Textbook New Keynesian Model w/NGDP Nominal Rate Rule

Model Environment:

- Goods Market:
  - Demand - households consume a basket of goods
  - Supply - firms produce different consumption goods.
  - Prices are assumed to be sticky in the form of Calvo:
    - In each period, each firm can adjust price with probability $(1-\phi_p)$
- Labor Market:
  - Demand - firms hire laborers
  - Supply - household supplies labor
  - Wages are assumed to be perfectly flexible
- Financial Markets:
  - Household optimally invests in a one-period risk-free bond.

① Households

Households maximize the following lifetime utility function:

$$\max_{\{C_t, L_t, B_t\}} E_t \sum_{j=0}^{\infty} \beta^{t+j} \left\{ \ln(C_{t+j}) - \frac{\chi L_t^{1+\eta}}{1+\eta} \right\}$$

subject to $P_t C_t + B_t \le (1+i_{t-1}) B_{t-1} + W_t L_t + DIV_t$

Write this as a Lagrangian:

$$\mathcal{L} = E_t \sum_{j=0}^{\infty} \beta^{t+j} \left\{ \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\chi L_t^{1+\eta}}{1+\eta} + \lambda_t \left(W_t L_t + (1+i_{t-1}) B_{t-1} - B_t - P_t C_t + DIV_t \right) \right\}$$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \cdot C_t^{-\sigma} - \beta^t P_t \lambda_t = 0 \longrightarrow C_t^{-\sigma} = \lambda_t P_t \qquad ①$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = -\beta^t \chi L_t^{\eta} + \beta^t \lambda_t W_t = 0 \longrightarrow \chi L_t^{\eta} = \lambda_t W_t \qquad ②$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = -\beta^t \lambda_t + \beta^{t+1} E_t\left[\lambda_{t+1}(1+i_t)\right] = 0 \longrightarrow \lambda_t = \beta E_t\left[\lambda_{t+1}(1+i_t)\right] \qquad ③$$

Sub ① into ②:

$$\chi L_t^{\eta} = (C_t P_t)^{-1} W_t \longrightarrow \chi L_t^{\eta} = C_t^{-\sigma} w_t$$

Sub ① into ③

$$C_t^{-\sigma} P_t^{-1} = \beta E_t \left[C_{t+1}^{-\sigma} P_{t+1}^{-1} (1+i_t) \right]$$

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-1} \Pi_{t+1}^{-1} (1+i_t) \right]$$

② Production

2.1 Final Goods Producer

The final goods producer aggregates production from the intermediate firm into the final good using a CES technology:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{for } \varepsilon > 0,\ j \in [0,1].$$

The final goods producer maximizes:

$$\max_{\{Y_t(j)\}} P_t \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{\varepsilon}{\varepsilon-1}} - \int_0^1 P_t(j) Y_t(j) dj$$

FOC:

$$\langle Y_t(j) \rangle \quad P_t \cancel{\frac{\varepsilon}{\varepsilon-1}} \cdot \cancel{\frac{\varepsilon-1}{\varepsilon}} Y_t(j)^{\frac{-1}{\varepsilon}} \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{1}{\varepsilon-1}} = P_t(j)$$

$$Y_t(j)^{\frac{-1}{\varepsilon}} \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{1}{\varepsilon-1}} = \frac{P_t(j)}{P_t}$$

$$Y_t(j) \left(\int_0^1 Y_t(j)^{\frac{\varepsilon-1}{\varepsilon}} dj \right)^{\frac{-\varepsilon}{\varepsilon-1}} = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon}$$

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\varepsilon} Y_t$$

Now we find the aggregate price index:

$$P_t Y_t = \int_0^1 P_t(j) Y_t(j) dj$$

$$P_t Y_t = \int_0^1 P_t(j) P_t(j)^{-\varepsilon} P_t^{\varepsilon} Y_t dj$$

$$P_t^{1-\varepsilon} = \int_0^1 P_t(j)^{1-\varepsilon} dj$$

$$P_t = \left[\int_0^1 P_t(j)^{1-\varepsilon} dj \right]^{\frac{1}{1-\varepsilon}}$$

2.2 Intermediate Producers

Intermediate producers produce $Y_t(j)$ using:

$$Y_t(j) = A_t L_t(j) \quad , \text{ where } A_t \text{ is productivity.}$$

Because the producers cannot adjust price each period, they minimize costs subject to demand for their product:

$$\min_{\{L_t(j)\}} W_t L_t(j) \quad \text{s.t.} \quad A_t L_t(j) \geq \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} Y_t$$

Putting this in a Lagrangian gives:

$$\mathcal{L} = -W_t L_t(j) + \varphi_t \left(A_t L_t(j) - \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} Y_t\right)$$

FOCs:

$$\langle L_t(j) \rangle \quad -W_t + \varphi_t A_t = 0 \longrightarrow W_t = \varphi_t A_t \longrightarrow \frac{W_t}{A_t} = \varphi_t$$

Note that φ_t is nominal marginal cost (wage divided by marginal product of labor). We can now write flow profits:

$$DIV_t(j) = P_t(j) Y_t(j) - W_t L_t(j) \qquad \text{Divide by } P_t:$$

$$div_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} L_t(j) \qquad \text{Sub in for } W_t:$$

$$div_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - \frac{\varphi_t A_t}{P_t} L_t(j)$$

$$div_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j) \qquad \text{where } mc_t = \frac{\varphi_t}{P_t}$$

Recall that intermediate firm can only change price each period with probability $(1-\phi_p)$. Then the firm's infinite horizon problem is:

$$\max_{\{P_t(j)\}} E_t \sum_{s=0}^{\infty} \phi_p^s \beta^s \frac{U'(C_{t+s})}{U'(C_t)} \left(\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}}\right)^{-\varepsilon} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}}\right)^{-\varepsilon} Y_{t+s}\right)$$

$$= \max_{\{P_t(j)\}} E_t \sum_{s=0}^{\infty} \phi_p^s \beta^s \frac{U'(C_{t+s})}{U'(C_t)} \left(P_t(j)^{1-\varepsilon} P_{t+s}^{\varepsilon-1} Y_{t+s} - mc_{t+s} P_t(j)^{-\varepsilon} P_{t+s}^{\varepsilon} Y_{t+s}\right)$$

where $\beta^s \frac{U'(C_{t+s})}{U'(C_t)}$ is the stochastic discount factor

FOC:

$$\langle P_t(j) \rangle \quad (1-\varepsilon) P_t(j)^{-\varepsilon} E_t \sum_{s=0}^{\infty} \phi^s \beta^s \frac{U'(C_{t+s})}{U'(C_t)} \left(P_{t+s}^{\varepsilon-1} Y_{t+s}\right) + \varepsilon P_t(j)^{-\varepsilon-1} E_t \sum_{s=0}^{\infty} \phi^s \beta^s \frac{U'(C_{t+s})}{U'(C_t)} mc_{t+s} P_{t+s}^{\varepsilon} Y_{t+s} = 0$$

$$\frac{\varepsilon}{\varepsilon-1} P_t(j)^{-1} = \frac{E_t \sum_{s=0}^{\infty} \phi_p^s \beta^s \frac{U'(C_{t+s})}{\cancel{U'(C_t)}} \left(P_{t+s}^{\varepsilon-1} Y_{t+s}\right)}{E_t \sum_{s=0}^{\infty} \phi_p^s \beta^s \frac{U'(C_{t+s})}{\cancel{U'(C_t)}} mc_{t+s} P_{t+s}^{\varepsilon} Y_{t+s}}$$

$$P_t(j) = \frac{\varepsilon}{\varepsilon - 1} \cdot \frac{E_t \sum_{s=0}^{\infty} \phi_p^s \beta^s U'(C_{t+s}) mc_{t+s} P_{t+s}^{\varepsilon} Y_{t+s}}{E_t \sum_{s=0}^{\infty} \phi_p^s \beta^s U'(C_{t+s}) P_{t+s}^{\varepsilon-1} Y_{t+s}}$$

Because the left-hand side is constant across firms, $P_t(j) = P_t^*$. Also note that the numerator and denominator can be written recursively as:

$$X_{1,t} = U'(C_t)\, MC_t\, P_t^{\varepsilon} Y_t + \phi_p \beta E_t\left[X_{1,t+1}\right]$$
$$X_{2,t} = U'(C_t)\, P_t^{\varepsilon-1} Y_t + \phi_p \beta E_t\left[X_{2,t+1}\right]$$

And so $P_t^* = \frac{\varepsilon}{\varepsilon-1} \cdot \frac{X_{1,t}}{X_{2,t}}$

④ Central Bank

I assume the central bank sets monetary policy according to:

$$i_t = (1-\rho_r) r^* + \rho_r i_{t-1} + (1-\rho_r)\phi_\pi\left(\log(\Pi_t) - \log(\Pi^*)\right)$$
$$+ (1-\rho_r)\phi_y\left(\log(Y_t) - \log(Y_{t-1})\right) + \sigma_r \varepsilon_r$$

in the standard version. In the NGDP version, the policy rule is:

$$i_t = (1-\rho_r) r^* + \rho_r i_{t-1} + (1-\rho_r)\phi_\pi\left(\log(\Pi_t) - \log(\alpha) + \log(Y_t) - \log(Y_{t-1})\right)$$
$$+ \sigma_r \varepsilon_r$$

⑤ Random Processes

Productivity:

$$\ln(A_t) = \rho_a \ln(A_{t-1}) + \sigma_a \varepsilon_{a,t}$$

⑥ Aggregation

From the household budget constraint:

$$W_t L_t + (1+i_{t-1}) B_{t-1} = B_t + P_t C_t - DIV_t \qquad \text{Divide by } P_t:$$

$$w_t L_t + div_t + (1+i_{t-1}) b_{t-1} = b_t + C_t \qquad \text{Impose bond market clearing:}$$

$$w_t L_t + div_t = C_t$$

Dividends comes from the intermediate firms:

$$div_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) - w_t L_t(j)\, dj$$
$$= \int_0^1 \frac{P_t(j)}{P_t} Y_t(j)\, dj - w_t L_t$$

Sub this into the budget constraint:

$$w_t L_t + \int_0^1 P_t(j) P_t^{-1} Y_t(j)\,dj - w_t L_t = C_t$$

$$\int_0^1 P_t(j) P_t^{-1} Y_t(j)\,dj = C_t$$ Plug in demand for $Y_t(j)$:

$$\int_0^1 P_t(j) P_t^{-1} \cdot \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} Y_t\,dj = C_t$$

$$P_t^{\varepsilon-1} Y_t \int_0^1 P_t(j)^{1-\varepsilon}\,dj = C_t$$ Use the aggregate price index:

$$P_t^{\varepsilon-1} Y_t P_t^{1-\varepsilon} = C_t$$

$$Y_t = C_t$$

Now find Y_t:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} Y_t$$

$$A_t L_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon} Y_t$$

$$\int_0^1 A_t L_t(j)\,dj = \int_0^1 P_t(j)^{-\varepsilon} P_t^{\varepsilon} Y_t\,dj$$

$$A_t \int_0^1 L_t(j)\,dj = Y_t \int_0^1 P_t(j)^{-\varepsilon} P_t^{\varepsilon}\,dj$$

$$A_t N_t = Y_t V_t^p$$

$$\frac{A_t N_t}{V_t^p} = Y_t$$, where V_t^p is a measure of price dispersion

Simplify V_t^p:

$$V_t^p = \int_0^{1-\phi_p} \left(\frac{P_t^*}{P_t}\right)^{-\varepsilon} dj + \int_{1-\phi_p}^{\phi_p} \left(\frac{P_{t-1}(j)}{P_t}\right)^{-\varepsilon} dj$$

$$V_t^p = \int_0^{1-\phi_p} \left(\frac{P_t^*}{P_{t-1}}\right)^{-\varepsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\varepsilon} dj + \int_{1-\phi_p}^{\phi_p} \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\varepsilon} \cdot \left(\frac{P_{t-1}}{P_t}\right)^{-\varepsilon} dj$$

$$= \int_0^{1-\phi_p} \Pi_t^{*-\varepsilon} \Pi_t^{\varepsilon}\,dj + \Pi_t^{\varepsilon} \int_{1-\phi_p}^{\phi_p} \left(\frac{P_{t-1}(j)}{P_{t-1}}\right)^{-\varepsilon} dj$$

$$= (1-\phi_p) \Pi_t^{*-\varepsilon} \Pi_t^{\varepsilon} + \Pi_t^{\varepsilon} \phi_p V_{t-1}^p$$

