Exercises for Chapter 1

Exercise 1. Show that 2 is the only prime number which is even.

Exercise 2. Show that if n^2 is even, then n is even, for n an integer.

Exercise 3. The goal of this exercise is to show that $\sqrt{2}$ is irrational. We provide a step by step way of doing so.

1. Suppose by contradiction that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{m}{n}$, for m and n integers with no common factor. Show that m has to be even, that is $m = 2k$.

2. Compute m^2, and deduce that n has to be even too, a contradiction.

Exercise 4. Let n be an integer greater than 1. Suppose that $a \equiv a' \mod n$ and $b \equiv b' \mod n$. Show that

1. $(a + b) \mod n \equiv (a' + b') \mod n$,

2. $(a \cdot b) \mod n \equiv (a' \cdot b') \mod n$.

Exercise 5. Compute the addition table and the multiplication tables for integers modulo 4.

Exercise 6. Show that $\frac{m(m+1)}{2} \equiv 0 \pmod{m}$ for m an odd number.

Exercise 7.

1. Compute $7 \cdot 8 \cdot 9 \cdot 10$ modulo 3.

2. Show that $n^3 - n$ is always divisible by 3, for n any positive integer.

Exercise 8. Compute 40^{1234} modulo 2.

Exercise 9. Consider the set S of odd natural numbers, with respective operator Δ.

- Let Δ be the multiplication. Is S closed under Δ? Justify your answer.

- Let Δ be the addition. Is S closed under Δ? Justify your answer.

Exercise 10. Consider the following sets S, with respective operator Δ.

- Let S be the set of rational numbers, and Δ be the multiplication. Is S closed under Δ? Justify your answer.

Exercise 1. Show that 2 is the only prime number which is even.

Prime numbers can only be divided by themselves & one

let even number set : $2(x)$, $x \in \mathbb{Z}$

$2x \bmod 2 = 0 \bmod 2$

Exercise 2. Show that if n^2 is even, then n is even, for n an integer.

2 possibilites of n

n = even

$n = 2k$, $k \in \mathbb{Z}$

$n^2 = 4k^2$

$= 2(2k^2)$

n = odd

$n = 2j+1$, $j \in \mathbb{Z}$

$n^2 = (2j+1)^2$

$n^2 = 4j^2 + 4j + 1$

$n^2 = 2(2j^2) + 2(2j) + 1$

$= 2(2j^2 + 2j) + 1$

Exercise 3. The goal of this exercise is to show that $\sqrt{2}$ is irrational. We provide a step by step way of doing so.

1. Suppose by contradiction that $\sqrt{2}$ is rational, that is $\sqrt{2} = \frac{m}{n}$, for m and n integers with no common factor. Show that m has to be even, that is $m = 2k$.
2. Compute m^2, and deduce that n has to be even too, a contradiction.

if $\sqrt{2}$ is rational, $\sqrt{2} = \frac{m}{n}$, $m, n \in \mathbb{Z}$

$\therefore 2 = \frac{m^2}{n^2}$

$2n^2 = m^2$

$\therefore m^2$ is even, and m is even

$m = 2k$, $k = \mathbb{Z}$

$m^2 = 2(2k^2)$

$\therefore 4k^2 = 2n^2$

$\therefore n$ should be even. However,

n & m should have no common factor

Exercise 4. Let n be an integer greater than 1. Suppose that $a \equiv a' \mod n$ and $b \equiv b' \mod n$. Show that

1. $(a+b) \mod n \equiv (a'+b') \mod n$,
2. $(a \cdot b) \mod n \equiv (a' \cdot b') \mod n$.

1. $(a+b) \bmod n = ([a' + n(x)] + [b' + n(y)]) \bmod n$

$= (a' + b') \bmod n$

2. $(a \cdot b) \bmod n \equiv ([a' + n(x)][b' + n(y)]) \bmod n$

$= (a'b' + a'ny + b'nx + n^2xy) \bmod n$

$= (a' \cdot b') \bmod n$

Exercise 5. Compute the addition table and the multiplication tables for integers modulo 4.

mod 4

$4 \to \{0, 1, 2, 3\}$

+4	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

4	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

Exercise 6. Show that $\frac{m(m+1)}{2} \equiv 0 \pmod{m}$ for m an odd number.

if m is odd,
$m+1$ is even
$m+1 = 2k$, $k \in \mathbb{Z}$

$$\frac{m(m+1)}{2} = \frac{m(\cancel{2}k)}{\cancel{2}} = m(k) \bmod m$$
$$\equiv 0 \bmod m,$$

Exercise 7. 1. Compute $7 \cdot 8 \cdot 9 \cdot 10$ modulo 3.

2. Show that $n^3 - n$ is always divisible by 3, for n any positive integer.

1. $7 \cdot 8 \cdot 9 \cdot 10 \bmod 3 \equiv 3(3) \cdot 7 \cdot 8 \cdot 10 \bmod 3$
$\equiv 0$

2. $n \equiv 0 \bmod 3$ $\quad n \equiv 1 \bmod 3$ $\quad n = 2 \bmod 3$

n	n^3	$n^3 - n$
0	0	0
1	1	0
2	$8 \equiv 2$	0

Exercise 8. Compute 40^{1234} modulo 2.

$$40 \times 40^{1233} \bmod 2 \equiv 2(20)(40^{1233}) \bmod 2$$
$$\equiv 0$$

Exercise 9. Consider the set S of odd natural numbers, with respective operator Δ.

- Let Δ be the multiplication. Is S closed under Δ? Justify your answer.
- Let Δ be the addition. Is S closed under Δ? Justify your answer.

1) $S = \{2k+1\}$ $k \in \mathbb{Z}$

$(2k+1)(2n+1) = 4kn + 2k + 2n + 1$
$= 2(2kn + k + n) \underline{+1}$ always odd

2) not closed, $3+3=6$

Exercise 10. Consider the following sets S, with respective operator Δ.

- Let S be the set of rational numbers, and Δ be the multiplication. Is S closed under Δ? Justify your answer.
- Let S be the set of natural numbers, and Δ be the subtraction. Is S closed under Δ? Justify your answer.
- Let S be the set of irrational numbers, and Δ be the addition. Is S closed under Δ? Justify your answer.

rational numbers $\rightarrow \frac{m}{n}$, $m, n \in \mathbb{Z}$

1) $\frac{m}{n} \cdot \frac{m'}{n'} = \frac{mm'}{nn'}$, yes, closed

2) $5 - 10 = -5$, non natural

- Let S be the set of natural numbers, and Δ be the subtraction. Is S closed under Δ? Justify your answer.

- Let S be the set of irrational numbers, and Δ be the addition. Is S closed under Δ? Justify your answer.

Exercises for Chapter 2

Exercise 11. Decide whether the following statements are propositions. Justify your answer.

1. $2 + 2 = 5$. Proposition
2. $2 + 2 = 4$. Proposition
3. $x = 3$. Not prop
4. Every week has a Sunday. Propositn
5. Have you read "Catch 22"? not proposition

Exercise 12. Show that

$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

This is the second law of De Morgan.

Exercise 13. Show that the second absorption law $p \wedge (p \vee q) \equiv p$ holds.

Exercise 14. These two laws are called distributivity laws. Show that they hold:

1. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.
2. Show that $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.

Exercise 15. Verify $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ by

- constructing a truth table,
- developing a series of logical equivalences.

