NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2016–2017

MH1812 – Discrete Mathematics

April 2017 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

QUESTION 1. **(15 marks)**

Decide whether or not the following argument is valid:

$$\begin{aligned}
&p \wedge q;\\
&r \to s;\\
&\neg r \to q;\\
&p \vee r;\\
&\therefore (p \vee q) \wedge r
\end{aligned}$$

Briefly justify your answer.

Solution: The argument is invalid. Indeed, set $p = T, q = T, r = F$. Then the assumptions are all true and the conclusion is false.

QUESTION 2. **(15 marks)**

Consider three sets S, T, and U where S is defined to be the set of all even integers, $T = \{n \in \mathbb{Z} \ : \ 3 \mid n\}$, and $U = \{n \in \mathbb{Z} \ : \ n \equiv 0 \pmod 6\}$.

(a) Prove the set equality $S \cap T = U$. **(8 marks)**

(b) Determine the truth value of the following proposition **(7 marks)**

$$\neg \left(\forall x \in U, \exists y \in T, \ x \cdot y \notin S \right),$$

where $\cdot$ denotes multiplication. Justify your answer.

Solution:

(a) $S \cap T \subseteq U$: Let $x \in S \cap T$. Then x is both a multiple of 2 and 3. Hence x is a multiple of 6 and therefore $x \in U$.

$U \subseteq S \cap T$: Let $x \in U$. Then $x = 6k$ for some $k \in \mathbb{Z}$. Hence $x = 2(3k)$ and $x = 3(2k)$. Therefore x is both a multiple of 2 and 3 as required.

(b) First note that

$$\neg \left(\forall x \in U, \exists y \in T, \ x \cdot y \notin S \right) \equiv \exists x \in U, \forall y \in T, \ x \cdot y \in S$$

This statement is true. Indeed, $6 \in U$ and $6 \cdot y$ is even for all $y \in T$.

QUESTION 3. **(30 marks)**

(a) Using the characteristic equation, solve the recurrence relation **(10 marks)**

$$a_0 = 2, a_1 = 3, \quad a_n = 7a_{n-1} - 12a_{n-2}.$$

(b) Consider the recurrence relation given by the initial conditions $D_0 = 1$, $D_1 = 0$, and $D_n = (n-1)(D_{n-1} + D_{n-2})$ for all $n \geqslant 2$. Prove the equality

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

(Hint: use induction.) **(20 marks)**

Solution:

(a) We have the characteristic equation

$$x^2 - 7x + 12 = (x-3)(x-4).$$

Hence $a_n = u3^n + v4^n$. Since $a_0 = u + v = 2$ and $a_1 = 3u + 4v = 3$. Therefore $u = 5$ and $v = -3$.

(b) Basis cases are OK. Let $P(n)$ be our inductive hypothesis $D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$. Suppose that $P(k)$ is true for all $k \leqslant n$. Consider $P(n+1)$. We have

$$\begin{aligned}
D_{n+1} &= n(D_n + D_{n-1}) \\
&= n\left(n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\right) \\
&= n\left(n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (-1)^n + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\right) \\
&= n(-1)^n + n\left((n+1)(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}\right) \\
&= n(-1)^n + (n+1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}.
\end{aligned}$$

It suffices to show that $n(-1)^n = (n+1)!\left(\frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!}\right)$. This follows since

$$(n+1)!(\frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!}) = (n+1)(-1)^n + (-1)^{n+1}$$

and $(-1)^n + (-1)^{n+1} = 0$ for all $n \in \mathbb{Z}$.

QUESTION 4. **(25 marks)**

(a) Consider the relation R on the set of integers $\mathbb{Z}$ given by

$$aRb \Longleftrightarrow b \equiv a^3 - a \pmod 3.$$

(i) Is R reflexive? **(5 marks)**

(ii) Is R symmetric? **(5 marks)**

(iii) Is R transitive? **(5 marks)**

Justify your answers.

(b) Let $S = \{1, 2, \ldots, n\}$. Determine

(i) the cardinality of the set T of all functions $f : S \to S$? **(5 marks)**

(ii) the cardinality of the set $U = \{f \in T \mid f \text{ is invertible}\}$? **(5 marks)**

Solution:

(a) (i) No. $(1, 1) \notin R$.

(ii) No. $(1, 0) \in R$ but $(0, 1) \notin R$.

(iii) Yes. Suppose $(x, y) \in R$ and $(y, z) \in R$. Then z is a multiple of 3. Now $(x, w) \in R$ for all w that is divisible by 3. So (x, z) is also in R.

(b) (i) Each function $f : S \to S$ must map each element of S to precisely one element in S. So each element can be mapped to n elements. Thus the cardinality is n^n.

(ii) Each invertible function $f : S \to S$ must map each element of S to an unique element in S. So the first element can be mapped to n elements; the second to $(n - 1)$ elements, and so on. Thus the cardinality is $n!$.

QUESTION 5.

Consider the two graphs in Figure 1.

Figure 1: The graphs X and Y.

(a) For each of the graphs X and Y

(i) determine whether or not it has an Euler path; **(4 marks)**

(ii) determine whether or not it has an Euler circuit; **(4 marks)**

(iii) determine whether or not it has an Hamilton circuit. **(4 marks)**

(b) Are the graphs X and Y are isomorphic? Justify your answer. **(3 marks)**

Solution:

(a) (i) Both X and Y have precisely two vertices of odd degree. Hence both have Euler paths.

(ii) Both X and Y have precisely two vertices of odd degree. Hence both do not have Euler circuits.

(iii) Both X and Y have Hamilton circuits.

(b) The graphs are not isomorphic. They have different degree sequences.

END OF PAPER