3. Logic Gates and Boolean Algebra

Digital circuits can be found in smart phones, computers, washing machines, cars, etc.

Logic gates are the basic building blocks of digital circuits.

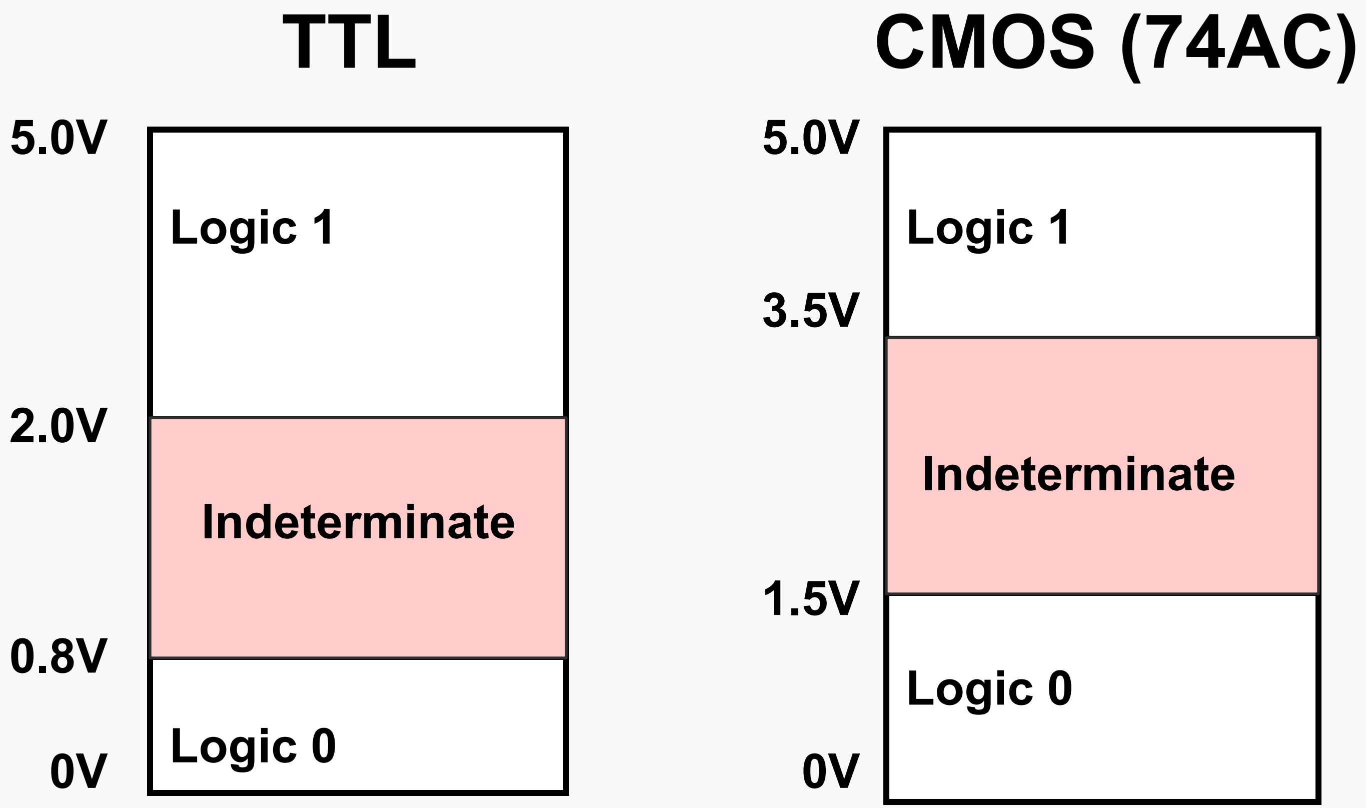
Boolean algebra is used to describe, design and simplify digital circuits.

Boolean Constants – only 2 values

- TRUE, FALSE
- Logic HIGH, Logic LOW
- HI, LO
- 1, 0

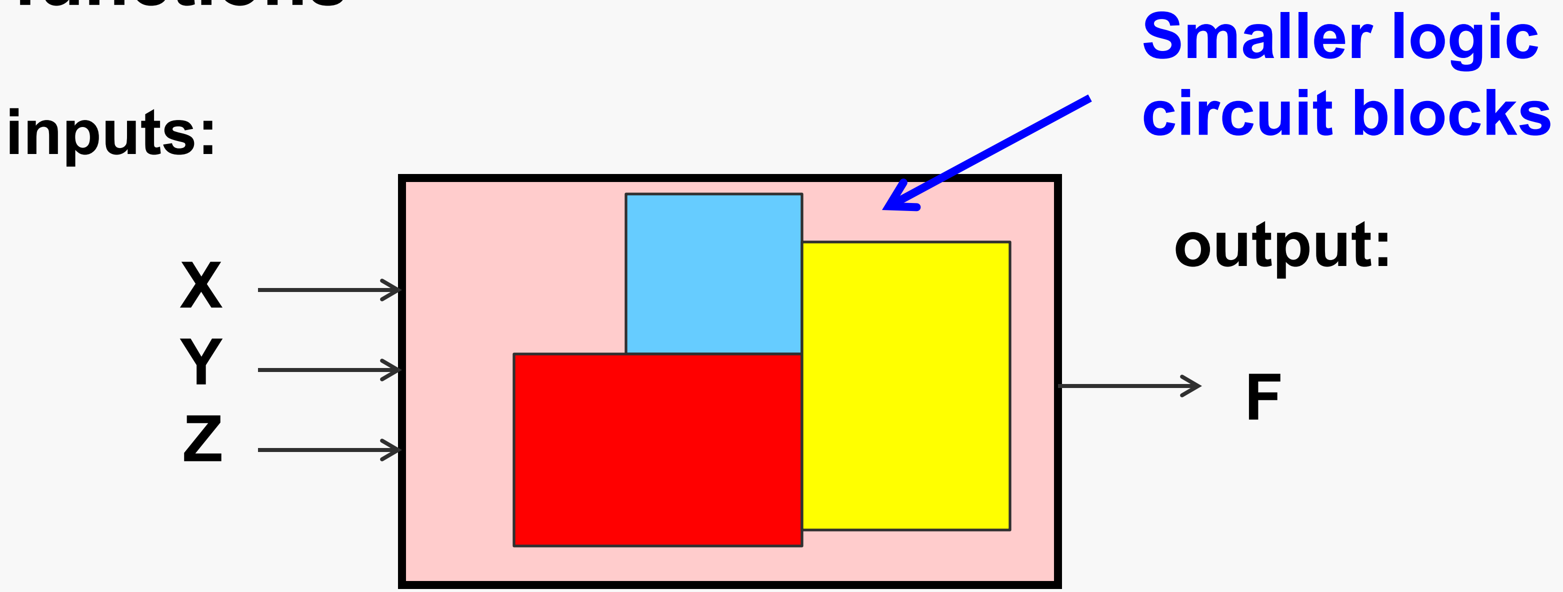
Boolean (logic) variables can only assume one of the two values

Common Logic-level voltage ranges



A typical logic circuit

- 1 or more logic inputs
- 1 or more logic outputs
- Outputs are related to inputs by logic functions



Truth table

- Logic function can be fully described by a **Truth Table**.
- The Truth table
  - shows how a logic circuit's output responds to various combinations of logic inputs
  - has 2^N number of input combinations for N inputs
  - lists all possible input combinations in the binary counting sequence

A typical N-input truth table

2^N rows

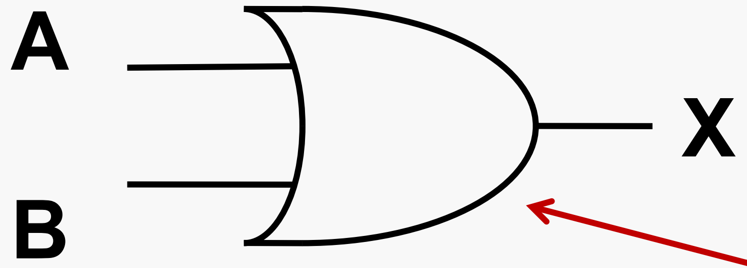
Input 1	Input 2	…	Input N-1	Input N	Output
0	0	…	0	0	1
0	0	…	0	1	0
0	.	…	1	0	0
0	.	…	1	1	.
.	.	…	.	.	.
.	.	…	.	.	.
1	.	…	0	0	.
1	.	…	0	1	.
1	1	…	1	0	.
1	1	…	1	1	.

