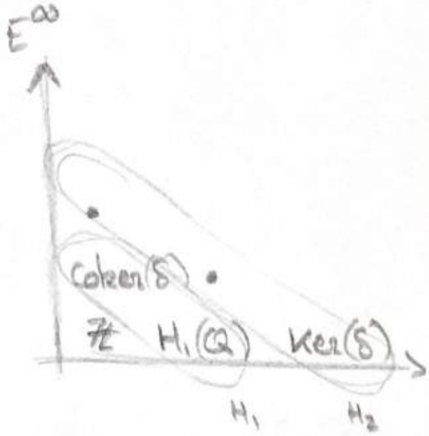
Rk similar exact seq. holds for homology of fiber bdle $F \hookrightarrow E \to B$ with same pf.

Pf There is a Hochschild - Serre spec. seq

$$E^2_{p,q} = H_p(Q, H_q(K)) \Rightarrow H_{p+q}(G)$$

Fragment of E^2





$$\begin{cases} 0 \to \mathrm{Coker}(\delta) \to H_1(G) \to H_1(Q) \to 0 \\ H_2(G) \to \mathrm{Ker}(\delta) \to 0 \end{cases}$$

$\hookrightarrow$ put together: $H_2(G) \to H_2(Q) \xrightarrow{\delta} \underbrace{H_0(Q, H_1(K))}_{= H_1(K)_Q} \to H_1(G) \to H_1(Q) \to 0$ ∎

Pf (of Hall thm) Consider a presentation $G = \langle x_1 \dots x_n \mid r_1 \dots r_m \rangle$ (F

WTS $n - m \leq b_1(G) - \mathrm{rk}\, H_2(G)$. Let $R \trianglelefteq F_n$ be normal closure of r_i
so $G = F_n/_R$. Have SES

$$1 \longrightarrow R \longrightarrow F_n \longrightarrow G \longrightarrow 1$$

5-Term exact seq. G-action induced by F_n-action

$$\begin{array}{ccccccccc} H_2(F_n) & \longrightarrow & H_2(G) & \longrightarrow & (R^{ab})_{F_n} & \longrightarrow & H_1(F_n) & \longrightarrow & H_1(G) \longrightarrow 0 \\ \| & & & & & & \| & & \\ 0 & & & & & & \mathbb{Z}^n & & \end{array}$$

Let $A = \mathrm{Im}\,\varphi \subseteq \mathbb{Z}^n$, so have:

$$0 \longrightarrow H_2(G) \longrightarrow (R^{ab})_{F_n} \longrightarrow A \longrightarrow 0 \qquad (*)$$

$$0 \longrightarrow A \longrightarrow \mathbb{Z}^n \longrightarrow H_1(G) \longrightarrow 0 \qquad (**)$$

$A \subseteq \mathbb{Z}^n$ free abelian, so $(*)$ splits and $(R^{ab})_{F_n} \cong H_2(G) \oplus A$
R is F_n-normally gen by $r_1 \dots r_m$, so in F_n-coinv $(R^{ab})_{F_n}$
the im of r_i gen and $\mathrm{rk}\,(R^{ab})_{F_n} \leq m \Rightarrow \mathrm{rk}\, H_2(G) + \mathrm{rk}\, A \leq m$.
$(**)$ does <u>not</u> split neccessarily.

Letting $A \cong \mathbb{Z}^\ell$, have $H_1(G) = \mathrm{coker}\,(\mathbb{Z}^\ell \hookrightarrow \mathbb{Z}^n)$

$\Rightarrow b_1(G) = \dim_{\mathbb{Q}} H_1(G) \otimes \mathbb{Q} = n - \ell$

Combining with $\mathrm{rk}\, H_2(G) + \underset{\ell}{\underset{\|}{\mathrm{rk}\, A}} \leq m$

get $n - m = (b_1(G) + \ell) - m \leq b_1(G) + \cancel{\ell} - (\mathrm{rk}\, H_2(G) + \cancel{\ell})$
$= b_1(G) - \mathrm{rk}\, H_2(G)$ ◻

<u>Expl</u> $\mathbb{Z}^n = \langle x_1 \dots x_n \mid [x_i, x_j] \; 1 \leq i < j \leq n \rangle$
$H_2(\mathbb{Z}^n) = \Lambda^2 \mathbb{Z}^n$ so $\mathrm{rk}\, H_2 \mathbb{Z}^n = \binom{n}{2}$

Thm gives $\operatorname{def} \mathbb{Z}^n \leq n - \binom{n}{2}$

since $\langle \text{—} \mid \text{---} \rangle$ has def $n - \binom{n}{2}$, have

$\operatorname{def} \mathbb{Z}^n = n - \binom{n}{2}$

In particular, $\operatorname{def}(\mathbb{Z}^n) < 0$ for $n \geq 4$

Exple A general f.g. ab. gp G of form

$$G \cong \mathbb{Z}^n \oplus \mathbb{Z}/k_1 \oplus \cdots \oplus \mathbb{Z}/k_\ell \quad \text{with } k_i \mid k_{i+1} \text{ for } 1 \leq i < \ell$$

$$G = \left\langle x_1 \ldots x_n, x_{n+1} \ldots x_{n+\ell} \;\middle|\; \begin{array}{l} [x_i, x_j] = 1 \quad 1 \leq i < j \leq n+\ell \\ x_{n+i}^{k_i} = 1 \quad 1 \leq i \leq \ell \end{array} \right\rangle$$

$$\text{def of pres} = (n+\ell) - \left(\binom{n+\ell}{2} + \ell \right)$$

$$b_1(G) = n, \quad H_2(G) \overset{\text{Kunneth}}{=} \bigoplus_{1 \leq i < j \leq n+\ell} \mathbb{Z}/a_i \otimes \mathbb{Z}/b_j$$

- a_i and b_j are 0 for $1 \leq i, j \leq n$
- equal $k_?$ for $n+1 \leq i, j \leq n+\ell$.

Since $k_i \mid k_{i+1}$ for all i, all those terms are non zero

$$\operatorname{rk} H_2(G) = \binom{n+\ell}{2}$$

Thm says $\operatorname{def} G \leq b_1(G) - \operatorname{rk} H_2(G) = n - \binom{n+\ell}{2}$

since our presentation has this, get equality

In particular:

Thm: only f.g. abelian gp with def ≥ 0 are $0, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z}/r, \mathbb{Z} \oplus \mathbb{Z}/r$

Thm M cpct connected 3-mfd with non empty bdry
$\Rightarrow \operatorname{def} \pi_1(M) \geq 1 - \chi(M)$

Lemma 1 M^n smooth connected cpct n-mfd
Then M^n has a CW cplx structure with 1 n-cell

Pf Start with triangulation delete interior $(n-1)$-simplices as needed to combine all n-simplices into single n-cell.

