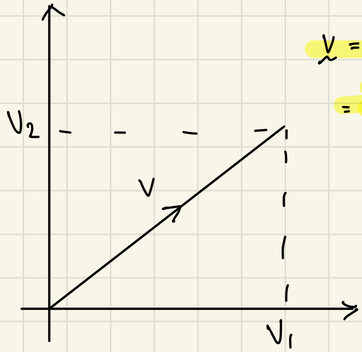
Meth
1810

Vectors

2D Vectors

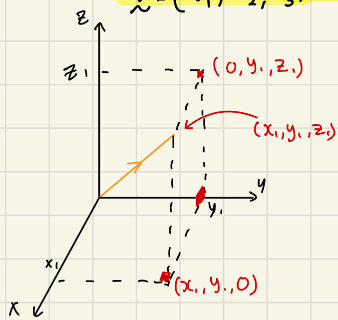


$\underset{\sim}{V} = (V_1, V_2)$

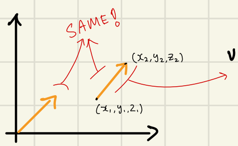
$= \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$

3D Vectors

$\underset{\sim}{V} = (V_1, V_2, V_3) = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$



Free Vectors



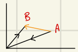
$v = \overrightarrow{P_1P_2} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$

$\therefore \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= \overrightarrow{OB} + \overrightarrow{AO}$

$= \overrightarrow{AO} + \overrightarrow{OB}$ "Bypass O"

$= \overrightarrow{AB}$



☆ length/magnitude/norm

$U = \sqrt{|V_1|^2 + |V_2|^2}$

or

$= \sqrt{|V_1|^2 + |V_2|^2 + |V_3|^2}$

☆ linear combination

$U = \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = V_1(i) + V_2(j) + V_3(k)$

Unit vector

↳ $\hat{U} = \frac{1}{|U|}(U)$

$\therefore U = K\hat{U}$

$= |U|\hat{U}$

Dot Product / Scalar product

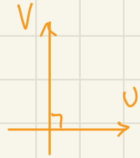
$U \cdot V = |U||V|\cos\theta$

∠ Btwn U & V [radians]

$U \cdot U = |U||U|\cos\theta$

$= |U|^2$

if $U \cdot V = 0$, $\cos(\frac{\pi}{2})$,



$(\lambda U) \cdot (\mu V) = (\lambda + \mu)(U \cdot V)$

$U \cdot (V + W) = U \cdot V + U \cdot W$

Dot/ Scalar Product Cont.

$$U = \begin{pmatrix}U_1\\U_2\\U_3\end{pmatrix} \quad V = \begin{pmatrix}V_1\\V_2\\V_3\end{pmatrix}$$

$$U \cdot V = (U_1 i + U_2 j + U_3 k)\cdot(V_1 i + V_2 j + V_3 k)$$

$$= U_1V_1\, i\cdot i + \cancel{U_1V_2\, i\cdot j}^{\,0} + \cancel{U_1V_3\, i\cdot k}^{\,0}$$

$$+ \cancel{U_2V_1\, j\cdot i}^{\,0} + U_2V_2\, j\cdot j + \cancel{U_2V_3\, j\cdot k}^{\,0}$$

$$+ \cancel{U_3V_1\, k\cdot i}^{\,0} + \cancel{U_3V_2\, k\cdot j}^{\,0} + U_3V_3\, k\cdot k$$

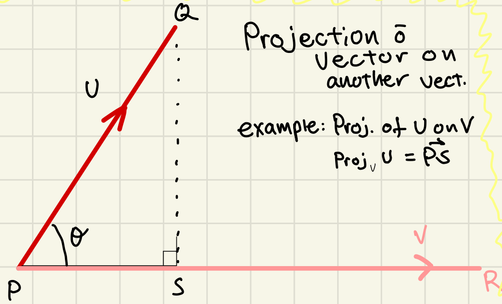
$$= U_1V_1\, i\cdot i + U_1V_1\, j\cdot j + U_1V_1\, k\cdot k$$

$$= U_1V_1 + U_2V_2 + U_3V_3$$

Projection of vector on another vect.

example: Proj. of U on V

$Proj_V U = \vec{PS}$



$$\frac{U\cdot V}{|U||V|} = \cos\theta = \frac{|PS|}{|U|}$$

$$|PS| = \frac{U\cdot V}{|V|} = U\cdot\frac{V}{|V|} = \frac{\cos\theta}{|U|} = U\cdot\hat{V}$$

Unit vector of V

$$PS = \left(\frac{1}{|V|}V\right)(|PS|)$$

$$= \left(\frac{V}{|V|}\right)(U\cdot\hat{V})$$

$$= (U\cdot\hat{V})\hat{V}$$

$U \perp V$, $U\cdot V = 0$ $\left[\cos\left(\frac{\pi}{2}\right)\right]$

$U \parallel V$ $\quad U\cdot V = |U||V|[\cos(0)]$

$$= |U||V|\ (1)$$

$$(U+V)\cdot(U+V) = (U\cdot U) + 2(U\cdot V) + (V\cdot V)$$

$$= ||U||^2 + ||V||^2 + 2(U\cdot V)$$

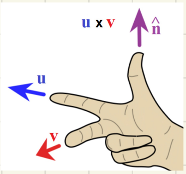
Cross Product of U x V

? = Perpendicular to U & V

① magnitude of $V\times U = |V||U|\sin\theta$

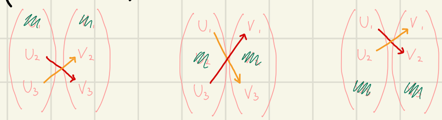
② $V\times U = ||V||\,||U||\sin\theta\ \hat{n}$

③ Shortcut method:



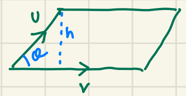
$$U\times V = \begin{pmatrix}U_1\\U_2\\U_3\end{pmatrix} \times \begin{pmatrix}V_1\\V_2\\V_3\end{pmatrix}$$

$$= (U_2V_3 - U_3V_2)i + (U_3V_1 - U_1V_3)j + (U_1V_2 - U_2V_1)k$$



red first orange second

Applications: Area of parallelogram

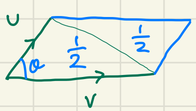


$$\text{Area} = |V| \times h$$

$$= |V|\cdot|U|\sin\theta$$

$$= |V\times U|$$

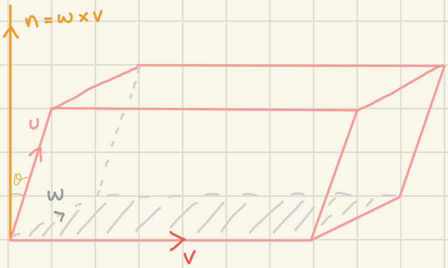
Area of triangle



$$\text{Area} = \frac{1}{2}\ \text{parallelogram}$$

$$= \frac{1}{2}|V\times U|$$

Scalar Triple Product



① Understand: Volume = height × base area.

② Find height : $h = \mathrm{Proj}_n u = (u \cdot \hat{n})$ ← Magnitude, not vector

$= \left(u \cdot \frac{w \times v}{|w \times v|}\right)$

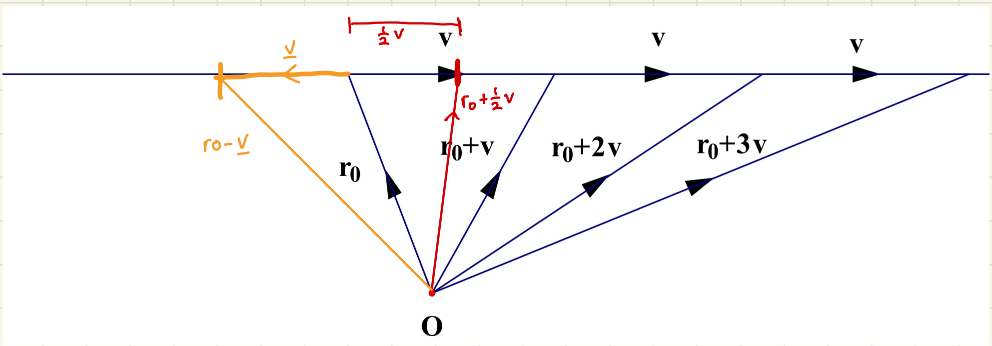
③ Find base area : From area of parallelogram

Area $= |w \times v|$

④ Volume : $h \times \text{Area} = \left(u \cdot \frac{w \times v}{\cancel{|w \times v|}}\right) \cancel{|w \times v|}$

TAKE ABSOLUTE $|-x| = x$ → $= \boxed{u \cdot (w \times v)}$

Vector Eq^n of lines



$l : r(t) = r_0 + tv \qquad t \in \mathbb{R}$

Direction vector

l passes through point P for $\vec{OP} = r_0$

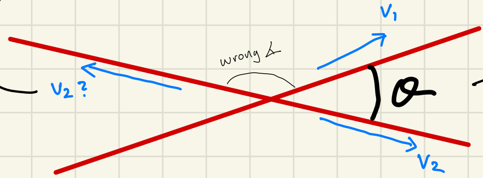
Cartesian form: $r = \underbrace{\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}}_{\text{Passes thru point}} + t \underbrace{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}}_{\text{direction}}, \quad t \in \mathbb{R}$