3 Basic Logic Operations

- **Logical addition - OR**
- **Logical multiplication - AND**
- **Logical complement or inversion - NOT**

In digital circuits, these are realised by electronic devices called logic gates.

Logical OR Operation



Logic symbol

- Logical OR operation
  - X = A + B
  - X = A OR B

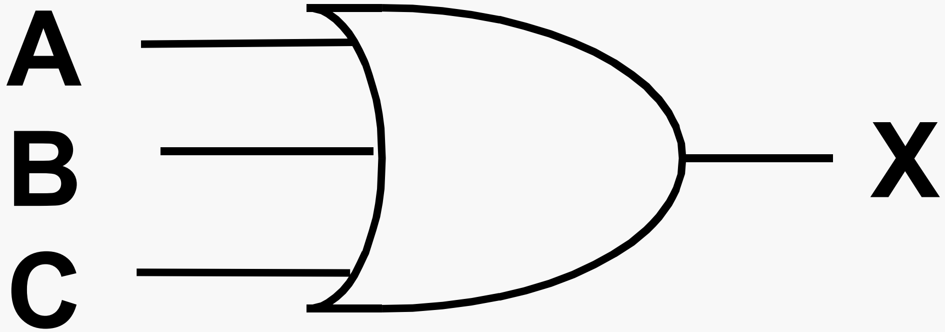
Logic expression

- 2-input OR gate: X = A + B

Truth table for a 2-input OR gate
(X = A+B)

Inputs		Output
A	B	X
0	0	0
0	1	1
1	0	1
1	1	1

X=1 if at least one input is = 1



3-input OR gate:

$$X = A + B + C$$

OR operation result will be 1 if at least one input is 1

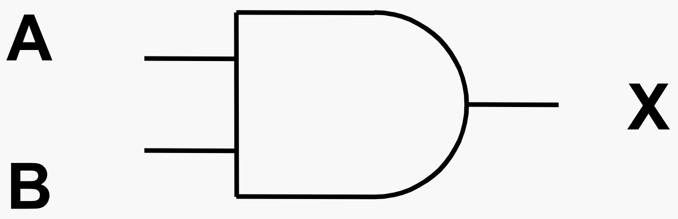
Truth Table for a 3-input OR gate ($X = A+B+C$)

Inputs			Output
A	B	C	X
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Logical AND operation

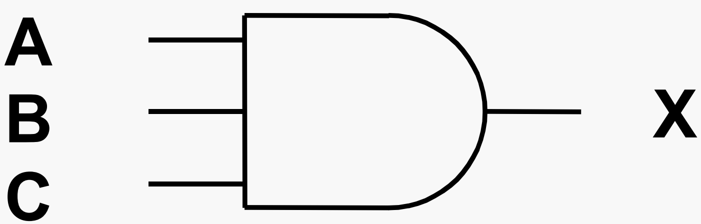
Logical AND operation

- $X = A \bullet B$
- X = A AND B
- $X = AB$



2-input AND gate:

- $X = AB$



3-input AND gate:

- $X = ABC$

AND operation result will be 1 only if all inputs are 1

Truth table for 2-input AND gate
(X = AB)

Inputs		Output
A	B	X
0	0	0
0	1	0
1	0	0
1	1	1

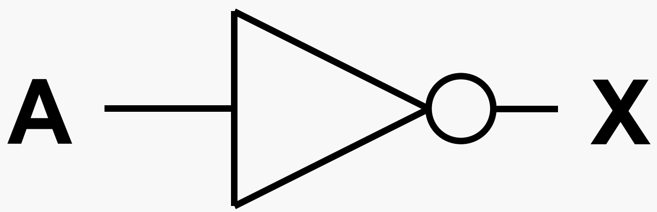
Output X=1 only if all inputs are 1

The Truth Table for a 3-input AND gate (X = ABC)

Inputs			Output
A	B	C	X
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Logical NOT operation

- NOT gate only has 1 input and it is commonly known as an inverter
- the output is the complement/inverse of the input
- $X = \overline{A}$



