NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2020–2021

MH1812 – Discrete Mathematics

May 2021 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

QUESTION 1. **(16 marks)**

(a) Prove or disprove the following logical equivalence.

$$\neg(q \to \neg p) \vee \neg(r \to \neg p) \equiv (\neg q \to r) \wedge p$$

(b) Decide whether or not the following argument is valid:

$$\begin{aligned}
&p \vee \neg q;\\
&\neg q \to r;\\
&r \vee p;\\
&r \wedge \neg s;\\
&\therefore p
\end{aligned}$$

Justify your answer.

QUESTION 2. **(16 marks)**

(a) Using the characteristic equation, solve the recurrence relation

$$a_0 = 1,\ a_1 = 2, \quad a_n = 5a_{n-2} + 4a_{n-1} \quad \text{for all } n \geqslant 2,$$

that is, write a_n in terms of n. Show each step of the characteristic equation method.

(b) Prove by induction that, for all integers $n \geqslant 2$,

$$\sum_{k=2}^{n} \binom{k}{2} = \frac{(n-1)n(n+1)}{6}.$$

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a) $x^2 = 4x + 5$

$x = 5,\ x = -1$

$a_n = U(5)^n + V(-1)^n$

$a_0 = 1 = U + V$

$a_1 = 2 = 5U - V \rightarrow$

$2 = 5U - (1-U)$

$2 = 6U - 1$

$6U = 3$

$U = \frac{1}{2} \quad V = \frac{1}{2}$

b) $\sum_{K=2}^{n} \binom{K}{2} = \frac{(n-1)n(n+1)}{6}$

Let $P(x) = \sum_{K=2}^{n} \binom{K}{2}$

Basic $P(2) = \sum_{K=2}^{2} \binom{K}{2} = 1 = 1$

Let $P(n)$ be T for $n \geq 2$

Inductive: $P(n+1) = \sum_{K=2}^{n} \binom{K}{2} + \binom{n+1}{2}$

$\frac{n(n+1)(n+2)}{6}$

$= \frac{(n-1)(n)(n+1)}{6} + \frac{n(n+1)}{2}$

$= \frac{(n-1)(n)(n+1) + 3n(n+1)}{6}$

$= \frac{(n+1)(n)((n-1)+2n)}{6} =$

$3n-1$

QUESTION 3. **(17 marks)**

A *bit string* is a sequence of 0s and 1s. How many bit strings of length 11 are there

(a) in total? $2^{11} =$

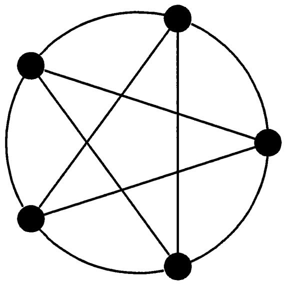
(b) that contain exactly two 0s? $11C2 = 55$

(c) that contain at $\boxed{\text{most}}$ three 0s and every 0 is followed immediately by a 1? (E.g., 01011111011 but not 10011111111.)

unit of '01', atmost 3. $1 + 10C_1 + 9C_2 + 8C_3 = 102 + 1$
$= 103$

QUESTION 4. **(12 marks)**

Consider the graph X:



(a) Is the graph X bipartite? Justify your answer.

(b) Does the graph X have

(i) an Euler path (with distinct starting and ending vertices)?

(ii) a Hamiltonian path?

(iii) an Euler circuit?

Justify your answers.

QUESTION 5. **(18 marks)**

Let $D = \mathbb{R} - \{0\}$ be the set of real numbers without 0. Let $f : D \to \mathbb{R}$ be given by $f(x) = (2x+1)/x$ and let $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x/(x^2+1)$.

(a) Show that f is one-to-one.

(b) Is f onto? If yes then prove it, if not then show that there exists an element in the codomain that does not have any preimages.

(c) Is g one-to-one? If yes then prove it, if not then find two distinct elements in the domain that have the same image.

QUESTION 6. **(21 marks)**

(a) Find the transitive closure of the relation $R = \{(1,2), (2,3), (3,1), (3,4)\}$.

(b) Let R be a relation on a set $A = \{1,2,3\}$. Suppose that R is anti-symmetric but not reflexive. Do there exist such relations for which

$$\exists\,(x,y) \in R,\ \big((x,y) \in R\big) \wedge \big((y,x) \in R\big)?$$

If so, give an example of such a relation; if not, explain why.

(c) For an integer $n \geqslant 5$, let $A = \{1,\ldots,n\}$. Consider the cartesian product $P = A \times A \times A \times A \times A$. How many elements $(x_1,\ldots,x_5) \in P$ satisfy

$$\sum_{i=1}^{5} x_i = n?$$

Justify your answer.

END OF PAPER

MH1812 DISCRETE MATHEMATICS

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.