QUESTION 1. **(15 marks)**

(a) [5 marks] Which integer $a \in \{0, 1, \ldots, 4\}$ is congruent to 2021 modulo 5?

(b) [5 marks] Which integer $a \in \{0, 1, \ldots, 9\}$ is congruent to 1812^{56} modulo 10? Justify your answer.

(c) [5 marks] Let $S = \{\text{integers congruent to 7 modulo 6}\}$ and Δ be multiplication. Is S closed under Δ? Justify your answer.

Solution:

(a) $a = 1$.

(b) $a = 6$.

Indeed, $1812 \equiv 2 \pmod{10}$. And $2^5 \equiv 2 \pmod{10}$.

$$\begin{aligned} 1812^{56} &\equiv 2^{56} \pmod{10} \\ &\equiv 2 \cdot (2^5)^{11} \pmod{10} \\ &\equiv 2 \cdot 2^{11} \pmod{10} \\ &\equiv 2^2 \cdot (2^5)^2 \pmod{10} \\ &\equiv 2^4 \equiv 6 \pmod{10}. \end{aligned}$$

[Distribution: 2 marks for $a = 2$ and 3 marks for the justification]

(c) Here S is closed under Δ. Indeed, for generic elements $x \in S$ and $y \in S$, we can write $x = 6p + 7 = 6(p + 1) + 1$ and $y = 6q + 7 = 6(q + 1) + 1$ for some integers p and q. Then

$$\begin{aligned} x \cdot y &= (6(p + 1) + 1)(6(q + 1) + 1) \\ &= 6^2(p + 1)(q + 1) + 6(p + 1) + 6(q + 1) + 1 \\ &= 6(6(p + 1)(q + 1) + p + q + 2) + 1, \end{aligned}$$

which is congruent to 1 modulo 6. Therefore

$$xy \equiv 1 \equiv 7 \pmod{6}$$

[Distribution: 2 marks for correctly identifying that S is closed under Δ and 3 marks for the justification]

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QUESTION 1. **(15 marks)**

(a) [5 marks] Which integer $a \in \{0, 1, \ldots, 4\}$ is congruent to 2021 modulo 5?

(b) [5 marks] Which integer $a \in \{0, 1, \ldots, 9\}$ is congruent to 1812^{56} modulo 10? Justify your answer.

(c) [5 marks] Let $S = \{$integers congruent to 7 modulo 6$\}$ and Δ be multiplication. Is S closed under Δ? Justify your answer.

a) $(2020 + 1)(\text{mod } 5) \equiv 1 \text{ mod } 5$

$a = 1$

b) $1812^{56} \text{ mod } 10 \equiv 2^{56} \text{ mod } 10$

$\equiv (2^7)^8 \text{ mod } 10$

$= (8)^8 \text{ mod } 10$

$= 6 \text{ mod } 10$

c) $7 \text{ mod } 6 = 1 \text{ mod } 6$

$\therefore S = 6(n) + 1, \quad n \in \mathbb{Z}, \quad m \in \mathbb{Z}$

$(6(n)+1)(6(m)+1) = 36nm + 6m + 6n + 1$

$= 6(6nm + m + n) + 1$

QUESTION 2. **(15 marks)**

Let $\mathbb{Q}$ denote the set of rational numbers. Consider the predicate $P(x, y, z) =$ "$x(y + z) = 2021$". Determine the truth value of the following statements. Justify your answers.

(i) [5 marks] $\forall x \in \mathbb{Q}, \exists y \in \mathbb{Q}, \exists z \in \mathbb{Q}, P(x, y, z)$;

(ii) [5 marks] $\exists x \in \mathbb{Q}, \forall y \in \mathbb{Q}, \exists z \in \mathbb{Q}, P(x, y, z)$;

(iii) [5 marks] $\exists x \in \mathbb{Q}, \exists y \in \mathbb{Q}, \forall z \in \mathbb{Q}, P(x, y, z)$.