Lemma 2 M^n smooth connected closed mfd then $H_{n-1}(M) = \mathbb{Z}^a \oplus (\mathbb{Z}/_2)^b$ for some $a, b \geq 0$

Pf Look all cellular chains from Lemma 1.

$$0 \longrightarrow \mathbb{Z} \xrightarrow{d_n} C_{n-1}(M) \xrightarrow{d_{n-1}} C_{n-2}(M) \longrightarrow \cdots$$

$$\overset{\shortparallel}{\mathbb{Z}}^?$$

$$H_{n-1} = \ker d_{n-1} / \operatorname{Im} d_n = \mathbb{Z}^{??} / \operatorname{Im} d_n$$

Since M has no bdry, each $(n-1)$-cell appears twice in bdry of single n-cell
(sometimes with same sign, sometimes with opposite sign)
Hence d_n(single n-cell) $= 2$ (sum of distinct $(n-1)$-cell) ∎

Pf (of Thm) M cpct connected 3-mfd with non empty bdry
Consider CW cplx from Lemma 1 with
1 3-cell, f 2-cells, e 1-cells, v 0-cells
By "push in", one 2-cell on nonempty bdry, M def retracts to CW cplx M' with $(f-1)$ 2-cells, e 1-cells, v 0-cells.

Get presentation for $\pi_1(M) = \pi_1(M')$ by

- gens come from 1-skeleton (a graph)
  - pick max tree T (has v vertices, $v-1$ edges)
  - gen for each edge not in T

$\#$ gens $= e-(v-1)$

1-skel.

Get relation for each 2-cell. $(f-1)$ relations

so def presentation is

$$(e-(v-1))-(f-1) = 1+(1-f+e-v) = 1-\chi(M) \quad \square$$

Lecture 17: November 3rd, 2021

Recall: $\mathrm{def}(G) = \max$ #(gen.) - #(rel.) among all presentations $\leq b_1(G) - rk(H_2(G))$.

theorem. M a compact 3-manifold with $\partial M \neq \emptyset$, then $\mathrm{def}(\pi_1(M)) \geq 1 - \chi(M)$.

theorem. M a closed 3-manifold $\Rightarrow H_2(M) = \mathbb{Z}^n \oplus (\mathbb{Z}/2)^m$ for some $n, m \geq 0$.

theorem. If M is a closed 3-manifold then $\mathrm{def}(M) \geq 0$.

pf. $M' = M \setminus$ open ball. The first theorem says $\mathrm{def}(\pi_1(M')) \geq 1 - \chi(M')$. But $\pi_1(M) = \pi_1(M')$ and $\chi(M') = \chi(M) + 1 \overset{\text{Poincaré duality}}{=} 0 + 1$. So $\mathrm{def}(\pi_1(M)) \geq 1 - \chi(M') = 1 - 1 = 0$. #

Not always an equality: $\pi_1(S^2 \times S^1) = \mathbb{Z}$, $\mathrm{def}(\mathbb{Z}) = 1$. remark: can be proved using Heegard splitting

theorem. Let M be an irreducible closed orientable 3-manifold, then $\mathrm{def}(\pi_1(M)) = 0$.

pf. It is enough to show $\mathrm{def}(\pi_1(M)) \leq 0$ as we have $\mathrm{def}(\pi_1(M)) \geq 0$ by the previous theorem.

case 1: $\pi_1(M)$ is finite

Since $b_1(\pi_1(M)) = 0$ due to finiteness, we have $\mathrm{def}(\pi_1(M)) \leq b_1(\pi_1(M)) - rk(H_2(\pi_1(M))) \leq 0$.

case 2: $\pi_1(M)$ is infinite

By sphere theorem, M is aspherical, i.e. the universal cover is contractible. So M is a $K(\pi_1(M), 1)$. Then $\mathrm{def}(\pi_1(M)) \leq b_1(M) - rk(H_2(M))$ $\leq b_1(M) - b_2(M) = 0$ by Poincaré duality. #

theorem. M a closed 3-manifold with $\pi_1(M)$ abelian, then $\pi_1(M) \in \{0 = \pi_1(S^3), \mathbb{Z} = \pi_1(S^2 \times S^1), \mathbb{Z}^3 = \pi_1(T^3), \mathbb{Z} \oplus \mathbb{Z}/2 = \pi_1(S^1 \times \mathbb{RP}^2), \mathbb{Z}/r = \pi_1(\text{lens space})\}$

pf. We can assume WLOG that M is prime since $G_1 * G_2$ is not abelian for $G_1, G_2 \neq 1$. Can also assume M irreducible since otherwise $M = S^2 \times S^1$ or $S^1 \times \mathbb{RP}^2$. Last time we showed that if A is a finitely generated abelian group with $\mathrm{def}(A) \geq 0$, then $A \in \{0, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z} \oplus \mathbb{Z}/r, \mathbb{Z}/r\}$.

We must rule out:

- $\mathbb{Z}^2$

Assume $\pi_1(M) = \mathbb{Z}^2$, thus infinite, then by the sphere theorem M is aspherical, i.e. a $K(\mathbb{Z}^2, 1)$. But $H_3(M; \mathbb{F}_2) = \mathbb{F}_2$, $H_3(M, \mathbb{F}_2) = H_3(\mathbb{Z}^2, \mathbb{F}_2) = 0$, a contradiction.

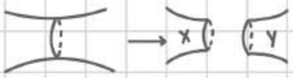
- $\mathbb{Z} \oplus \mathbb{Z}/r$ for $r \geq 3$

Assume $\pi_1(M) = \mathbb{Z} \oplus \mathbb{Z}/r$ for $r \geq 2$, we will prove $r = 2$. Since $\pi_1(M)$ is infinite, sphere theorem says M is aspherical, but $\mathbb{Z}/r = H_2(\mathbb{Z} \oplus \mathbb{Z}/r) = H_2(M)$ which can only have 2-torsion. #

theorem. M is a compact 3-manifold with $\pi_1(M)$ abelian, then $\pi_1(M)$ is in the above list or is $\mathbb{Z}^2$. assume orientable for simplicity

pf. If any components of ∂M is S^2, gluing a disk to them does not change $\pi_1(M)$. So we can assume no components of ∂M are S^2. and that the boundary is nonempty. We will prove that $\chi(M) = 0$ so $\mathrm{def}(\pi_1(M)) \geq 1 - \chi(M) = 1$ and it follows from classification of abelian group with $\mathrm{def} = 1$ that $\pi_1(M) = \mathbb{Z}$ or $\mathbb{Z}^2$. We know that all components of the boundary have positive genus. What we will show is that all are T^2 which will imply the theorem. Assume component $S \subseteq \partial M$ has genus ≥ 2. We know that $\pi_1(S)$ is not injective, so there exists nontrivial compressing disk.

case 1: compression disk is separating, $\pi_1(M) = \pi_1(X) * \pi_1(Y)$



By Poincaré conj., both X and Y have nontrivial π_1, so $\pi_1(M)$ is not abelian.