Parametric form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + tv_1 \\ y_0 + tv_2 \\ z_0 + tv_3 \end{pmatrix}, \; t \in \mathbb{R}$

$\therefore \quad x = x_0 + tv_1, \; y = y_0 + tv_2, \; z = z_0 + tv_3, \; t \in \mathbb{R}$

Angle between two lines

If V_2 goes this way, U.V will be -ve Take abs(U.V)

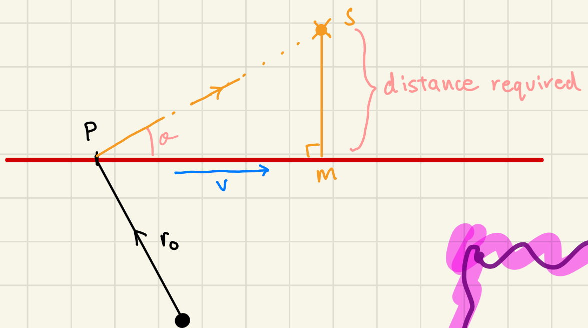


Take acute angle

$$\text{abs}(V_1 \cdot V_2) = |V_1||V_2| \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{\text{abs}(V_1 \cdot V_2)}{|V_1||V_2|}\right)$$

Distance of a point to a line



$l : r = r_0 + t\underset{\sim}{V}$

1. Take any point on the line (P)
2. Construct $\vec{PS}$
3. FORMULA:

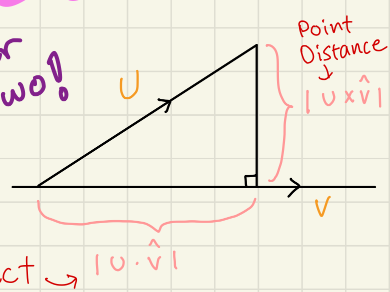
$$\sin\theta = \frac{d}{|PS|} \qquad |PS \times V| = |PS|\,|v| \sin\theta$$

$$\frac{d}{\cancel{|PS|}} = \frac{|PS \times V|}{\cancel{|PS|}\,|V|}$$

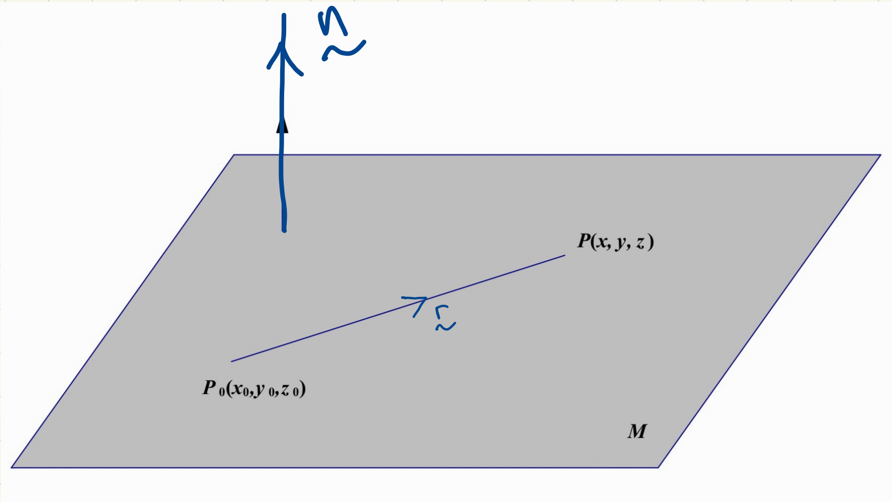
$$d = \frac{|PS \times v|}{|V|}$$

$$= |PS \times \hat{V}|$$

Remember these two!



Planes



This is a point

$n \cdot \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$

constant

Vector equation of a plane : $\underset{\sim}{r} \cdot \underset{\sim}{n} = d$

Cartesian/Scalar equation of a plane : $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0$

Parametric

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

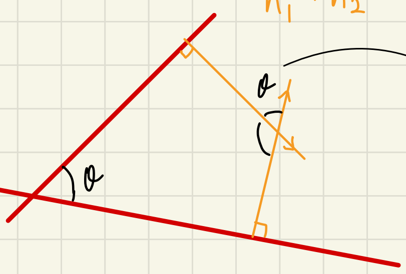
$ax + by + cz = ax_0 + by_0 + cz_0$

$ax + by + cz = d$

☆ Angle between two planes

↳

$n_1 \cdot n_2 = |n_1||n_2| \cos\theta$



You want to get acute ∡

therefore, $\theta = \cos^{-1}\left(\frac{abs(n_1 \cdot n_2)}{|n_1||n_2|}\right)$

Matrices

m×n matrix → $\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \cdot & \cdot & \cdots & a_{2n} \\ a_{31} & \cdot & \cdot & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

Zero matrix = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

row / column matrix $A = (a_1 \; a_2 \; a_3 \; \dots \; a_n)$

$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$

when m=n

m×n is square matrix = $\begin{bmatrix} 1 & 3 & 4 \\ 2 & 6 & 0 \\ 1 & 8 & 7 \end{bmatrix}$

Identity matrix = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Diagonal Matrix = $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Upper/lower triangular matrix = $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ / $\begin{bmatrix} 3 & 0 & 0 \\ 2 & 6 & 0 \\ 7 & 1 & 9 \end{bmatrix}$

Matrix multiplication

$\left.\begin{aligned} A &= m \times r \\ B &= r \times n \end{aligned}\right\} AB = m \times n$

$$AB = \begin{bmatrix} \boxed{A_{11} \quad A_{12}} \\ \boxed{A_{21} \quad A_{22}} \\ A_{31} \quad A_{32} \\ A_{41} \quad A_{42} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} B_{11} \\ B_{21} \end{matrix}} & \boxed{\begin{matrix} B_{12} \\ B_{22} \end{matrix}} & \begin{matrix} B_{13} \\ B_{23} \end{matrix} & \begin{matrix} B_{14} \\ B_{24} \end{matrix} \end{bmatrix}$$

$$= \begin{bmatrix} (A_{11}B_{11} + A_{12}B_{21}) & (A_{11}B_{12} + A_{12}B_{22}) & \cdots \\ (A_{21}B_{11} + A_{22}B_{21}) & & \\ \cdots & & \end{bmatrix}$$

Transpose

3×2 $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$ 2×3 $A^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$

$$(AB)^T = B^T A^T$$

$$(AB)^{-1} = B^{-1} A^{-1}$$

Invertible

$AB = I_n$ and $BA = I_n$

singular = non-invertible B is an inverse of A

Singular eg: $\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & b & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & a \\ 1 & 2 & b \\ 1 & 2 & c \end{bmatrix}}_{\text{Determinant} = 0}$

also singular ↴

$G = \begin{pmatrix} a & b & c \\ d & e & f \\ 2a - 5d & 2b - 5e & 2c - 5f \end{pmatrix}.$

Power Matrix

$A^2 = AA$ $A^3 = AAA$ $(A^{-1})^n = A^{-n} = (A^n)^{-1}$

The inverse matrix

$AA^{-1} = I \qquad (A^{-1})^{-1} = A$

2x2

for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $ad - bc$ = determinant

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Diagonal Matrix

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix}$$

Adjugate : Adj $= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Cofactor

i^{th} row

i^{th} row of $AA = [i^{th}$ row of $A]\,A$

i^{th} Column of $AA = A\,[i^{th}$ Column of $A]$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix} \qquad C_{31} = (1)(i)\begin{vmatrix} b & c & d \\ f & g & h \\ n & o & p \end{vmatrix}$$

Determinant

2x2

$\det(A) = ad - bc$

mxn

minor of A: determinant of submatrix

$\det(B) = M_{ij}$ with i row and j column deleted

$$\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$$

$M_{ij} \to$ minor of A

$$A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$M_{11} = \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} = 2 - 2 = 0$$

$$C_{11} = (-1)^{1+1} M_{11} = 0$$

$$M_{23} = \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} = 2 - 5 = -3$$

$$C_{23} = (-1)^{2+3}(-3) = -3$$

$$M_{12} = \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} = \det\begin{bmatrix} -3 & 1 \\ 1 & 1 \end{bmatrix} = -4$$

$$C_{12} = (-1)^{1+2}(-4) = 4$$

$n \times n \ (n>2)$

$$A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$$

$$= a_{11} \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot (-1)^{1+1} + a_{12} \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot (-1)^{1+2} + a_{13} \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot (-1)^{1+3}$$

$$= 1(0)(1) + 5(-4)(-1) + 0(-8)(1)$$

$$= 20$$

☆ You can perform co-factor expansion along any row

$$\det(A) = a_{21}C_{21} + a_{22}C_{22} + \dots + a_{2n}C_{2n}$$

$$= a_{21} \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot (-1)^{2+1} + a_{22} \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot (-1)^{2+2} + a_{23} \det\begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot (-1)^{2+3}$$

Choose the least zeros

$$= -3(5)(-1) + 2(1)(1) + 1(-3)(-1)$$

$$= 15 + 2 + 3 = 20$$

☆ You can perform co-factor expansion along any column

$n \times n \ (n>2)$, Checkerboard

4×4, $S = \begin{bmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{bmatrix}$

3×3, $S = \begin{bmatrix} +1 & -1 & +1 \\ -1 & +1 & -1 \\ +1 & -1 & +1 \end{bmatrix}$ $\rightsquigarrow$