Now, find P_t:

$$P_t^{1-\varepsilon} = \int_0^1 P_t(j)^{1-\varepsilon}\,dj$$

$$= \int_0^{1-\phi_p} P_t^{*\,1-\varepsilon}\,dj + \int_{1-\phi_p}^1 P_{t-1}(j)^{1-\varepsilon}\,dj$$

$$P_t^{1-\varepsilon} = (1-\phi_p) P_t^{*\,1-\varepsilon} + \phi_p P_{t-1}^{1-\varepsilon}$$

$$\left(\frac{P_t}{P_{t-1}}\right)^{1-\varepsilon} = (1-\phi_p)\left(\Pi_t^*\right)^{1-\varepsilon} + \phi_p$$

$$\Pi_t^{1-\varepsilon} = (1-\phi_p)\,\Pi_t^{*\,1-\varepsilon} + \phi_p$$

Lastly, define P_t^* in real terms:

$$P_t^* = \frac{\varepsilon}{\varepsilon-1}\frac{X_{1,t}}{X_{2,t}}$$

$$\frac{X_{1,t}}{P_t^{\varepsilon}} = x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta E_t\left[\frac{P_{t+1}^{\varepsilon}}{P_t^{\varepsilon}} \frac{X_{1,t+1}}{P_{t+1}^{\varepsilon}}\right]$$

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta E_t\left[\Pi_{t+1}^{\varepsilon} x_{1,t+1}\right]$$

$$\frac{X_{2,t}}{P_t^{\varepsilon-1}} = x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta E_t\left[\frac{P_{t+1}^{\varepsilon-1}}{P_t^{\varepsilon-1}} \frac{X_{2,t+1}}{P_{t+1}^{\varepsilon-1}}\right]$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta E_t\left[\Pi_{t+1}^{\varepsilon-1} x_{2,t+1}\right]$$

So $P_t^* = \frac{\varepsilon}{\varepsilon-1} P_t \frac{x_{1,t}}{x_{2,t}} \longrightarrow \Pi_t^* = \frac{\varepsilon}{\varepsilon-1} \Pi_t \frac{x_{1,t}}{x_{2,t}}$

7 Equilibrium Equations

$C_t^{-\sigma} = \beta E_t\left[C_{t+1}^{-\sigma}(1+i_t)\,\Pi_{t+1}^{-1}\right]$	Euler Equation
$\chi L_t^{\eta} = C_t^{-\sigma} w_t$	Labor Supply
$mc_t = \frac{w_t}{A_t}$	Real Marginal Cost
$C_t = Y_t$	Aggregate Budget Constraint
$Y_t = \frac{A_t L_t}{v_t^p}$	Aggregate Production Function
$v_t^p = (1-\phi_p)\Pi_t^{*\,-\varepsilon}\Pi_t^{\varepsilon} + \Pi_t^{\varepsilon}\phi_p v_{t-1}^p$	Price Dispersion
$\Pi_t^{1-\varepsilon} = (1-\phi_p)\Pi_t^{*\,1-\varepsilon} + \phi_p$	Real Price Dynamics
$\Pi_t^* = \frac{\varepsilon}{\varepsilon-1}\Pi_t \frac{x_{1,t}}{x_{2,t}}$	Real Optimal Reset Price
$x_{1t} = C_t^{-\sigma} mc_t Y_t + \phi_p\beta E_t\left[\Pi_{t+1}^{\varepsilon} x_{1t+1}\right]$	Auxiliary Variable 1
$x_{2t} = C_t^{-\sigma} Y_t + \phi_p\beta E_t\left[\Pi_{t+1}^{\varepsilon-1} x_{2t+1}\right]$	Auxiliary Variable 2
$\ln(A_t) = \rho_a \ln(A_{t-1}) + \varepsilon_{a,t}$	Production AR(1)
$i_t = (1-\rho_i)\bar{i} + \rho_i i_{t-1} + (1-\rho_i)\phi_n(\pi_t + y_t - y_{t-1} - \alpha) + \varepsilon_{i,t}$	NGDP Rule

8 Steady State

$A = 1$

$\Pi = 1$

$$C^{-\sigma} = \beta C^{-\sigma}\Pi^{-1}(1+\bar{i}) \longrightarrow (1+\bar{i}) = \frac{\Pi}{\beta}$$

$$\Pi^{1-\varepsilon} = (1-\phi_p)\Pi^{*\,1-\varepsilon} + \phi_p \longrightarrow \Pi^* = \left(\frac{\Pi^{1-\varepsilon}-\phi_p}{1-\phi_p}\right)^{\frac{1}{1-\varepsilon}}$$

$$v^p = (1-\phi_p)\Pi^{*-\varepsilon}\Pi^{\varepsilon} + \Pi^{\varepsilon}\phi_p v^p \longrightarrow v^p = \frac{(1-\phi_p)\Pi^{*-\varepsilon}\Pi^{\varepsilon}}{1-\Pi^{\varepsilon}\phi_p}$$

$$X_2 = C^{-\sigma}Y + \phi_p\beta\Pi^{\varepsilon-1}X_2 \longrightarrow X_2 = C^{-\sigma}Y(1-\phi_p\beta\Pi^{\varepsilon-1})^{-1}$$

$$X_1 = C^{-\sigma}mcY + \phi_p\beta\Pi^{\varepsilon}X_1 \longrightarrow X_1 = C^{-\sigma}mcY(1-\phi_p\beta\Pi^{\varepsilon})^{-1}$$

Divide X_1 by X_2: $\frac{X_1}{X_2} = \frac{mc(1-\phi_p\beta\Pi^{\varepsilon-1})}{1-\phi_p\beta\Pi^{\varepsilon}}$

$\Pi^* = \frac{\varepsilon}{\varepsilon-1}\Pi\frac{X_1}{X_2} \longrightarrow \frac{X_1}{X_2} = \frac{\varepsilon-1}{\varepsilon}\frac{\Pi^*}{\Pi}$ Set equal:

$$\frac{mc(1-\phi_p\beta\Pi^{\varepsilon-1})}{1-\phi_p\beta\Pi^{\varepsilon}} = \frac{\varepsilon-1}{\varepsilon}\frac{\Pi^*}{\Pi}$$

$$mc = \frac{\varepsilon-1}{\varepsilon}\frac{\Pi^*}{\Pi}\frac{1-\phi_p\beta\Pi^{\varepsilon}}{1-\phi_p\beta\Pi^{\varepsilon-1}}$$

$$\frac{w}{A} = mc \longrightarrow w = mcA$$

$$\chi L^{\eta} = C^{-\sigma}w \longrightarrow L = \left(\frac{C^{-\sigma}w}{\chi}\right)^{\frac{1}{\eta}}$$

$$Y = \frac{AL}{v^p} \longrightarrow Y = \frac{A}{v^p}\left(\frac{C^{-\sigma}w}{\chi}\right)^{\frac{1}{\eta}}$$

$$C = Y \longrightarrow C = \frac{A}{v^p}\left(\frac{C^{-\sigma}w}{\chi}\right)^{\frac{1}{\eta}}$$

$$C = \left[\frac{A}{v^p}\left(\frac{w}{\chi}\right)^{\frac{1}{\eta}}\right]^{\frac{\eta}{\sigma+\eta}}$$

$$Y = C$$

$$L = \left(\frac{C^{-\sigma}w}{\chi}\right)^{\frac{1}{\eta}}$$

$$X_1 = C^{-\sigma}mcY(1-\phi_p\beta\Pi^{\varepsilon})^{-1}$$

$$X_2 = C^{-\sigma}Y(1-\phi_p\beta\Pi^{\varepsilon-1})^{-1}$$

(9) Flexible Prices

Impose $\phi_p = 0$. Then:

$$\Pi_t^f = \Pi_t^{*f} \longrightarrow v_t^{p,f} = \left(\frac{\Pi_t^*}{\Pi}\right)^{-\varepsilon} = 1 \longrightarrow Y_t^f = A_t L_t^f$$

$$\frac{X_{1,t}^f}{X_{2,t}^f} = \frac{\varepsilon-1}{\varepsilon}$$

$$X_{1,t}^f = (C_t^f)^{-\sigma}mc_t^f Y_t^f$$

$$X_{2,t}^f = (C_t^f)^{-\sigma}Y_t^f$$

$$mc_t^f = \frac{\varepsilon-1}{\varepsilon}$$

$$w_t^f = A_t\frac{\varepsilon-1}{\varepsilon}$$

$$\chi(L_t^f)^{\eta} = (Y_t^f)^{-\sigma}\frac{\varepsilon-1}{\varepsilon}A_t \longrightarrow \chi(L_t^f)^{\eta} = (A_t L_t^f)^{-\sigma}\frac{\varepsilon-1}{\varepsilon}A_t \longrightarrow L_t^f = \left[\frac{1}{\chi}\frac{\varepsilon-1}{\varepsilon}A_t^{1-\sigma}\right]^{\frac{1}{\sigma+\eta}}$$

$$Y_t^f = \left[\frac{1}{\chi}\frac{\varepsilon-1}{\varepsilon}\right]^{\frac{1}{\sigma+\eta}} A_t^{\frac{1+\eta}{\sigma+\eta}}$$

⑩ Log-Linearization

Start with the Euler Equation:

$$C_t^{-\sigma} = \beta E_t\left[C_{t+1}^{-\sigma}\Pi_{t+1}^{-1}(1+i_t)\right]$$ Take logs:

$$-\sigma \ln(C_t) = \ln\beta - \sigma E_t[\ln C_{t+1}] + i_t - E_t[\pi_{t+1}]$$ Use $C_t = Y_t$

$$-\sigma \ln(Y_t) = \ln\beta - \sigma E_t[\ln Y_{t+1}] + i_t - E_t[\pi_{t+1}]$$ Do a first-order Taylor Approximation:

$$-\sigma \frac{1}{y}(y_t - y) = -\sigma\frac{1}{y}\left(E_t[y_{t+1} - y]\right) + (i_t - \bar{i}) - E_t[\pi_{t+1} - \bar{\pi}]$$

$$-\sigma \tilde{y}_t = -\sigma E_t[\tilde{y}_{t+1}] + \tilde{i}_t - E_t[\tilde{\pi}_{t+1}]$$

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}\left(\tilde{i}_t - E_t[\tilde{\pi}_{t+1}]\right)$$ OIS