- X = A' ⟵ we will use this notation

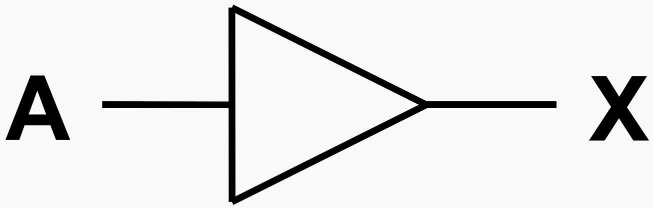
Logical NOT operation

- Its truth table is very simple

input	output
A	X = A’
0	1
1	0

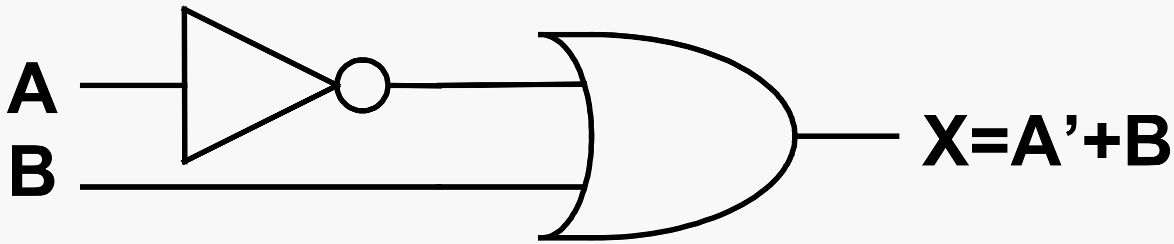
Buffer

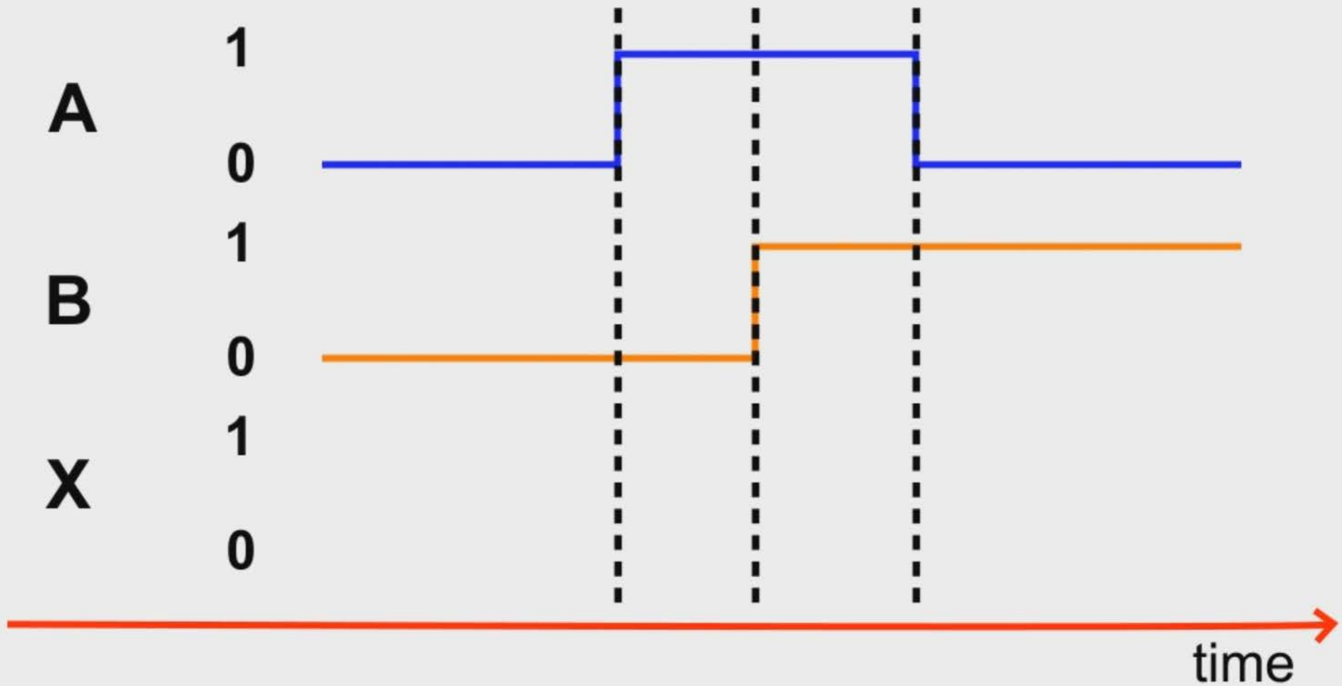
- Its truth table is also very simple
- No change in logic



input	output
A	X = A
0	0
1	1

- **Example: Sketch the logic waveform of X**





- **Timing diagram with more realistic appearance, showing propagation delays, rise time and fall time**

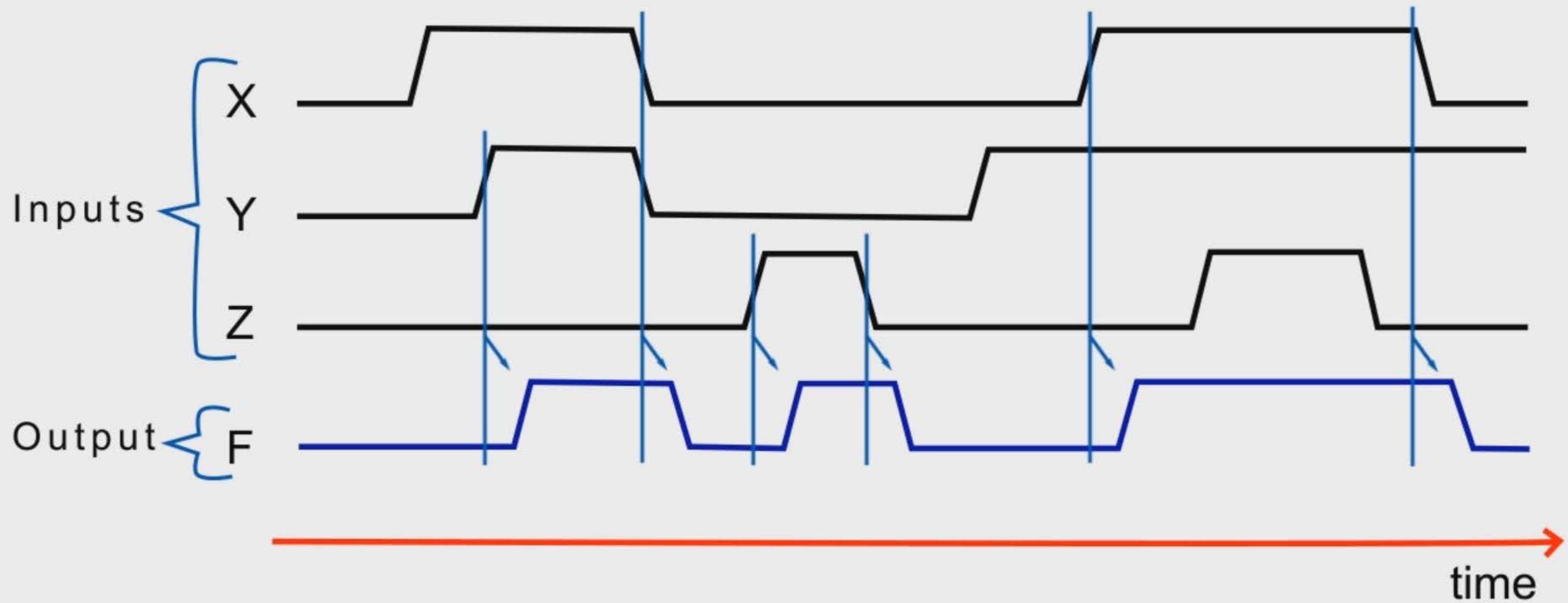


Fig. 3.17 (taken from Wakerly)

Boolean Algebra

- Helps to analyse logic circuits.
- Express operations mathematically.
- Similar to normal algebra but much simpler.
- It does not have fraction or negative number.

Order of Precedence in Boolean Algebra:

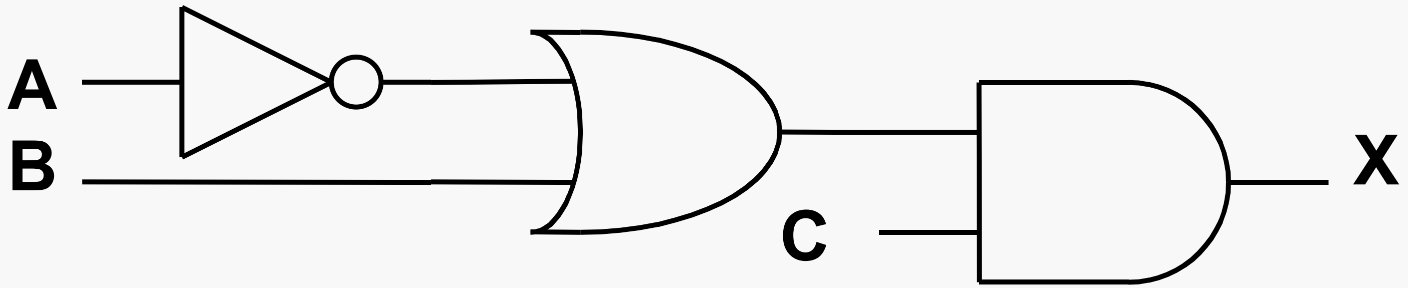
- Complement over a single variable (inversion)
- Expression within parentheses
- AND
- OR

Examples:

1. $Y = A + B\,C'$
2. $Y = (A + B)\,C'$
3. $Y = A + (B\,C)'$

Describing logic circuits algebraically

- **AND, OR and NOT operations**
  - **are basic building blocks of digital system**
  - **can completely describe any logic circuit**
- **Example: express output X in terms of inputs A, B and C**



Evaluating logic circuit outputs

From a Boolean expression, the logic level of an output can be determined for any values of the circuit inputs.

