NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2018-2019

MH1812 - DISCRETE MATHEMATICS

December 2018 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the end of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This **IS NOT** an **OPEN BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1.

(a) Prove that $\neg p \to \neg q$ and its inverse are not logically equivalent. **(10 marks)**

Solution:Truth values differ when p is false and q is true.
An alternative solution is to use the truth table. $\square$

(b) Prove that $(q \wedge (p \to \neg q)) \to \neg p$ is a tautology using propositional equivalence and the laws of logic. **(10 marks)**

Solution: $(q \wedge (p \to \neg q)) \to \neg p \equiv (q \wedge (\neg p \vee \neg q)) \to \neg p \equiv ((q \wedge \neg p) \vee (q \wedge \neg q)) \to \neg p \equiv (q \wedge \neg p) \to \neg p \equiv \neg(q \wedge \neg p) \vee \neg p \equiv (\neg q \vee p) \vee \neg p \equiv \neg q \vee (p \vee \neg p)$, which is always true. $\square$

QUESTION 2.

Prove that $\sum_{j=n}^{2n-1}(2j+1) = 3n^2$ for all positive integers n. **(12 marks)**

Solution:The basis step holds since $\sum_{j=1}^{1}(2j+1) = 3 = 3 \cdot 1^2$. Now assume that $\sum_{j=k}^{2k-1}(2j+1) = 3k^2$. It follows that $\sum_{j=k+1}^{2(k+1)-1}(2j+1) = \sum_{j=k}^{2k-1}(2j+1) - (2k+1) + (4k+1) + (4k+3) = 3k^2 + 6k + 3 = 3(k+1)^2$. $\square$

QUESTION 3.

Find the solution to the recurrence relation $a_n = a_{n-1} + 2n + 1$ with $a_0 = 2$. **(10 marks)**

Solution: $a_n = a_{n-1} + 2n + 1 = a_{n-2} + 2((n-1) + n) + 2 = a_{n-3} + 2((n-2) + (n-1) + n) + 3 = a_{n-n} + 2(1 + 2 + \ldots (n-1) + n) + n = 2 + n(n+1) + n = n^2 + 2n + 2$. $\square$

QUESTION 4.

(a) $x_1, x_2, \ldots, x_k$ are positive integers such that $\sum_{i=1}^{k} x_i = n$, for some positive integers k, n and $n \geq k$. How many distinct tuples of $(x_1, x_2, \ldots, x_k)$ are there? **(6 marks)**

Solution: Assume there are n '1's, and we are to place $k-1$ separators to split the n '1's into k blocks, with the number of '1's in each block corresponding to x_i. Hence $k-1$ separators to be placed in $n-1$ possible positions, $\binom{n-1}{k-1}$. □

(b) How many distinct tuples of $(x_1, x_2, \ldots, x_k)$ are there for the above question if $x_1, x_2, \ldots, x_k$ are non-negative integers, rather than positive integers ? **(4 marks)**

Solution: This corresponds to the above question with $x'_i = x_i + 1$ (so x'_i are positive numbers) and $n' = n + k$, hence $\binom{n+k-1}{k-1}$. □

(c) How many bit strings contain exactly 5 '0's and 9 '1's if every '0' must be immediately followed by a '1' ? **(4 marks)**

Solution: This is an instance of the question above. The 5 '0's must be followed by '1's, hence there are 5 '01's, and extra 4 '1's ($n = 4$), which are to be inserted into $5 + 1$ places ($k = 6$). $\binom{4+6-1}{6-1} = \binom{9}{5}$. □

QUESTION 5.

Prove by the method of membership table that

$$\overline{(A - B) \cup (B - A)} = (A \cap B) \cup (\overline{A} \cap \overline{B}).$$

(14 marks)

Solution:

A	B	$A-B$	$B-A$	$(A-B)\cup(B-A)$	$\overline{(A-B)\cup(B-A)}$	$(A\cap B)$	$\overline{A}\cap\overline{B}$	$(A\cap B)\cup(\overline{A}\cap\overline{B}$
0	0	0	0	0	1	0	1	1
0	1	0	1	1	0	0	0	0
1	0	1	0	1	0	0	0	0
1	1	0	0	0	1	1	0	1

It is easy to see the two columns for $\overline{(A-B)\cup(B-A)}$ and $(A\cap B)\cup(\overline{A}\cap\overline{B})$ are identical.

$\square$

QUESTION 6.

Let $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and define the relation R as follows: $\forall x, y \in A$, $x\,R\,y$ *iff* $3^x \equiv 3^y \mod 5$.

(a) Prove R is an equivalence relation. **(8 marks)**

Solution: Reflexive: $\forall x \in A$, $3^x - 3^x = 0 = 0 \times 5$, i.e., $3^x \equiv 3^x \mod 5$, hence $x\,R\,x$.
Symmetric: $\forall x, y \in A$, if $(x, y) \in R$, then $3^x \equiv 3^y \mod 5$, $3^y \equiv 3^x \mod 5$, hence $y\,R\,x$
Transitive: $\forall x, y, z \in A$, if $3^x \equiv 3^y \mod 5$ and $3^y \equiv 3^z \mod 5$, then $3^x \equiv 3^z \mod 5$, i.e., $x\,R\,z$.
Hence, the relation is an equivalence relation. □

(b) List all the equivalence classes and all the elements in each class. **(8 marks)**

Solution:
$[1] = \{1, 5, 9\}$
$[2] = \{2, 6\}$
$[3] = \{3, 7\}$
$[0] = \{0, 4, 8\}$ □

QUESTION 7.

Define a function $f : D \to \mathbb{Z}$ by $f(x) = x^2 + 5$, where $D = \{-4, -3, -2, -1, 0\}$.

(a) Find the range of the function. **(8 marks)**

Solution:$R = \{21, 14, 9, 6, 5\}$. □

(b) Find f^{-1}. **(6 marks)**

Solution:$f^{-1}(x) = -\sqrt{x - 5}$. □

END OF PAPER