+ve -ve +ve

$$A = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= +\left[\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}(1)\right] - \left[\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}(5)\right] + \left[\begin{vmatrix} -3 & 1 \\ 1 & 1 \end{vmatrix}(0)\right]$$

$$= 20$$

3×3 shortcut

copy

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12} \end{matrix}$$

Example: 4x4

$$S = \begin{bmatrix} +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \\ +1 & -1 & +1 & -1 \\ -1 & +1 & -1 & +1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 5 & -1 \\ 2 & 0 & 1 \end{bmatrix} \begin{matrix} 1 & -2 \\ 1 & 5 \\ 2 & 0 \end{matrix} = 5+4+0-10-0+2 = 1$$

Compute $\begin{vmatrix} 3 & 1 & 2 & 2 \\ 1 & 0 & -2 & 1 \\ 1 & 0 & 5 & -1 \\ 2 & 0 & 0 & 1 \end{vmatrix}$

Choose 2nd column

$$= (1)\begin{vmatrix} 1 & -2 & 1 \\ 1 & 5 & -1 \\ 2 & 0 & 1 \end{vmatrix}(-1) + 0\begin{vmatrix} \cdots \end{vmatrix} + 0\begin{vmatrix} \cdots \end{vmatrix}(-1) + 0\begin{vmatrix} \cdots \end{vmatrix}$$

$$= (1)(1)(-1)$$

$$= -1$$

Triangular Matrix

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 7 & 3 & 0 \\ 2 & 3 & 6 \end{bmatrix} = 1 \times 3 \times 6$$

Determinant ō Product

$$\det(AB) = \det(A) \times \det(B)$$

If A is invertible,

$$A \times A^{-1} = I$$

$$\det(AA^{-1}) = \det(I)$$

$$\det(A)\det(A^{-1}) = 1$$

$$\therefore \det(A) \neq 0 \quad \text{and} \quad \det(A^{-1}) = \frac{1}{\det(A)}$$

Adjugate : Adj $= \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A\,\mathrm{adj}(A) = \det(A)I$$

Linear Equations

$$\begin{aligned} 2u - v + w &= 3 \\ u + v - 3w &= 5 \\ 5u - 4v + 9w &= 4 \end{aligned}$$

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -3 \\ 5 & -4 & 9 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$ ← how to solve?

$\times A^{-1}$ $\qquad \times A^{-1}$

$$A^{-1}A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$I\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

A^{-1} is hard to find

System ō linear eq^n

The linear system

$$\begin{aligned} 2u &- v &+ w &= 3 \\ u &+ v &- 3w &= 5 \\ 5u &- 4v &+ 9w &= 4 \end{aligned}$$

is equivalent to

$$\begin{pmatrix} 2 & -1 & 1 \\ 1 & 1 & -3 \\ 5 & -4 & 9 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

Cramer's Rule: $Ax=b$ can be solved when A is square matrix, $\det(A) \neq 0$

$$X_j = \frac{\det(A_j)}{\det(A)}$$

The rules for 3×3 matrices are similar. Given

$$\begin{cases} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{cases}$$

which in matrix format is

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

Then the values of x, y and z can be found as follows:

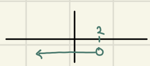
$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \text{ and } z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Limits

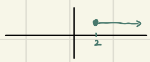
$$\lim_{x \to a} f(x)$$

One sided limit

Left hand: $\lim_{x \to 2^-} f(x) = -1$ Approach from Left side



Right hand: $\lim_{x \to 2^+} f(x) = 1$ Approach from Right Side



☆ If $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$, the limit $\lim_{x \to a} f(x)$ DNE (does not exist)

☆ $\lim_{x \to c} \frac{p(x)}{q(x)}$

1) $= \frac{a}{b}$, $b \neq 0$, $a \neq 0$ Simple!

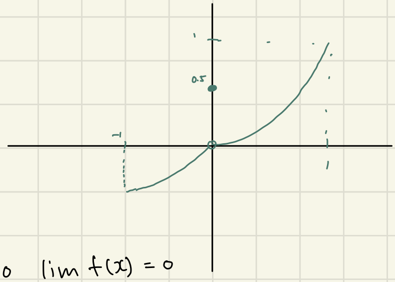
2) $= \frac{0}{0}$, $b = 0$, $a = 0$ further analysis → factorize

3) $= \frac{a}{0}$, $a \neq 0$, $b = 0$ → DNE, ∞, $-\infty$

Let g be the function defined by

$$g(x) = \begin{cases} x^2 & \text{if } 0 < x \leq 1, \\ 0.5 & \text{if } x = 0, \\ \sin x & \text{if } -1 \leq x < 0 \end{cases}$$

Does $\lim_{x \to 0} g(x)$ exist?



$\lim_{x \to 0^+} f(x) = 0 \quad \lim_{x \to 0^-} f(x) = 0$

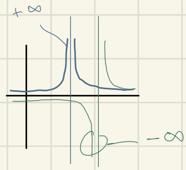
$\therefore \lim_{x \to 0} f(x) = 0.$ Don't care abt $f(0) = 0.5$

Infinite limits

$$\lim_{x \to a} f(x) = \infty / -\infty$$

NOTE:

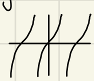
-



Continous Function

- When you have a "line", it is continous.

a.k.a.



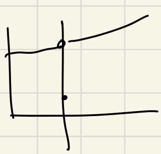
- Use domain : $f(x) = \frac{x^2 - 1}{x - 1}$ $\text{dom}(f) = \{ x \in \mathbb{R} : x \neq 1 \}$

$f(x) = \sin x$ is continuous at x for each $x \in \mathbb{R}$.
$g(x) = \ln x$ is continuous at x for each $x \in \mathbb{R}^+ = (0, \infty)$
$h(x) = \tan x$ is continuous at x for each $x \in \mathbb{R} - \{ \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots \}$

How to check

To check if a function f is continuous at a point $x = c$, we have to check that $\lim_{x \to c} f(x) = f(c)$. This means that

(i) $f(c)$ is defined.
(ii) the limit $\lim_{x \to c} f(x)$ exists.
(iii) $f(c)$ and $\lim_{x \to c} f(x)$ are equal.



Let f be a function defined on an interval I and let c be an interior point of I. We say that f is continuous at $x = c$ if

$$\lim_{x \to c} f(x) = f(c).$$

In words, the definition tells us that

- The function f is continuous at $x = c$ means that the limit $\lim_{x \to c} f(x)$ can be obtained by substituting $x = c$ into $f(x)$.
- The function f is continuous at $x = c$ means that the we may interchange the order of "$\lim_{x \to c}$" and "f", i.e.,

$$\lim_{x \to c} f(x) = f\left(\lim_{x \to c} x\right) = f(c).$$

(we can bring "$\lim_{x \to c}$" into $f(\cdot)$)

One sided function

Continous to the left

$$\lim_{x\to c^-} f(x) = f(c)$$

Continous to the right

$$\lim_{x\to c^+} f(x) = f(c)$$

Zero Denominator

1) Eliminate common factor by factorizing

$$\lim_{x\to -3} \frac{x^3+4x^2+4x+3}{-x^3-2x^2+5x+6} = \frac{\cancel{(x+3)}(x^2+x+1)}{(-1)(x-2)(x+1)\cancel{(x+3)}}$$

$$= \frac{\lim_{x\to -3} x^2+x+1}{\lim_{x\to -3}(-1)(x-2)(x+1)} = -\frac{7}{10}$$

$$\begin{array}{r|l} & x^2 + x + 1 \\ x+3 & x^3+4x^2+4x+3 \\ & -(x^3+3x^2) \\ \hline & x^2+4x \\ & -(x^2+3x) \\ \hline & x+3 \end{array}$$

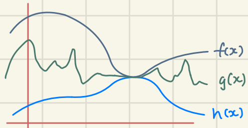
$$\lim_{x\to 1^+} \frac{\sqrt{2-x}-1}{x-1} = \lim_{x\to 1^+} \frac{\sqrt{2-x}-1}{x-1}\left(\frac{\sqrt{2-x}+1}{\sqrt{2-x}+1}\right)$$

$$= \lim_{x\to 1^+} \frac{2-x-1}{(x-1)(\sqrt{2-x}+1)}$$

$$= \lim_{x\to 1^+} \frac{(-1)\cancel{(x-1)}}{\cancel{(x-1)}(\sqrt{2-x}+1)}$$

$$= -\frac{1}{2}$$

Squeeze Theorem



$$h(x) \le g(x) \le f(x)$$

$$\lim_{x\to 0} x^2 \sin\left(\tfrac{1}{x^2}\right) = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\left(\tfrac{1}{x^2}\right)$$

$$= 0 \cdot ?$$

X wrong solution

Using squeeze, $-1 \le \sin\theta \le 1$ for all θ

$-1 \le \sin\frac{1}{x^2} \le 1$ for all $x \in \mathbb{R}$

$-x^2 \le \sin\frac{1}{x^2} \le x^2$ when $x^2 \ge 0$

Let $f(x) = -x^2$, $g(x) = \sin\frac{1}{x^2}$, $h(x) = \sin\frac{1}{x^2}$

$f(x) \le g(x) \le h(x)$, $\lim_{x\to 0} f(x) = 0 = \lim_{x\to 0} h(x)$

$\therefore \lim_{x\to 0} g(x) = 0$ //

∞

$$c \times \infty = \begin{cases} \infty, & c>0 \\ -\infty, & c<0 \end{cases}$$

$\frac{1}{\infty} = 0$ $\quad$ $\infty - \infty$ = undefined

$\frac{\infty}{\infty}$ = undefined $\quad$ $0 \times \infty$ = undefined

For lim involving ∞, care +/- ∞

$\sinh x = \frac{e^x - e^{-x}}{2}$ $\quad$ $\cosh x = \frac{e^x + e^{-x}}{2}$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

when $\lim_{x\to a} f(x) = 0$

$\lim_{x\to a} \frac{1}{f(x)}$ DNE $\quad$ $\lim_{x\to 0} \ln x = -\infty$ $\quad$ $\lim_{x\to\infty} \ln x = \infty$