Now go to the labor supply condition

$$\chi L_t^{\eta} = C_t^{-\sigma} w_t$$ Use $Y_t = C_t$

$$\chi L_t^{\eta} = Y_t^{-\sigma} w_t$$ Log-linearize:

$$\eta \frac{1}{L}(L_t - L) = \frac{-\sigma}{y}(y_t - y) + \frac{1}{w}(w_t - w)$$

$$\eta \tilde{L}_t = -\sigma \tilde{y}_t + \tilde{w}_t$$

Next, go to real marginal cost:

$$mc_t = \frac{w_t}{A_t}$$ Log-linearize

$$\tilde{mc}_t = \tilde{w}_t - \tilde{A}_t$$

$$\tilde{w}_t = \tilde{mc}_t + \tilde{A}_t$$

Plug this into the labor supply condition:

$$\eta \tilde{L}_t = -\sigma \tilde{y}_t + \tilde{mc}_t + \tilde{A}_t$$

Now log-linearize the production function:

$$Y_t = \frac{A_t L_t}{v_t^p}$$ Log-linearize:

$$\tilde{y}_t = \tilde{A}_t + \tilde{L}_t - \tilde{v}_t^p$$

Go to the price dispersion term:

$$v_t^p = (1-\phi_p)\Pi_t^{*-\varepsilon}\Pi_t^{\varepsilon} + \Pi_t^{\varepsilon}\phi_p v_{t-1}^p$$ Take logs:

$$\ln(v_t^p) = \ln\left((1-\phi_p)\Pi_t^{*-\varepsilon}\Pi_t^{\varepsilon} + \Pi_t^{\varepsilon}\phi_p v_{t-1}^p\right)$$ Linearize:

$$\tilde{v}_t^p = \frac{1}{(1-\phi_p)\Pi^{*-\varepsilon}\Pi^{\varepsilon} + \Pi^{\varepsilon}\phi_p v^p}\left[-\varepsilon(1-\phi_p)\Pi^{*-\varepsilon-1}\Pi^{\varepsilon}(\pi_t^* - \pi^*) + \varepsilon(1-\phi_p)\Pi^{*-\varepsilon}\Pi^{\varepsilon-1}(\pi_t - \bar{\pi}) + \varepsilon\Pi^{\varepsilon-1}\phi_p(\pi_t - \bar{\pi}) + \Pi^{\varepsilon}\phi_p(v_{t-1}^p - v^p)\right]$$ Impose $\bar{\pi} = 0$. Then $\Pi^* = \Pi = 1 \rightarrow v^p = 1$. So:

$$\tilde{v}_t^p = -\varepsilon(1-\phi_p)\tilde{\pi}_t^* + \varepsilon(1-\phi_p)\tilde{\pi}_t + \varepsilon\phi_p\tilde{\pi}_t + \phi_p\tilde{v}_{t-1}^p$$

$$\tilde{v}_t^p = -\varepsilon(1-\phi_p)\tilde{\pi}_t^* + \varepsilon\tilde{\pi}_t + \phi_p \tilde{v}_{t-1}^p$$

Next, go to the real price dynamics

$$\Pi_t^{1-\varepsilon} = (1-\phi_p)\Pi_t^{*\,1-\varepsilon} + \phi_p \qquad \text{Log-linearize:}$$

$$(1-\varepsilon)\tilde{\pi}_t = \frac{1}{(1-\phi_p)\Pi^{*\,1-\varepsilon}+\phi_p}(1-\varepsilon)(1-\phi_p)\Pi^{*\,-\varepsilon}(\pi_t^* - \pi^*)$$

$$(1-\varepsilon)\tilde{\pi}_t = \Pi^{\varepsilon-1}(1-\varepsilon)(1-\phi_p)\Pi^{*\,-\varepsilon}(\pi_t^* - \pi^*)$$

$$(1-\varepsilon)\tilde{\pi}_t = (1-\varepsilon)(1-\phi_p)\tilde{\pi}_t^*$$

$$\tilde{\pi}_t = (1-\phi_p)\tilde{\pi}_t^*$$

Go back to the price dispersion equation:

$$\tilde{v}_t^p = -\varepsilon(1-\phi_p)\tilde{\pi}_t^* + \varepsilon(1-\phi_p)\tilde{\pi}_t^* + \phi_p \tilde{v}_{t-1}^p$$

$$\tilde{v}_t^p = \phi_p \tilde{v}_{t-1}^p \qquad \text{Impose initial steady-state:}$$

$$\tilde{v}_t^p = \phi_p(0) = 0. \qquad \text{Price dispersion, in a first-order approximation, is zero.}$$

Now to the production function again:

$$\tilde{y}_t = \tilde{A}_t + \tilde{L}_t \qquad \text{As } \tilde{v}_t^p = 0.$$

$$\tilde{L}_t = \tilde{y}_t - \tilde{A}_t$$

Back to the Labor Supply condition:

$$\eta(\tilde{y}_t - \tilde{A}_t) = -\sigma\tilde{y}_t + \widetilde{mc}_t + \tilde{A}_t$$

$$(\sigma+\eta)\tilde{y}_t - (1+\eta)\tilde{A}_t = \widetilde{mc}_t$$

Now log-linearize flexible output:

$$y_t^f = \left[\frac{1}{\chi}\frac{\varepsilon-1}{\varepsilon}\right]^{\frac{1}{\sigma+\eta}} A_t^{\frac{1+\eta}{\sigma+\eta}}$$

$$\tilde{y}_t^f = \frac{1+\eta}{\sigma+\eta}\tilde{A}_t$$

$$\tilde{A}_t = \frac{\sigma+\eta}{1+\eta}\tilde{y}_t^f$$

Plug this into the labor supply condition

$$(\sigma+\eta)\tilde{y}_t - (1+\eta)\frac{\sigma+\eta}{1+\eta}\tilde{y}_t^f = \widetilde{mc}_t$$

$$(\sigma+\eta)(\tilde{y}_t - \tilde{y}_t^f) = \widetilde{mc}_t$$

Next, move to real optimal reset price:

$$\Pi_t^* = \frac{\varepsilon}{\varepsilon-1}\Pi_t\frac{X_{1,t}}{X_{2,t}} \qquad \text{Log-linearize:}$$

$$\tilde{\pi}_t^* = \tilde{\pi}_t + \tilde{X}_{1t} - \tilde{X}_{2t}$$

Rewrite X_{1t}:

$$X_{1t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta E_t\left[\Pi_{t+1}^{\varepsilon} X_{1t+1}\right]$$

Use $Y_t = C_t$

$$X_{1t} = Y_t^{1-\sigma} mc_t + \phi_p \beta E_t\left[\Pi_{t+1}^{\varepsilon} X_{1t+1}\right]$$

Log-linearize

$$\tilde{X}_{1t} = \frac{1}{X_1}\left[(1-\sigma)Y^{-\sigma} mc(y_t - y) + Y^{1-\sigma}(mc_t - mc) + \phi_p \beta \varepsilon \Pi^{\varepsilon-1} X_1 E_t\left[\pi_{t+1} - \bar{\pi}\right] + \phi_p \beta \Pi^{\varepsilon} E_t\left[X_{1t+1} - X_1\right]\right]$$

$$\tilde{X}_{1t} = \frac{(1-\sigma)Y^{1-\sigma} mc}{Y X_1}(y_t - y) + \frac{Y^{1-\sigma}}{X_1 mc}(mc_t - mc) + \phi_p \beta \varepsilon \Pi^{\varepsilon-1} E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta \Pi^{\varepsilon} E_t\left[\tilde{X}_{1t+1}\right]$$

Impose $\Pi = 1$:

$$\tilde{X}_{1t} = \frac{(1-\sigma)Y^{1-\sigma} mc}{X_1}\tilde{y}_t + \frac{Y^{1-\sigma} mc}{X_1}\widetilde{mc}_t + \phi_p \beta \varepsilon E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{1t+1}\right]$$

Sub-in X_1:

$$\tilde{X}_{1t} = (1-\sigma)(1-\phi_p\beta)\tilde{y}_t + (1-\phi_p\beta)\widetilde{mc}_t + \phi_p \beta \varepsilon E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{1t+1}\right]$$

Rewrite X_{2t}:

$$X_{2t} = C_t^{-\sigma} Y_t + \phi_p \beta E_t\left[\Pi_{t+1}^{\varepsilon-1} X_{2t+1}\right]$$

Use $Y_t = C_t$

$$X_{2t} = Y_t^{1-\sigma} + \phi_p \beta E_t\left[\Pi_{t+1}^{\varepsilon-1} X_{2t+1}\right]$$

Log-linearize

$$\tilde{X}_{2t} = \frac{1}{X_2}\left[(1-\sigma)Y^{-\sigma}(y_t - y) + \phi_p \beta(\varepsilon-1)\Pi^{\varepsilon-2} X_2 E_t\left[\pi_{t+1} - \bar{\pi}\right] + \phi_p \beta \Pi^{\varepsilon-1} E_t\left[X_{2t+1} - X_2\right]\right]$$

$$\tilde{X}_{2t} = \frac{(1-\sigma)Y^{1-\sigma}}{X_2 Y}(y_t - y) + \phi_p \beta(\varepsilon-1) E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{2t+1}\right]$$

$$\tilde{X}_{2t} = \frac{(1-\sigma)Y^{1-\sigma}}{X_2}\tilde{y}_t + \phi_p \beta(\varepsilon-1) E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{2t+1}\right]$$

$$\tilde{X}_{2t} = (1-\sigma)(1-\phi_p\beta)\tilde{y}_t + \phi_p \beta(\varepsilon-1) E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{2t+1}\right]$$

Subtract $\tilde{X}_{2t}$ from $\tilde{X}_{1t}$:

$$\tilde{X}_{1t} - \tilde{X}_{2t} = (1-\sigma)(1-\phi_p\beta)\tilde{y}_t + (1-\phi_p\beta)\widetilde{mc}_t + \phi_p \beta \varepsilon E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{1t+1}\right] - (1-\sigma)(1-\phi_p\beta)\tilde{y}_t - \phi_p \beta(\varepsilon-1) E_t\left[\tilde{\pi}_{t+1}\right] - \phi_p \beta E_t\left[\tilde{X}_{2t+1}\right]$$

$$\tilde{X}_{1t} - \tilde{X}_{2t} = (1-\phi_p\beta)\widetilde{mc}_t + \phi_p \beta E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{1t+1} - \tilde{X}_{2t+1}\right]$$

From the optimal reset price:

$$\tilde{\pi}_t^* = \tilde{\pi}_t + \tilde{X}_{1t} - \tilde{X}_{2t} \longrightarrow \tilde{X}_{1t} - \tilde{X}_{2t} = \tilde{\pi}_t^* - \tilde{\pi}_t$$

Plug in the real price dynamics:

$$\tilde{X}_{1t} - \tilde{X}_{2t} = \frac{\tilde{\pi}_t}{1-\phi_p} - \tilde{\pi}_t$$

$$\tilde{X}_{1t} - \tilde{X}_{2t} = \frac{\phi_p}{1-\phi_p}\tilde{\pi}_t$$

Sub this into the difference in the log-linear auxiliary variables above:

$$\frac{\phi_p}{1-\phi_p}\tilde{\pi}_t = (1-\phi_p\beta)\widetilde{mc}_t + \phi_p \beta E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{X}_{1t+1} - \tilde{X}_{2t+1}\right]$$

Isolate $\tilde{\pi}_t$ and forward $\tilde{X}_{1t} - \tilde{X}_{2t}$:

$$\tilde{\pi}_t = \frac{(1-\phi_p\beta)(1-\phi_p)}{\phi_p}\widetilde{mc}_t + (1-\phi_p)\beta E_t\left[\tilde{\pi}_{t+1}\right] + \phi_p \beta E_t\left[\tilde{\pi}_{t+1}\right]$$

Sub-in for $\widetilde{mc}_t$:

$$\tilde{\pi}_t = \frac{(1-\phi_p\beta)(1-\phi_p)}{\phi_p}(\sigma+\eta)(\tilde{y}_t - \tilde{y}_t^f) + \beta E_t\left[\tilde{\pi}_{t+1}\right]$$

Let $\kappa = \frac{(1-\phi_p\beta)(1-\phi_p)}{\phi_p}(\sigma+\eta)$

$$\tilde{\pi}_t = \kappa(\tilde{y}_t - \tilde{y}_t^f) + \beta E_t\left[\tilde{\pi}_{t+1}\right]$$

NKPC

Now go to the technology process:

$$\ln(A_t) = \rho_a \ln(A_{t-1}) + \varepsilon_{a,t}$$ Linearize:

$$\tilde{A}_t = \rho_a \tilde{A}_{t-1} + \varepsilon_{a,t}$$ Productivity AR(1)

To eliminate $\tilde{A}_t$, go to flexible output:

$$\tilde{y}_t^F = \frac{1+\eta}{\sigma+\eta}\tilde{A}_t$$ Sub-in the AR(1) process

$$\tilde{y}_t^F = \frac{1+\eta}{\sigma+\eta}\rho_a \tilde{A}_{t-1} + \frac{1+\eta}{\sigma+\eta}\varepsilon_{a,t}$$

$$\tilde{y}_t^F = \rho_a \tilde{y}_{t-1}^F + \frac{1+\eta}{\sigma+\eta}\varepsilon_{a,t}$$ Flexible Output AR(1)