Exercise 16. Using a truth table, show that:

$$\neg q \to \neg p \equiv p \to q.$$

Exercise 17. Show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$.

Exercise 18. Are $(p \to q) \vee (q \to r)$ and $p \to r$ equivalent statements?

Exercise 12. Show that
$$\neg(p \vee q) \equiv \neg p \wedge \neg q.$$

$\neg(p \vee q)$ $\begin{cases} F, & \text{at least one of } p \text{ \& } q \text{ is T} \\ T, & \text{otherwise} \end{cases}$

$\neg p \wedge \neg q$ $\begin{cases} F, & \text{at least } p \text{ \& } q \text{ is T} \\ T, & \text{otherwise} \end{cases}$

Exercise 13. Show that the second absorption law $p \wedge (p \vee q) \equiv p$ holds.

p	q	$(p \vee q)$	$p \wedge (p \vee q)$
T	T	T	T
T	F	T	T
F	T	T	F
F	F	F	F

✓

Exercise 14. These two laws are called distributivity laws. Show that they hold:

1. Show that $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$.
2. Show that $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$.

$(p \wedge q) \vee r = \begin{cases} F, & \text{both } (p \wedge q) \text{ and } r \text{ are F} \\ T, & \text{otherwise} \end{cases}$

$= \begin{cases} F, & r \text{ is F \& at least one of } p \text{ \& } q \text{ are F} \\ T, & \text{otherwise} \end{cases}$

$(p \vee r) \wedge (q \vee r) = \begin{cases} F, & \text{at least one of } p \vee r \text{ \& } q \vee r \text{ is F} \\ T, & \text{otherwise} \end{cases}$

$= \begin{cases} F, & r \text{ is F \& at least one } p \text{ and } q \text{ is F} \\ T, & \text{otherwise} \end{cases}$

$(p \vee q) \wedge r = \begin{cases} T, & \text{both } p \vee q \text{ and } r \text{ are T} \\ F, & \text{otherwise} \end{cases}$

$= \begin{cases} T, & r \text{ is T and at least one of } p \text{ and } q \text{ is T} \\ F, & \text{otherwise} \end{cases}$

$(p \wedge r) \vee (q \wedge r) = \begin{cases} T, & \text{at least one of } p \wedge r \text{ and } q \wedge r \text{ is T} \\ F, & \text{otherwise} \end{cases}$

$= \begin{cases} T, & r \text{ is T and at least one of } p \text{ and } q \text{ is T} \\ F, & \text{otherwise} \end{cases}$

Exercise 15. Verify $\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv \neg p$ by

- constructing a truth table,
- developing a series of logical equivalences.

$$\neg(p \vee \neg q) \vee (\neg p \wedge \neg q) \equiv (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\equiv \neg p \wedge (q \vee \neg q)$$
$$\equiv \neg p$$

Exercise 16. Using a truth table, show that:
$$\neg q \to \neg p \equiv p \to q.$$

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \to \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Exercise 17. Show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$.

$$(p \vee q) \to r \equiv \neg(p \vee q) \vee r$$
$$\equiv (\neg p \wedge \neg q) \vee r$$
$$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$$
$$\equiv (p \to r) \wedge (q \to r)$$

Exercise 18. Are $(p \to q) \vee (q \to r)$ and $p \to r$ equivalent statements?

p	q	r	$p \to r$	$p \to q$	$q \to r$	$(p \to q) \vee (q \to r)$
T	T	T	T	T	T	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	T	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

Exercise 17. Show that $p \vee q \to r \equiv (p \to r) \wedge (q \to r)$.

$$(p \vee q) \to r \equiv (p \to r) \wedge (q \to r)$$

$(p \vee q) \to r = \begin{cases} F, & p \vee q \text{ is T and } r \text{ is F} \\ T, & \text{otherwise} \end{cases}$

$= \begin{cases} F, & r \text{ is F and at least one of } p \text{ and } q \text{ is T} \\ T, & \text{otherwise} \end{cases}$

$(p \to r) \wedge (q \to r) = \begin{cases} F, & \text{at least one of } p \to r \text{ and } q \to r \text{ is F} \\ T, & \text{otherwise} \end{cases}$

$p \wedge (\neg(\neg q \wedge r)) \equiv p \wedge (q \wedge \neg r)$
$\equiv p \wedge q \wedge \neg r$

$p \Rightarrow r = \neg p \wedge r$
$\therefore$ disproved

$\neg p \Rightarrow F$ is True when $\neg p$ is false
$\therefore$ p is true.
premise & conclusion T

Premise is $\neg p \rightarrow C$
$\neg p \Rightarrow C$ is true when

Exercise 19. Prove or disprove the following statement:

$$p \wedge (\neg(q \to r)) \equiv (p \to r).$$

Exercise 20. Show that this argument is valid:

$$\boxed{\neg p \to F; \therefore p.}$$

?

Exercise 21. Show that this argument is valid, where C denotes a contradiction.

$$\boxed{\neg p \to C; \therefore p.}$$

Exercise 22. 1. Prove or disprove the following statement:

$$(p \wedge q) \to p \equiv T.$$

2. Decide whether the following argument is valid.

$\neg d \to h;$
$\neg h \to d;$
$\therefore \neg d \vee \neg h$

Exercise 23. Determine whether the following argument is valid:

$$\begin{array}{l} \neg p \to r \wedge \neg s \\ t \to s \\ u \to \neg p \\ \neg w \\ u \vee w \\ \therefore t \to w. \end{array}$$

Exercise 24. Determine whether the following argument is valid:

$$\begin{array}{l} p \\ p \vee q \\ q \to (r \to s) \\ t \to r \\ \therefore \neg s \to \neg t. \end{array}$$

Exercise 25. Decide whether the following argument is valid:

$(p \vee q) \to \neg r;$
$\neg r \to s;$
$p;$
$\therefore s$

Exercise 22. 1. Prove or disprove the following statement:

$$(p \wedge q) \to p \equiv T.$$

2. Decide whether the following argument is valid.

$\neg d \to h$;
$\neg h \to d$;
$\therefore \neg d \vee \neg h$

$\neg(p \wedge q) \vee p \equiv T$

$(\neg p \vee \neg q) \vee p \equiv T$

$\underbrace{(\neg p \vee p)}_{\text{always } T} \vee (\neg q \vee p) \equiv T$

2)

d	h	¬d	¬h	¬d→¬h	¬h→d	¬d∨¬h
T	T	F	F	T	T	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	F	T

Exercise 23. Determine whether the following argument is valid:

$\neg p \to r \wedge \neg s$
$t \to s$
$u \to \neg p$
$\neg w$
$u \vee w$
$\therefore t \to w.$

$u \vee w$ is T ; $\neg w$, w is F, u is T

$u \to \neg p$ is T, u is T, $\neg p$ is T

$\neg p \to r \wedge \neg s$ is T, $\neg p$ is T, $r \wedge \neg s$ is T

$\neg s$ is T, s is F

$t \to s$ is T, t is F

$t \to w$ must be T

Valid

Exercise 24. Determine whether the following argument is valid:

p
$p \vee q$
$q \to (r \to s)$
$t \to r$
$\therefore \neg s \to \neg t.$

p is T

q can be T / F

if q is F, $q \to (r \to s)$ is T

if q is T, $q \to (r \to s)$ is T when $(r \to s)$ is T

if $t \to r$ is T, t cannot be T when r is F

q	r	s	t
F	T	T	
F	T	F	
F	F	T	
F	F	F	
T	T	T	
T	F	T	
T	F	F	

Exercises for Chapter 3

Exercise 26. Consider the predicates $M(x, y) =$ "x has sent an email to y", and $T(x, y) =$ "x has called y". The predicate variables x, y take values in the domain $D = \{\text{students in the class}\}$. Express these statements using symbolic logic.