$\lim_{x\to a^+} \frac{1}{f(x)} = +\infty$

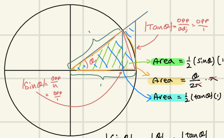
$\lim_{x\to a^-} \frac{1}{f(x)} = \infty$

$\lim_{x\to\infty} e^x = \infty$ $\quad$ $\lim_{x\to\infty} \sinh x = +\infty$ $\quad$ $\lim_{x\to -\infty} \sinh x = -\infty$

$\lim_{x\to\infty}$ = horizontal Asymp $\quad$ $\lim_{x\to\infty} \cosh x = +\infty$ $\quad$ $\lim_{x\to -\infty} \cosh x = +\infty$

$\lim_{x\to\infty} \tanh x = +1$ $\quad$ $\lim_{x\to -\infty} \tanh x = -1$

Another Squeeze, $\lim_{x\to 0} \frac{\sin x}{x}$ ...



$$\frac{|\sin\theta|}{2} \le \frac{|\theta|}{2} \le \frac{|\tan\theta|}{2}$$

$$|\sin\theta| \le |\theta| \le \frac{|\sin\theta|}{|\cos\theta|}$$

$$\frac{\cancel{|\sin\theta|}}{\cancel{|\sin\theta|}} \le \frac{|\theta|}{|\sin\theta|} \le \frac{\cancel{|\sin\theta|}}{|\cos\theta|} \cdot \frac{1}{\cancel{|\sin\theta|}}$$

$$1 \ge \frac{|\sin\theta|}{|\theta|} \ge |\cos\theta|$$

Using proof...

$$1 \ge \lim_{x\to 0} \frac{\sin x}{x} \ge \lim_{x\to 0} \cos\theta$$

$$1 \ge \lim_{x\to 0} \frac{\sin x}{x} \ge 1$$

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{x\to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x\to\infty} \frac{x+4}{x^2-6x+5} \div \frac{x^2}{x^2} = \frac{\frac{1}{x}+\frac{4}{x^2}}{1-\frac{6}{x}+\frac{5}{x^2}} = \frac{0+0}{1-0-0} = 0$$

$$\lim_{x\to\infty} \frac{x^3+4x-5}{7x^3-6x+5} \div \frac{x^3}{x^3} = \frac{1+\frac{4}{x^2}-\frac{5}{x^3}}{7-\frac{6}{x^2}+\frac{5}{x^3}} = \frac{1}{7}$$

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} \div \frac{x}{x} = \frac{\sqrt{\frac{2x^2+1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{\sqrt{2}}{3}$$

$$\lim_{x\to-\infty} \frac{\sqrt{2x^2+1}}{3x-5} \div \frac{x}{x} = \frac{\frac{\sqrt{2x^2+1}}{-\sqrt{x^2}}}{3-\frac{5}{x}} = \frac{-\sqrt{2+\frac{1}{x^2}}}{3-\frac{5}{x}} = \frac{-\sqrt{2}}{3}$$

(when $x<0$, $x=-\sqrt{x^2}$)

☆ $\frac{a^2-b^2}{a-b} = \frac{\cancel{(a-b)}(a+b)}{\cancel{a-b}}$

1) Identify: what causes 0.
if $a-b$, try convert to $a+b$

☆ $\ln a - \ln b = \ln\frac{a}{b}$

Intervals & Endpoints

$[a, b)$ → a is included
→ b is not included

aka, $f(x)$ is continous for $a \le x < b$

$(a, b), [a, b), (a, b], [a, b], (a, \infty), [a, \infty), (-\infty, b), (-\infty, b]$.

The points a and b are called endpoints.

The point a is a left endpoint and b is a right endpoint.

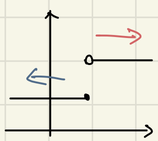
A point x in an interval is called an interior point of the interval if it is not an endpoint.

If functions

Domain = Input

Range = Output

Heavyside Function



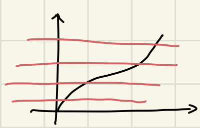
Continous for ⇄

can use $(-\infty, 1] \cup (1, \infty)$

for exceptions, $/\{5, -2\}$

Inverse

One - One function can be inversed



← only cut once

Trigo inverse

$\sin x$ for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

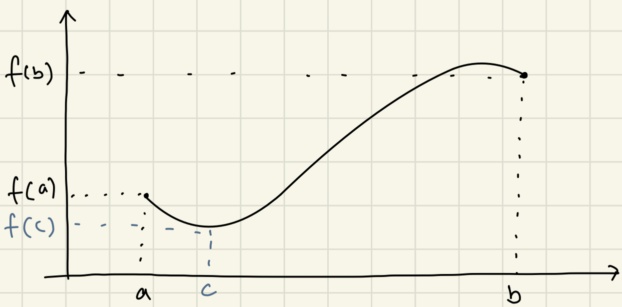
$\cos x$ for $x \in [0, \pi]$

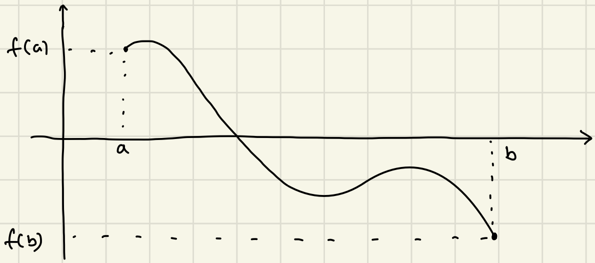
$\tan x$ for $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

Intermediate Value Theorem

☆ Only Close & Bounded, $[a, b]$, $a \neq b$

when $f(c)=0$, you find a root!



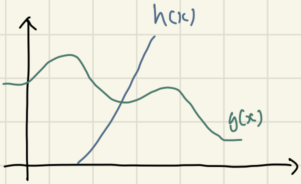


If $f(a)$ & $f(b)$ sign opposite, graph pass through 0

IVI steps

1) Find $f(a)$ & $f(b)$
2) Check for 0 inbetween
3) By IVI, there exists $c \in (0,1)$ such that $f(c)=0$

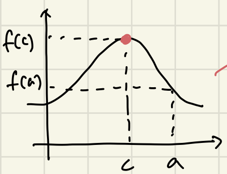
IVI intersect curve



1) Let $f(x) = h(x) - g(x)$
2) Sub in value, find 0
3) $h(c) - g(c) = 0$

   $h(c) = g(c)$

Extreme Value Theorem



If $f(c) \geq f(x)$ for all in domain, Maxima

☆ Same for minimum

Definition ō derivative

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

→ h is a random point

$f(x) = x^2$ @ $x = 1$

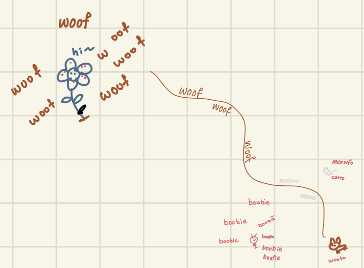
$$\lim_{x \to 1} = \frac{x^2 - 1}{x - 1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = 2$$

$f(x) = x^{\frac{1}{3}}$ @ $x = 0$

$\frac{a-b}{a^2-b^2}$, $a = y^{\frac{1}{3}}$, $b = x^{\frac{1}{3}}$

$$\lim_{x \to 0} = \frac{y^{\frac{1}{3}} - x^{\frac{1}{3}}}{y - x} = \lim_{x \to 0} \frac{\cancel{y^{\frac{1}{3}} - x^{\frac{1}{3}}}}{(\cancel{y^{\frac{1}{3}} - x^{\frac{1}{3}}})(y^{\frac{2}{3}} + y^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}})}$$

$$= \frac{1}{3x^{\frac{2}{3}}} = 3x^{-\frac{2}{3}}$$



Check for Differentiability

1) $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists

2) For piecewise function, at the joining part if the gradient ($f'(c)$) is the same, it is continuous and can be differentiated

Let $f(x) = \begin{cases} x^3 + 2 & \text{if } x > 1 \\ 3x & \text{if } x \leq 1 \end{cases}$, find $f'(x)$.

$\rightarrow f'(x) = 3x^2$

$\rightarrow f'(x) = 3$

@ $x=1$, both are 3.