Lastly, linearize the NGDP Rule:

$$i_t = (1-\rho_i)\bar{i} + \rho_i i_{t-1} + (1-\rho_i)\phi_N(\pi_t + y_t - y_{t-1} - \alpha) + \varepsilon_{i,t}$$ (This is already in logs)

$$\tilde{i}_t = \rho_i(i_{t-1} - \bar{i}) + (1-\rho_i)\phi_N(\pi_t - \bar{\pi} + y_t - y - y_{t-1} + y) + \varepsilon_{i,t}$$

$$\tilde{i}_t = \rho_i \tilde{i}_{t-1} + (1-\rho_i)\phi_N(\tilde{\pi}_t + \tilde{y}_t - \tilde{y}_{t-1}) + \varepsilon_{i,t}$$ NGDP Rule

11 Three Equation New Keynesian Model

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\tilde{\pi}_{t+1}])$$ DIS

$$\tilde{\pi}_t = \kappa(\tilde{y}_t - \tilde{y}_t^F) + \beta E_t[\tilde{\pi}_{t+1}]$$ NKPC

$$\tilde{i}_t = \rho_i \tilde{i}_{t-1} + (1-\rho_i)\phi_N(\tilde{\pi}_t + \tilde{y}_t - \tilde{y}_{t-1}) + \varepsilon_{i,t}$$ NGDP Rule

$$\tilde{y}_t^F = \rho_a \tilde{y}_{t-1}^F + \frac{1+\eta}{\sigma+\eta}\varepsilon_{a,t}$$ Potential Output AR(1)

12 Solving the Model

Assume that $\rho_i = 0$ for simplicity. Then the model becomes:

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\tilde{\pi}_{t+1}])$$ DIS

$$\tilde{\pi}_t = \kappa(\tilde{y}_t - \tilde{y}_t^F) + \beta E_t[\tilde{\pi}_{t+1}]$$ NKPC

$$\tilde{i}_t = \phi_N(\tilde{\pi}_t + \tilde{y}_t - \tilde{y}_{t-1})$$ NGDP Rule

$$\tilde{y}_t^F = \rho_a \tilde{y}_{t-1}^F + \frac{1+\eta}{\sigma+\eta}\varepsilon_{a,t}$$ Potential Output AR(1)

Model Summary:

Jump variables: $\tilde{y}_t, \tilde{\pi}_t$

State variables: $\tilde{y}_{t-1}, \tilde{y}_t^f$

So, conjecture that the jump variables are linear in the state variables:

$$\tilde{y}_t = \lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f \qquad \tilde{\pi}_t = \lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f$$

Starting with the NKPC:

$$\tilde{\pi}_t = \varkappa(\tilde{y}_t - \tilde{y}_t^f) + \beta E_t[\tilde{\pi}_{t+1}]$$

$$\lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f = \varkappa(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f - \tilde{y}_t^f) + \beta E_t[\lambda_{2a}\tilde{y}_t + \lambda_{2b}\tilde{y}_{t+1}^f]$$

$$\lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f = \varkappa\lambda_{1a}\tilde{y}_{t-1} + \varkappa\lambda_{1b}\tilde{y}_t^f - \varkappa\tilde{y}_t^f + \beta\lambda_{2a}(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f) + \beta\lambda_{2b}\rho_a\tilde{y}_t^f$$

$$\underline{\lambda_{2a}\tilde{y}_{t-1}} + \underline{\lambda_{2b}\tilde{y}_t^f} - \underline{\varkappa\lambda_{1a}\tilde{y}_{t-1}} - \underline{\varkappa\lambda_{1b}\tilde{y}_t^f} + \underline{\varkappa\tilde{y}_t^f} - \underline{\beta\lambda_{2a}\lambda_{1a}\tilde{y}_{t-1}} - \underline{\beta\lambda_{2a}\lambda_{1b}\tilde{y}_t^f} - \underline{\beta\lambda_{2b}\rho_a\tilde{y}_t^f} = 0$$

$$[\lambda_{2a} - \varkappa\lambda_{1a} - \beta\lambda_{2a}\lambda_{1a}]\tilde{y}_{t-1} + [\lambda_{2b} - \varkappa\lambda_{1b} + \varkappa - \beta\lambda_{2a}\lambda_{1b} - \beta\lambda_{2b}\rho_a]\tilde{y}_t^f = 0$$

Therefore:

$$\lambda_{2a} - \varkappa\lambda_{1a} - \beta\lambda_{2a}\lambda_{1a} = 0 \qquad (1)$$

$$\lambda_{2b} - \varkappa\lambda_{1b} + \varkappa - \beta\lambda_{2a}\lambda_{1b} - \beta\lambda_{2b}\rho_a = 0 \qquad (2)$$

Now go to the DIS equation:

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\tilde{\pi}_{t+1}])$$

$$\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f = E_t[\lambda_{1a}\tilde{y}_t + \lambda_{1b}\tilde{y}_{t+1}^f] - \frac{1}{\sigma}(\phi_N\tilde{\pi}_t + \phi_N\tilde{y}_t - \phi_N\tilde{y}_{t-1} - E_t[\lambda_{2a}\tilde{y}_t + \lambda_{2b}\tilde{y}_{t+1}^f])$$

$$\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f = \lambda_{1a}(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f) + \lambda_{1b}\rho_a\tilde{y}_t^f - \frac{1}{\sigma}(\phi_N(\lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f) + \phi_N(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f) - \phi_N\tilde{y}_{t-1} - \lambda_{2a}(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f) - \lambda_{2b}\rho_a\tilde{y}_t^f)$$

$$\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f = \lambda_{1a}^2\tilde{y}_{t-1} + \lambda_{1a}\lambda_{1b}\tilde{y}_t^f + \lambda_{1b}\rho_a\tilde{y}_t^f - \frac{1}{\sigma}(\phi_N\lambda_{2a}\tilde{y}_{t-1} + \phi_N\lambda_{2b}\tilde{y}_t^f + \phi_N\lambda_{1a}\tilde{y}_{t-1} + \phi_N\lambda_{1b}\tilde{y}_t^f - \phi_N\tilde{y}_{t-1} - \lambda_{2a}\lambda_{1a}\tilde{y}_{t-1} - \lambda_{2a}\lambda_{1b}\tilde{y}_t^f - \lambda_{2b}\rho_a\tilde{y}_t^f)$$

$$\underline{\lambda_{1a}\tilde{y}_{t-1}} + \underline{\lambda_{1b}\tilde{y}_t^f} - \underline{\lambda_{1a}^2\tilde{y}_{t-1}} - \underline{\lambda_{1a}\lambda_{1b}\tilde{y}_t^f} - \underline{\lambda_{1b}\rho_a\tilde{y}_t^f} + \underline{\frac{\phi_N}{\sigma}\lambda_{2a}\tilde{y}_{t-1}} + \underline{\frac{\phi_N}{\sigma}\lambda_{2b}\tilde{y}_t^f} + \underline{\frac{\phi_N}{\sigma}\lambda_{1a}\tilde{y}_{t-1}} + \underline{\frac{\phi_N}{\sigma}\lambda_{1b}\tilde{y}_t^f} - \underline{\frac{\phi_N}{\sigma}\tilde{y}_{t-1}} - \underline{\frac{\lambda_{2a}\lambda_{1a}}{\sigma}\tilde{y}_{t-1}} - \underline{\frac{\lambda_{2a}\lambda_{1b}}{\sigma}\tilde{y}_t^f} - \underline{\frac{\lambda_{2b}\rho_a}{\sigma}\tilde{y}_t^f} = 0$$

$$\left[\lambda_{1a} - \lambda_{1a}^2 + \frac{\phi_N}{\sigma}\lambda_{2a} + \frac{\phi_N}{\sigma}\lambda_{1a} - \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}\lambda_{1a}}{\sigma}\right]\tilde{y}_{t-1} + \left[\lambda_{1b} - \lambda_{1a}\lambda_{1b} - \lambda_{1b}\rho_a + \frac{\phi_N}{\sigma}\lambda_{2b} + \frac{\phi_N}{\sigma}\lambda_{1b} - \frac{\lambda_{2a}\lambda_{1b}}{\sigma} - \frac{\lambda_{2b}\rho_a}{\sigma}\right]\tilde{y}_t^f = 0$$

Therefore:

$$\lambda_{1a} - \lambda_{1a}^2 + \frac{\phi_N}{\sigma}\lambda_{2a} + \frac{\phi_N}{\sigma}\lambda_{1a} - \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}\lambda_{1a}}{\sigma} = 0 \qquad (3)$$

$$\lambda_{1b} - \lambda_{1a}\lambda_{1b} - \lambda_{1b}\rho_a + \frac{\phi_N}{\sigma}\lambda_{2b} + \frac{\phi_N}{\sigma}\lambda_{1b} - \frac{\lambda_{2a}\lambda_{1b}}{\sigma} - \frac{\lambda_{2b}\rho_a}{\sigma} = 0 \qquad (4)$$

We have four unknowns in four equations. Starting with equation ①:

$$\lambda_{2a} - x\lambda_{1a} - \beta\lambda_{2a}\lambda_{1a} = 0$$

$$\lambda_{1a}(x + \beta\lambda_{2a}) = \lambda_{2a}$$

$$\lambda_{1a} = \frac{\lambda_{2a}}{x + \beta\lambda_{2a}} \qquad ⑤$$

Then moving to equation ③:

$$\lambda_{1a} - \lambda_{1a}^2 + \frac{\phi_N}{\sigma}\lambda_{2a} + \frac{\phi_N}{\sigma}\lambda_{1a} - \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}\lambda_{1a}}{\sigma} = 0$$

$$\frac{\lambda_{2a}}{x+\beta\lambda_{2a}} - \frac{\lambda_{2a}^2}{(x+\beta\lambda_{2a})^2} + \frac{\phi_N}{\sigma}\lambda_{2a} + \frac{\phi_N}{\sigma}\frac{\lambda_{2a}}{(x+\beta\lambda_{2a})} - \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^2}{\sigma(x+\beta\lambda_{2a})} = 0$$

Here, an analytical solution does not exist. I parameterize the model at this point, using $x = 0.1717$, $\phi_N = 1.5$, $\sigma = 1$, and $\beta = 0.99$. λ_{2a} becomes approximately $\frac{1}{3}$. Continuing onward, I use λ_{2a}^* to designate the optimal value of λ_{2a}.