Restart proof for an easier proof!

pf. Let boundary components of M be $S_1, \ldots, S_n$. Let $g_i = \mathrm{genus}(S_i)$. Glue a genus g_i handle body $\mathcal{H}_{g_i}$ to S_i to get a closed 3-manifold M', $0 = \chi(M') = \chi(M) + \sum \chi(\mathcal{H}_{g_i}) - \sum \chi(S_i)$. Thus $\chi(M) = \sum \chi(S_i) - \sum \chi(\mathcal{H}_{g_i}) = \sum[(2 - 2g_i) - (1 - g_i)] = \sum(1 - g_i) \leq 0$ since none of the $g_i = 0$. Thus $\mathrm{def}(\pi_1(M)) \geq 1$, so we know that the abelian subgroup $\pi_1(M)$ must be $\mathbb{Z}$ or $\mathbb{Z}^2$. #

theorem. If M is a 3-manifold and $A < \pi_1(M)$ is a finitely generated group then $A \in \{0, \mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^3, \mathbb{Z} \oplus \mathbb{Z}/r, \mathbb{Z}/r\}$.

follows from:

theorem. (Scott Core). For a 3-manifold M and a f.g. $G \leq \pi_1(M)$, then there is a cpt submanifold M' of M with $\pi_1(M') = G$.

cor. For M a 3-manifold $\pi_1(M)$ is coherent, i.e. all f.g. subgroups are finite presented.

examples. • abelian groups are coherent

• Free groups are coherent since all subgroups are free

• For closed surfaces S, $\pi_1(S)$ is coherent

pf. consider f.g. subgroup $G < \pi_1(S)$. This corresponds to a cover, $T \to S$ with $\pi_1(T) = G$.

case 1: $T \to S$ is finite cover

$\Rightarrow$ T is compact surface so $G = \pi_1(T)$ is f.p.

case 2: $T \to S$ is an infinite cover

so T is noncompact, it is classical that π_1(non-compact surface) is a free group, i.e. $\pi_1(G)$ is f.g. free group. ※

theorem. $F_2 \times F_2$ is <u>not</u> coherent.

part of pf. $F_2 \times F_2 = \langle a,b,c,d \mid [a,c] = [a,d] = [b,c] = [b,d] = 1 \rangle$. Let N = subgroup generated by $\{a, c, bd = db\}$.

claim: N is not f.p. In fact, $H_2(N, \mathbb{Q})$ is ∞-dimensional.

First prove N is a normal subgroup.

• clearly closed under conjugation by a, c.

• closed under conjugation by b:

- $bcb^{-1} = c \in N$
- $bab^{-1} = bd^{-1}add^{-1}b^{-1} = (bd)(a)(bd)^{-1} \in N$
- $bbdb^{-1} = bd \in N$

• closed under conjugation by d through same proof.

Next, consider $F_2 \times F_2 / N = \langle b, d \mid [b,d] = 1, bd = 1 \rangle = \mathbb{Z}$. So we have short exact sequence $1 \to N \to F_2 \times F_2 \to \mathbb{Z} \to 1$.

We can show $H_2(N, \mathbb{Q})$ must be infinite dimensional by studying the spec. sequence of extension. ※

Open Question by Serre: Is $SL_3 \mathbb{Z}$ coherent?

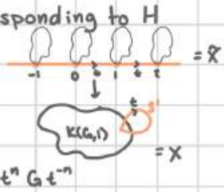
Lecture 18: November 8th, 2021

Kurosh Subgroup Theorem

theorem. Let H be a subgroup of $G_1 * G_2$. Then $H = H_1^1 * \dots * H_n^1 * H_1^2 * \dots * H_m^2 * F$ with H_i^j conjugate to a subgroup of G_i and F free. remark: n, m can be infinity

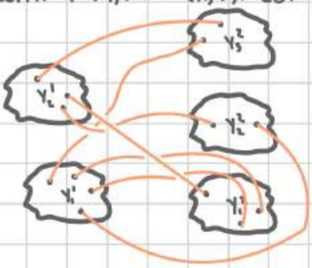
examples.

1. G = any group, H = normal closure of G in $G * \mathbb{Z}$.
   Q: What is H?
   $\hat{X}$ cover corresponding to H
   $\downarrow$
   $X = K(G,1) \vee S^1$
   $\pi_1(X) = G * \mathbb{Z}$
   $H = \pi_1(\hat{X}) = \underset{n \in \mathbb{Z}}{*}\, t^n G t^{-n}$



2. $H = [G, G]$, $G = \mathbb{Z} * \mathbb{Z} = F_2$.
   $G = \pi_1(\infty)$,
   $H = \pi_1(\#)$ = infinite rank free group
   no conjugate of a^n or b^n are in it.

pf. $(X_i, v_i) = $ a $K(G_i, 1)$. $X = \{X_1 v_1\} \overset{e}{—} \{v_2 X_2\}$. So $(X, v) =$ a $K(G_1 * G_2, 1)$. Let (Y, ω) = cover of (X, v) corresponding to H, with $f : (Y, \omega) \to (X, v)$. Let $\{Y_i^j\}_{i \in I_j}$ = components of $f^{-1}(X_j)$. (Y, ω) looks like:



($f^{-1}(e)$ = a bunch of disjoint edges mapping a Y_i^1 to a Y_i^2)

$Y \simeq (\bigvee_{i \in I_1} Y_i^1 \vee (\bigvee_{i \in I_2} Y_i^2) \vee$ (graph). $H = \pi_1(Y) \cong (\underset{i \in I_1}{*} \pi_1(Y_i^1)) * (\underset{i \in I_2}{*} \pi_1(Y_i^2)) * (\pi_1(\text{graph}))$ ← free group

Y_i^j cover of X_j, so $\pi_1(Y_i^j) \cong$ subgroup of $\pi_1(X_j) = G_j$. Under our identifications, $\pi_1(Y_i^j)$ maps to a $\underbrace{\text{conjugate}}_{\text{due to base point issues}}$ of a subgroup of $\pi_1(X_j) = G_j$. ⁂

def. A group G is indecomposable if you can't write $G = G_1 * G_2$ with $G_1, G_2 \neq 1$.