$\lim_{x \to 1^+} = \lim_{x \to 1^-}$ are also 3

Differentiation

Power Law $\quad ax^n \rightarrow anx^{n-1}$

Product Rule: $\quad f$ & g diff @ c

$$fg'(x) = f'(x)g(x) + f(x)g'(x)$$

Quotient Rule: $\quad f$ & g diff @ c, $g(c) \neq 0$

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

Reciprocal Rule: $\quad g(c) \neq 0$

$$\left(\frac{1}{g(c)}\right)'(x) = \frac{-g'(c)}{(g(c))^2}$$

Chain Rule: $\quad f(g(x)) = F(x)$

$$F'(x) = (f'g(x)) \cdot g'(x)$$

or

$$\frac{dy}{dx} = \frac{d}{du}y \times \frac{du}{dx} = \frac{f'(x)}{g'(x)} \times g'(x)$$

$\cos(4x)$

$= -4\sin(4x)$

Proof:

$$fg'(x) = \lim_{x \to c} \frac{fg(x) - fg(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x - c}$$

$$= \lim_{x \to c} \left[f(x) \cdot \frac{g(x) - g(c)}{x - c} + \frac{f(x) - f(c)}{x - c} \cdot g(c) \right]$$

$$= \lim_{x \to c} f(x) \cdot \lim_{x \to c} \left(\frac{g(x) - g(c)}{x - c} \right) + \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) + g(c)$$

$$= f(c)g'(c) + f'(c)g(c)$$

Proof; quotient:

$$\left(\frac{f}{g}\right)'(x) = \lim_{x \to c} \left[\frac{\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}}{x - c} \right]$$

$$= \lim_{x \to c} \left[\frac{\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}}{x - c} \right]$$

$$= \lim_{x \to c} \left[\frac{f(x)g(c) - f(x)g(x) + f(x)g(x) - f(c)g(x)}{g(x)g(c)(x - c)} \right.$$

Trigo Diff:

sec	$\frac{1}{\cos}$
cosec	$\frac{1}{\sin}$
cot	$\frac{1}{\tan}$

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}\sin(Ax+B) = A\cos(Ax+B)$$
$$\frac{d}{dx}\cos(Ax+B) = -A\sin(Ax+B)$$
$$\frac{d}{dx}\tan(Ax+B) = A\sec^2(Ax+B)$$
$$\frac{d}{dx}\sin\left(f(x)\right) = f'(x)\cos\left(f(x)\right)$$

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(a^x) = a^x \ln a,\ a>0\ \&\ a \neq 1.$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a},\ a>0\ \&\ a\neq 1$$
$$\frac{d}{dx}e^{f(x)} = e^{f(x)} \cdot f'(x)$$
$$\frac{d}{dx}\ln\ f(x) = \frac{f'(x)}{f(x)},\ \text{where } f(x)>0$$

Parametric Diff

$y = U(t)$

$x = V(t)$

$$\frac{dy}{dx} = \frac{d}{dt}[y] \times \frac{dt}{dx} = \frac{U'(t)}{V'(t)}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$
$$= \frac{d}{dt}\left[\frac{dy}{dx}\right] \times \frac{dt}{dx}$$

$x = 9(t - \sin t)$ & $y = 9(1 - \cos t)$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cancel{9}\sin t}{\cancel{9}(1-\cos t)} = \frac{\cancel{2}\sin\frac{t}{2}\cos\frac{t}{2}}{\cancel{1-1}+\cancel{2}\sin^2(\frac{t}{2})}$$
$$= \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \cot\left(\frac{t}{2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right] \times \frac{dt}{dx} = \frac{-\frac{1}{2}\csc^2(\frac{t}{2})}{9(1-\cos t)}$$
$$= -\frac{1}{18}\frac{\csc^2(\frac{t}{2})}{\cancel{1-1}+2\sin^2(\frac{t}{2})}$$
$$= -\frac{1}{36\sin^4(\frac{t}{2})}$$
$$= -\frac{1}{36}\csc^4\left(\frac{t}{2}\right)$$

Implicit Differentiation

x^2	$2x$
y	$\frac{dy}{dx}$ or y'
y^2	$2y\frac{dy}{dx}$
xy	$x\frac{dy}{dx} + y(1)$
x^2y	$x^2\frac{dy}{dx} + 2xy$

Diff ⤳ $3x^4y^2 - 7xy^3 = 4 - 8y$

$$3x^4 2y\frac{dy}{dx} + 3(4)x^3y^2 - 7x3y^2\frac{dy}{dx} - 7y^3 + 8\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\left(6x^4y - 21xy^2 + 8\right) = 7y^3 - 12x^3y^2$$
$$\frac{dy}{dx} = \frac{7y^3 - 12x^3y^2}{6x^4y - 21xy^2 + 8}$$

Logarithmic Differentiation

$y = x^r \quad \ln y = r\ln x$

$$\frac{1}{y}\frac{dy}{dx} = r\left(\frac{1}{x}\right)$$
$$\frac{dy}{dx} = rx^{-1}y = rx^{-1}(x^r)$$
$$= rx^{r-1}$$

$$\frac{d}{dx}(x^\pi - \pi^x) = \pi x^{\pi-1} - \ln\pi(\pi^x)$$

$\frac{d}{dx}(-\pi^x) = ?$ Let $y = \pi^x$

$$\ln y = x\ln\pi$$
$$\frac{1}{y}\frac{dy}{dx} = \ln\pi$$
$$\frac{dy}{dx} = \ln\pi(\pi^x)$$

$$\frac{d}{dx}(x^x) = (x^x)(1 + \ln x)$$

Let $y = x^x$

$$\ln y = x\ln x$$
$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x$$
$$\frac{dy}{dx} = (1 + \ln x)(x^x)$$

Derivative ō inverse

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))} = \frac{1}{f'(x_0)}$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}},\ \textit{for } -1 < x < 1.$$

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}},\ \textit{for } -1 < x < 1.$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2},\ \textit{for } x \in \mathbb{R}.$$

$$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2},\ \textit{for } x \in \mathbb{R}.$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{|x|\sqrt{x^2-1}},\ \textit{for } x < -1 \textit{ or } x > 1.$$

$$\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}},\ ,\ \textit{for } x < -1 \textit{ or } x > 1$$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \pm\sqrt{1-\sin^2 y}$

$f(x) = \cos x \qquad x \in (0, \pi)$

$(f^{-1})'(0) = \dfrac{1}{f'(\frac{\pi}{2})} = \dfrac{1}{-\sin(\frac{\pi}{2})} = -1$

what gives $\cos(x) = 0$

$x_0 = \frac{\pi}{2}$

$y = \sin^{-1}(x) \qquad -1 < x < 1$

$x = \sin(y) \qquad -\frac{\pi}{2} < y < \frac{\pi}{2}$

1st & 4th quadrant

$1 = \frac{dy}{dx}\cos(y)$

$\frac{dy}{dx} = \dfrac{1}{\cos(y)}$

$\frac{dy}{dx} = \dfrac{1}{+\sqrt{1-\sin^2 y}}$

+ve only, as $\cos > 0$ for S | A / T | C

$\frac{dy}{dx} = \dfrac{1}{\sqrt{1-\sin^2(\sin^{-1}x)}}$

$= \dfrac{1}{\sqrt{1-x^2}}$

$\frac{d}{dx}(\tan^{-1}x)$

$y = \tan^{-1}x$

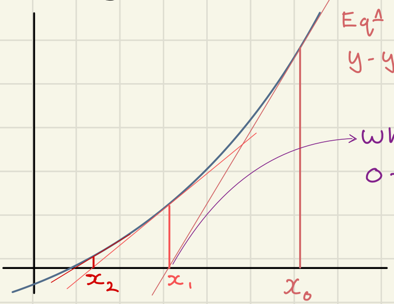
$x = \tan y$

$1 = \sec^2 y \frac{dy}{dx}$

$\frac{dy}{dx} = \dfrac{1}{\sec^2 y} = \dfrac{1}{1+\tan^2 y}$

$= \dfrac{1}{1+x^2}$

Newton's Method



Eq^n ō line through (y_0, x_0)

$y - y_0 = f'(x_0)(x - x_0)$

when $x = x_1,\ y = 0$

$0 - y_0 = f'(x_0)(x_1 - x_0)$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

⋮

until converge on point

Example:

$x^3 - x + 1 = 0$ by newton diff

Let $f(x) = x^3 - x + 1 \qquad f'(x) = 3x^2 - 1$

$f(0) = 1$, $f(-2) = -5$ } By IVT, there is a root

let $x_0 = -2 \qquad x_1 = (-2) - \dfrac{f(-2)}{f'(-2)} = -\dfrac{17}{11}$

$x_2 = -1.359614$

$x_3 = \ldots\ldots \quad x_4 = \ldots\ldots$

To 5 d.p.

Typical usage, reciprocol

Our aim: Given $\alpha \neq 0$, estimate x where $x = \frac{1}{\alpha}$.

Note that $x = \frac{1}{\alpha}$ is equivalent to $\frac{1}{x} = \alpha$, i.e., $\frac{1}{x} - \alpha = 0$.

Thus, we use the function

$$f(x) = \frac{1}{x} - \alpha. \qquad f'(x) = \frac{-1}{x^2}$$

☆ Estimating $\frac{1}{\alpha}$, $\alpha \in \mathbb{R}$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(\frac{1}{x_n} - \alpha)}{(-\frac{1}{(x_n)^2})}$$

$$= x_n + x_n^2\left(\frac{1}{x_n} - \alpha\right)$$

$$= 2x_n - \alpha x_n^2$$

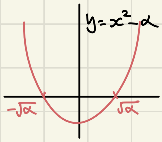
Division turns into multiplication, easier for computers

To 5 d.p.