Plugging λ_{2a}^* into equation ⑤:

$$\lambda_{1a} = \frac{\lambda_{2a}^*}{x + \beta\lambda_{2a}^*}$$

Moving to equation ②:

$$\lambda_{2b} - x\lambda_{1b} + x - \beta\lambda_{2a}\lambda_{1b} - \beta\lambda_{2b}p_a = 0$$

$$\lambda_{2b} - x\lambda_{1b} + x - \beta\lambda_{2a}^*\lambda_{1b} - \beta\lambda_{2b}p_a = 0$$

$$\lambda_{2b}(1 - \beta p_a) = \lambda_{1b}(x + \beta\lambda_{2a}^*) - x$$

$$\lambda_{2b} = \lambda_{1b}\left(\frac{x + \beta\lambda_{2a}^*}{1 - \beta p_a}\right) - \frac{x}{1 - \beta p_a} \qquad ⑥$$

Now sub-in to equation ④:

$$\lambda_{1b} - \lambda_{1a}\lambda_{1b} - \lambda_{1b}p_a + \frac{\phi_N}{\sigma}\lambda_{2b} + \frac{\phi_N}{\sigma}\lambda_{1b} - \frac{\lambda_{2a}\lambda_{1b}}{\sigma} - \frac{\lambda_{2b}p_a}{\sigma} = 0$$

$$\lambda_{1b} - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*}\lambda_{1b} - p_a\lambda_{1b} + \frac{\phi_N}{\sigma}\left[\lambda_{1b}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right) - \frac{x}{1-\beta p_a}\right] + \frac{\phi_N}{\sigma}\lambda_{1b} - \lambda_{2a}^*\frac{\lambda_{1b}}{\sigma} - \frac{p_a}{\sigma}\left[\lambda_{1b}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right) - \frac{x}{1-\beta p_a}\right] = 0$$

$$\lambda_{1b} - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*}\lambda_{1b} - p_a\lambda_{1b} + \frac{\phi_N}{\sigma}\lambda_{1b}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right) - \frac{\phi_N x}{1-\beta p_a} + \frac{\phi_N}{\sigma}\lambda_{1b} - \frac{\lambda_{2a}^*}{\sigma}\lambda_{1b} - \frac{p_a}{\sigma}\lambda_{1b}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right) + \frac{p_a x}{\sigma(1-\beta p_a)} = 0$$

$$\lambda_{1b}\left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - p_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{p_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right)\right] = \frac{x}{1-\beta p_a}\left[\phi_N - p_a\right]$$

$$\lambda_{1b} = \left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - p_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{p_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right)\right]^{-1}\frac{x}{1-\beta p_a}(\phi_N - p_a)$$

Designate the optimal λ_{1b} as λ_{1b}^*. Sub this into equation ⑥:

$$\lambda_{2b} = \lambda_{1b}^*\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta p_a}\right) - \frac{x}{1-\beta p_a}$$

We have verified that the policy functions are linear in the state variables.

$$\tilde{y}_t = \lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f$$

$$\tilde{y}_t = \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*}\tilde{y}_{t-1} + \left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - \rho_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{\rho_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1}\frac{x}{1-\beta\rho_a}(\phi_N - \rho_a)\tilde{y}_t^f$$

$$\tilde{\pi}_t = \lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f$$

$$\tilde{\pi}_t = \lambda_{2a}^*\tilde{y}_{t-1} + \lambda_{1b}^*\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\tilde{y}_t^f - \frac{x}{1-\beta\rho_a}\tilde{y}_t^f$$

Now we can plug this into the NGDP Rule:

$$\tilde{i}_t = \phi_N(\tilde{\pi}_t + \tilde{y}_t - \tilde{y}_{t-1})$$

$$\tilde{i}_t = \phi_N(\lambda_{2a}^*\tilde{y}_{t-1} + \lambda_{2b}^*\tilde{y}_t^f + \lambda_{1a}^*\tilde{y}_{t-1} + \lambda_{1b}^*\tilde{y}_t^f - \tilde{y}_{t-1})$$

$$= \phi_N\left[\underbrace{(\lambda_{2a}^* + \lambda_{1a}^* - 1)}_{\equiv A}\tilde{y}_{t-1} + \underbrace{(\lambda_{2b}^* + \lambda_{1b}^*)}_{\equiv B}\tilde{y}_t^f\right]$$

Go back to the equation for λ_{2a}. Impose $\sigma = 1$. Then:

$$\frac{\lambda_{2a}}{x+\beta\lambda_{2a}} - \frac{\lambda_{2a}^2}{(x+\beta\lambda_{2a})^2} + \phi_N\lambda_{2a} + \frac{\phi_N\lambda_{2a}}{x+\beta\lambda_{2a}} - \phi_N - \frac{\lambda_{2a}^2}{x+\beta\lambda_{2a}} = 0$$

$$\frac{1}{(x+\beta\lambda_{2a})^2}\left[\lambda_{2a}(x+\beta\lambda_{2a}) - \lambda_{2a}^2 + \phi_N\lambda_{2a}(x+\beta\lambda_{2a}) + \phi_N\lambda_{2a}(x+\beta\lambda_{2a}) - \phi_N(x+\beta\lambda_{2a})^2 - \lambda_{2a}^2(x+\beta\lambda_{2a})\right] = 0$$

$$\rightarrow \lambda_{2a}^* = \frac{\sqrt{\beta^2 + 2\beta(x-1) + (x+1)^2} + \beta - x - 1}{2\beta}$$

(from Wolfram Alpha - there are two other solutions that do not satisfy equations ① - ④)

Now solve A :

$$\lambda_{2a}^* + \lambda_{1a}^* - 1 = \frac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1}{2\beta} + \frac{\frac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1}{2\beta}}{x + \frac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1}{2}} - 1$$

$$= \frac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1}{2\beta} + \frac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1}{2\beta}\frac{2}{2x + \sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1} - 1$$

$$= \frac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1}{2\beta} + \frac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1}{\beta(x + \sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - 1)} - 1$$

$$= \sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1 + \frac{2\left(\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1\right)}{x + \sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - 1} - 2\beta$$

$$= \left(\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1\right)\left(x + \sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - 1\right) + 2\left(\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - x - 1\right) - 2\beta\left(x + \sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + \beta - 1\right)$$

$$= \cancel{x\sqrt{\beta^2+2\beta(x-1)+(x+1)^2}} + x(\beta - x - 1) + \beta^2 + 2\beta(x-1) + (x+1)^2 + (\cancel{\beta - x - 1})\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + (\cancel{\beta - 1})\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} + (\beta - x - 1)(\beta - 1)$$

$$\cancel{+ 2\sqrt{\beta^2+2\beta(x-1)+(x+1)^2}} + 2(\beta - x - 1) - 2\beta(\beta + x - 1) \cancel{- 2\beta\sqrt{\beta^2+2\beta(x-1)+(x+1)^2}}$$

$$= \cancel{x\beta} \cancel{- x^2} \cancel{- x} \cancel{+ \beta^2} \cancel{+ 2\beta x} \cancel{- 2\beta} \cancel{+ x^2} \cancel{+ 2x} \cancel{+ 1} \cancel{+ \beta^2} \cancel{- \beta} \cancel{- x\beta} \cancel{+ x} \cancel{- \beta} \cancel{+ 1} \cancel{+ 2\beta} \cancel{- 2x} \cancel{- 2} \cancel{- 2\beta^2} \cancel{- 2\beta x} \cancel{+ 2\beta}$$

$$= 0$$

Next, solve B:

$$\lambda_{2b}^* + \lambda_{1b}^* = \lambda_{1b}^*\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) - \frac{x}{1-\beta\rho_a} + \left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - \rho_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{\rho_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1} \frac{x}{1-\beta\rho_a}(\phi_N - \rho_a)$$

$$= \left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - \rho_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{\rho_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1} \frac{x}{1-\beta\rho_a}(\phi_N - \rho_a)\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a} - \frac{x}{1-\beta\rho_a}$$

$$+ \left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - \rho_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{\rho_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1} \frac{x}{1-\beta\rho_a}(\phi_N - \rho_a)$$

$$= \underbrace{\left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - \rho_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{\rho_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1}}_{\equiv C} \underbrace{\frac{x}{1-\beta\rho_a}}_{\equiv D} \underbrace{(\phi_N - \rho_a)\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a} + 1\right)}_{\equiv E} \underbrace{- \frac{x}{1-\beta\rho_a}}_{\equiv F}$$

Focus on C:

$$\left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - \rho_a + \frac{\phi_N}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}^*}{\sigma} - \frac{\rho_a}{\sigma}\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1}$$ Recall that $\sigma = 1$:

$$\left[1 - \frac{\lambda_{2a}^*}{x+\beta\lambda_{2a}^*} - \rho_a + \phi_N\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \phi_N - \lambda_{2a}^* - \rho_a\left(\frac{x+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1}$$

$$\left[\frac{1}{x+\beta\lambda_{2a}^*}\left(x+\beta\lambda_{2a}^* - \lambda_{2a}^* - \rho_a(x+\beta\lambda_{2a}^*) + \phi_N\left(\frac{(x+\beta\lambda_{2a}^*)^2}{1-\beta\rho_a}\right) + \phi_N(x+\beta\lambda_{2a}^*) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*) - \rho_a\left(\frac{(x+\beta\lambda_{2a}^*)^2}{1-\beta\rho_a}\right)\right)\right]^{-1}$$

$$\left[\frac{1}{(x+\beta\lambda_{2a}^*)(1-\beta\rho_a)}\left((x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) + \phi_N(x+\beta\lambda_{2a}^*)^2 + \phi_N(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)^2\right)\right]^{-1}$$

$$(x+\beta\lambda_{2a}^*)(1-\beta\rho_a)\frac{1}{(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) + \phi_N(x+\beta\lambda_{2a}^*)^2 + \phi_N(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)^2}$$

Multiply by D:

$$x(x+\beta\lambda_{2a}^*)\frac{1}{(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) + \phi_N(x+\beta\lambda_{2a}^*)^2 + \phi_N(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)^2}$$

Multiply by E:

$$\frac{x(x+\beta\lambda_{2a}^*)^2(1-\beta\rho_a)^{-1}(\phi_N - \rho_a)}{(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) + \phi_N(x+\beta\lambda_{2a}^*)^2 + \phi_N(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)^2}$$

$$+ \frac{x(x+\beta\lambda_{2a}^*)(\phi_N - \rho_a)}{(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) + \phi_N(x+\beta\lambda_{2a}^*)^2 + \phi_N(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)^2}$$

Give F a common denominator and subtract it:

$$\frac{x(x+\beta\lambda_{2a}^*)^2(1-\beta\rho_a)^{-1}(\phi_N - \rho_a) + x(x+\beta\lambda_{2a}^*)(\phi_N - \rho_a) - x(1-\beta\rho_a)^{-1}\left((x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) + \phi_N(x+\beta\lambda_{2a}^*)^2 + \phi_N(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)^2\right)}{(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) + \phi_N(x+\beta\lambda_{2a}^*)^2 + \phi_N(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \lambda_{2a}^*(x+\beta\lambda_{2a}^*)(1-\beta\rho_a) - \rho_a(x+\beta\lambda_{2a}^*)^2}$$

Working just with the numerator:

$$\chi(\chi+\beta\lambda^*_{2a})^2(1-\beta\rho_a)^{-1}(\phi_\pi-\rho_a)+\chi(\chi+\beta\lambda^*_{2a})(\phi_\pi-\rho_a)-\chi(1-\beta\rho_a)^{-1}\Big((\chi+\beta\lambda^*_{2a})(1-\beta\rho_a)-\lambda^*_{2a}(1-\beta\rho_a)-\rho_a(\chi+\beta\lambda^*_{2a})(1-\beta\rho_a)+\phi_\pi(\chi+\beta\lambda^*_{2a})^2+\phi_\pi(\chi+\beta\lambda^*_{2a})(1-\beta\rho_a)-\lambda^*_{2a}(\chi+\beta\lambda^*_{2a})(1-\beta\rho_a)-\rho_a(\chi+\beta\lambda^*_{2a})^2\Big)$$

$$\chi(\chi^2+2\chi\beta\lambda^*_{2a}+\beta^2\lambda^{*2}_{2a})(1-\beta\rho_a)^{-1}(\phi_\pi-\rho_a)+(\chi^2+\beta\chi\lambda^*_{2a})(\phi_\pi-\rho_a)-\chi(\chi+\beta\lambda^*_{2a})+\chi\lambda^*_{2a}+\chi\rho_a(\chi+\beta\lambda^*_{2a})-\chi\phi_\pi(\chi+\beta\lambda^*_{2a})^2(1-\beta\rho_a)^{-1}-\chi\phi_\pi(\chi+\beta\lambda^*_{2a})+\chi\lambda^*_{2a}(\chi+\beta\lambda^*_{2a})+\rho_a\chi(\chi+\beta\lambda^*_{2a})^2(1-\beta\rho_a)^{-1}$$