1. There are at least two students in the class such that one student has sent the other an email, and the second student has called the first student.

2. There are some students in the class who have emailed everyone.

Exercise 27. Consider the predicate $C(x, y) =$ "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $D = \{\text{courses}\}$. Express each statement by an English sentence.

1. $\exists\ x \in S,\ C(x, \text{MH1812})$.

2. $\exists\ y \in D,\ C(\text{Carol}, y)$.

3. $\exists\ x \in S,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.

4. $\exists\ x \in S,\ \exists\ x' \in S,\ \forall y \in D,\ ((x \neq x') \wedge (C(x, y) \leftrightarrow C(x', y)))$.

Exercise 28. Consider the predicate $P(x, y, z) =$ "$xyz = 1$", for $x, y, z \in \mathbb{R}$, $x, y, z > 0$. What are the truth values of these statements? Justify your answer.

1. $\forall\ x,\ \forall\ y,\ \forall\ z,\ P(x, y, z)$.

2. $\exists\ x,\ \exists\ y,\ \exists\ z,\ P(x, y, z)$.

3. $\forall\ x,\ \forall\ y,\ \exists\ z,\ P(x, y, z)$.

4. $\exists\ x,\ \forall\ y,\ \forall\ z,\ P(x, y, z)$.

Exercise 29. Consider the domains $X = \{2, 3\}$ and $Y = \{2, 4, 6\}$, and the predicate $P(x, y) =$ "x divides y". What are the truth values of these statements:

a) $\exists x \in X,\ \forall y \in Y,\ P(x, y)$.

Exercise 26. Consider the predicates $M(x, y)$ = "x has sent an email to y", and $T(x, y)$ = "x has called y". The predicate variables x, y take values in the domain D = {students in the class}. Express these statements using symbolic logic.

1. There are at least two students in the class such that one student has sent the other an email, and the second student has called the first student.

2. There are some students in the class who have emailed everyone.

1. $\exists x \in D, \exists y \in D, M(x,y) \wedge T(y,x)$

also, x & y have to be distinct

$\therefore \exists x \in D, \exists y \in D, ((x \neq y) \wedge M(x,y) \wedge T(y,x))$

2. $\exists x \in D, \forall y \in D, M(x,y)$

$\forall y \in D, \exists x \in D, M(x,y)$ why is this wrong

Exercise 27. Consider the predicate $C(x, y)$ = "x is enrolled in the class y", where x takes values in the domain S = {students}, and y takes values in the domain D = {courses}. Express each statement by an English sentence.

1. $\exists\, x \in S,\ C(x, \text{MH1812})$.
2. $\exists\, y \in D,\ C(\text{Carol}, y)$.
3. $\exists\, x \in S,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
4. $\exists\, x \in S,\ \exists\, x' \in S,\ \forall y \in D,\ ((x \neq x') \wedge (C(x, y) \leftrightarrow C(x', y)))$.

1. Some students are enrolled in class MH1812
2. Carol is enrolled in some courses
3. Some students are enrolled in MH1812 and CZ2002
4. Some distinct student x and x', for all courses, are enrolled in the same class

Exercise 28. Consider the predicate $P(x, y, z)$ = "$xyz = 1$", for $x, y, z \in \mathbb{R}$, $x, y, z > 0$. What are the truth values of these statements? Justify your answer.

1. $\forall\, x,\ \forall\, y,\ \forall\, z,\ P(x, y, z)$.
2. $\exists\, x,\ \exists\, y,\ \exists\, z,\ P(x, y, z)$.
3. $\forall\, x,\ \forall\, y,\ \exists\, z,\ P(x, y, z)$.
4. $\exists\, x,\ \forall\, y,\ \forall\, z,\ P(x, y, z)$.

1. If $x=1, y=1, z \neq 1$, F
2. Let $x=1, y=1, z=1$, $xyz=1$, T
3. for any x & any y, if $z = \frac{1}{xy}$, $=1$
4. for some of x, not any y & z can make $xyz=1$
This is because x has already been chosen

b) $\neg(\exists x \in X,\ \exists y \in Y,\ P(x,y))$.

Exercise 30. 1. Express

$$\neg(\forall\ x,\ \forall\ y,\ P(x,y))$$

in terms of existential quantification.

2. Express

$$\neg(\exists\ x,\ \exists\ y,\ P(x,y))$$

in terms of universal quantification.

Exercise 31. Consider the predicate $C(x,y) =$ "x is enrolled in the class y", where x takes values in the domain $S = \{\text{students}\}$, and y takes values in the domain $C = \{\text{courses}\}$. Form the negation of these statements:

1. $\exists\ x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
2. $\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \leftrightarrow C(y,z)))$.

Exercise 32. Show that $\forall x \in D,\ P(x) \rightarrow Q(x)$ is equivalent to its contrapositive.

Exercise 33. Show that

$$\neg(\forall\ x, P(x) \rightarrow Q(x)) \equiv \exists x,\ P(x) \wedge \neg Q(x).$$

Exercise 34. Let y, z be positive integers. What is the truth value of "$\exists y,\ \exists z,\ (y = 2z \wedge (y \text{ is prime}))$".

Exercise 35. Consider the domains $X = \{2,4,6\}$ and $Y = \{2,3\}$, and the predicate $P(x,y) =$ "x is a multiple of y". What are the truth values of these statements:

1. $\forall x \in X,\ \exists y \in Y,\ P(x,y)$.
2. $\neg(\forall x \in X,\ \forall y \in Y,\ P(x,y))$.

Exercise 36. Write in symbolic logic "Every SCE student studies discrete mathematics. Jackson is an SCE student. Therefore Jackson studies discrete mathematics".

Exercise 37. Here is an optional exercise about universal generalization. Consider the following two premises: (1) for any number x, if $x > 1$ then $x - 1 > 0$, (2) every number in D is greated than 1. Show that therefore, for every number x in D, $x - 1 > 0$.

Exercise 29. Consider the domains $X = \{2, 3\}$ and $Y = \{2, 4, 6\}$, and the predicate $P(x, y)$ = "x divides y". What are the truth values of these statements:

a) $\exists x \in X,\ \forall y \in Y,\ P(x, y)$.

b) $\neg(\exists x \in X,\ \exists y \in Y,\ P(x, y))$.

a) when $x = 2$, all can divide,
$x = 3$, all can be divide,
$\therefore T$

b) $\neg(\exists x \in X,\ \exists y \in Y,\ P(x,y))$
$\forall x \in X,\ \forall y \in Y,\ \neg(P(x,y))$

when $x = 2$, $y = 4$, $4/2 = 2$.

$\therefore F$ as x divides y

Exercise 30. 1. Express

$$\neg(\forall\ x,\ \forall\ y,\ P(x, y))$$

in terms of existential quantification.

2. Express

$$\neg(\exists\ x,\ \exists\ y,\ P(x, y))$$

in terms of universal quantification.