Typical Use, square roots

Note that the root of $x = \sqrt{\alpha}$ is equivalent to the positive root of $x^2 = \alpha$ i.e., $x^2 - \alpha = 0$.
Let $f(x) = x^2 - \alpha$.

$f'(x) = 2x$



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^2 - \alpha}{2x_n}$$
$$= x_n - \tfrac{1}{2}x_n + \frac{\alpha}{2x_n}$$
$$= \tfrac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$$

☆ Estimating $\sqrt{\alpha}$, $\alpha \in \mathbb{R}$

Example: Find the approx of $\sqrt{3}$

$\alpha = 3$ let $x_0 = 1.5$

$x_1 = \frac{1}{2}\left(1.5 + \frac{3}{1.5}\right) = 1.75$

$x_2 = \frac{1}{2}\left(1.75 + \frac{3}{1.75}\right) = 1.73214$

$x_3 = 1.732050$ $x_4 = 1.7320508$

An approx root to 5 d.p is $x_3 = x_4 = 1.73205$

To 5 d.p.

Linearization (approximate function w. line)

Aim: To approximate $f(x)$ near $x = a$ by a linear function $L(x)$ through $x = a$.

Definition
The linearization of f at a is the linear function

$$L(x) = f(a) + (x - a)f'(a)$$

It appears!

at P, $x = a$, $y = f(a)$

$\frac{dy}{dx} = f'(a)$

Eq^n of tangent, $y - f(a) = f'(a)(x-a)$

$\therefore \quad y = \boxed{f(a) + f'(a)(x-a)}$

Example: $f(x) = \ln(1+x)$ linear @ $a=0$
to approx $\ln(1.01)$

$L(x) = f(a) + f'(a)(x-a)$
$= \ln(1) + \frac{1}{1+0}(x)$
$= x$

$\ln(1.01) = \ln(1+0.01) = f(0.01)$
$\approx L(0.01) = 0.01$

$f(x) = \sqrt{x}$ use $a=4$ to approx $\sqrt{4.001}$

$L(x) = f(a) + f'(a)(x-a)$
$= 2 + \frac{1}{2}(4)^{-\frac{1}{2}}(x-4)$
$= 2 + \frac{1}{2}\left(\frac{1}{\sqrt{4}}\right)(x-4)$
$= 2 + \frac{1}{4}x - 1 = 1 + \frac{1}{4}x$

$\sqrt{4.001} = f(4.001) \approx L(4.001)$
$= \frac{8001}{4000}$

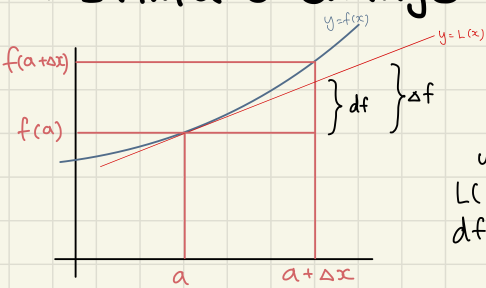
Approx $\sqrt[3]{7.99}$

$f(x) = (x)^{\frac{1}{3}}$ $x = 8$

$L(x) = f(a) + f'(a)(x-a)$
$= 2 + \frac{1}{3}(8)^{-\frac{2}{3}}(x-8)$
$= 2 + \frac{1}{12}(x-8)$

$f(7.99) = L(7.99) \simeq 1.99916$

Estimate Change using $\frac{d}{dx}$



differential of f

$df \simeq \Delta f$
when Δx is small

$L(x) = f(a) + f'(a)(x-a)$
$df = L(a+\Delta x) - L(a)$
$= f'(a)(x-a)$
$= f'(a)\Delta x$

	Actual Change	Estimated change
Absolute change	Δf	df
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$100\frac{\Delta f}{f(a)}$	$100\frac{df}{f(a)}$

The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2 cm.

(a) Use the differentials to estimate the maximum error in the calculated area of the disk.

(b) What is the relative error? What is the percentage error?

$A(r) = \pi r^2$
want to find ΔA $\therefore$ use dA

$dA = f'(r)\,\Delta r$
$dA = (2\pi r)(\Delta r)$
$dA = 9.6\pi$

relative error
$\hookrightarrow \frac{9.6\pi}{f(a)} = \frac{9.6\pi}{\pi(24)^2} = \frac{1}{60}$ //

Closed Interval Method (find max, min)

Critical Point

Stationary Points — Singular Point

$f'(c) = 0$ — $f'(c)$ DNE

Example: Find max/min of

$f(t) = \sqrt[3]{t}\,(8-t)$ on $[-1, 8]$

$f'(t) = \frac{1}{3}t^{-\frac{2}{3}}(8-t) - (t)^{\frac{1}{3}}$

$= (3t^{\frac{2}{3}})^{-1}(8-t) - (t)(t^{-\frac{2}{3}})$

$= \frac{8-t}{3t^{\frac{2}{3}}} - \frac{t}{t^{\frac{2}{3}}} = \frac{8-4t}{3t^{\frac{2}{3}}}$

Stationary $f'(t) = 0 \Rightarrow 8 - 4t = 0$

$t = 2$

$f(2) = \sqrt[3]{2}\,(8-2) = 6\sqrt[3]{2}$

Singular $f'(t)$ DNE $\Rightarrow 3t^{\frac{2}{3}} = 0$

$t = 0$

$f(0) = 0$

Values > $f(2) = 6\sqrt[3]{2}$ → Biggest

$f(0) = 0$

$f(-1) = -9$ → Smallest

$f(8) = 0$

Example: Find min/max of

$f(x) = (x^2-1)^{\frac{2}{3}}$ $[-3, 3]$

$f'(x) = \frac{2}{3}(x^2-1)^{-\frac{1}{3}}(2x)$

$= \frac{4x}{3(\sqrt[3]{x^2-1})}$

$f'(x) = 0$, $x = 0$

$f'(x)$ DNE, $\sqrt[3]{x^2-1} = 0$

$x = 1$, $x = -1$

$f(0) = 1$

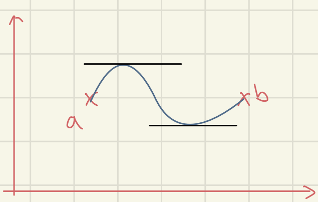
$f(1) = 0$ min $= 0$

$f(-1) = 0$ max $= 4$

$f(-3) = 4$

$f(3) = 4$

Rolle's Theorem



If $f(a) = f(b)$

& continous,

$f'(x) = 0$ exists.

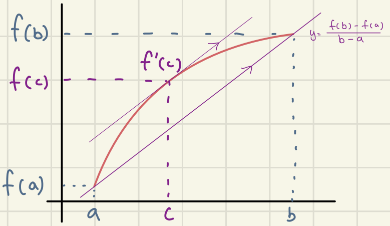
Mean value theorem

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Instant rate
$\bar{o}$ change

"Average rate of change"

This is basically
the gradient
aka, there exists a point C
with gradient of the ab line



Example: Show $f(x) = e^x - 1$ only 1 root

$f(-1) = e^{-1} - 1 < \frac{1}{2} - 1$ $\quad$ $f(1) = e^1 - 1 > 2 - 1$

< 0 $\quad$ > 0

by IVT, $f(x)$ has roots

If multiple roots, $f(a) = f(b) = 0$

and by Rolle's Theorem, there is a $f'(c) = 0$

$f'(x) = e^x$

$e^x \neq 0$ for $x \in \mathbb{R}$

Example: $f(0) = -3$ $\quad$ $f'(x) \leq 5$

Biggest $f(2)$ value

$$\frac{f(b) - f(a)}{b - a} \leq 5$$

$$\frac{f(2) + 3}{2 - 0} \leq 5$$

$$f(2) \leq 7$$

Example: Use MVT, estimate $\sqrt[3]{65}$

$f(x) = \sqrt[3]{x}$

Apply MVT on $[64, 65]$

1) continous? yes

2) Differentiable? yes

by MVT, $f'(x_0) = \dfrac{f(65) - f(64)}{65 - 64}$

$$\frac{1}{3}(x_0)^{-2/3} = f(65) - 4$$

$$\sqrt[3]{65} = \frac{1}{3x_0^{2/3}} + 4 \rightarrow 4 < \sqrt[3]{65} < 4 + \frac{1}{3x_0^{2/3}} \quad ①$$

$$64 \leq x_0 \leq 65$$

$$64^{2/3} \leq x_0^{2/3} \leq 65^{2/3}$$

$$16 \leq x_0^{2/3} \leq 65^{2/3} \rightarrow \frac{1}{16} \geq \frac{1}{x_0^{2/3}} \quad ②$$

① $\div$ ② $\quad 4 < \sqrt[3]{65} < 4 + \dfrac{1}{3(16)}$

$$4 < \sqrt[3]{65} < 4\frac{1}{48}$$

L'Hospital Rule

when $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$ or $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = \pm\infty$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example: $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+}\frac{\overbrace{\ln x}^{-\infty}}{\underbrace{\frac{1}{x}}_{+\infty}}$

$= \lim\limits_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to 0^+} -x = 0$

Example: $\lim\limits_{x\to 1}\frac{\ln x}{x-1} = \lim\limits_{x\to 1}\frac{\frac{1}{x}}{1}$

$= 1$

Example: $\lim\limits_{x\to\infty}\frac{e^x}{x^2} = \lim\limits_{x\to\infty}\frac{e^x}{2x}$

$= \lim\limits_{x\to\infty}\frac{e^x}{2}$

$= \infty$

Example: $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{x\ln x}$

$= e^{\lim\limits_{x\to 0^+} x\ln x}$

$= e^0 = 1$

Maximum / Minimum

If $f'(x) > 0$ on (a, b), then f is *increasing* on $[a, b]$.
If $f'(x) < 0$ on (a, b), then f is *decreasing* on $[a, b]$.