$$\cancel{(\chi^3+2\chi^2\beta\lambda^*_{2a}+\beta^2\chi\lambda^{*2}_{2a})(1-\beta\rho_a)^{-1}(\phi_\pi-\rho_a)}+\phi_\pi\chi^2+\beta\phi_\pi\chi\lambda^*_{2a}-\rho_a\chi^2-\beta\rho_a\chi\lambda^*_{2a}-\chi^2-\chi\beta\lambda^*_{2a}+\chi\lambda^*_{2a}+\chi^2\rho_a+\chi\rho_a\beta\lambda^*_{2a}\cancel{-\chi\phi_\pi(\chi^2+2\beta\chi\lambda^*_{2a}+\beta^2\lambda^{*2}_{2a})(1-\beta\rho_a)^{-1}}-\chi^2\phi_\pi-\chi\phi_\pi\beta\lambda^*_{2a}+\chi^2\lambda^*_{2a}+\chi\beta\lambda^{*2}_{2a}\cancel{+\rho_a\chi(\chi^2+2\chi\beta+\beta^2\lambda^{*2}_{2a})(1-\beta\rho_a)^{-1}}$$

$$=\phi_\pi\chi^2+\beta\phi_\pi\chi\lambda^*_{2a}-\rho_a\chi^2-\beta\rho_a\chi\lambda^*_{2a}-\chi^2-\chi\beta\lambda^*_{2a}+\chi\lambda^*_{2a}+\chi^2\rho_a+\chi\rho_a\beta\lambda^*_{2a}-\chi^2\phi_\pi-\chi\phi_\pi\beta\lambda^*_{2a}+\chi^2\lambda^*_{2a}+\chi\beta\lambda^{*2}_{2a}$$

$$=(\cancel{\phi_\pi\chi^2}\cancel{-\rho_a\chi^2}-\chi^2\cancel{+\rho_a\chi^2}\cancel{-\chi^2\phi_\pi})+(\cancel{\beta\phi_\pi\chi}\cancel{-\beta\rho_a\chi}-\chi\beta+\chi\cancel{+\chi\rho_a\beta}\cancel{-\chi\phi_\pi\beta}+\chi^2)\lambda^*_{2a}+\chi\beta\lambda^{*2}_{2a}$$

$$=\chi\beta\lambda^{*2}_{2a}+(\chi^2-\chi\beta+\chi)\lambda^*_{2a}-\chi^2$$

$$=\chi\beta\left(\frac{\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\beta-\chi-1}{2\beta}\right)^2+(\chi^2-\chi\beta+\chi)\left(\frac{\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\beta-\chi-1}{2\beta}\right)-\chi^2$$

$$=\frac{\chi}{4\beta}\Big[\beta^2+2\beta(\chi-1)+(\chi+1)^2+2(\beta-\chi-1)\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+(\beta-\chi-1)^2\Big]+\frac{\chi^2}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{\chi}{2}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}$$
$$+\frac{\chi}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\frac{1}{2\beta}(\beta-\chi-1)(\chi^2-\chi\beta+\chi)-\chi^2$$

$$=\frac{\chi\beta}{4}+\frac{\chi}{2}(\chi-1)+\frac{\chi(\chi+1)^2}{4\beta}+\frac{2(\beta-\chi-1)\chi}{4\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\frac{(\beta^2-\beta\chi-\beta-\beta\chi+\chi^2+\chi-\beta+\chi+1)\chi}{4\beta}+\frac{\chi^2}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{\chi}{2}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}$$
$$+\frac{\chi}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\frac{1}{2\beta}(\beta-\chi-1)(\chi^2-\chi\beta+\chi)-\chi^2$$

$$=\frac{\chi\beta}{4}+\cancel{\frac{\chi^2}{2}}-\frac{\chi}{2}+\frac{\chi(\chi^2+2\chi+1)}{4\beta}+\frac{2\chi}{4}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{2\chi^2}{4\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{2\chi}{4\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}$$
$$+\frac{\chi\beta}{4}\cancel{-\frac{\chi^2}{4}}-\frac{\chi}{4}\cancel{-\frac{\chi^2}{4}}+\frac{\chi^3}{4\beta}+\frac{\chi^2}{4\beta}-\frac{\chi}{4}+\frac{\chi^2}{4\beta}+\frac{\chi}{4\beta}+\frac{\chi^2}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{\chi}{2}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}$$
$$+\frac{\chi}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\frac{1}{2\beta}(\beta\chi^2-\chi\beta^2+\beta\chi-\chi^3+\chi^2\beta-\chi^2-\chi^2+\chi\beta-\chi)-\chi^2$$

$$=\frac{\chi\beta}{4}-\frac{\chi}{2}+\frac{\chi^3}{4\beta}+\frac{2\chi^2}{4\beta}+\frac{\chi}{4\beta}+\frac{2\chi}{4}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{2\chi^2}{4\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{2\chi}{4\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\frac{\chi\beta}{4}-\frac{\chi}{4}$$
$$+\frac{\chi^3}{4\beta}+\frac{\chi^2}{4\beta}-\frac{\chi}{4}+\frac{\chi^2}{4\beta}+\frac{\chi}{4\beta}+\frac{\chi^2}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{\chi}{2}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\frac{\chi}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}$$
$$+\frac{\chi^2}{2}-\frac{\chi\beta}{2}+\frac{\chi}{2}-\frac{\chi^3}{2\beta}+\frac{\chi^2}{2}-\frac{\chi^2}{\beta}+\frac{\chi}{2}-\frac{\chi}{2\beta}-\chi^2$$

$$=\cancel{\frac{\chi\beta}{4}}\cancel{-\frac{\chi}{2}}\cancel{+\frac{\chi^3}{4\beta}}\cancel{+\frac{2\chi^2}{4\beta}}\cancel{+\frac{\chi}{4\beta}}+\frac{2\chi}{4}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{2\chi^2}{4\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{2\chi}{4\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}\cancel{+\frac{\chi\beta}{4}}\cancel{-\frac{\chi}{4}}$$
$$\cancel{+\frac{\chi^3}{4\beta}}\cancel{+\frac{\chi^2}{4\beta}}\cancel{-\frac{\chi}{4}}\cancel{+\frac{\chi^2}{4\beta}}\cancel{+\frac{\chi}{4\beta}}+\frac{\chi^2}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}-\frac{\chi}{2}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}+\frac{\chi}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}$$
$$\cancel{+\chi^2}\cancel{-\frac{\chi\beta}{2}}\cancel{+\chi}\cancel{-\frac{\chi^3}{2\beta}}\cancel{-\frac{\chi^2}{\beta}}\cancel{-\frac{\chi}{2\beta}}\cancel{-\chi^2}$$

$$=\cancel{\frac{\chi}{2}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}}\cancel{-\frac{\chi^2}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}}\cancel{-\frac{\chi}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}}$$
$$\cancel{+\frac{\chi^2}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}}\cancel{-\frac{\chi}{2}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}}\cancel{+\frac{\chi}{2\beta}\sqrt{\beta^2+2\beta(\chi-1)+(\chi+1)^2}}$$

$$=0$$

So $\tilde{i}_t=\phi_\pi(0\cdot\tilde{y}_{t-1}+0\cdot\tilde{y}^f_t)=0$

$$\tilde{y}_t = \lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f + \theta_1 u_t \qquad \tilde{\pi}_t = \lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f + \theta_2 u_t$$

NKPC: $\tilde{\pi}_t = \varkappa(\tilde{y}_t - \tilde{y}_t^f) + \beta E_t[\tilde{\pi}_{t+1}] + u_t$

$$\lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f + \theta_2 u_t = \varkappa(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f + \theta_1 u_t - \tilde{y}_t^f) + \beta E_t[\lambda_{2a}\tilde{y}_t + \lambda_{2b}\tilde{y}_{t+1}^f + \theta_1 u_{t+1}]$$

$$\lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f + \theta_2 u_t = \varkappa\lambda_{1a}\tilde{y}_{t-1} + \varkappa\lambda_{1b}\tilde{y}_t^f - \varkappa\tilde{y}_t^f + \theta_1\varkappa u_t + \beta\lambda_{2a}(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f + \theta_1 u_t) + \beta\lambda_{2b}\rho_a\tilde{y}_t^f$$

$$\lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f - \varkappa\lambda_{1a}\tilde{y}_{t-1} - \varkappa\lambda_{1b}\tilde{y}_t^f + \varkappa\tilde{y}_t^f - \beta\lambda_{2a}\lambda_{1a}\tilde{y}_{t-1} - \beta\lambda_{2a}\lambda_{1b}\tilde{y}_t^f - \beta\lambda_{2b}\rho_a\tilde{y}_t^f + \theta_2 u_t - \theta_1\varkappa u_t - \beta\theta_1\lambda_{2a}u_t = 0$$

$$[\lambda_{2a} - \varkappa\lambda_{1a} - \beta\lambda_{2a}\lambda_{1a}]\tilde{y}_{t-1} + [\lambda_{2b} - \varkappa\lambda_{1b} + \varkappa - \beta\lambda_{2a}\lambda_{1b} - \beta\lambda_{2b}\rho_a]\tilde{y}_t^f + [\theta_2 - \theta_1\varkappa - \beta\theta_1\lambda_{2a} - 1]u_t = 0$$

Now go to the DIS equation:

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(i_t - E_t[\tilde{\pi}_{t+1}])$$

$$\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f + \theta_1 u_t = E_t[\lambda_{1a}\tilde{y}_t + \lambda_{1b}\tilde{y}_{t+1}^f] - \frac{1}{\sigma}(\phi_N\tilde{\pi}_t + \phi_N\tilde{y}_t - \phi_N\tilde{y}_{t-1} - E_t[\lambda_{2a}\tilde{y}_t + \lambda_{2b}\tilde{y}_{t+1}^f])$$

$+\theta_2 u_t$ $\quad +\theta_1 u_t$ $\quad +\theta_1 u_t$

$$\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f + \theta_1 u_t = \lambda_{1a}(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f + \theta_1 u_t) + \lambda_{1b}\rho_a\tilde{y}_t^f - \frac{1}{\sigma}(\phi_N(\lambda_{2a}\tilde{y}_{t-1} + \lambda_{2b}\tilde{y}_t^f) + \phi_N(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f) - \phi_N\tilde{y}_{t-1} - \lambda_{2a}(\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f) - \lambda_{2b}\rho_a\tilde{y}_t^f)$$

$$\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f = \lambda_{1a}^2\tilde{y}_{t-1} + \lambda_{1a}\lambda_{1b}\tilde{y}_t^f + \lambda_{1b}\rho_a\tilde{y}_t^f - \frac{1}{\sigma}(\phi_N\lambda_{2a}\tilde{y}_{t-1} + \phi_N\lambda_{2b}\tilde{y}_t^f + \phi_N\lambda_{1a}\tilde{y}_{t-1} + \phi_N\lambda_{1b}\tilde{y}_t^f - \phi_N\tilde{y}_{t-1} - \lambda_{2a}\lambda_{1a}\tilde{y}_{t-1} - \lambda_{2a}\lambda_{1b}\tilde{y}_t^f - \lambda_{2b}\rho_a\tilde{y}_t^f)$$

$$\lambda_{1a}\tilde{y}_{t-1} + \lambda_{1b}\tilde{y}_t^f - \lambda_{1a}^2\tilde{y}_{t-1} - \lambda_{1a}\lambda_{1b}\tilde{y}_t^f - \lambda_{1b}\rho_a\tilde{y}_t^f + \frac{\phi_N}{\sigma}\lambda_{2a}\tilde{y}_{t-1} + \frac{\phi_N}{\sigma}\lambda_{2b}\tilde{y}_t^f + \frac{\phi_N}{\sigma}\lambda_{1a}\tilde{y}_{t-1} + \frac{\phi_N}{\sigma}\lambda_{1b}\tilde{y}_t^f - \frac{\phi_N}{\sigma}\tilde{y}_{t-1} - \frac{\lambda_{2a}\lambda_{1a}}{\sigma}\tilde{y}_{t-1} - \frac{\lambda_{2a}\lambda_{1b}}{\sigma}\tilde{y}_t^f - \frac{\lambda_{2b}\rho_a}{\sigma}\tilde{y}_t^f = 0$$

$$\left[\lambda_{1a} - \lambda_{1a}^2 + \frac{\phi_N}{\sigma}\lambda_{2a} + \frac{\phi_N}{\sigma}\lambda_{1a} - \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}\lambda_{1a}}{\sigma}\right]\tilde{y}_{t-1} + \left[\lambda_{1b} - \lambda_{1a}\lambda_{1b} - \lambda_{1b}\rho_a + \frac{\phi_N}{\sigma}\lambda_{2b} + \frac{\phi_N}{\sigma}\lambda_{1b} - \frac{\lambda_{2a}\lambda_{1b}}{\sigma} - \frac{\lambda_{2b}\rho_a}{\sigma}\right]\tilde{y}_t^f = 0$$