Grushko $\Rightarrow rk(G_1 * G_2) = rk(G_1) + rk(G_2)$.

cor. If G is a finitely generated group then we can write $G = G_1 * \dots * G_n$ with G_i indecomposable

theorem. If $G = G_1 * \dots * G_n$ and $G = G_1' * \dots * G_m'$ with each G_i and G_i' indecomposable. Then:
- $n = m$
- up to reordering, $G_i \cong G_i'$
- after reordering each G_i' that is not $\mathbb{Z}$ is conjugate to G_i.

pf. By induction on n. ($n=1$) trivial. ($n>1$) If each G_i is $\mathbb{Z}$, so G = free group $\Rightarrow G_i'$ is also free so since they are indecomposable, we must have $G_i' = \mathbb{Z}$. All we have to prove now is that $n = m$. But $G \cong F_n \cong F_m$ so $\mathbb{Z}^n \cong F_n^{ab} \cong F_m^{ab} = \mathbb{Z}^m$ and $n = m$. If we can find a G_i that is not $\mathbb{Z}$, the same must be true for some G_i'. Assume $G_1' \ncong \mathbb{Z}$. Since $G_1' < G_1 * \dots * G_n$ and G_1' is indecomposable and not $\mathbb{Z}$, Krurosh gives G_1' is conjugate to a subgroup of some G_i. Conjugate everything. We can assume $G_1' < G_1$ and Kurosh again implies that G_1 is conjugate to a subgroup of some G_j' must have $G_1' = G_1$. Then we must have $G_2 * \dots * G_n \cong G_2' * \dots G_m'$ since each is a quotient of G by normal closure of $G_1 = G_1'$. Done by induction. ⁂

Scott Core Theorem

theorem. Let M be a 3-manifold and G a finitely generated subgroup of $\pi_1(M)$, then there exists a compact 3-manifold with boundary M' with boundary M' with $G = \pi_1(M')$.

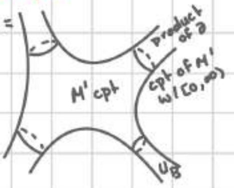
pf. By induction on $rk(G)$. Base case: $rk(G)=1$. So G is cyclic, i.e. $G = \mathbb{Z} = \pi_1(S^2 \times S^1)$ or $\mathbb{Z}/r = \pi_1$(lens sequence), done. Now assume $n = rk(G) \geq 2$, and that the theorem is true for all groups generated by $\leq n-1$ elements. If G is not indecomposable, then $G = G_1 * G_2$ with $G_i \neq 1$. We have $rk(G_1), rk(G_2) < n$ by Grushko so we can find compact 3-manifold M_i with $\pi_1(M_i) = G_i$, $G = \pi_1(M_1 \# M_2)$ and done. So we can assume G is indecomposable. Passing to cover of M corresponding to G, we can assume $\pi_1(M) = G$ So enough to prove:

<u>Scott Core's core</u>: Let M be a 3-manifold with $G = \pi_1(M)$ finitely generated. Assume G is indecomposable and that all subgroups of G generated by $< rk(G)$ elements are finitely presented $\Rightarrow$ there exists compact submanifolds $M' \subseteq M$ with $\pi_1(M') = G$. (can prove that M deformation retracts to M' and that M' exists without indecomposability by hypothesis, M' = scott core of M).

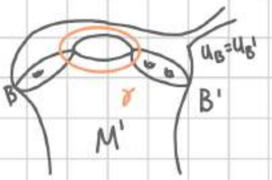
comments:

It's easy to find compact submanifold M' of M s.t. $\pi_1(M') \twoheadrightarrow \pi_1(M)$ (just choose a compact submanifold containing loops corresponding to generators of $G = \pi_1(M)$.) Hard part: Arranging $\pi_1(M') \to \pi_1(M)$ to be injective.

example.

M =


Injectivity Criterion: Let M be a 3-manifold and let $M' \subseteq M$ be a compact submanifold s.t. $\pi_1(M') \twoheadrightarrow \pi_1(M)$ is a surjection. For each boundary component B of M', let $U_B \subset M$ be compact of $M \setminus int(M')$ containing B. Assume that $\pi_1(B) \to \pi_1(U_B)$ injective for all B. Then $\pi_1(M') \to \pi_1(M)$ is injective.



pf. claim that $U_B \neq U_{B'}$ for $B \neq B'$, otherwise we could find loop γ in M s.t. $\gamma \cap M'$ is an arc connecting B to B'. The map $H_2(M) \to \mathbb{Z}$, $[\omega] \longmapsto$ algebraic intersection of ω with γ is nontrivial on $[B] \in H_2(M)$. Thus $[\gamma] \neq 0$ in $H_1(M)$. Since $\pi_1(M') \twoheadrightarrow \pi_1(M)$, γ is homotopic to loop in M' but all such loops have algebraic intersection 0 with B, a contradiction. Now using Van Kampen, $\pi_1(M) = \pi_1(M') * (\mathop{*}\limits_{B \text{ cpt of } \partial M'} \pi_1(U_B))$ / amalgamate over $\pi_1(B)$. Since $\pi_1(B) \hookrightarrow \pi_1(U_B)$ for all B, structure theorem for free product with amalgamation says that $\pi_1(M') \hookrightarrow \pi_1(M)$ as desired.

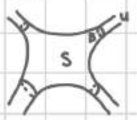
Lecture 19: November 10th, 2021

Core of Scott's Core Theorem

Let M be a 3-manifold. Assume $\pi_1(M)$ is finitely generated, indecomposable and not cyclic. Also, assume all subgroups of $\pi_1(M)$ generated by $\leq rk(\pi_1(M))-1$ are finitely presented. Then there exists a compact submanifold $M' \subseteq M$ with $\pi_1(M') = \pi_1(M)$.

pf. Let $\varphi: G \to \pi_1(M)$ be surjective with G finitely presented (we will later impose hypotheses on G). Let K be a compact 2-dimensional simplicial complex with $\pi_1(K) = G$ and let $f: K \to M$ be a map realizing $\pi_1(K) = G \overset{\varphi}{\twoheadrightarrow} \pi_1(M)$. Set S = regular neighborhood of $f(K)$ so S = compact submanifold of M. Letting $H = \text{im}(G = \pi_1(K) \to \pi_1(S))$. We also have that $H \to \pi_1(M)$ is surjective. We want to modify S s.t. $\pi_1(S) \to \pi_1(M)$ is injective (already know it is surjective, in fact $H \leq \pi_1(S)$ surjects).