1. $\neg(\forall x, \forall y,\ P(x,y))$ let $\forall y, P(x,y)$ be $Q(x)$

$\therefore\ \neg(\forall x,\ Q(x)) \equiv \exists x,\ \neg(Q(x))$
$\equiv \exists x,\ \neg(\forall y,\ P(x,y))$
$\equiv \exists x,\ \exists y,\ \neg(P(x,y))$

2. $\neg(\exists x,\ \exists y,\ P(x,y))$ let $\exists y, P(x,y)$ be $Q(x)$

$\therefore\ \neg(\exists x,\ Q(x)) \equiv \forall x,\ \neg(Q(x))$
$\equiv \forall x,\ \forall y,\ \neg(P(x,y))$

Exercise 31. Consider the predicate $C(x, y)$ = "x is enrolled in the class y", where x takes values in the domain S = {students}, and y takes values in the domain C = {courses}. Form the negation of these statements:

1. $\exists\ x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$.
2. $\exists\ x\ \exists\ y,\ \forall z,\ ((x \neq y) \wedge (C(x, z) \leftrightarrow C(y, z)))$.

1. $\neg(\exists x,\ (C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))) \equiv \forall \in x,\ \neg(C(x, \text{MH1812}) \wedge C(x, \text{CZ2002}))$
$\equiv \forall \in x,\ \neg C(x, \text{MH1812}) \vee \neg C(x, \text{CZ2002})$

2. $\neg(\exists x,\ \exists y,\ \forall z,\ ((x \neq y) \wedge (C(x,z) \longleftrightarrow C(y,z))))$

$\equiv \forall x,\ \forall y,\ \exists z,\ \neg((x \neq y) \wedge (C(x,z) \longleftrightarrow C(y,z)))$
$\equiv \forall x,\ \forall y,\ \exists z,\ (x = y) \vee \neg(C(x,z) \longleftrightarrow C(y,z))$
$\equiv \forall x,\ \forall y,\ \exists z,\ (x = y) \vee (C(x,z) \wedge \neg C(y,z)) \vee (C(y,z) \wedge \neg C(x,z))$

$\neg(C(x,z) \longleftrightarrow C(y,z))$
$\equiv \neg((\neg C(x,z) \vee C(y,z)) \wedge (\neg C(y,z) \vee C(x,z)))$
$\equiv (C(x,z) \wedge \neg C(y,z)) \vee (C(y,z) \wedge \neg C(x,z))$

Exercise 32. Show that $\forall x \in D,\ P(x) \to Q(x)$ is equivalent to its contrapositive.

Contrapositive is $\forall x \in D,\ \neg Q(x) \to \neg P(x)$
$\therefore\ \forall x \in D,\ P(x) \to Q(x)$

$\neg Q(x) \to \neg P(x) \equiv \neg(\neg Q(x)) \vee \neg P(x)$
$\equiv Q(x) \vee \neg P(x)$
$\equiv P(x) \to Q(x)$

Exercise 33. Show that

$$\neg(\forall\ x, P(x) \to Q(x)) \equiv \exists x,\ P(x) \wedge \neg Q(x).$$

$\neg(\forall x,\ P(x) \to Q(x)) \equiv \exists x,\ \neg(\neg P(x) \vee Q(x))$
$\equiv \exists x,\ P(x) \wedge \neg Q(x)$ //

Exercise 34. Let y, z be positive integers. What is the truth value of "$\exists y,\ \exists z,\ (y = 2z \wedge (y \text{ is prime}))$".

when $y = 2$, $z = 1$, value is T

Exercise 35. Consider the domains $X = \{2, 4, 6\}$ and $Y = \{2, 3\}$, and the predicate $P(x, y)$ = "x is a multiple of y". What are the truth values of these statements:

1. $\forall x \in X,\ \exists y \in Y,\ P(x, y)$. for any x, there exists some y
2. $\neg(\forall x \in X,\ \forall y \in Y,\ P(x, y))$.

1) $x = 2$, $y = 2$, T
$x = 4$, $y = 2$, T
$x = 6$, $y = 2, 3$ T
Exhausted all possibilities

2) $\neg(\forall x \in X,\ \forall y \in Y,\ P(x,y)) \equiv \exists x \in X,\ \exists y \in Y,\ \neg(P(x,y))$

There exists $x = 4$, $Y = 2$, is multiple, therefore F

Exercise 36. Write in symbolic logic "Every SCE student studies discrete mathematics. Jackson is an SCE student. Therefore Jackson studies discrete mathematics".

let $D = \{ \text{scse student} \}$

$P(x) = x$ studies MH1812

$\therefore \quad \forall x \in D,\ P(x)$

Jackson is an scse student,

$\therefore \quad \text{Jackson} \in D$

$\therefore \quad P(\text{Jackson}) = T$

$\therefore$ The logic is, $\forall x \in D, P(x)$, Jackson $\in D$, $P(\text{Jackson})$

Exercise 37. Here is an optional exercise about universal generalization. Consider the following two premises: (1) for any number x, if $x > 1$ then $x - 1 > 0$, (2) every number in D is greated than 1. Show that therefore, for every number x in D, $x - 1 > 0$.

Premise 1: Let $P(x) =$ "$x > 1$"
$Q(x) =$ "$x - 1 > 0$"

$$[\forall x(P(x) \longrightarrow Q(x)) \wedge \forall x \in D,\ P(x)] \Rightarrow \forall x \in D,\ Q(x)$$

Exercise 38. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Let $P(q) =$ "q is irrational" $\quad q \in \mathbb{R}^+$
$Q(q) =$ "$\sqrt{q}$ is irrational" $\quad P(q) \to Q(q)$

By Contrapositive, $\neg Q(q) \Rightarrow \neg P(q)$

If $\sqrt{q}$ is rational, $\sqrt{q} = \frac{a}{b}$, $a \in \mathbb{Z}, b \in \mathbb{Z}, b \neq 0$

$$q = \frac{a^2}{b^2}$$

This implies that q is rational $\quad \therefore \neg Q(q) \Rightarrow \neg P(q)$ is T

$P(q) \Rightarrow Q(q)$ is T

Exercise 39. Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Prove $n^2 = \sum_{i=0}^{n} (2i+1)$ ... Let $P(n) = \sum_{i=0}^{n} (2i+1)$ for $i \in \mathbb{Z}^+$

Basic step: when $P(1)$, $1^2 = 2(1) - 1$
$1 = 1$

Inductive step: Assuming $P(k)$ is T, $\sum_{i=0}^{k} (2i+1) = k^2$

{like dominos}

$$P(k+1) = \sum_{i=0}^{k} (2i+1) + 2(k+1) - 1$$

sub

$$= k^2 + 2(k+1) - 1$$
$$= k^2 + 2k + 1$$
$$= (k+1)^2$$

$\therefore P(k+1)$ is T, $P(n)$ is T for all n

Exercises for Chapter 4

Exercise 38. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Exercise 39. Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Exercise 40. Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

when $n = 2k$, $k \in +\mathbb{Z}$, $(n^3 - n) \bmod (3)$

$P(n) =$ "3 divides $n^3 - n$"