Therefore:

$$\lambda_{1a} - \lambda_{1a}^2 + \frac{\phi_N}{\sigma}\lambda_{2a} + \frac{\phi_N}{\sigma}\lambda_{1a} - \frac{\phi_N}{\sigma} - \frac{\lambda_{2a}\lambda_{1a}}{\sigma} = 0 \qquad (3)$$

$$\lambda_{1b} - \lambda_{1a}\lambda_{1b} - \lambda_{1b}\rho_a + \frac{\phi_N}{\sigma}\lambda_{2b} + \frac{\phi_N}{\sigma}\lambda_{1b} - \frac{\lambda_{2a}\lambda_{1b}}{\sigma} - \frac{\lambda_{2b}\rho_a}{\sigma} = 0 \qquad (4)$$

$$\theta_1 u_t - \lambda_{1a}\theta_1 u_t + \phi_N\theta_2 u_t + \phi_N\theta_1 u_t - \lambda_{2a}\theta_1 u_t = 0$$

$\lambda_{1a} + \lambda_{2a} = 1 \rightarrow \lambda_{1a} = 1 - \lambda_{2a}$

$$(\theta_1 - \lambda_{1a}\theta_1 + \phi_N\theta_2 + \phi_N\theta_1 - \lambda_{2a}\theta_1)u_t = 0$$

From above, we know $\lambda_{1a} = 1 - \lambda_{2a}$

$$(\theta_1 + \phi_N\theta_2 + \phi_N\theta_1 - (\lambda_{1a} + \lambda_{2a})\theta_1)u_t = 0$$

$$(\theta_1 + \phi_N\theta_2 + \phi_N\theta_1 - \theta_1)u_t = 0$$

$$\theta_1 = -\theta_2$$

$$\theta_2 - \theta_1\varkappa - \beta\theta_1\lambda_{2a} - 1$$

$$-\theta_1 - \theta_1\varkappa - \beta\theta_1\lambda_{2a} = 1 \rightarrow \theta_1 + \theta_1\varkappa + \beta\theta_1\lambda_{2a} = -1$$

$$\theta_1(1 - \varkappa - \beta\lambda_{2a}) = -1 \qquad \theta_1(1 + \varkappa + \beta\lambda_{2a}) = -1$$

$$\theta_1^* = \frac{-1}{1 + \varkappa + \beta\lambda_{2a}^*}$$

$$\theta_2^* = \frac{+1}{1 + \varkappa + \beta\lambda_{2a}^*}$$

$$\tilde{i}_t = \phi_N(\tilde{\pi}_t + \tilde{y}_t - \tilde{y}_{t-1})$$

$$= \phi_N(\lambda_{2a}^*\tilde{y}_{t-1} + \lambda_{2b}^*\tilde{y}_t^f + \theta_2^* u_t + \lambda_{1a}^*\tilde{y}_{t-1} + \lambda_{1b}^*\tilde{y}_t^f + \theta_1^* u_t - \tilde{y}_{t-1})$$

$$= \phi_N((\cancelto{0}{\lambda_{2a}^* + \lambda_{1a}^* - 1})\tilde{y}_{t-1} + (\cancelto{0}{\lambda_{2b}^* + \lambda_{1b}^*})\tilde{y}_t^f + (\theta_2^* + \theta_1^*)u_t)$$

$$= \phi_N(\theta_2^* + \theta_1^*)u_t$$

$$= \phi_N(-\theta_1^* + \theta_1^*)u_t$$

$$= 0$$

(13) Intuition?

Start with the OIS equation:

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(\tilde{i}_t - E_t[\tilde{\pi}_{t+1}])$$

Use the fact that $\tilde{\pi}_t = \ln(\Pi_t) = \ln\left(\frac{P_t}{P_{t-1}}\right) = \ln(P_t) - \ln(P_{t-1})$

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}(\tilde{i}_t - E_t[\tilde{p}_{t+1}] + \tilde{p}_t)$$

Impose $\sigma = 1$ and solve for $\tilde{i}_t$

$$\tilde{i}_t = E_t[\tilde{y}_{t+1}] - \tilde{y}_t + E_t[\tilde{p}_{t+1}] - \tilde{p}_t$$

$$\tilde{i}_t = \Big[E_t[\tilde{y}_{t+1}] + E_t[\tilde{p}_{t+1}]\Big] - \Big[\tilde{y}_t + \tilde{p}_t\Big]$$

The nominal rate in the growth rate of NGDP. Rewrite with inflation:

$$\tilde{i}_t = E_t[\tilde{y}_{t+1}] - \tilde{y}_t + E_t[\tilde{\pi}_{t+1}]$$

Consider the NGDP targeting rule though:

$$\tilde{i}_t = \phi_N(\tilde{\pi}_t + \tilde{y}_t - \tilde{y}_{t-1}).$$ Forward this:

$$E_t[\tilde{i}_{t+1}] = \phi_N\big(E_t[\tilde{\pi}_{t+1}] + E_t[\tilde{y}_{t+1}] - y_t\big)$$

$$E_t[\tilde{i}_{t+1}] = \phi_N \tilde{i}_t.$$

For a solution to exist, $\phi_N > 1$. But if $\phi_N > 1$, $E_t[\tilde{i}_{t+1}]$ explodes unless $\tilde{i}_t = 0$.

Therefore, for an equilibrium to exist, $\tilde{i}_t = 0$.

Textbook New Keynesian Model

Assume that $\rho_i = 0$ for simplicity. Then the model becomes:

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}\left(\tilde{i}_t - E_t[\tilde{\pi}_{t+1}]\right) \qquad \text{DIS}$$

$$\tilde{\pi}_t = \kappa(\tilde{y}_t - \tilde{y}_t^f) + \beta E_t[\tilde{\pi}_{t+1}] \qquad \text{NKPC}$$

$$\tilde{i}_t = \phi_\pi \tilde{\pi}_t \qquad \text{Inflation Targeting}$$

$$\tilde{y}_t^f = \rho_a \tilde{y}_{t-1}^f + \frac{1+\eta}{\sigma+\eta}\varepsilon_{a,t} \qquad \text{Potential Output AR(1)}$$

Conjecture that the jump variables, $\tilde{y}_t$ and $\tilde{\pi}_t$, are linear in the state variable $\tilde{y}_t^f$:

$$\tilde{y}_t = \lambda_1 \tilde{y}_t^f \qquad \tilde{\pi}_t = \lambda_2 \tilde{y}_t^f$$

Starting with the NKPC:

$$\tilde{\pi}_t = \kappa(\tilde{y}_t - \tilde{y}_t^f) + \beta E_t[\tilde{\pi}_{t+1}]$$

$$\lambda_2 \tilde{y}_t^f = \kappa(\lambda_1 \tilde{y}_t^f - \tilde{y}_t^f) + \beta E_t[\lambda_2 \tilde{y}_{t+1}^f]$$

$$\lambda_2 \tilde{y}_t^f = \kappa \tilde{y}_t^f(\lambda_1 - 1) + \beta \lambda_2 \rho_a \tilde{y}_t^f$$

$$\lambda_2 \tilde{y}_t^f - \kappa(\lambda_1 - 1)\tilde{y}_t^f - \beta\rho_a\lambda_2\tilde{y}_t^f = 0$$

$$\left[\lambda_2 - \kappa(\lambda_1 - 1) - \beta\rho_a\lambda_2\right]\tilde{y}_t^f = 0$$

This provides one equation:

$$\lambda_2 - \kappa(\lambda_1 - 1) - \beta\rho_a\lambda_2 = 0 \qquad (1)$$

Now go to the DIS equation:

$$\tilde{y}_t = E_t[\tilde{y}_{t+1}] - \frac{1}{\sigma}\left(\tilde{i}_t - E_t[\tilde{\pi}_{t+1}]\right)$$

$$\lambda_1 \tilde{y}_t^f = \lambda_1 \rho_a \tilde{y}_t^f - \frac{1}{\sigma}\left(\phi_\pi \tilde{\pi}_t - \lambda_2 \rho_a \tilde{y}_t^f\right)$$

$$\lambda_1 \tilde{y}_t^f - \lambda_1 \rho_a \tilde{y}_t^f + \frac{1}{\sigma}\left(\phi_\pi \lambda_2 \tilde{y}_t^f - \lambda_2 \rho_a \tilde{y}_t^f\right) = 0$$

$$\lambda_1 \tilde{y}_t^f - \lambda_1 \rho_a \tilde{y}_t^f + \frac{\phi_\pi}{\sigma}\lambda_2 \tilde{y}_t^f - \frac{\lambda_2\rho_a}{\sigma}\tilde{y}_t^f = 0$$

$$\left[\lambda_1 - \lambda_1\rho_a + \frac{\phi_\pi}{\sigma}\lambda_2 - \frac{\lambda_2\rho_a}{\sigma}\right]\tilde{y}_t^f = 0$$

This provides another equation:

$$\lambda_1 - \lambda_1\rho_a + \frac{\phi_\pi}{\sigma}\lambda_2 - \frac{\lambda_2\rho_a}{\sigma} = 0 \qquad (2)$$

Solve ② for λ_1:

$$\lambda_1 - \lambda_1 \rho_a + \frac{\phi_\pi}{\sigma}\lambda_2 - \frac{\lambda_2 \rho_a}{\sigma} = 0$$

$$\lambda_1(1-\rho_a) = \lambda_2\left(\frac{\rho_a}{\sigma} - \frac{\phi_\pi}{\sigma}\right)$$

$$\lambda_1 = \frac{\lambda_2}{1-\rho_a}\left(\frac{\rho_a}{\sigma} - \frac{\phi_\pi}{\sigma}\right)$$

Plug this into ①:

$$\lambda_2 - \kappa(\lambda_1 - 1) - \beta\rho_a\lambda_2 = 0$$

$$\lambda_2 - \kappa\left(\frac{\lambda_2}{\sigma(1-\rho_a)}(\rho_a - \phi_\pi) - 1\right) - \beta\rho_a\lambda_2 = 0$$

$$\lambda_2(1-\beta\rho_a) - \lambda_2\frac{\kappa(\rho_a-\phi_\pi)}{\sigma(1-\rho_a)} + \kappa = 0$$

$$\lambda_2\left[(1-\beta\rho_a) - \frac{\kappa(\rho_a-\phi_\pi)}{\sigma(1-\rho_a)}\right] = -\kappa$$

$$\lambda_2 = \frac{-\kappa}{\frac{\sigma(1-\beta\rho_a)(1-\rho_a)-\kappa(\rho_a-\phi_\pi)}{\sigma(1-\rho_a)}}$$

$$\lambda_2^* = \frac{-\sigma\kappa(1-\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)-\kappa(\rho_a-\phi_\pi)}$$

Plug λ_2^* into λ_1:

$$\lambda_1 = \frac{\lambda_2^*}{1-\rho_a}\left(\frac{\rho_a}{\sigma} - \frac{\phi_\pi}{\sigma}\right)$$

$$\lambda_1 = \lambda_2^* \frac{\rho_a - \phi_\pi}{\sigma(1-\rho_a)}$$

$$\lambda_1 = \frac{-\cancel{\sigma}\kappa\cancel{(1-\rho_a)}}{\sigma(1-\beta\rho_a)(1-\rho_a)-\kappa(\rho_a-\phi_\pi)} \cdot \frac{\rho_a-\phi_\pi}{\cancel{\sigma(1-\rho_a)}}$$

$$\lambda_1^* = \frac{-\kappa(\rho_a-\phi_\pi)}{\sigma(1-\beta\rho_a)(1-\rho_a)-\kappa(\rho_a-\phi_\pi)}$$