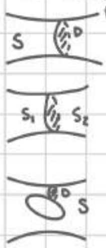
Picture:



Last time we showed that it is enough to show:

1. ∂S incompressible
2. for each compact B of ∂S and U = component of $M \setminus \text{int}(S)$ with $B \subseteq \partial U$.

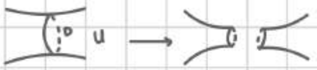
The map $\pi_1(B) \to \pi_1(U)$ is an injection. Try to change S s.t. ① holds. If $\pi_1(B) \to \pi_1(S)$ is not injective, then loop then gives compressing disk D. Then either D separates and $\pi_1(S) = \pi_1(S_1) * \pi_1(S_2)$ or D is non-separating. Let $S' = S$ cut along D and $\pi_1(S) = \pi_1(S') * \mathbb{Z}$. We want to replace S by S_1 or S_2 in case 1 and S' in case 2. Problem: How to ensure it doesn't mess up $\pi_1(S) \twoheadrightarrow \pi_1(M)$. Need: The subgroup $H \subseteq \pi_1(S)$ that surjects onto $\pi_1(M)$ to be indecomposable. Then if D is separating and $\pi_1(S) = \pi_1(S_1) * \pi_1(S_2)$, Kürosh subgroup theorem and indecomposible implies H conjugates to subgroup of $\pi_1(S_1)$ or $\pi_1(S_2)$. So we can pass to either S_1 or S_2 and still have the subgroup H (changed by conjugation) surjecting onto $\pi_1(M)$. Q: How to ensure H is indecomposible? All we know is $G \twoheadrightarrow H \to \pi_1(M)$ (with $G \twoheadrightarrow \pi_1(M)$). We need to choose G in such a way that all H are indecomposible.



Group Theory of Scott Core:

Let Γ = finitely generated indecomposible group with $rk(\Gamma) = n \geq 2$. Assume all subgroups of Γ with $rk \leq n-1$ are finitely presented G and surjection $\varphi: G \to \Gamma$ s.t. φ factors as $G \twoheadrightarrow H \to \Gamma$ then H is indecomposible. (we will prove this later).

Assuming this statement to be true, we can ensure ∂S is incompressible. Other part is easier. If $B \subseteq \partial S$ is a component, we need $B \to \pi_1(U)$ injective. If not, there exists a compression disk $D \subseteq U$. Just add to S:



All this does is kill off loops in $\pi_1(S)$, just do it over and over. #

theorem. (improved Grushko). Let $G = G_1 * \dots * G_r * F$ with F free, $H = H_1 * \dots * H_s$, $f: G \to H$ surjective, and G finitely generated. Assume $f(G_i)$ conjugate to a subgroup of some H_j for each i. Then $G = K_1 * \dots * K_s$ with $f(K_i) = H_i$ for all i.

pf. very similar to original proof of Grushko so we omit it.

Kürosh subgroup theorem says for any finitely generated group G, we can write $G = G_1 * \dots * G_r * Z_1 * \dots * Z_s$ with

- each G_i is indecomposable, not $\mathbb{Z}$.
- each $Z_j \cong \mathbb{Z}$

and the G_i is unique up to conjugation and number of Z_j's is unique. Call this the Kürosh decomposition of G. Define the complexity of G to be $c(G) = (r+s, s) \in \mathbb{Z}^2$. Order these lexicographically: $(n,m) \geq (n',m')$ if $n > n'$ or $n = n'$ and $m > m'$.

def. If $G = G_1 * \dots * G_r * Z_1 * \dots * Z_s$ is Kürosh's decomposition, a homomorphism $f: G \to H$ is semi-injective if $f|_{G_i}$ is injective for all i (if G is free, this is no condition at all).

Key lemma. Let G, H be finitely generated groups and let $f: G \to H$ be semi-injective and surjective. Then either f is isomorphism or $c(G) > c(H)$

pf. Let Kürosh decompositions be $G = G_1 * \dots * G_r * \mathbb{Z}_1 * \dots * \mathbb{Z}_s$ and $H = H_1 * \dots * H_{r'} * \mathbb{Z}_1' * \dots * \mathbb{Z}_{s'}'$. $f(G_i) \cong G_i$ by semi-injectivity so $f(G_i)$ is a indecomposable subgroup of H with $f(G_i) \ncong \mathbb{Z}$. The Kürosh subgroup theorem implies $f(G_i)$ is conjugate to some H_j. Can apply improved Grushko to write $G = K_1 * \dots * K_{r'} * L_1 * \dots * L_{s'}$ with $f(K_i) = H_i$ and $f(L_i) = \mathbb{Z}_i'$. We can get Kürosh decomposition of G by further decomposing this, so $r+s \geq r'+s'$. If $r+s > r'+s'$ then we are done. So assume $r+s = r'+s'$, then this decomposition must be the Kürosh decomposition for G. So we know that up to conjugation the G_i's appear in $\{K_1, \dots, K_{r'}, L_1, \dots, L_{s'}\}$. But can't appear among L_i's since then they would be $\mathbb{Z}$ by semi-injectivity (since $f(L_i) \leq \mathbb{Z}_i' = \mathbb{Z}$). So $r \geq r'$ which implies that $s' \leq s$. If $s' = s$ then since f is surjective we must have $f(L_i) = \mathbb{Z}_i'$ and f is an isomorphism. #

Group Theory of Scott Core:

Let Γ = finitely generated indecomposible group with $rk(\Gamma) = n \geq 2$. Assume all subgroups of Γ with $rk \leq n-1$ are finitely presented G and surjection $\varphi: G \to \Gamma$ s.t. φ factors as $G \twoheadrightarrow H \to \Gamma$ then H is indecomposible.

pf. We can definitely find finitely presented groups A with semi-injective maps $f: A \twoheadrightarrow \Gamma$, for example take A = free group of rank $rk(\Gamma)$. Pick such an $f: A \twoheadrightarrow \Gamma$ with complexity $c(A)$ as small as possible. If isomorphism then done. So assume f is not an isomorphism. Pick $r \in \ker(f)$, nontrivial, and set $G = A / \langle\langle r \rangle\rangle$. f factors as $\varphi: G \to \Gamma$. We claim this is what we want. Consider an intermediate subgroup H s.t.:

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & \Gamma \\ & H \arrow[u] \end{tikzcd}$$

Assume for sake of contradiction that H is decomposible: $H = H_1 * H_2$, $H_i \neq 1$. Let C_i = image of H_i in Γ so $H_1 * H_2 \to \Gamma$ factors as $C_1 * C_2 \to \Gamma$, $rk(C_i) \leq rk(H_i) < rk(H)$ so by our hypothesis C_i is finitely presented and thus $C_1 * C_2$ is as well.

recap image:

$$\begin{tikzcd} A \arrow[r, "\text{semi-inj.}", "f"'] \arrow[dr] \arrow[ddrr, bend right, "\text{semi-inj., not iso.}"'] & \Gamma & \\ & G_1 \arrow[u] \arrow[dr] & \\ & & C_1 * C_2 \arrow[uul, "\text{semi-inj.}"'] \end{tikzcd}$$

By Key lemma, $c(A) > c(C_1 * C_2)$, contradicting minimality of $c(A)$. #

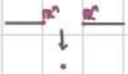
Lecture 20: November 15th, 2021

Lecture 21: November 17th, 2021

Q: What are all orientable n-dimensional vector bundles on a connected graph X?
A: Only the trivial one!

pf. Let T = maximal tree, Z = edges not in T. For $z \in Z$, pick pt. p_z in interior of z.

Pick any orientable $\mathbb{R}^n$-bundle $E \to X$. Let $X' = X$ cut open along p_z and $E' \to X'$ pullback of E along gluing map $X' \to X$, X' contractible $\Rightarrow$ E' trivial! Pick trivialization $E' = X' \times \mathbb{R}^n$. For $z \in Z$, Let $p_z', p_z'' = 2$ points in X' glued to $p_z \in X$. To assemble E, have to choose isomorphism $E'_{p_z'} = \mathbb{R}^n$ with $E'_{p_z''} = \mathbb{R}^n$, i.e. linear isomorphism $\varphi_z: \mathbb{R}^n \to \mathbb{R}^n$. E is orientable $\Rightarrow$ $\varphi_z \in GL_n^{>0}\mathbb{R}$, moving φ_z in $GL_n\mathbb{R}$ does not change anything. $GL_n^{>0}\mathbb{R}$ connected $\Rightarrow$ we can assume $\varphi_z = \mathrm{id}$. ⌗



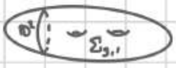
cor. If Σ is a surface which is not closed then all orientable $\mathbb{R}^n$-bundles on Σ are trivial.

pf. $\Sigma \overset{he}{\simeq}$ connected graph. ⌗

Q. How about closed orientable surfaces?

theorem. Σ_g = closed orientable genus $g \geq 0$ surface. Then the set of orientable $\mathbb{R}^n$-bundles on Σ_g is
- (n=1) just trivial one
- (n=2) $\mathbb{Z}$, classified by euler class $\in H^2(\Sigma_g) = \mathbb{Z}$
- ($n \geq 3$) $\mathbb{Z}/2$, classified by 2^{nd} stiefel-Whitney class $w_2 \in H^2(\Sigma_g, \mathbb{F}_2) = \mathbb{F}_2$.

pf. Write $\Sigma_g = \mathbb{D}^2 \cup_{S^1} \Sigma_{g,1}$ ($\Sigma_{g,1} = \Sigma_g \setminus$ open ball)



Let $E \to \Sigma_g$ be orientable $\mathbb{R}^n$-bundle $E|_{\mathbb{D}^2}$ and $E|_{\Sigma_{g,1}}$ trivial. We have to glue $\mathbb{D}^2 \times \mathbb{R}^n$ to $\Sigma_{g,1} \times \mathbb{R}^n$ along boundary circles so for $p \in S^1$ we need orientation-preserving isomorphism $\varphi_p: p \times \mathbb{R}^n \to p \times \mathbb{R}^n$. i.e. need gluing maps $\varphi: S^1 \to GL_n^{>0}(\mathbb{R})$. $GL_n^{>0}(\mathbb{R}) \overset{he}{\simeq} SO(n)$ (Gram-Schmidt), $\pi_1(SO(n)) = \begin{cases} 1 & n=1 \\ \mathbb{Z} & n=2 \\ \mathbb{Z}/2 & n \geq 3 \end{cases}$. So these parameterize gluings. So, we get surjective map $\left\{\begin{smallmatrix} 1 \\ \mathbb{Z} \\ \mathbb{Z}/2 \end{smallmatrix}\right\} \twoheadrightarrow \left\{\begin{smallmatrix} \text{orientable} \\ \mathbb{R}^n \\ \text{bundles} \end{smallmatrix}\right\}$. The above char. classes form an inverse, so this is a bijection. ⌗

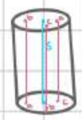
Alternate construction of unit $\mathbb{R}$-vector bundle with euler # $p \in \mathbb{Z}$ over Σ_g:
Start with trivial bundle $\Sigma_g \times S^1$. Consider a disk $\mathbb{D} \subseteq \Sigma_g$. Cut out solid torus $\mathbb{D} \times S^1 \subseteq \Sigma_g \times S^1$ and reglue, i.e. preform surgery on knot $p \times S^1$ with p = center of $\mathbb{D}$. If you do this correctly, you will get unit vector bundle of $\mathbb{R}^2$-bundle / Σ_g with euler # p. The regluing map is an orientation preserving diffeo. $T^2 = \partial\mathbb{D} \times S^1 \to \partial\Sigma_{g,1} \times S^1 = T^2$, i.e. element of mapping class group $SL_2\mathbb{Z}$ of T^2. For this to give unit vector bundle it needs to preserve fibers $x_0 \times S^1$, i.e. must be matrix of form $\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$. This one works.

Q. What would other matrices give you?
in fact: orientable Seifert fibered space can be obtained by:
start with $\Sigma_g \times S^1$. Pick some points $x_1, \dots, x_n \in \Sigma_g$. Perform Dehn surgery on vertical knots $x_i \times S^1 \subseteq \Sigma_g \times S^1$.

Q. Why does this give Seifert fiber space?

theorem. Let M^3 be orientable seifert space. Let $S \in M^3$ be one of the circles then there exists neighborhood N of S with $N \cong \mathbb{D}^2 \times S^1$ and S = center $\times S^1$ s.t. the circles on N are:
- start with $\mathbb{D}^2 \times [0,1]$
- for some relatively prime p and q, glue $\mathbb{D}^2 \times 1$ to $\mathbb{D}^2 \times 0$ via $\mathbb{D}^2 \to \mathbb{D}^2$, rotate by $2\pi \, p/q$ (gives a solid torus N)
- circles are images of vertical lines in $\mathbb{D}^2 \times [0,1]$.