$P(1) = \frac{1^3 - 1}{3} = \frac{0}{3}$

Asuming $P(k) = T$, $\frac{k^3 - k}{3}$ is T

$$P(k+1) = \frac{(k+1)^3 - (k+1)}{3} = \frac{k^3 + 3k^2 + 3k + 1 - (k+1)}{3}$$

$$= \frac{k^3 + 3k^2 + 2k}{3}$$

$$= \frac{(k^3 - k) + 3k^2 + 3k}{3}$$

$$= \frac{(k^3 - k)}{3} + \frac{3(k^2 + k)}{3}$$

$P(k+1)$ is divisible by 3

$\therefore$ $P(n)$ is T for all n

$n^3 - n$ is divisible by 3

basic step: when $n = 1$, $n^3 - n = 0$ is divisible by 3

$\therefore$ $n = 1$ is T

inductive step: assume it's T when $n = k$, $k = \mathbb{Z}^+$

Prove it's also T when $n = k+1$

$\therefore$ $n = k$, $k^3 - k$ is divisible by 3

$n = k+1$, $(k+1)^3 - (k+1) = (k^3 - k) + 3(k^2 + k)$ is divisible by 3

$\therefore$ T by math induction

Exercise 41. Prove by mathematical induction that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

let $p(n) = \text{"}1^2+2^2+\ldots+n^2 = \frac{1}{6}n(n+1)(2n+1)\text{"}$

Basic step: $P(1)$: $1^2 = \frac{1}{6}(1)(1+1)(2+1)$

$1 = 1$

Inductive Step Assuming $P(k)$ is T, $1^2+2^2+\ldots+k^2 = \frac{1}{6}(k)(k+1)(2k+1)$

$P(k+1)$: $1^2+2^2+\ldots k^2+(k+1)^2 = \frac{1}{6}(k+1)(k+1+1)(2(k+1)+1)$

$\frac{1}{6}(k)(k+1)(2k+1)+(k+1)^2 = \frac{1}{6}(k+1)(k+2)(2k+3)$

$\cancel{(k+1)}\left(\frac{1}{6}(k)(2k+1)+(k+1)\right) = \frac{1}{6}\cancel{(k+1)}(k+2)(2k+3)$

$\frac{1}{3}k^2+\frac{7}{6}k+1 = \left(\frac{1}{6}k+\frac{1}{3}\right)(2k+3)$

$\frac{1}{3}k^2+\frac{7}{6}k+1 = \frac{1}{3}k^2+\frac{1}{2}k+\frac{2}{3}k+1$

$\frac{1}{3}k^2+\frac{7}{6}k+1 = \frac{1}{3}k^2+\frac{7}{6}k+1$

$\therefore P(n)$ is T

Exercises for Chapter 5

Exercise 42. A set menu proposes 2 choices of starters, 3 choices of main dishes, and 2 choices of desserts. How many possible set menus are available?

Exercise 43.
- In a race with 30 runners where 8 trophies will be given to the top 8 runners (the trophies are distinct, there is a specific trophy for each place),in how many ways can this be done?
- In how many ways can you solve the above problem if a certain person, say Jackson, must be one of the top 3 winners?

Exercise 44. In how many ways can you pair up 8 boys and 8 girls?

Exercise 45. How many ternary strings of length 4 have zero ones?

Exercise 46. How many permutations are there of the word "repetition"?

42) $2 \times 3 \times 2 = 12$

43) a) $P(30, 8) = \frac{30!}{(30-8)!}$

b) $\underbrace{P(29,2)}_{\text{select other top 2}} \times \underbrace{P(27,5)}_{\text{select the rest}} \times 3\}$ Jackson can be $1^{st}, 2^{nd}, 3^{rd}$

44) $8!$ → the first guy got 8 choice, second got 7, 3^{rd} got 6 ...
$8 \times 7 \times 6 \times 5 \dots$

45) $2 \times 2 \times 2 \times 2 = 16$

46) 10 letters, $e \times 2$, $t \times 2$, $i \times 2$

$$\frac{10!}{2! \cdot 2! \cdot 2!} = 453600$$

Exercises for Chapter 6

Exercise 47. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

Exercise 48. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Exercise 49. Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Exercise 50. Solve the following two recurrence relations:

$$a_n = 3a_{n-1},\ a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2},\ b_1 = 0,\ b_2 = 12.$$

Exercise 51. Solve the following linear recurrence relation:

$$b_n = 4b_{n-1} - b_{n-2},\ b_0 = 2,\ b_1 = 4.$$

$x^2 = 4x - 1$

$x = 2+\sqrt{3}$, $x = 2-\sqrt{3}$

$b_n = U(2+\sqrt{3})^n + V(2-\sqrt{3})^n$

$b_0 = 2 = U+V \Rightarrow U = 2-V$

$b_1 = 4 = U(2+\sqrt{3}) + V(2-\sqrt{3})$

$4 = (2-V)(2+\sqrt{3}) + V(2-\sqrt{3})$

$4 = 2(2+\sqrt{3}) - V(2+\sqrt{3}) + V(2-\sqrt{3})$

$\cancel{4} = \cancel{4} + 2\sqrt{3} - \cancel{2V} - V\sqrt{3} + \cancel{2V} - V\sqrt{3}$

$-2\sqrt{3} = -2V\sqrt{3}$

$\therefore V = 1$, $U = 1$

$\therefore b_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

Exercise 47. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

a)
$$a_n = 2a_{n-1} - a_{n-2}$$
$$a_n - a_{n-1} = a_{n-1} - a_{n-2}$$
$$\vdots$$
$$= 3 - 0$$
$$= 3$$
$$a_n = a_{n-1} + 3 \;//$$

b)
$$a_n = 2a_{n-1} - a_{n-2}$$
$$x^2 = 2x - 1$$
$$x^2 - 2x + 1 = 0 \qquad (x-1)^2 = 0 \qquad x = 1$$
$$\text{Eq}^{n}: \; a_n = U(1)^n + V \cdot n \cdot (1)^n$$
$$a_0 = U + V(0)$$
$$= U = 0$$
$$a_1 = U(1) + V(1)$$
$$= U + V = 3$$
$$\therefore \; a_n = 3n \;//$$

Exercise 48. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

$$a_n = a_{n-1} + 2a_{n-2}$$
$$x^2 = x + 2$$
$$x^2 - x - 2 = 0$$
$$x = 2, \; x = -1$$
$$\therefore \; a_n = U(2)^n + V(-1)^n$$
$$a_0 = 2 = U + V \qquad U = 3$$
$$a_1 = 7 = 2U - V \qquad V = -1$$
$$\therefore \; a_n = 3(2)^n - (-1)^n \;//$$

Exercise 49. Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Proofs here

Exercise 50. Solve the following two recurrence relations:

$$a_n = 3a_{n-1}, \; a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2}, \; b_1 = 0, \; b_2 = 12.$$

a)
$$a_n = 3a_{n-1}$$
$$= 3(3a_{n-2})$$
$$= 3^3 a_{n-3}$$
$$= 3^4 a_{n-4}$$
$$= 3^i a_{n-i} \leftarrow \text{when } i = n-1 \ldots$$
$$= 3^{n-1} a_1 \qquad n-i = n-(n-1) = 1$$
$$= 3^{n-1}(4) \;//$$

b)
$$b_n = 4b_{n-1} - 3b_{n-2}$$
$$x^2 = 4x - 3$$
$$x^2 - 4x + 3 = 0$$
$$x = 3, \; x = 1$$
$$b_n = U(3)^n + V$$
$$b_1 = 0 = 3U + V \qquad V = -6$$
$$b_2 = 12 = 9U + V \qquad U = 2$$
$$\therefore \; b_n = 2(3)^n - 6 \;//$$

Exercises for Chapter 7

Exercise 52. 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \; n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Exercise 53. Consider the set $A = \{1, 2, 3\}$, $P(A)$ =power set of A.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.
- Compute the cardinality of $P(A)$ using the counting approach.