We've thus confirmed the guess for the policy functions. The nominal rate is thus:

$$\tilde{i}_t = \phi_\pi \lambda_2^* \tilde{y}_t^f$$

$$\tilde{i}_t = \phi_\pi \frac{\sigma\kappa(1-\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)-\kappa(\rho_a-\phi_\pi)} \tilde{y}_t^f$$

Comparing the Policy Functions:

① Comparing $\tilde{y}_t$ (Suppose $\sigma = 1$)

Inflation:

$$\tilde{y}_t = \frac{-\varkappa(\rho_a - \phi_\pi)}{(1-\beta\rho_a)(1-\rho_a) - \varkappa(\rho_a - \phi_\pi)}\tilde{y}_t^f = \frac{\varkappa(\phi_\pi - \rho_a)}{(1-\beta\rho_a)(1-\rho_a) - \varkappa(\rho_a - \phi_\pi)}\tilde{y}_t^f$$

NGDP (with $\phi_N = \phi_\pi$)

$$\tilde{y}_t = \frac{\lambda_{2a}^*}{\underbrace{\varkappa + \beta\lambda_{2a}^*}_{\text{Smooth transition back to SS?}}}\tilde{y}_{t-1} + \left[1 - \frac{\lambda_{2a}^*}{\varkappa + \beta\lambda_{2a}^*} - \rho_a + \phi_\pi\left(\frac{\varkappa + \beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \phi_\pi - \lambda_{2a}^* - \rho_a\left(\frac{\varkappa+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right]^{-1}\frac{\varkappa}{1-\beta\rho_a}(\phi_\pi - \rho_a)\tilde{y}_t^f$$

First, look at the numerator on the coefficient of $\tilde{y}_t^f$:

$$\left.\begin{aligned} \text{numerator}_\pi &= \varkappa(\phi_\pi - \rho_a) \\ \text{numerator}_{NGDP} &= \varkappa(\phi_\pi - \rho_a) \end{aligned}\right] \text{ These are the same, so which denominator is bigger?}$$

The numerator are the same. So looking at the denominator:

$$\text{denominator}_\pi = (1-\beta\rho_a)(1-\rho_a) - \varkappa(\rho_a - \phi_\pi) = (1-\beta\rho_a)(1-\rho_a) + \varkappa(\phi_\pi - \rho_a)$$

$$\begin{aligned}
\text{denominator}_{NGDP} &= \left[1 - \frac{\lambda_{2a}^*}{\varkappa+\beta\lambda_{2a}^*} - \rho_a + \phi_\pi\left(\frac{\varkappa+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + \phi_\pi - \lambda_{2a}^* - \rho_a\left(\frac{\varkappa+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right)\right](1-\beta\rho_a) \\
&= \left[1 - \lambda_{1a}^* + (\phi_\pi - \rho_a)\left(\frac{\varkappa+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + (\phi_\pi - \rho_a)\right](1-\beta\rho_a) \\
&= \left[\cancel{\lambda_{2a}^*} + (\phi_\pi - \rho_a)\left(\frac{\varkappa+\beta\lambda_{2a}^*}{1-\beta\rho_a}\right) + (\phi_\pi - \rho_a) \cancel{- \lambda_{2a}^*}\right](1-\beta\rho_a) \\
&= \left[(\phi_\pi - \rho_a)\left(\frac{\varkappa + \beta\lambda_{2a}^* + 1 - \beta\rho_a}{1-\beta\rho_a}\right)\right](1-\beta\rho_a) \\
&= (\phi_\pi - \rho_a)(\varkappa + \beta\lambda_{2a}^* + 1 - \beta\rho_a)
\end{aligned}$$

Compare denominators:

$$(1-\beta\rho_a)(1-\rho_a) + \varkappa(\phi_\pi - \rho_a) \leq \overbrace{(\phi_\pi - \rho_a)}^{>0}(\varkappa + \beta\lambda_{2a}^* + 1 - \beta\rho_a)$$

$$\frac{(1-\beta\rho_a)(1-\rho_a)}{(\phi_\pi - \rho_a)} + \varkappa \leq (\varkappa + \beta\lambda_{2a}^* + 1 - \beta\rho_a)$$

$$\frac{(1-\beta\rho_a)(1-\rho_a)}{(\phi_\pi - \rho_a)} \leq \beta\lambda_{2a}^* + 1 - \beta\rho_a \qquad (\text{A1: RHS} > 0)$$

$$\frac{(1-\beta\rho_a)(1-\rho_a)}{\beta\lambda_{2a}^* + 1 - \beta\rho_a} + \rho_a \leq \phi_\pi$$

$$\lambda_{2a}^{*} = \frac{\overbrace{\sqrt{\beta^2 + 2\beta(\kappa-1) + (\kappa+1)^2}}^{\geq 0} + \overbrace{\beta - \kappa - 1}^{\leq 0}}{2\beta}$$

② Comparing $\tilde{\pi}_t$ (Impose $\sigma = 1$)

Inflation:

$$\tilde{\pi}_t = \frac{-\sigma\kappa(1-\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a) - \kappa(\rho_a - \phi_\pi)}\tilde{y}_t^f = \frac{\kappa(1-\rho_a)}{(1-\beta\rho_a)(\rho_a-1) - \kappa(\phi_\pi - \rho_a)}\tilde{y}_t^f$$

NGDP:

$$\tilde{\pi}_t = \lambda_{2a}^{*}\tilde{y}_{t-1} + \lambda_{1b}^{*}\left(\frac{\kappa+\beta\lambda_{2a}^{*}}{1-\beta\rho_a}\right)\tilde{y}_t^f - \frac{\kappa}{1-\beta\rho_a}\tilde{y}_t^f$$

$$= \lambda_{2a}^{*}\tilde{y}_{t-1} + \tilde{y}_t^f\left[\lambda_{1b}^{*}\frac{\kappa+\beta\lambda_{2a}^{*}}{1-\beta\rho_a} - \frac{\kappa}{1-\beta\rho_a}\right]$$

First look at the numerator on $\tilde{y}_t^f$:

$$\text{numerator}_\pi = \kappa(1-\rho_a)$$

$$\text{numerator}_N = \lambda_{1b}^{*}(\kappa+\beta\lambda_{2a}^{*}) - \kappa$$

$$= \left[(\phi_\pi-\rho_a)(\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a)\right]^{-1}\kappa(\phi_\pi-\rho_a)(\kappa+\beta\lambda_{2a}^{*}) - \kappa$$

$$= \frac{\kappa\cancel{(\phi_\pi-\rho_a)}(\kappa+\beta\lambda_{2a}^{*})}{\cancel{(\phi_\pi-\rho_a)}(\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a)} - \kappa$$

$$= \frac{\kappa(\kappa+\beta\lambda_{2a}^{*})}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a} - \frac{\kappa(\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a)}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a}$$

$$= \frac{\cancel{\kappa^2}\cancel{+\kappa\beta\lambda_{2a}^{*}}\cancel{-\kappa^2}\cancel{-\kappa\beta\lambda_{2a}^{*}} - \kappa + \kappa\beta\rho_a}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a}$$

$$= \frac{\kappa(\beta\rho_a-1)}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a}$$

Simplify the NGDP policy rule:

$$\frac{\cancel{-}\kappa\cancel{(\beta\rho_a-1)}}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a}\cdot\frac{1}{\cancel{1-\beta\rho_a}} = \frac{-\kappa}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a}$$

Compare the two policy rules:

$$\frac{-\kappa}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a} \leq \frac{\kappa(1-\rho_a)}{(1-\beta\rho_a)(\rho_a-1) - \kappa(\phi_\pi-\rho_a)}$$

(A2, LHS Denom > 0)
$$\frac{-1}{\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a} \leq \frac{1-\rho_a}{\underbrace{(1-\beta\rho_a)(\rho_a-1) - \kappa(\phi_\pi-\rho_a)}_{<0}}$$

$$\kappa(\phi_\pi-\rho_a) - (1-\beta\rho_a)(\rho_a-1) \geq (1-\rho_a)(\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a)$$

$$\kappa(\phi_\pi-\rho_a) \geq (1-\rho_a)(\kappa+\beta\lambda_{2a}^{*}+1-\beta\rho_a) + (1-\beta\rho_a)(\rho_a-1)$$

$$\phi_\pi - \rho_a \geq \frac{1-\rho_a}{\kappa}\left[\kappa+\beta\lambda_{2a}^{*}\cancel{+1}\cancel{-\beta\rho_a}\cancel{-1}\cancel{+\beta\rho_a}\right]$$

$$\phi_\pi \geq \frac{(1-\rho_a)(\kappa+\beta\lambda_{2a}^{*})}{\kappa} + \rho_a$$

Verify A1:

Suppose $\lambda_{2a}^{*} = \dfrac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2}+\beta-x-1}{2\beta} < 1$. Then:

$$\sqrt{\beta^2+2\beta(x-1)+(x+1)^2}+\beta-x-1 < 2\beta$$

$$\sqrt{\beta^2+2\beta(x-1)+(x+1)^2}-x < \beta+1$$

$$\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} < \beta+x+1$$ Both sides are positive, so I can square the inequality

$$\beta^2+2\beta(x-1)+(x+1)^2 < (\beta+x+1)(\beta+x+1)$$

$$\cancel{\beta^2}+\cancel{2\beta x}-2\beta+\cancel{x^2}+\cancel{2x}+\cancel{1} < \cancel{\beta^2}+\cancel{x\beta}+\beta+\cancel{x\beta}+\cancel{x^2}+\cancel{x}+\beta+\cancel{x}+\cancel{1}$$

$$-2\beta < 2\beta$$

$$0 < 4\beta$$

$$0 < \beta$$ So as long as $\beta > 0$, $\lambda_{2a}^{*} < 1$.

We are given that $\beta > 0$, so $\lambda_{2a}^{*} < 1$. Now suppose that

$\lambda_{2a}^{*} = \dfrac{\sqrt{\beta^2+2\beta(x-1)+(x+1)^2}+\beta-x-1}{2\beta} > 0$. Then:

$$\sqrt{\beta^2+2\beta(x-1)+(x+1)^2} > x+1-\beta$$ Both sides are positive, so I can square the inequality.

$$\beta^2+2\beta(x-1)+(x+1)^2 > (x+1-\beta)(x+1-\beta)$$

$$\cancel{\beta^2}+2\beta x-\cancel{2\beta}+\cancel{x^2}+\cancel{2x}+\cancel{1} > \cancel{x^2}+\cancel{x}-\beta x+\cancel{x}+\cancel{1}-\cancel{\beta}-\beta x-\cancel{\beta}+\cancel{\beta^2}$$

$$2\beta x > -2\beta x$$

$$4\beta x > 0$$

$$\beta x > 0$$ which is also given.

Therefore, $0 < \lambda_{2a}^{*} < 1. \longrightarrow 0 < \beta\lambda_{2a}^{*} < \beta \longrightarrow 0 < \beta\lambda_{2a}^{*} < 1.$ (1)

Now take $0 < \beta < 1. \longrightarrow 0 < \beta p_a < p_a \longrightarrow 0 < \beta p_a < 1$

$$-1 < \beta p_a - 1 < 0$$

$$0 < 1-\beta p_a < 1 \qquad (2)$$

Add (1) and (2):

$0 < \beta\lambda_{2a}^{*}+1-\beta p_a < 2. \longrightarrow \beta\lambda_{2a}^{*}+1-\beta p_a > 0$, which completes the proof #

Verify A2:

Start from A1:

$$\beta \lambda_{2a}^{*} + 1 - \beta p_a > 0$$

I know that $x > 0$ as well. Add these two together to get:

$$x + \beta \lambda_{2a}^{*} + 1 - \beta p_a > 0$$

Which is what I wanted to show #