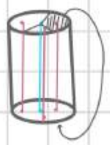
Picture



nb: The S^1-action preserves N and is just vertical translation. For a point $z \in S$, the S^1-stabilizer of z is $\mathbb{Z}/q \subseteq S^1$.

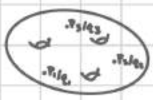
observation: For any Seifert fiber space M^3, the quotient space Σ that collapses circles is a surface

Picture



Every circle passes uniquely through this so leaf space is

Points where local model involves $2\pi P/q$ twist are exceptional fibers. Isolated neighborhood of red circle above is $\mathbb{D}^2 \times S^1$ with trivial circles $p \times S^1$ so the surface Σ = space of circles has finitely many exceptional fibers lying over interior points.



this is not unique but non-uniqueness is understood

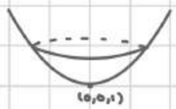
theorem. (Seifert-fibered Space Conjecture) Let M be a compact irreducible 3-manifold with infinite π_1. Them M is Seifert-fibered $\iff$ $\pi_1(M)$ has nontrivial center. (the center is loop around fiber).

Lecture 22: November 22nd 2021

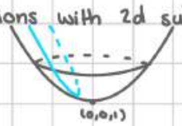
Hyperbolic Geometry

Lecture 23: November 29^{th}, 2021

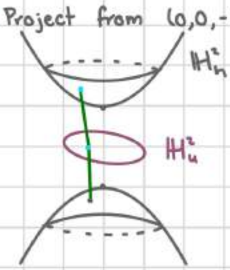
Recall: $\mathbb{H}^2_h$ = (-1)-sphere in $(\mathbb{R}^3, x^2+y^2-z^2)$ with $z>0$



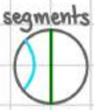
Special curves in $\mathbb{H}^2_h$ preserved by $\mathrm{Isom}^+(\mathbb{H}^2_h)$:
intersections with 2d subspaces:



Alternate model: unit disk $\mathbb{H}^2_u = \{p \in \mathbb{R}^2 \mid \|z\| < 1\}$
related: Project from $(0,0,-1)$



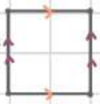
These project to circles in $\mathbb{H}^2_u$ that are orthoganol to $\partial\mathbb{H}^2_u = S^1$ (or straight line segments through the origin).

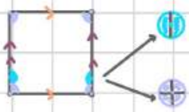


lemma. These are the geodesics for hyperbolic metric.
pf. First, consider straight line ℓ through 0: 

The metric is rotationally symmetries so reflection $r: \mathbb{H}^2_u \to \mathbb{H}^2_u$ in ℓ is an isometry. For $a,b \in \ell$, if γ is a geodesic from a to b then $r(\gamma)$ = another geodesic from a to b. If a is close enough to b then there is a unique geodesic, so $r(\gamma)=\gamma$, thus γ = segment of ℓ from a to b $\Rightarrow$ ℓ is a geodesic (and the only one connecting points on ℓ). $\mathrm{Isom}^+(\mathbb{H}^2_u)$-orbit of ℓ are also geodesics. Easy: $\mathrm{Isom}^+(\mathbb{H}^2_h)$ acts transitively on subspaces of $\mathbb{R}^3$ intersecting $\mathbb{H}^2_h$ $\Rightarrow$ $\mathrm{Isom}^+(\mathbb{H}^2_h)$-orbit of ℓ as above. #

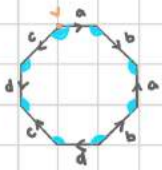
Q: How to construct hyperbolic metric on a surface?
Easier Q: Flat metric on torus.

$T^2 =$  Give this metric coming from $\mathbb{R}^2$.
Check this works on edges and vertex:
no singularities!



What is going on at a vertex?
each corner has angle $\pi/2$ and 4 of them meet at a vertex, giving a total angle of 2π.

Non-Example:

 = 

This gives a flat metric on Σ_2, except at vertex v.
 around v, we have a $8 \cdot \frac{3\pi}{4} = 6\pi$ angle so the metric does not extend to v.

We can fix this! with hyperbolic geometry
lemma. In $\mathbb{H}^2_u$, there exists a regular octagon with angle $\pi/4$.

pf. For $0<r<1$, let O_r = the octagon in $\mathbb{H}^2_u$ with vertices $re^{2\pi i/8}$ and geodesic sides:
 Let θ_r = vertex angle of O_r. claim: $\lim_{r\to 1} \theta_r = 0$.
In tiny neighborhood of (0,0) metric is very close to Euclidean metric.
So angles degenerate to Euclidean angle $3\pi/4$. Lemma now follows from intermediate value theorem. #

We can now do previous example with hyperbolic regular octagons with angle $3\pi/4$ and metric extends continuously to v.
Similarly, we get a hyperbolic metric on Σ_g for $g \geq 2$.

theorem. (Fricke-klein). For $g \geq 2$, the "sphere" of hyperbolic metrics on Σ_g is $\cong \mathbb{R}^{6g-6}$.

Parameter Count:

$\{$space of hyperbolic metrics on $\Sigma_g\} = \{\varphi : \pi_1(\Sigma_g) \to PSL_2\mathbb{R} \mid \varphi$ faithful, $Im(\varphi)$ discrete$\}/PSL_2\mathbb{R}$

given φ, get surface $\mathbb{H}^2/Im(\varphi)$. This has $\pi_1 = \pi_1(\Sigma_g)$ so is genus g

$PSL_2\mathbb{R} = 3d$ real Lie group $\Rightarrow$ $Hom(F(a_1,b_1,\dots,a_g,b_g), PSL_2\mathbb{R}) = (PSL_2\mathbb{R})^{2g}$ is a $6g$-dimensional manifold.
$Hom(\pi_1(\Sigma_g), PSL_2\mathbb{R})$ = subset satisfying $[a_1,b_1]\cdots[a_g,b_g]=1$. (3-conditions: the entries of single matrix in $PSL_2\mathbb{R}$ must be $\begin{pmatrix}1&0\\0&1\end{pmatrix}$).
$\Rightarrow$ Intuitively, $Hom(\pi_1(\Sigma_g), PSL_2\mathbb{R})$ is $6g-3$ dimensional. Modding out by $PSL_2\mathbb{R}$-conjugate, cut down dimension by 3, so $Hom(\pi_1(\Sigma_g), PSL_2\mathbb{R})/PSL_2\mathbb{R}$ is $(6g-6)$-dimensional (in fact space of metrics is connected component).