Exercise 54. Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B), \; P(A), \; P(A \cup B), \; P(A \times B).$$

Exercise 55. Prove by contradiction that for two sets A and B

$$(A - B) \cap (B - A) = \varnothing.$$

Exercise 56. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B.$$

Exercise 52. 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \le k \le l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

You will need 1. for this!

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

1

$$= \frac{n!(n-k+1)}{k!\,(n-k+1)!} + \frac{n!\,(k)}{k!\,(n-k+1)!}$$

$$= \frac{n!\big((n-k+1)+k\big)}{k!\,(n-k+1)!} = \frac{n!\,(n+1)}{k!\,(n-k+1)!}$$

$$= \frac{(n+1)!}{k!\,(n-k+1)!}$$

/

2 let $P(n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$ for $1 \le k \le 2$

Basic Step: when $P(1)$, $(x+y)^1 = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1$

$$x+y = (1)\,x + (1)\,y$$

$$x+y = x+y$$

Inductive Step: Assuming $P(i)$ is true, $(x+y)^i = \sum_{k=0}^{i} \binom{i}{k} x^{i-k} y^k$ for $1 \le k \le i$

$$P(i+1) = (x+y)^{i+1}$$

$$= (x+y)\,(x+y)^i$$

$$= (x+y) \sum_{k=0}^{i} \binom{i}{k} x^{i-k} y^k$$

$$= x\left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k} y^k\right) + y\left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k} y^k\right)$$

$$= \left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k+1} y^k\right) + \left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k} y^{k+1}\right)$$ ← change varible $j = k+1$

$$= \left(\sum_{k=0}^{i} \binom{i}{k} x^{i-k+1} y^k\right) + \left(\sum_{j=1}^{i+1} \binom{i}{j-1} x^{i-j+1} y^j\right)$$

Look @ solution !

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

$|S| = n$

$S = \{a, b, c \ldots\ldots, d\}$

2 possibilities for $P(S)$, either include a or exclude a.

for n items, $|P(S)| = 2^n$

Exercise 53. Consider the set $A = \{1, 2, 3\}$, $P(A)$ =power set of A.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.
- Compute the cardinality of $P(A)$ using the counting approach.

a) $|P(A)|$ = empty set + one element + two element + whole set

$$= 1 + \binom{3}{1} + \binom{3}{2} + 1 = 8$$

Exercise 54. Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B),\ P(A),\ P(A \cup B),\ P(A \times B).$$

a) $A \cap B = \{2\}$, $P(A \cap B) = \{\emptyset, \{2\}\}$

$A = \{1,2\}$, $P(A) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\}$

$A \cup B = \{1, 2, 3\}$, $P(A \cup B) =$ - - - -

$A \times B = \{\{1,2\}, \{1,3\}, \{2,2\}, \{2,3\}\} =$ - - -

Exercise 55. Prove by contradiction that for two sets A and B

$$(A - B) \cap (B - A) = \varnothing.$$

By contradiction, $(A-B) \cap (B-A) \neq \emptyset$

$\therefore$ There exists x in A for $x \in A-B$

$\overset{?}{=}$ there does not exist x in A for $x \in A-B$

Exercise 56. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B.$$

$P(A) = P(B) \rightarrow A = B$

If power set the same, all sets that contain one element will be the same.

$\overset{?}{=}$

$A = B \rightarrow P(A) = P(B)$

If sets are the same, $P(A) = P(B)$

Exercise 57. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Exercise 58. Show that the empty set is a subset of all non-null sets.

Let $Y = \emptyset$, Let X = some non-null set

Show $Y \subseteq X$

$\forall x (x \in Y \rightarrow x \in X)$

$x \in Y$ is False, as cannot take any x in null set

$\therefore \forall x (x \in Y \rightarrow x \in X)$ is T

Exercise 57. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Exercise 58. Show that the empty set is a subset of all non-null sets.

Exercise 59. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

Exercise 60. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

Exercise 61. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

Exercise 62. How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.

Exercise 63. Prove the following set equality:

$$\{12a + 25b, a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

Exercise 64. Let A, B, C be sets. Prove or disprove the following set equality:

$$A - (B \cup C) = (A - B) \cap (A - C).$$

Exercise 65. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

Exercise 66. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Exercise 59. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

$\exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)]$

$$\begin{aligned}
A \neq B &= \neg \forall x (x \in A \longleftrightarrow x \in B) \\
&= \exists x \ \neg (x \in A \rightarrow x \in B \ \wedge \ x \in B \rightarrow x \in A) \\
&= \exists x [\neg (x \in A \Rightarrow x \in B) \vee \neg (x \in B \Rightarrow x \in A)] \\
&= \exists x [\neg (x \notin A \vee x \in B) \vee \neg (x \notin B \vee x \in A)] \\
&= \exists x [(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)]
\end{aligned}$$

Exercise 60. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

LHS, can have $A \times B$ & $C \times D$
RHS, can have $A \times B$, $A \times D$, $C \times B$ & $C \times D$

Exercise 61. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

$(A_1 \cup A_2) \times (B_1 \cup B_2) = \{\{A_1, B_1\}, \{A_1, B_2\}, \{A_2, B_1\}, \{A_2, B_2\}\}$
$(A_1 \times B_1) \cup (A_2 \times B_2) = \{\{A_1, B_1\}, \{A_2, B_2\}\}$
Does not.

Exercise 62. How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.

?

Exercise 63. Prove the following set equality:

$$\{12a + 25b, a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

Exercise 64. Let A, B, C be sets. Prove or disprove the following set equality:

$$A - (B \cup C) = (A - B) \cap (A - C).$$

$$\begin{aligned}
A - (B \cup C) &= A \cap (\overline{B \cup C}) \\
&= A \cap (\bar{B} \cap \bar{C}) \\
&= (A \cap \bar{B}) \cap (A \cap \bar{C}) \\
&= (A - B) \cap (A - C)
\end{aligned}$$

Exercise 65. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

$$\begin{aligned}
\overline{(A - B) - (B - C)} &= \overline{(A - B) \cap \overline{(B - C)}} \\
&= \overline{(A \cap \bar{B}) \cap \overline{(B \cap \bar{C})}} \\
&= \overline{(A \cap \bar{B})} \cup (B \cap \bar{C}) \\
&= (\bar{A} \cup B) \cup (B \cap \bar{C}) \\
&= \bar{A} \cup B
\end{aligned}$$

Exercise 66. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

LHS: $x \in (A \cup B) \cap \overline{(A \cap B)} \equiv [(x \in A \vee x \in B) \wedge \overline{(A \cap B)}]$
$\equiv [(x \in A \vee x \in B) \wedge \overline{((x \in A) \wedge (x \in B))}]$
$\equiv [(x \in A \vee x \in B) \wedge ((x \notin A) \vee (x \notin B))]$

for $(x \in A) \wedge ((x \notin A) \vee (x \notin B)) \equiv (x \in A) \wedge (x \notin A) \vee (x \in A) \wedge (x \notin B)$
$\equiv F \vee (x \in A) \wedge (x \notin B)$
$\equiv A - B$

for $(x \in B) \wedge ((x \notin A) \vee (x \notin B)) \equiv ((x \in B) \wedge (x \notin A)) \vee ((x \in B) \wedge (x \notin B))$
$\equiv B - A \vee F$

RHS: $x \in (A - B) \cup (B - A) \equiv (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)$

for $(x \in A \wedge x \notin B)$, $x \in A \cup B$ & $x \notin A \cap B$

for $(x \in B \wedge x \notin A)$, $x \in A \cup B$ & $x \notin A \cap B$

Exercise 67. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.