Can also check for one-one function

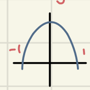
if $f'(x) < 0$ or > 0 for entire function,

there will not be any turning point

Example: $f(x) = 2 + 3x - x^3$, increasing interval

$f'(x) = -3x^2 + 3$

$f(x)\uparrow$ when $f'(x) > 0$ $(-1, 1)$



Example: show $f(x)$ $\sin x$ $[-\pi/2, \pi/2]$ is one-one

$f'(x) = \cos x$

S	A
T	C

for 1^{st} & 4^{th} quadrant $\cos x > 0$

This means $f(x)$ is always increasing

Example: construct metal can with volume V (constant)

Volume $= (h)(\pi r^2)$ — ①

Area $= (h)(2\pi r) + 2\pi r^2$ — ②

① & ② $A = \left(\frac{V}{\pi r^2}\right)(2\pi r) + 2\pi r^2$

$= \frac{2V}{r} + 2\pi r^2$

$\frac{dA}{dr} = -\frac{2V}{r^2} + 4\pi r = 0$

$r\left(-\frac{2V}{r^3} + 4\pi\right) = 0$

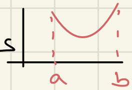
$-\frac{2V}{r^3} = -4\pi$

$\frac{2V}{4\pi} = r^3$

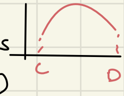
$r = \sqrt[3]{\frac{V}{2\pi}}$

Concavity

Graph Concaves upwards
$f''(x) > 0$ for (a,b)



Graph Concaves Downwards
$f''(x) < 0$ for (C,D)



Example: $f(x) = 2 + 3x - x^3$ Find when graph Concave upwards / downwards & intersection

$f'(x) = 3 - 3x^2$
$f''(x) = -6x$
$x < 0$, $f''(x) > 0$ Concave up
$x > 0$, $f''(x) < 0$ concave down
$f(0) = 2$, inflection @ $(0, 2)$

It can also tell us about global max/min
If $f'(c) = 0$,
when $f''(x) > 0$ for all values, $f(c)$ is minimum
when $f''(x) < 0$ for all values, $f(c)$ is maximum

Example: Continued from above can eg.
$A' = 4\pi r - \frac{2V}{r^2}$ → Stationary Pt. $r = \sqrt[3]{\frac{V}{2\pi}}$

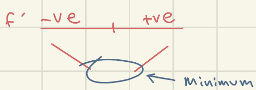
$A'' = 4\pi + \frac{4V}{r^3}$ } more than zero

∴ it is a global minimum

$f(x) = (x-1)^{2/3}$ $f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}}$

f' is never 0, no Stationary point

when $x < 1$, $x - 1 < 0 \Rightarrow f'(x) < 0$
$x > 0$, $f'(x) > 0$



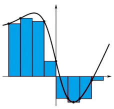
Reimann's Sum

Let f be a function on $[a,b]$ and

$$x_k = a + k\left(\frac{b-a}{n}\right) \quad \text{for } k = 0, 1, 2, \ldots n.$$

With $x_k^* \in [x_{k-1}, x_k]$, the finite sum

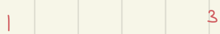
$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*),$$



is called a Riemann sum of f on $[a,b]$.

Riemann's Sum of $f(x) = x^2$ $[1, 3]$

① Partition into equal subintervals



$\frac{b-a}{n} = \frac{2}{n}$

$x_0 = 1 = 1$, $x_1 = 1 + \Delta x = 1 + \frac{2}{n}$, $x_2 = 1 + 2\Delta x = 1 + 2\left(\frac{2}{n}\right)$

$x_k^* = 1 + k\left(\frac{2}{n}\right)$

$x_k = a + k\frac{(b-a)}{n}$

$\sum_{k=1}^{n} f(x_k^*)\Delta x = \sum_{k=1}^{n} f\left(1 + \frac{2k}{n}\right)\left(\frac{2}{n}\right)$

Reimann $= \sum_{k=1}^{n} \left(1 + \frac{2k}{n}\right)^2 \left(\frac{2}{n}\right)$

Intergration

$x^2 \rightsquigarrow 2x$

$x^2+1 \rightsquigarrow 2x$

integrant

$\therefore \underbrace{\int 2x\,dx}_{\text{integral}} = x^2 + c$

Example

Prove $\int \frac{1}{x}\,dx = \ln|x| + c$

let $F(x) = \ln|x|$

$= \begin{cases} \ln(x), & x>0 \\ \ln(-x), & x<0 \end{cases}$

$\left.\begin{array}{l} \text{if } x>0,\ F'(x) = \frac{1}{x} \\ x<0,\ F'(x) = \frac{-1}{-x} = \frac{1}{x} \end{array}\right\} F'(x) = \frac{1}{x}$

$\therefore \int \frac{1}{x}\,dx = F(x) + c$

$= \ln|x| + c$

Area Under Curve, Reimann

$$\sum_{k=1}^{n} \frac{b-a}{n} f(x_k^*)$$ $\rightsquigarrow$ When n to ∞, this Remaud's Sum is area under curve

- If $a > b$, we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

Left

- If $a = b$, we define

$$\int_a^b f(x)\,dx = 0.$$

Example: $\int_1^3 x^2\,dx = \lim_{x\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x$

$\approx \lim_{x\to\infty} \sum_{k=1}^{n} \left(1 + \frac{4k}{n} + \frac{4k^2}{n^2}\right)\left(\frac{2}{n}\right)$

$\approx \lim_{x\to\infty} \left[n + \frac{4}{n}\left(\frac{n(n+1)}{2}\right) + \frac{4}{n^2}\left(\frac{n(n+1)(2n+1)}{6}\right)\right]\left(\frac{2}{n}\right)$

$\approx$ ...

$\approx 26/3$

Even continous function

Suppose f is an even continuous function.

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

Examples:

(a) $\int_{-5}^{5} x^2\,dx = 2\int_0^5 x^2\,dx$

(b) $\int_{-\pi}^{\pi} \cos x\,dx = 2\int_0^{\pi} \cos x\,dx$

Odd continous function

$$\int_{-5}^{5} f(x)\,dx = 0$$

FTC

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$\frac{d}{dx}\int_a^{u(x)} f(t)\,dt = u'(x)\cdot f(u(x))$$

Example: $f'(x)$ of $f(x)=\int_{x^2}^{x^3} e^{-t^2}dt$

$= \int_0^{x^3} e^{-t^2}dt - \int_0^{x^2} e^{-t^2}dt$

$= \frac{d}{dx}\int_0^{x^3} e^{-t^2}dt - \frac{d}{dx}\int_0^{x^2} e^{-t^2}dt$

$= 3x^2 e^{-(x^3)^2} - 2x e^{-(x^2)^2}$

Example: $g(x) = \int_1^x \frac{\sin t}{t}dt, \quad 1\le x\le b$

by FTC, $\frac{d}{dx}g(x) = \frac{\sin x}{x}$

Example: $\frac{d}{dx}\int_1^{\sin x} \ln(t^2+1)\,dt$

let $u = \sin x$

$\frac{d}{dx}\int_1^{\sin x}\ln(t^2+1)\,dt = \frac{d}{dx}\int_1^u \ln(t^2+1)\,dt$

$= \underbrace{\frac{d}{du}\int_1^u \ln(t^2+1)\,dt}\cdot\frac{du}{dx}$

$= \ln(u^2+1)\cdot\frac{du}{dx}$

$= \ln(\sin^2 x+1)\cos x$

Substitution Rule

$$\int f(u(x))\,\underbrace{u'(x)\,dx}_{du} = \int f(u)\,du$$

Example: $\int \frac{x}{x^2+1}dx$ Let $u = x^2$, $\frac{du}{dx} = 2x$

$= \frac{1}{2}\int \frac{2x}{x^2+1}dx$

$= \frac{1}{2}\int \frac{1}{u+1}\frac{du}{dx}\,\cancel{dx}$

$= \frac{1}{2}\int \frac{1}{u+1}du$

$= \frac{1}{2}\ln|u+1|+c$

$= \frac{1}{2}\ln|x^2+1|+c = \ln\sqrt{x^2+1}+c$

Example: $\int \sin^3 x\cos x\,dx$ Let $u = \sin x$, $\frac{du}{dx} = \cos x$

$= \int u^3\frac{du}{dx}dx$

$= \frac{1}{4}u^4 + c = \frac{1}{4}\sin^4 x + c$

Sub Rule for Definite Int

$$\int_b^a f(u(x))\,u'(x)\,dx = \int_{u(b)}^{u(a)} f(u)\,du$$

Example: $\int \frac{e^{3x}}{\sqrt{1-e^{6x}}}dx$ Let $u = e^{3x}$, $\frac{du}{dx} = 3e^{3x}$