i) rational $= \frac{a}{b}$, $a \in \mathbb{Z}$, $b \in \mathbb{Z}$, $b \neq 0$

for $x = \frac{a}{b}$, $2021 = \frac{a}{b}(y + z)$

$z = 0$, $y = 2021\left(\frac{b}{a}\right)$. True

ii) for some x, all y, some z

$2021 = \frac{a_1}{b_1}\left(\frac{a_2}{b_2} + z\right)$

$z = \dfrac{\frac{-a_1}{b_2} + \frac{b_1(2021)}{a_1}}{1}$

$x = 1$, for any y,

$z = (2021 - y)$

iii) $\exists x \in Q, \exists y \in Q, \forall z \in Q, P(x, y, z)$

negation, $\forall x \in Q, \forall y \in Q, \exists z \in Q, \neg P(x, y, z)$

when $x = 0$, $\neg P(x, y, z)$ is T

$x \neq 0$, for fixed y, $z = -y$, $\neg P(x, y, z)$ is T

If negation T, $\exists x \in Q, \exists y \in Q, \forall z \in Q, P(x, y, z)$ is F

QUESTION 2. **(15 marks)**

Let $\mathbb{Q}$ denote the set of rational numbers. Consider the predicate $P(x, y, z) =$ "$x(y + z) = 2021$". Determine the truth value of the following statements. Justify your answers.

(i) [5 marks] $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ P(x, y, z)$;

(ii) [5 marks] $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ P(x, y, z)$;

(iii) [5 marks] $\exists x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \forall z \in \mathbb{Q},\ P(x, y, z)$.

i) False. $P(0, y, z)$ is F for all y & z

ii) True. $P(1, y, 2021-y)$ is T for all y

iii) Take negation:

$\neg(\exists x \in Q, \exists y \in Q, \forall z \in Q, P(x,y,z))$

$= \forall x \in Q, \forall y \in Q, \exists z \in Q, \neg P(x,y,z)$

negation is T when $P(0, y, z)$

negation is also true when $P(x, y, -y)$

$\therefore$ statement is F

QUESTION 2. **(15 marks)**

Let $\mathbb{Q}$ denote the set of rational numbers. Consider the predicate $P(x, y, z) =$ "$x(y + z) = 2021$". Determine the truth value of the following statements. Justify your answers.

(i) [5 marks] $\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ P(x, y, z)$;

(ii) [5 marks] $\exists x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ P(x, y, z)$;

(iii) [5 marks] $\exists x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ \forall z \in \mathbb{Q},\ P(x, y, z)$.

Solution:

(i) False: counterexample when $x = 0$

(ii) True: Take $x = 1$ and for any fixed y, take $z = 2021 - y$

(iii) False. Consider the negation:

$$\forall x \in \mathbb{Q},\ \forall y \in \mathbb{Q},\ \exists z \in \mathbb{Q},\ \neg P(x, y, z).$$

If $x = 0$ then $\neg P(x, y, z)$ is true. Otherwise, for nonzero $x \in \mathbb{Q}$, and fixed $y \in \mathbb{Q}$, take $z = -y$, whence $\neg P(x, y, z)$ is true.

[Distribution: for each part, 2 marks for correctly identifying the truth value and 3 marks for the justification]

QUESTION 3. **(20 marks)**

(a) [5 marks] Use a truth table to prove or disprove the following equivalence.

$$(p \vee (p \to F)) \wedge q \equiv p \to q$$

(b) [5 marks] Prove the following equivalence using conversion theorem, De Morgan's law, double negation, and distributivity (noting where each is used)

$$p \vee (\neg(q \to r)) \equiv (p \vee q) \wedge (\neg p \to \neg r)$$

(c) [10 marks] Decide whether or not the following argument is valid:

$$\begin{aligned} &q \wedge r \to p; \\ &T \to p \wedge r; \\ &p \to (\neg r \to s); \\ &r \to \neg s; \\ \therefore\, &s \vee q. \end{aligned}$$

Briefly justify your answers.

Solution:

(a)

p	q	$p \to F$	$p \vee (p \to F)$	$(p \vee (p \to F)) \wedge q$	$p \to q$
T	T	F	T	T	T
T	F	F	T	F	F
F	T	T	T	T	T
F	F	T	T	F	T

The truth table disproves the equivalence!