2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

1. What the fuck

2.

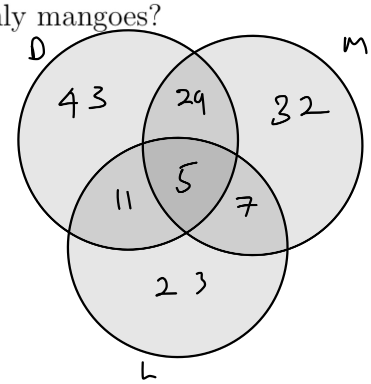
A	B	$A \oplus B$	$(A \oplus B) \oplus B$
F	F	F	F
F	T	T	F
T	F	T	T
T	T	F	T

Exercise 68. In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

$|D| = 88$ $\quad |D \cap M| = 34$ $\quad |D \cap M \cap L| = 5$

$|M| = 73$ $\quad |D \cap L| = 16$

$|L| = 46$ $\quad |M \cap L| = 12$



50 no fruit

32 mango

Exercise 69. Let A, B, C be sets. Prove or disprove the following set equality:

$$A \times (B - C) = (A \times B) - (A \times C).$$

LHS: $A \times (B - C) = x = (x_1, x_2)$, $x_1 \in A$, $x_2 \in B - C$, $x_2 \in B \wedge x_2 \notin C$

$\therefore (x_1, x_2) = (A \times B) - (A \times C)$

Exercises for Chapter 8

Exercise 70. Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \iff (x - y)$ is even. Compute the inverse relation R^{-1}. Compute its matrix representation.

Exercise 71. Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \iff x \mid y$. Compute the matrix of the inverse relation R^{-1}.

Exercise 72. Let R be a relation from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $xRy \leftrightarrow 2|(x - y)$. Show that if n is odd, then n is related to 1.

Exercise 73. This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

   $$R = \{(a_1, b_1), (a_1, b_2)\},\ S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

   What is the matrix of $R \circ S$?

2. In general, what is the matrix of $R \circ S$?

Exercise 74. Consider the relation R on $\mathbb{Z}$, given by $aRb \iff a - b$ divisible by n. Is it symmetric?

Exercise 75. Consider a relation R on any set A. Show that R symmetric if and only if $R = R^{-1}$.

Exercise 76. Consider the set $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Is this relation reflexive? symmetric? transitive?

Exercise 77. Consider the set $A = \{0, 1, 2\}$ and the relation $R = \{(0, 2), (1, 2), (2, 0)\}$. Is R antisymmetric?

Exercise 78. Are symmetry and antisymmetry mutually exclusive?

Exercise 79. Consider the relation R given by divisibility on positive integers, that is $xRy \leftrightarrow x|y$. Is this relation reflexive? symmetric? antisymmetric? transitive? What if the relation R is now defined over non-zero integers instead? x divides y

Reflexive yes $\frac{x}{x}$ ✓

Symmetric NO $\frac{6}{3}$ $\frac{3}{6}$

antisymmetric yes $\frac{x}{y}$, $\frac{y}{x}$, $y = x$

transitive $\frac{x}{y}$ $\therefore x = ay$

$\frac{y}{z}$ $\therefore y = bz$

$\therefore x = abz$

$\therefore \frac{x}{z}$ yes

Exercise 70. Consider the sets $A = \{1, 2\}$, $B = \{1, 2, 3\}$ and the relation $(x, y) \in R \iff (x - y)$ is even. Compute the inverse relation R^{-1}. Compute its matrix representation.

Relation R : $(1,1)$ $(1,3)$ $(2,2)$

inverse R^{-1} : $(1,1)$ $(3,1)$ $(2,2)$

matrix : $\begin{pmatrix} T & F \\ F & T \\ T & F \end{pmatrix}$

Exercise 71. Consider the sets $A = \{2, 3, 4\}$, $B = \{2, 6, 8\}$ and the relation $(x, y) \in R \iff x \mid y$. Compute the matrix of the inverse relation R^{-1}.

x divides y

$R = (2,2)$ $(2,6)$ $(2,8)$ $(3,6)$ $(4,8)$

$R^{-1} = (2,2)$ $(6,2)$ $(8,2)$ $(6,3)$ $(8,4)$

$R^{-1} = \begin{pmatrix} T & F & F \\ T & T & F \\ T & F & T \end{pmatrix}$

Exercise 72. Let R be a relation from $\mathbb{Z}$ to $\mathbb{Z}$ defined by $xRy \leftrightarrow 2|(x - y)$. Show that if n is odd, then n is related to 1.

$n = 2x - 1$

$n + 1 = 2x$

n is related to 1

Exercise 73. This exercise is about composing relations.

1. Consider the sets $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $C = \{c_1, c_2, c_3\}$ with the following relations R from A to B, and S from B to C:

$$R = \{(a_1, b_1), (a_1, b_2)\}, \ S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}.$$

What is the matrix of $R \circ S$?

2. In general, what is the matrix of $R \circ S$?

$R = (a_1, b_1)$ (a_1, b_2)

$S = (b_1, c_1)$ (b_2, c_1) (b_1, c_3) (b_2, c_2)

$\therefore$ $R \circ S = (a, c_1)$ (a, c_3) (a, c_2)

① write matrix R, & S

$\begin{pmatrix} T & T \\ F & F \end{pmatrix} \circ \begin{pmatrix} T & F & T \\ T & T & F \end{pmatrix}$

② find $(a, c) \in R \circ S$, where $aRb \wedge bSc$ for $b \in B$

Exercise 74. Consider the relation R on $\mathbb{Z}$, given by $aRb \iff a - b$ divisible by n. Is it symmetric?

$a - b$ divisible by n

$b - a = -(a - b)$ which is divisible by n

$\therefore$ Symmetric

Exercise 75. Consider a relation R on any set A. Show that R symmetric if and only if $R = R^{-1}$.

Relation R: (x, y)

R^{-1}: (y, x)

if $R = R^{-1}$, $x = y$ & $y = x$

$\therefore$ Symmetric.

Exercise 76. Consider the set $A = \{a, b, c, d\}$ and the relation

$$R = \{(a, a), (a, b), (a, d), (b, a), (b, b), (c, c), (d, a), (d, d)\}.$$

Is this relation reflexive? symmetric? transitive?

$\begin{matrix} T & T & F & T \\ T & T & F & F \\ F & F & T & F \\ T & F & F & T \end{matrix}$

→ Reflexive, every element related to itself ✓

Symmetric ✓

Not transitive.

$bRa \circ aRd$

$= (b, d) \notin R$

Exercise 77. Consider the set $A = \{0, 1, 2\}$ and the relation $R = \{(0, 2), (1, 2), (2, 0)\}$. Is R antisymmetric?

NO. $(0, 2) \in R$

$(2, 0) \in R$

$2 \neq 0$

Exercise 80. Consider the set $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Show that the relation $xRy \leftrightarrow 2|(x - y)$ is an equivalence relation.

Exercise 81. Show that given a set A and an equivalence relation R on A, then the equivalence classes of R partition A.