Example

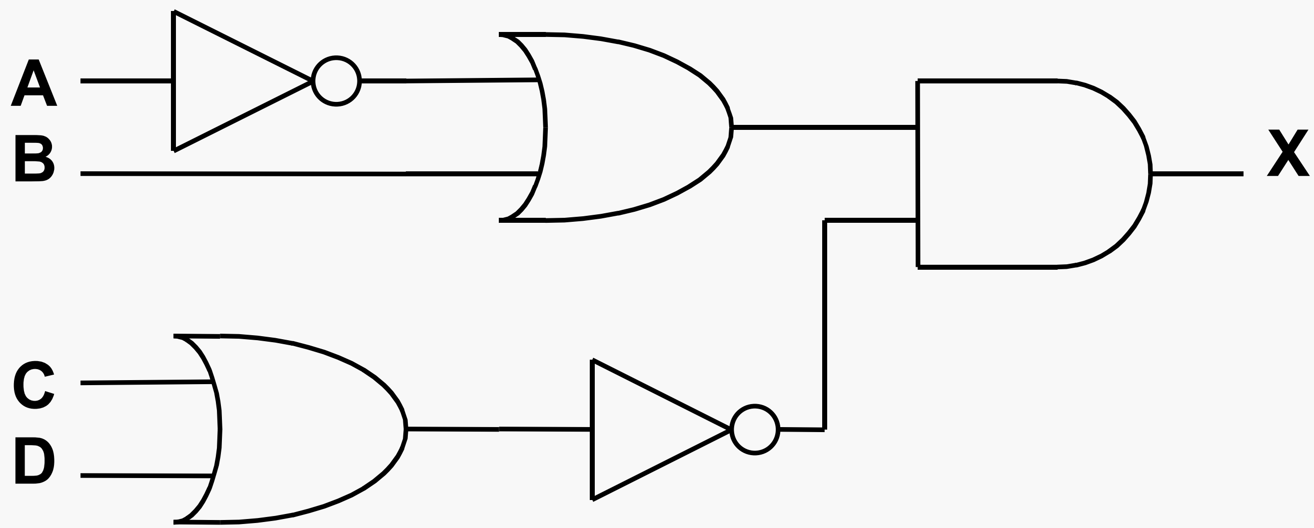
X = A'(B+C)(A+D)'

If inputs A,B,C,D = 0,1,1,0
X = ?

If inputs A,B,C,D = 1,1,1,1
X = ?

If inputs A,B,C,D = 0,0,0,0
X = ?

Determining Instantaneous Output Level from a Logic Circuit Diagram



e.g. if A=1, B=1, C=0, D=0, then X=?

Implementing Circuits from Boolean expressions

Example Y = AC + BC' + A'BC

Boolean Theorems

- Many of the theorems are similar to those in normal algebra.
- The theorems can be used to simplify logic expressions and therefore can help to simplify logic circuits.
- Simpler circuits cost less to build and are less prone to failure.

Boolean Theorems

Axioms:

$X = 0$ if $X \neq 1$

$X = 1$ if $X \neq 0$

$0 \cdot 0 = 0$

$1 \cdot 1 = 1$

$0 \cdot 1 = 1 \cdot 0 = 0$

$1 + 1 = 1$

$0 + 0 = 0$

$1 + 0 = 0 + 1 = 1$

Single variable theorems:

$X \bullet 0 = 0$	$X + 1 = 1$
$X \bullet 1 = X$	$X + 0 = X$
$X \bullet X = X$	$X + X = X$
$X \bullet X' = 0$	$X + X' = 1$
	$(X')' = X$

Duality: any theorem or identity in switching algebra remains true if 0 and 1 are swapped and · and + are swapped throughout

Single variable theorems:

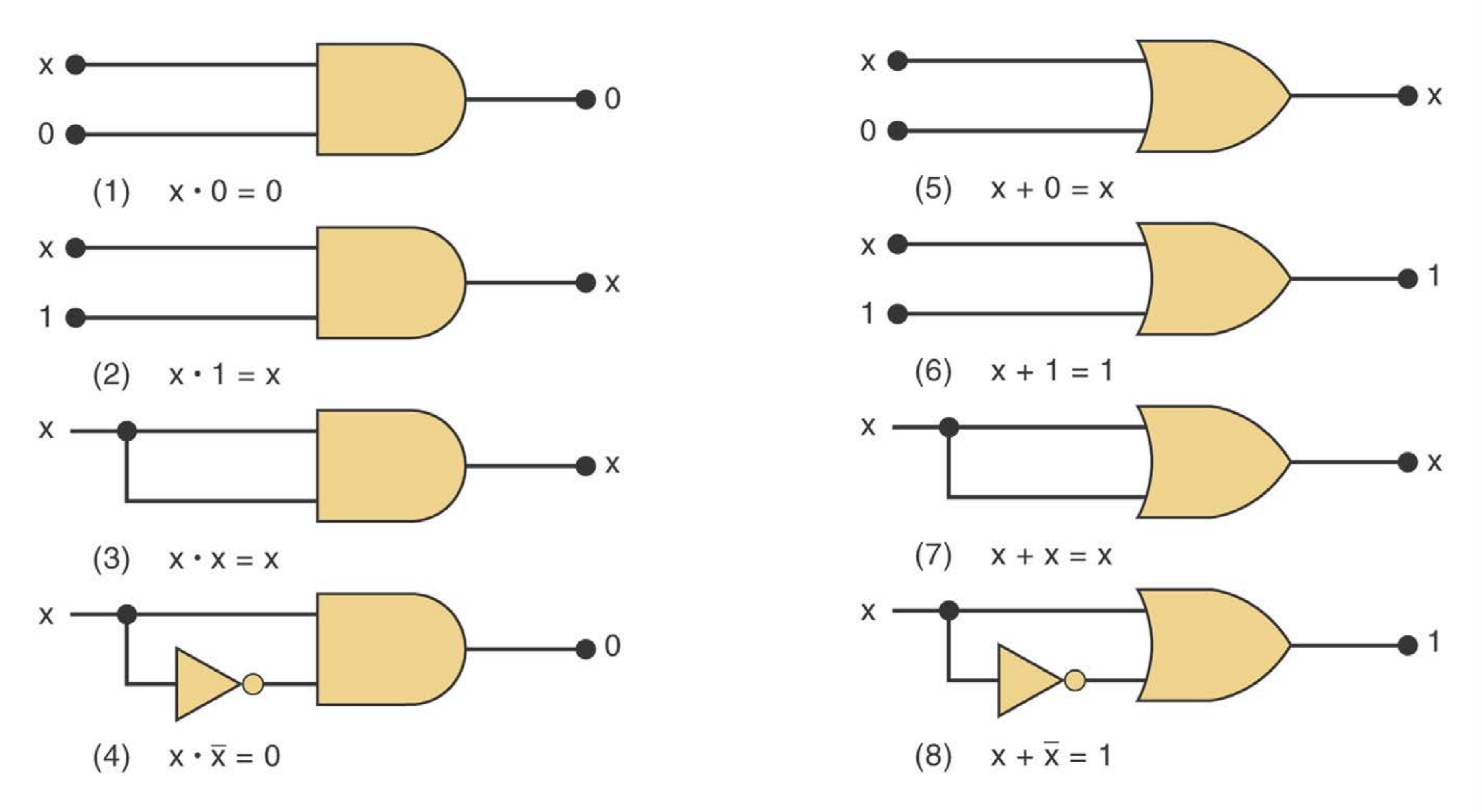


Fig. 3-25 (Tocci, 10th ed. Pg. 77)

Multivariable theorems:

- **Commutative laws:**

  $$A + B = B + A$$

  $$A \bullet B = B \bullet A$$

- **Associative laws:**

  $$A + (B + C) = (A + B) + C = A + B + C$$

  $$A(BC) = (AB)C = ABC$$

- **Distributive laws:**

  $$A(B + C) = AB + AC$$

  $$(A + B)(C + D) = AC + BC + AD + BD$$

Absorption laws

$$A + AB = A$$

$$A + A'B = A + B$$

proof:

$$\begin{aligned} A + AB &= A(1 + B) \\ &= A \bullet 1 = A \end{aligned}$$

proof:

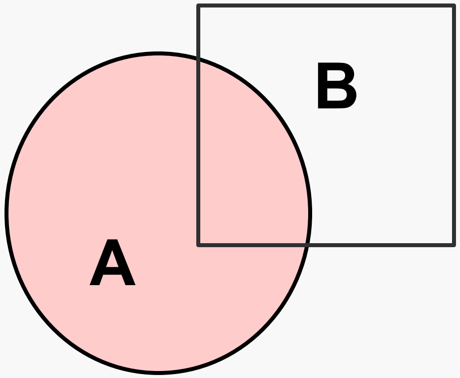
$$\begin{aligned} A + A'B &= A + AB + A'B \\ &= A + (A + A')B \\ &= A + (1)\ B \\ &= A + B \end{aligned}$$

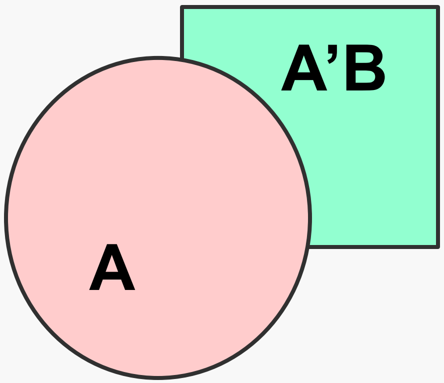
A technique commonly used in algebraic simplification

Absorption laws (Venn diagrams)

A + AB = A

A + A’B = A + B





Consensus

$$AB + A'C + BC = AB + A'C$$

proof:

$BC = ABC + A'BC$

Thus $AB + A'C + BC = AB + A'C + ABC + A'BC$

$= AB + ABC + A'C + A'BC$

$= AB\,(1+C) + A'C(1+B)$

$= AB + A'C$

DeMorgan's Theorems

- $(A + B)' = A' \bullet B'$

proof:

A	B	$(A+B)'$	$A' \bullet B'$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

DeMorgan's Theorems

- (AB)' = A' + B'

proof:

A	B	(AB)'	A' + B'
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

DeMorgan's Theorems generalise to many variables

$$(A+B+C+D+\ldots)' = A' \bullet B' \bullet C' \bullet D' \bullet \ldots$$

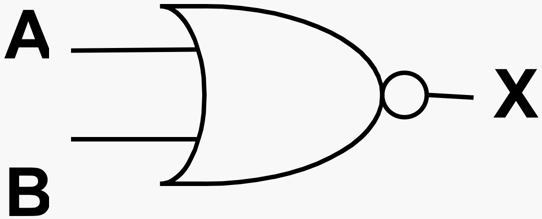
$$(ABCD\ldots)' = A'+B'+C'+D'+ \ldots$$

- Add or remove inverter to each variable
- Interchange AND with OR

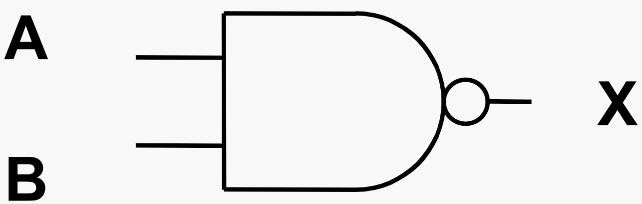
Example
Simplify [A (B + C')' D]'

NOR gate & NAND gate

- Combines basic operations of OR & AND with NOT
- NOR: $X = (A+B)'$



- NAND: $X = (AB)'$



Truth table for 2-input NOR gate
X = A NOR B

Inputs		Output
A	B	X
0	0	1
0	1	0
1	0	0
1	1	0

Output X=1 only when all inputs are 0

Truth table for 2-input NAND gate
X = A NAND B

Inputs		Output
A	B	X
0	0	1
0	1	1
1	0	1
1	1	0

Output X=0 only when all inputs are 1

Truth table for 3-input NOR gate
$X = (A+B+C)'$

Inputs			Output
A	B	C	X
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Truth table for 3-input NAND gate
X = (ABC)’

Inputs			Output
A	B	C	X
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

Summary

- **OR**: output is 1 when <u>any</u> of the inputs is 1
- **AND**: output is 1 when <u>all</u> the inputs are 1
- **NOT**: output is 0 when input is 1 and vice versa
- **NOR**: output is 0 when <u>any</u> of the inputs is 1
- **NAND**: output is 0 when <u>all</u> the inputs are 1

Universality of NAND gates and NOR gates

- NAND gates can be used to form AND gate, OR gate and NOT gate
- Therefore, NAND gates can be used to implement any Boolean function
- Similarly for NOR gates
- Equivalence can be proved by DeMorgan’s theorems

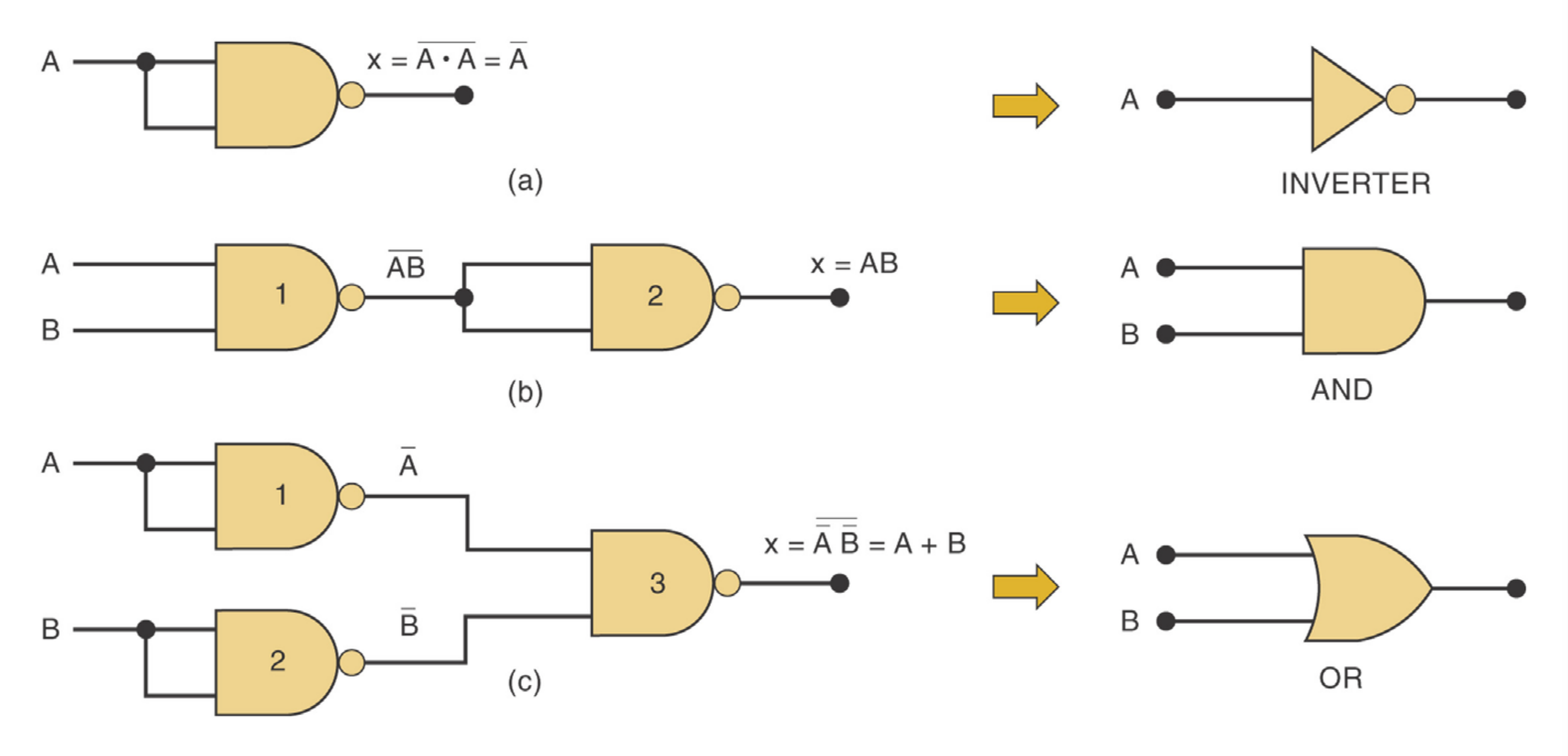


Fig. 3-29 Basic gates from NAND

(Tocci, 10th ed. Pg. 84)