[Distribution: 2 marks for correctly identifying nonequivalence of the statements and 3 marks for the justification]

(b)

$$\begin{aligned} p \vee (\neg(q \to r)) &\equiv p \vee (\neg(\neg q \vee r)) && \text{conversion theorem} \\ &\equiv p \vee (\neg\neg q \wedge \neg r)) && \text{De Morgan} \\ &\equiv p \vee (q \wedge \neg r)) && \text{double negation} \\ &\equiv (p \vee q) \wedge (p \vee \neg r) && \text{distributivity} \\ &\equiv (p \vee q) \wedge (\neg p \to \neg r) && \text{conversion theorem} \end{aligned}$$

(a) [5 marks] Use a truth table to prove or disprove the following equivalence.

$$(p \vee (p \to F)) \wedge q \equiv p \to q$$

(b) [5 marks] Prove the following equivalence using conversion theorem, De Morgan's law, double negation, and distributivity (noting where each is used)

$$p \vee (\neg(q \to r)) \equiv (p \vee q) \wedge (\neg p \to \neg r)$$

(c) [10 marks] Decide whether or not the following argument is valid:

$$q \wedge r \to p;$$
$$T \to p \wedge r;$$
$$p \to (\neg r \to s);$$
$$r \to \neg s;$$
$$\therefore s \vee q.$$

Briefly justify your answers.

~~a)~~ b) $p \vee (\neg(q \Rightarrow r)) = p \vee (\neg(\neg q \vee r))$

$= p \vee (q \wedge \neg r)$

$= (p \vee q) \wedge (p \vee \neg r)$

c) $T \Rightarrow p \wedge r$;

$p \wedge r$ from (2)

p

r

$r \Rightarrow \neg s$

$\neg s$

s is false

q can be false

if q is F

$q \wedge r$ is F

$(q \wedge r) \Rightarrow p$ is T

$\therefore$ $s \vee p$ is F, p, r is T

(a) [5 marks] Use a truth table to prove or disprove the following equivalence.

$$(p \vee (p \rightarrow F)) \wedge q \equiv p \rightarrow q$$

(b) [5 marks] Prove the following equivalence using conversion theorem, De Morgan's law, double negation, and distributivity (noting where each is used)

$$p \vee (\neg(q \rightarrow r)) \equiv (p \vee q) \wedge (\neg p \rightarrow \neg r)$$

(c) [10 marks] Decide whether or not the following argument is valid:

$$\begin{aligned} &q \wedge r \rightarrow p; \\ &T \rightarrow p \wedge r; \\ &p \rightarrow (\neg r \rightarrow s); \\ &r \rightarrow \neg s; \\ &\therefore s \vee q. \end{aligned}$$

$T \Rightarrow (\quad F \Rightarrow F$

Briefly justify your answers.

p	q	$p \Rightarrow q$	$p \vee (p \Rightarrow F)$	$(p \vee (p \Rightarrow F)) \wedge q$
T	T	T	T	T
T	F	F	T	F
F	T	T	T	T
F	F	T	T	F

a) disproved.

b) $p \vee (\neg(q \Rightarrow r)) \equiv (p \vee q) \wedge (\neg p \Rightarrow \neg r)$

$\equiv (p \vee q) \wedge (\neg\neg p \vee \neg r)$ conversion

$\equiv (p \vee q) \wedge (p \vee \neg r)$ doub. neg

$\equiv (p \vee (q \wedge \neg r))$ dist

$\equiv p \vee \neg(\neg q \vee r)$ de morg

$\equiv p \vee \neg(q \Rightarrow r)$ //

c) $T \Rightarrow p \wedge r$

$\therefore$ $p \wedge r$

p

r

$\neg s$

invalid. if s & q F

p & r still T

$q \wedge r \Rightarrow p$ still T

$\neg s$ is T

$r \Rightarrow \neg s$ is T

$\neg r$ is F

$\neg r \Rightarrow s$ is T

$p \Rightarrow (\neg r \Rightarrow s)$ is T

[Distribution: 1 mark for each line]

(c) The argument is invalid. Counterexample:

p	q	r	s
T	F	T	F

[Distribution: 5 marks for correctly identifying the argument is invalid and 5 marks for the justification]