$= \int \frac{\frac{1}{3}}{\sqrt{1-u^2}}\frac{du}{dx}dx$

$= \frac{1}{3}\sin^{-1}(u)+c$

$= \frac{1}{3}\sin^{-1}(e^{3x})+c$

Example: $\int_0^8 \frac{\cos\sqrt{x+1}}{\sqrt{x+1}}dx$ Let $u = \sqrt{x+1}$, $\frac{du}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+1}}$

$= \int_0^8 (2\cos\sqrt{x+1})\left(\frac{1}{2\sqrt{x+1}}\right)dx$

$= \int_{\sqrt{1}}^{\sqrt{9}} 2\cos u\,\frac{du}{dx}dx$

$= \left[2\sin u\right]_{\sqrt{1}}^{\sqrt{9}} = 2\left[\sin 3 - \sin 1\right]$

$[x]_b^a = a - b$

Intergration by Parts

$$\int U(x)\underbrace{v'(x)\,dx}_{dv} = U(x)v(x) - \int v(x)\underbrace{U'(x)\,dx}_{du}$$

aka. $\int u\,dv = uv - \int v\,du$

Diff
- Ln
- Inverse trigo
- Algebra
- Trigo
- Exponent

aka. $\int U\,v' = uv - \int v\,U'$

Example: $\int x\cos x\,dx$

let $v' = \cos x$, $U = x$

$v = \sin x$, $U' = 1$

a) $= UV - \int vU'$

$= x\sin x - \int (1)\sin x\,dx$

$= x\sin x + \cos x + C$

Example: $\int x^2 \ln x\,dx$

$v' = x^2$, $U = \ln x$

$v = \frac{x^3}{3}$, $U' = \frac{1}{x}$

$= x^2\ln x - \int \frac{x^2}{3}\,dx$

$= x^2\ln x - \frac{1}{3}\int x^2\,dx$

$= x^2\ln x - \frac{1}{9}x^3 + C$

Example: $\int (t+1)e^t\,dt$

$v' = e^t$, $u = (1+t)$

$v = e^t$, $u' = 1$

$= e^t(t+1) - \int (e^t)\,dt$

$= t e^t + C$

Example: $\int \tan^{-1}x\,dx$

$v' = 1$, $U = \tan^{-1}x$

$v = x$, $U' = \frac{1}{x^2+1}$

$= \tan^{-1}(x) - \int \frac{x}{x^2+1}\,dx$

$= \tan^{-1}(x) - \frac{1}{2}\ln|x^2+1| + C$

Reduction Formula

$$I_n = x^n e^x - n I_{n-1}$$

Example: Let $I_n = \int x^n e^x\,dx$, $n \geq 0$, use reduction formula

Partial Fractions

factors of $Q(x)$	corresponding partial fractions
$(ax+b)^k$	$\frac{A_r}{(ax+b)^r}, r = 1, 2, 3, \ldots, k$
$(ax^2+bx+c)^k$	$\frac{A_r x + B_r}{(ax^2+bx+c)^r}, r = 1, 2, 3, \ldots, k$

$$\int \frac{1}{x(x-1)} = \int \frac{A}{x} + \int \frac{B}{x-1}$$

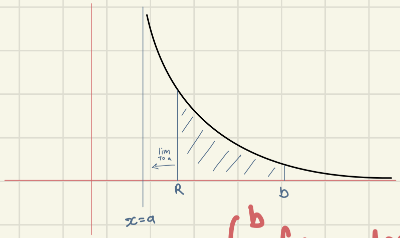
$$\int \frac{1}{x(x-1)^2} = \int \frac{A}{x} + \int \frac{B}{x-1} + \int \frac{C}{(x-1)^2}$$

$$\int \frac{1}{x(x^2-1)} = \frac{A}{x} + \frac{Bx+C}{x^2-1}$$

$$\int \frac{1}{x^2+4x+5} = \int \frac{1}{(x+2)^2+1} \rightsquigarrow \frac{1}{x^2+a^2} \rightsquigarrow \tan^{-1}x$$

$$\int \frac{2ax+b}{ax^2+bx+c} = \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Unbounded integrand



$$\int_R^b f(x)\,dx$$

Example: $\int_{-1}^{0} \frac{1}{x^2}\,dx$

$$\text{Area}(t) = \int_{-1}^{t} \frac{1}{x^2}\,dx$$
$$= \left[-\frac{1}{x}\right]_{-1}^{t}$$
$$= 1 - \frac{1}{t}$$

$$\int_{-1}^{0} \frac{1}{x^2}\,dx = \lim_{t\to 0^-} \int_{-1}^{t} \frac{1}{x^2}\,dx = \lim_{t\to 0^-}\left(1 - \frac{1}{t}\right)$$
$$= +\infty$$

Example: $\int_0^1 \frac{1}{\sqrt{2x - x^2}}\,dx$

$$\int_t^1 \frac{1}{\sqrt{2x-x^2}}\,dx = \int_t^1 \frac{1}{\sqrt{1-(x-1)^2}}\,dx$$
$$= \left[\sin^{-1}(x-1)\right]_t^1$$
$$= \sin^{-1}0 - \sin^{-1}(t-1)$$
$$= 0 - \sin^{-1}(t-1)$$

$$\therefore \int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx = \lim_{t\to 0}\left[-\sin^{-1}(t-1)\right]$$
$$= \sin^{-1}(-1)$$
$$= \frac{\pi}{2}$$

If $\int_R^\infty f(x)\,dx$,

Same! except $\lim_{t\to\infty}$!

P - integrals

Example: $\int_1^\infty \frac{1}{x^{3/2}+1}\,dx$

$$x^{\frac{3}{2}}+1 \geq x^{\frac{3}{2}}$$
$$\frac{1}{x^{\frac{3}{2}}+1} \leq \frac{1}{x^{\frac{3}{2}}}$$

for $\int_1^\infty \frac{1}{x^{\frac{3}{2}}}\,dx$, $p = \frac{3}{2}$, $p > 1$

Converges $\frac{1^{1-p}}{p-1}$

by comparison test, $\frac{1}{x^{\frac{3}{2}}+1}$ converges

The following improper integrals known as p-integrals are useful in comparison test.

Theorem (p-integrals)

Suppose $a \in \mathbb{R}$ and $a > 0$. Then

(a)

$$\int_a^\infty \frac{1}{x^p}\,dx = \begin{cases} \dfrac{a^{1-p}}{p-1} & \text{for } p > 1, \\ \infty & \text{for } p \leq 1. \end{cases}$$

(b)

$$\int_0^a \frac{1}{x^p}\,dx = \begin{cases} \dfrac{a^{1-p}}{1-p} & \text{for } p < 1, \\ \infty & \text{for } p \geq 1. \end{cases}$$

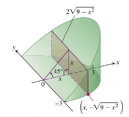
Volume

1) Find cross sectional area (one slice)

2) $\int_b^a A(x)\,dx$ for area

Example.

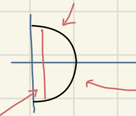
A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° at the centre of the cylinder. Find the volume of the wedge.



Circle Eq^n: $y^2 + x^2 = 3^2$

$y^2 = 9 - x^2$

$y = \sqrt{9-x^2}$ $\quad$ $y = -\sqrt{9-x^2}$



each slice

height $= x$.

width $= 2\sqrt{9-x^2}$

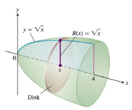
area $= 2x\sqrt{9-x^2}$

$\therefore \int_0^3 2x\sqrt{9-x^2}\,dx = 18$ unit^3

Revolution

Example:

Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$, $0 \le x \le 4$.



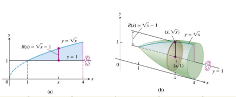
a slice is radius $\sqrt{x}$

Area $= \pi(\sqrt{x})^2 = \pi x$

Volume $= \int_0^4 \pi x\,dx = 8\pi$ unit^3

Example:

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.



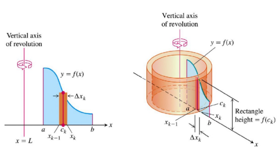
Typical slice is a disk of radius $\sqrt{x} - 1$

Area $= \pi(\sqrt{x}-1)^2 = \pi(x - 2\sqrt{x} + 1)$

Volume $= \int_1^4 \pi(x - 2\sqrt{x} + 1)\,dx = \frac{7}{6}\pi$

Cylindrical Shell

We note that : Volume of a cylindrical shell with radius r, height h and thickness t is approximated by $(2\pi r)ht$.



The solid generated by revolution may be interpreted as the "sum" of cylindrical shells.

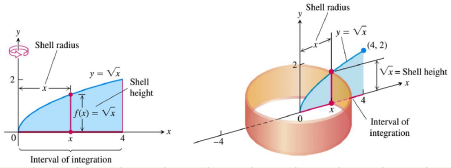
Theorem

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \ge 0$, $a \le x \le b$ and a vertical line is

$$V = 2\pi \int_a^b (\text{shell radius})\,(\text{shell height})\,dx.$$

Example:

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



Shell radius $= x$

height of the shell $= \sqrt{x}$

$$V = 2\pi \int_a^b (\text{shell } r)(\text{shell } h)\, dx$$

$$= 2\pi \int_0^4 x\sqrt{x}\, dx$$

$$= 2\pi \left(\frac{64}{5}\right)$$