In contrast, (Mostow Rigidity) For $n \geq 3$, let (M_1, μ_1) and (M_2, μ_2) be closed hyperbolic n-manifolds and let $f: M_1 \to M_2$ be a homotopy equivalence, then f is homotopic to an isometry.
remark: Also true for finite volume metrics on open manifolds but not for ∞-volume

cor. For a 3-manifold M with a finite-volume hyperbolic metric, any metric property (eg. volume) is a topological invariant.
remark: very powerful for knot programs like snappea can calculate hyperbolic volume to any precision you want.

Lecture 24: December 1st, 2021

Recall: (Mostow Rigidity) For $n \geq 3$, let (M_1, μ_1) and (M_2, μ_2) be closed hyperbolic n-manifolds and let $f: M_1 \to M_2$ be a homotopy equivalence, then f is homotopic to an isometry.

lemma. If M is a closed hyperbolic manifold then $\mathrm{Isom}(M)$ is finite

cor. If M is a closed hyperbolic n-manifold with $n \geq 3$, then $\pi_0(\mathrm{Diff}(M)) \cong \mathrm{Isom}(M)$ finite.

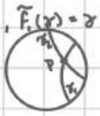
pf. immediate from Mostow Rigidity and lemma

remark: For 3-manifolds we also have:

theorem. (Gabai) If M is closed hyperbolic 3-manifold then $\mathrm{Diff}_0(M)$ is contractible.

remark: Also true for hyperbolic 2-manifold (i.e. Σ_g $g > 2$) by Earle-Eells.

pf of lemma. $\mathrm{Isom}(M)$ is a compact Lie group so we must show it is discrete, i.e. we must show that if $f: M \to M$ is an isometry and there exists homotopy $f_t: M \to M$ s.t. $f_0 = id$ then $f_1 = id$. (nb. this is better than discrete since the f_t need not be diffeo's much less an isometry). Lift f_t to $\widetilde{f}_t: \mathbb{H}^n \to \mathbb{H}^n$ with $\widetilde{f}_0 = id$ and $\widetilde{f}_1$ an isometry. Will show that for any geodesic γ in $\mathbb{H}^n$, $\widetilde{f}_1(\gamma) = \gamma$. This will imply $\widetilde{f}_1 = id$ since for all $p \in \mathbb{H}^n$, there exists geodesics γ_1 and γ_2 with $\gamma_1 \cap \gamma_2 = p$. Then we have:



$\widetilde{f}_1(p) = \widetilde{f}_1(\gamma_1 \cap \gamma_2) = \widetilde{f}_1(\gamma_1) \cap \widetilde{f}_1(\gamma_2) = \gamma_1 \cap \gamma_2 = p$. We know that $\widetilde{f}_1(\gamma)$ is a geodesic since $\widetilde{f}_1$ is an isometry.
Since M is compact, for $x \in M$ the path $t \mapsto f_t(x)$ has length $\leq C$ for some constant C independent of x.
$\Rightarrow$ For all $p \in \mathbb{H}^n$ we have $d_{\mathbb{H}^n}(p, \widetilde{f}_1(p)) \leq C$. In particular, for all t we have $d_{\mathbb{H}^n}(\gamma(t), \widetilde{f}_1(\gamma)(t))$. Since $\gamma(t)$ and $\widetilde{f}_1(\gamma)(t)$ go to infinity as $t \to \infty$, we have $d_{\mathbb{R}^n}(\gamma(t), \widetilde{f}_1(\gamma)(t)) \to 0$ as $t \to \infty$.

euclidean distance $\sim \frac{1}{1-r} \cdot$ (hyperbolic distance) $\Rightarrow \gamma$ and $\widetilde{f}_1(\gamma)$ end at same point of $\partial\mathbb{H}^n$.
Same argument shows the start is the same $\Rightarrow \gamma$ and $\widetilde{f}_1(\gamma)$ are geodesics with the same endpoints on $\partial\mathbb{H}^n$ $\Rightarrow \gamma = \widetilde{f}_1(\gamma)$. ✗

Construct a Finite Volume hyperbolic 3-manifold

We constructed hyperbolic metric on Σ_2 by finding a regular octagon with angles $2\pi/8$ and gluing

Angle $= \frac{2\pi}{8}$ implies metric on octagon induces a metric on Σ_2 (in particular, no singularity at vertex.

Q: What conditions on 3-d hyperbolic polyhedra needed to let glue to get hyperbolic 3-manifold?

- faces to be isometric polygons
- no problem in interior of face
- on interiors of edges need sum of dihedral angles to be 2π.

what happens at vertice?

we will avoid caring about vertices by deleting them by putting them at ∞.

example. (dim 2)

vertices at ∞, edges have ∞ length
cool fact: Area $= \pi$.

ideal triangle this is a limit on ordinary hyp triangle  area $= \pi - (a+b+c)$, becomes ideal as vertices $\to \partial$ and angles $\to 0$.

Using ideal triangles, we can find hyperbolic metrics of finite volume on punctured surfaces

"Regular" ideal tetrahedron in $\mathbb{H}^3$.
start with regular tetrahedron in 3d Euclidean space

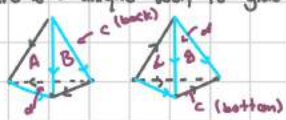
scale to make all points lie on unit sphere, take $\mathbb{H}^3$-convex hull of those vertices

the four faces are ideal triangles.

Easy to show the tetrahedron has finite volume.
basic fact: The dihedral angle is $2\pi/6 = \pi/3$.

Gluing: Take two and glue together.

there is a unique way to glue s.t. colors and directions match up.



All of the blue edges are identified, and all of the gray, $\overbrace{\text{vertices}}^{\text{at }\infty}$ are identified.
Around the blue edge there are six triangular pieces with each dihedral angle = $2\pi/6$ for a total of 2π angles.
$\Rightarrow$ no singularity of metric on blue edges. Exactly the same on the gray edge. Thus we get a hyperbolic 3-manifold with finite volume.

Q: What if you put vertex back?
If you put vertex back then you get a non-manifold neighborhood of new point = cone(torus).
$\Rightarrow$ M = Int(compact 3-manifold with torus boundary.
In fact, $M \cong S^3 \setminus$ (figure 8-knot)