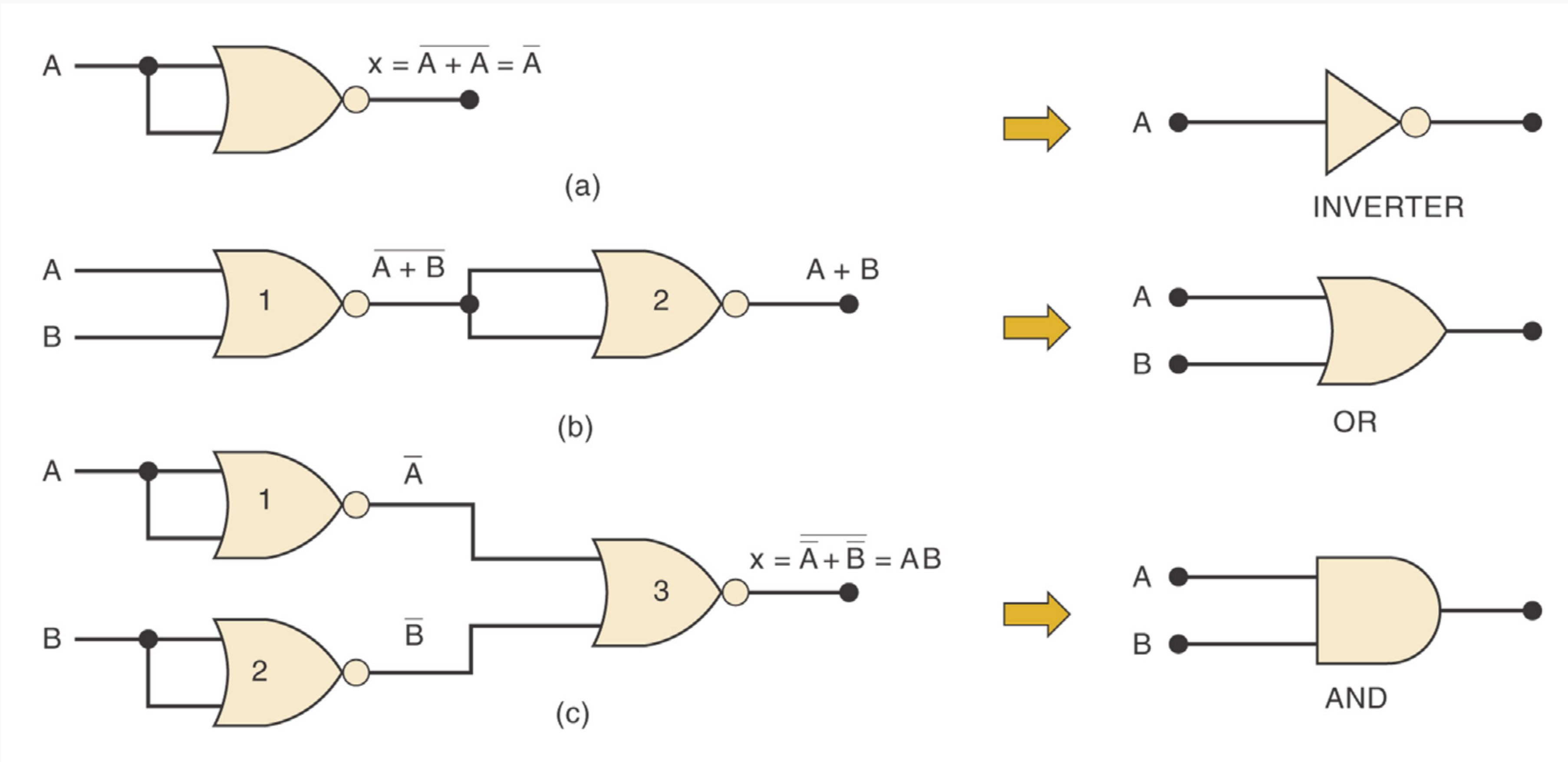
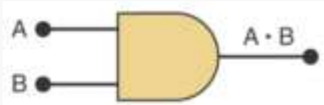
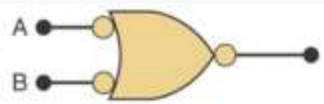
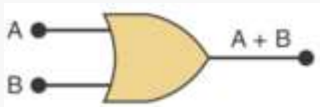
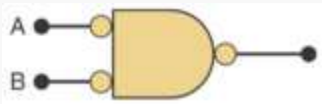
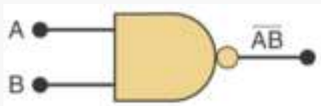
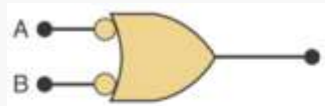
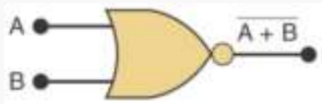
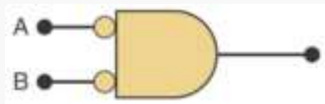


Fig. 3-30 Basic gates from NOR

(Tocci, 10th ed. Pg. 84)

Alternate Logic gate representations

- **The alternate symbol is obtained from the standard symbol by applying DeMorgan's theorems**

Logic operator	Standard symbol	Logic expression	Alternate Symbol
AND		A.B = (A' + B')'	
OR		A+B = (A'.B')'	
NAND		(A.B)' = A'+B'	
NOR		(A+B)' = A'.B'	

- Modification of standard to alternate symbol (and vice-versa) for the **same logic operator**:

	standard ⟷	**alternate**
Bubble at input or output	No	Add
	Yes	Remove
Symbol shape	"and"	Replace with "or"
	"or"	Replace with "and"

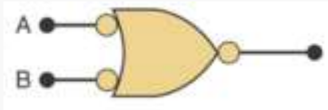
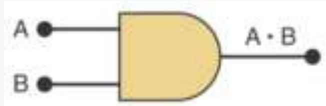
Note that a pair of standard and alternate symbols describes the **same logic gate** with the same truth table.

- **The standard symbol of a NOT gate is similarly modified to its alternate symbol (and vice-versa) but there is no change in symbol shape:**

We interpret the above symbol this way: Output=0 when input=1

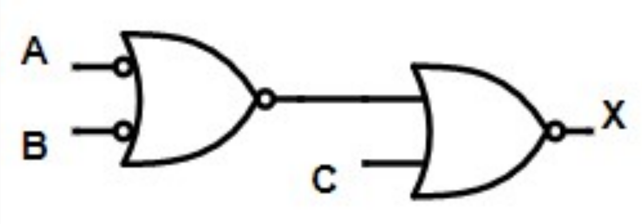
NOT gate truth table	
Input A	Output A'
0	1
1	0

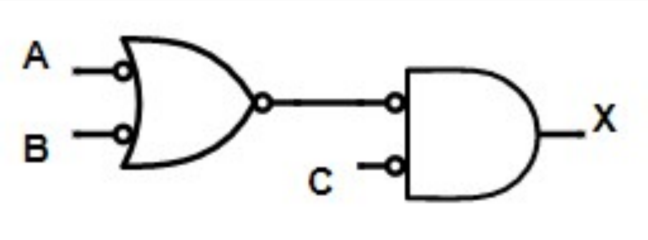
We interpret the above symbol this way: Output=1 when input=0