Exercise 82. Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation

$$xRy \leftrightarrow \exists c \in \mathbb{Z},\ y = cx.$$

Is R an equivalence relation? is R a partial order?

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80) Equivalence 1) Reflec
2) Symm
3) Transitive

Reflexive, $2|(x-x)$ ✓

Symmetric, $2|(x-y)$
$x-y=2n$
$y-x=-(x-y)$
$=-2n$ ✓

Transistive, $2|(x-a)$
$x-a=2n$, $x-2n=a$

$2|(a-y)$
$a-y=2m$
$x-2n-y=2m$
$x-y=2m+2n$
$x-y=2(m+n)$ ✓

Exercise 81. Show that given a set A and an equivalence relation R on A, then the equivalence classes of R partition A.

Exercise 82. Consider the set $A = \{2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the relation

$$xRy \leftrightarrow \exists c \in \mathbb{Z},\ y = cx.$$

Is R an equivalence relation? is R a partial order?

reflexive: $xRx \longleftrightarrow \exists c \in \mathbb{Z},\ y = cy$, $c = 1$

symmetric: $xRy \Leftrightarrow \exists c \in \mathbb{Z},\ y = cx$, but for yRx $x = cy$

$y = c(cy)$

$y = c^2 y$ if $c = -1/1$, OK

but when $c \neq -1/1$, then not sym.

$\therefore$ X equivalence

Partial order?

anti symmetric: xRy, $y = cx$; yRx, $x = cy$ $\Big\}$ $y = c(cy)$ If $c = 1$

Implies $y = x(1)$

$y = x$

Transitive: yes ?

Exercises for Chapter 9

Exercise 83. Consider the set $A = \{a, b, c\}$ with power set $P(A)$ and $\cap$: $P(A) \times P(A) \to P(A)$. What is its domain? its co-domain? its range? What is the cardinality of the pre-image of $\{a\}$?

Exercise 84. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one.

Exercise 85. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not onto, but $\sin : \mathbb{R} \to [-1, 1]$ is.

Exercise 86. Is $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 4n - 1$, onto (surjective)?

Exercise 87. Is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, a bijection (one-to-one correspondence)?

Exercise 88. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 5$. What is $g \circ f$? What is $f \circ g$?

Exercise 89. Consider $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$. What is $g \circ f$? What is $f \circ g$?

Exercise 90. Given two functions $f : X \to Y$, $g : Y \to Z$. If $g \circ f : X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counter-example.

Exercise 91. Show that if $f : X \to Y$ is invertible with inverse function $f^{-1} : Y \to X$, then $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$.

Exercise 92. Prove or disprove $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$, for x, y two real numbers.

Exercise 93. If you pick five cards from a deck of 52 cards, prove that at least two will be of the same suit.

Exercise 94. If you have 10 black socks and 10 white socks, and you are picking socks randomly, you will only need to pick three to find a matching pair.

Exercise 95. Prove that the set of all integers is countable.

Exercise 83. Consider the set $A = \{a, b, c\}$ with power set $P(A)$ and $\cap$: $P(A) \times P(A) \to P(A)$. What is its domain? its co-domain? its range? What is the cardinality of the pre-image of $\{a\}$?

Exercise 84. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not one-to-one.

$y = f(x)$, $f(x) = \sin(x)$

$\sin(0) = 0$ $\quad 0 \neq 2\pi$

$\sin(2\pi) = 0$

Exercise 85. Show that $\sin : \mathbb{R} \to \mathbb{R}$ is not onto, but $\sin : \mathbb{R} \to [-1, 1]$ is.

$-1 \leq \sin(x) \leq 1$ for $x \in \mathbb{R}$

for $\exists y \in \mathbb{R}$, $y = 2$, $f(x) \neq 2$

Exercise 86. Is $h : \mathbb{Z} \to \mathbb{Z}$, $h(n) = 4n - 1$, onto (surjective)?

$y = f(x)$

$y = 4x - 1$

$\frac{y+1}{4} = x$

$x = \frac{1}{4} + \frac{1}{4}y$ $\quad \therefore$

when $y = 1$, $x = \frac{1}{2}$, $x \notin \mathbb{Z}$ not surjective

Exercise 87. Is $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, a bijection (one-to-one correspondence)?

injective

$f(x_1) = f(x_2)$

$x_1^3 = x_2^3$

$x_1 = x_2$ ✓

surj

$y = x^3$

$x = \sqrt[3]{y}$ $\quad \sqrt[3]{y} \in \mathbb{R}$ ✓

$\therefore$ Bij

Exercise 88. Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x + 5$. What is $g \circ f$? What is $f \circ g$?

$g \circ f = g(f(x))$

$= x^2 + 5$

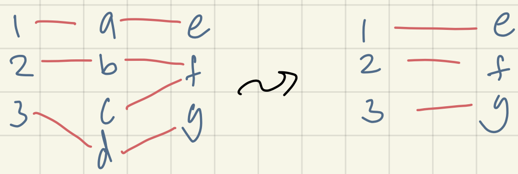
$f \circ g = f(g(x))$

$= (x+5)^2$

Exercise 89. Consider $f : \mathbb{Z} \to \mathbb{Z}$, $f(n) = n + 1$ and $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$. What is $g \circ f$? What is $f \circ g$?

$g \circ f = (n+1)^2$ $\quad f \circ g = n^2 + 1$

Exercise 90. Given two functions $f : X \to Y$, $g : Y \to Z$. If $g \circ f : X \to Z$ is one-to-one, must both f and g be one-to-one? Prove or give a counter-example.



Exercise 91. Show that if $f : X \to Y$ is invertible with inverse function $f^{-1} : Y \to X$, then $f^{-1} \circ f = i_X$ and $f \circ f^{-1} = i_Y$.

take $x \in X$

$y = f(x)$

$f^{-1}(y) = f^{-1}(f(x))$

Exercise 92. Prove or disprove $\lceil x+y \rceil = \lceil x \rceil + \lceil y \rceil$, for x, y two real numbers.

let $x = 1/2$, $y = 1/2$

$\lceil x+y \rceil = \lceil 1 \rceil = 1$

$\lceil x \rceil + \lceil y \rceil = \lceil 1/2 \rceil + \lceil 1/2 \rceil = 2$

Exercise 93. If you pick five cards from a deck of 52 cards, prove that at least two will be of the same suit.

Pigeonhole principle

Exercises for Chapter 10

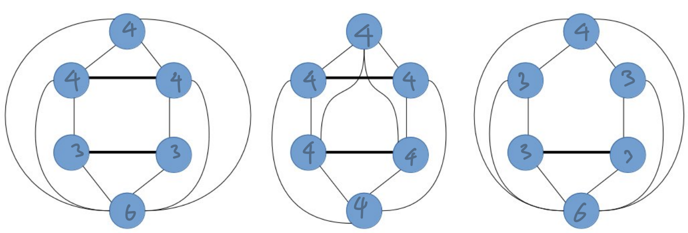
Exercise 96. Prove that if a connected graph G has exactly two vertices which have odd degree, then it contains an Euler path.

Exercise 97. Draw a complete graph with 5 vertices.

Exercise 98. Show that in every graph G, the number of vertices of odd degree is even.

Exercise 99. Show that in very simple graph (with at least two vertices), there must be two vertices that have the same degree.

Exercise 100. Decide whether the following graphs contain a Euler path/cycle.



P	yes 2 odd	y	N
Cy.	NO some odd exist	y	N