A	B	A AND B	symbol
0	0	0	
0	1	0	
1	0	0	
1	1	1	

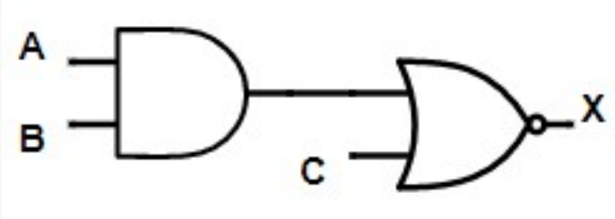
- **Both standard symbols and alternate symbols may be used in the same diagram to help describe logic flow**
- **Comply with bubble-to-bubble matching**
- **Say "0" when there is a bubble, otherwise say "1"**

Example 1:



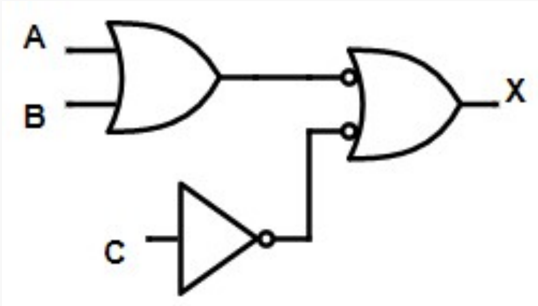


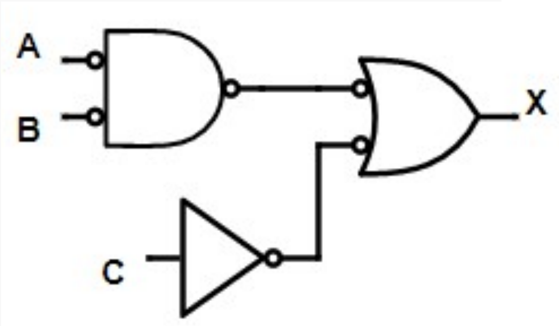
We say:
Output X=1
when inputs
C=0 <u>and</u> at the
same time either
A=0 <u>or</u> B=0

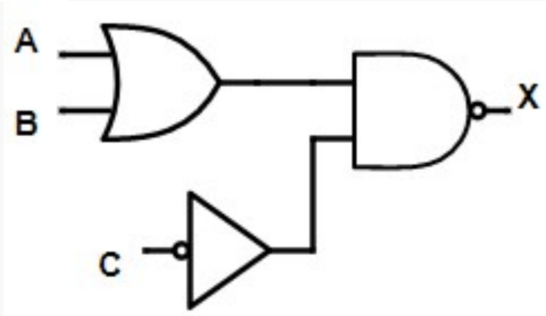


We say:
Output X=0
when inputs
C=1 <u>or</u>
A <u>and</u> B are
both=1

Example 2:







We say:
Output X=1
when inputs
C=1 or both A and
B=0

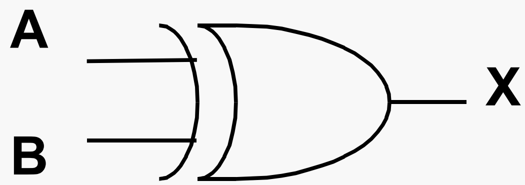
We say:
Output X=0
when inputs
C=0 and at the same
time either A=1 or
B=1

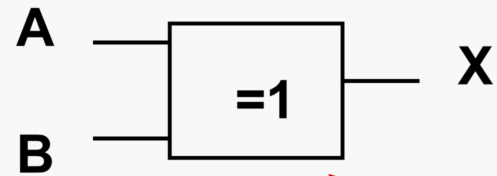
Exclusive-OR gate

Ex-OR (XOR)

$X = AB' + A'B$

$X = A \oplus B$





IEEE symbol

A	B	X
0	0	0
0	1	1
1	0	1
1	1	0

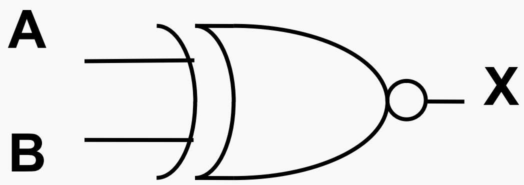
Different from OR

Exclusive-NOR gate

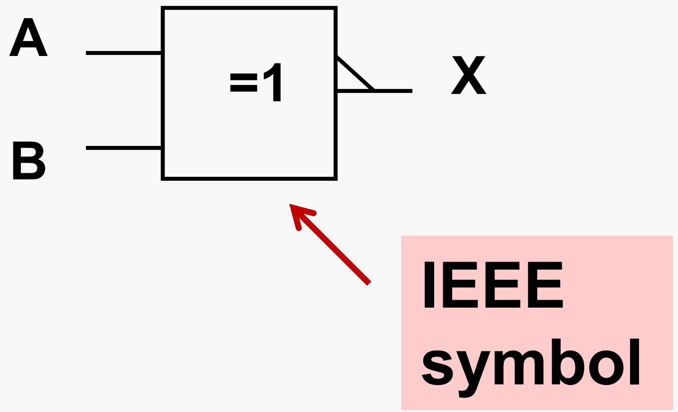
Ex-NOR (XNOR)

$X = AB + A'B'$

$X = (A \oplus B)'$



A	B	X
0	0	1
0	1	0
1	0	0
1	1	1



Application of XOR

- Bit-wise comparator
  - output is 1 if the two multi-bit inputs are different

XOR with multiple inputs

- Essentially an odd-function generator
- Output is 1 if there is an odd number of 1's among all the inputs
- E.g. for 3-input XOR, the output is 1 if there are 1 or 3 bits of 1 among the inputs

- $A \oplus B \oplus C = (A \oplus B) \oplus C$
- $= A \oplus (B \oplus C)$

Logic devices

- Different ways to create a physically functioning logic circuit
- Examples: use standard logic integrated circuits (ICs), application-specific ICs (ASICs), programmable logic devices
- Small-scale integrated logic devices: AND, OR, NOT, NAND, NOR, XOR, XNOR
- We will use some of these in lab experiment 1

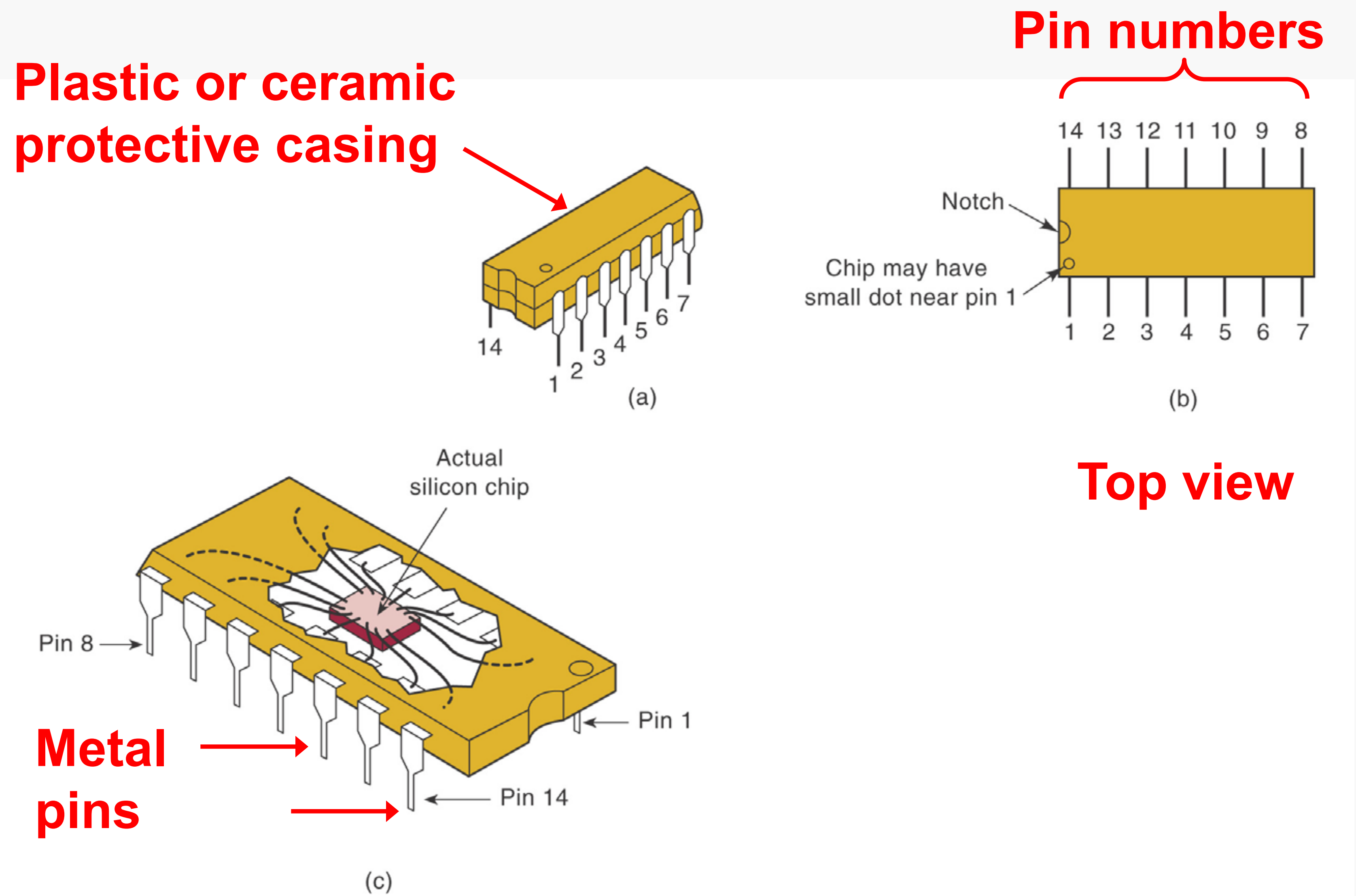
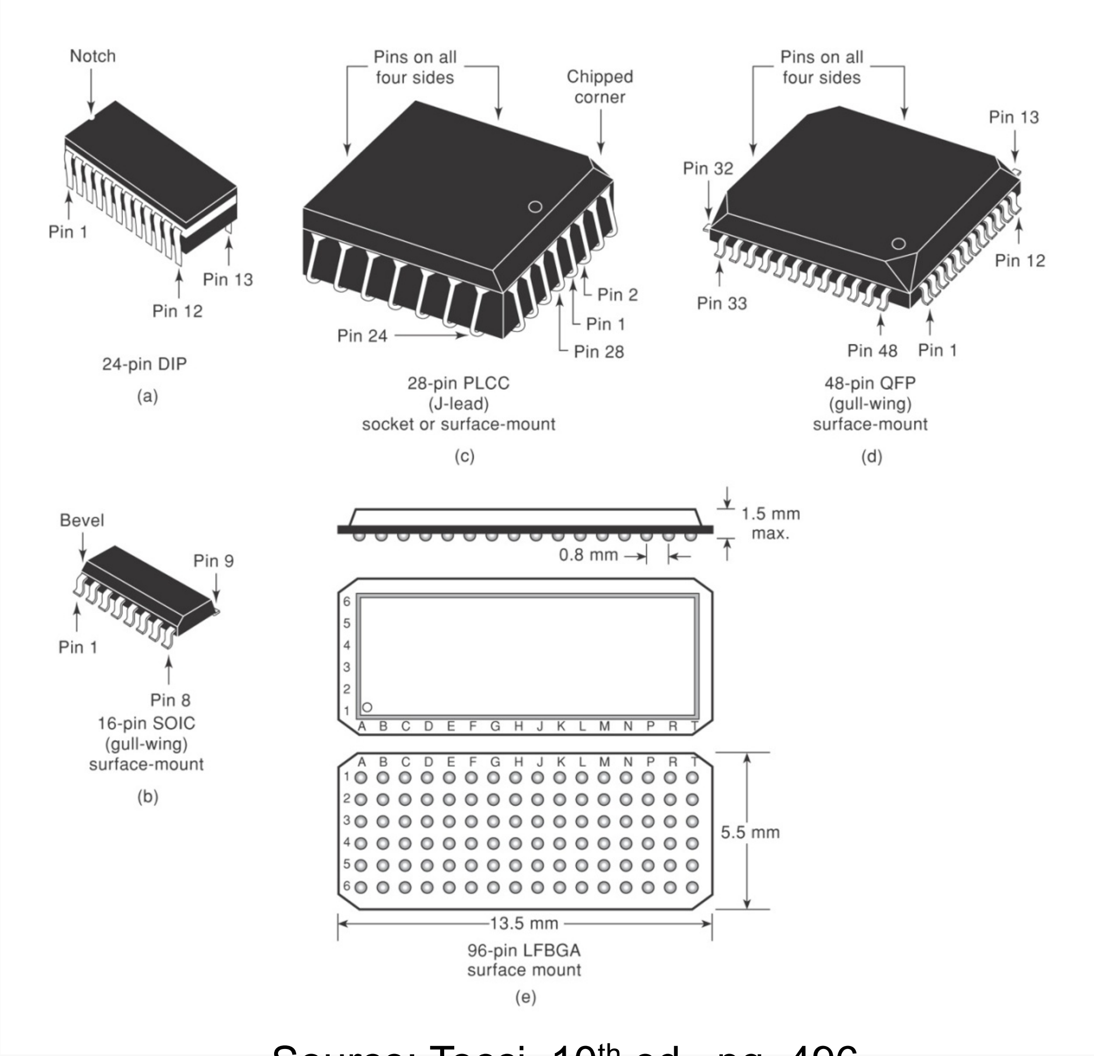


Figure 4.29: (Tocci 10th Ed) Dual-in-line Package

Common IC packaging

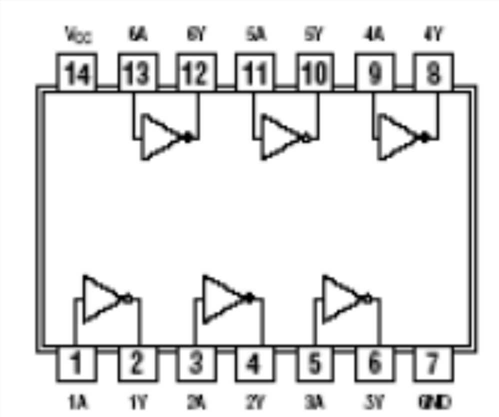


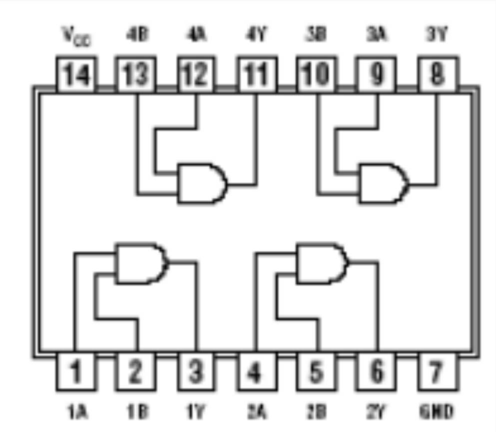
Source: Tocci, 10th ed., pg. 496

Logic circuit connections

- **Implement Y = AB'**

7404
Hex-NOT





7408
Quad-AND

Circuit connection diagram