Discrete Mathematics

MH1812

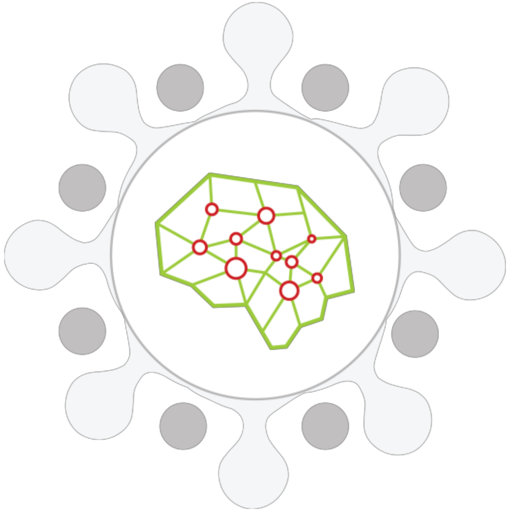
Topic 2.1 - Propositional Logic I

Dr. Gary Greaves

Topic Overview

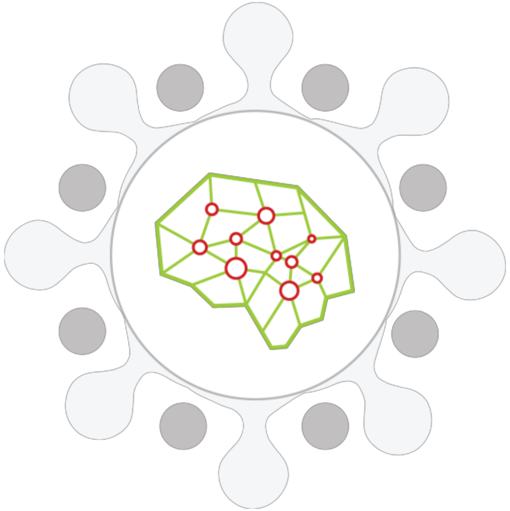
What’s in store…

- **P**roposition and Paradox
- **L**ogical Operators
- **D**e Morgan’s Laws
- **C**ontradiction and Tautology
- **E**quivalent Expressions



By the end of this lesson, you should be able to…

- Explain what is a proposition and a paradox.
- Use logical operators to combine statements.
- Apply De Morgan’s Laws.
- Explain what is a contradiction and a tautology.
- Identify equivalent expressions.
- Demonstrate that two expressions are equivalent.



Proposition and Paradox

Proposition and Paradox: Logic

- Accepted rules for making precise statements

- Logic for computer science:
  - Programming
  - Artificial intelligence
  - Logic circuits
  - Database

- Logic:
  - Represents knowledge precisely
  - Helps to extract information (inference)

Proposition and Paradox: Proposition

A **proposition** is a declarative statement that is either **true** or **false**.

Examples of propositions

- "$1 + 1 = 2$". . . True
- "$1 + 1 > 3$". . . False
- "Singapore is in Europe.". . . False

```
gap> (5>3);
true
gap> (1>3);
false
gap>
```

Proposition and Paradox: Proposition

A **proposition** is a declarative statement that is either **true** or **false**.

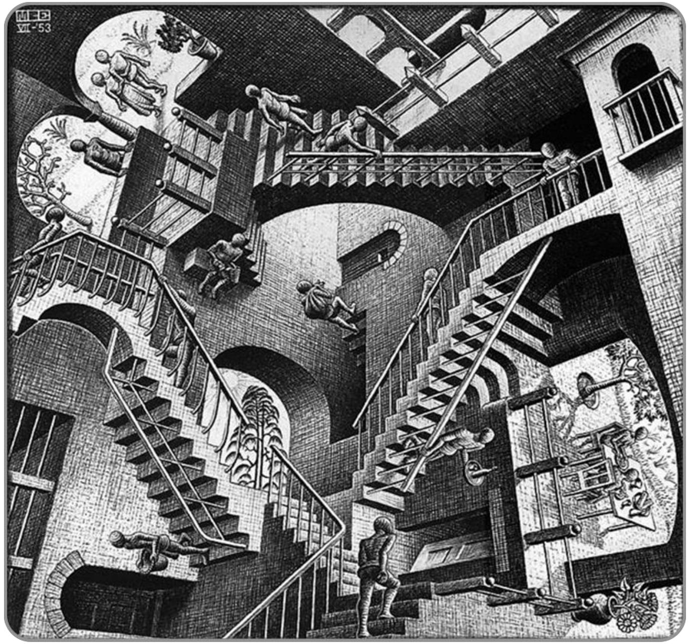
Examples that are not propositions

- "$1 + 1 > x$". . . X
- "What a great book!". . . X
- "Is Singapore in Asia?". . . X

Proposition and Paradox: Paradox

A declarative statement that **cannot** be assigned a **truth value** is called a **paradox**.

- A paradox is not a proposition.
- E.g., the liar paradox:
  "This statement is false".



Relativity Lattice (M.C. Escher)

Retrieved Feb 19, 2018 from https://www.wikiart.org/en/m-c-escher/relativity-lattice

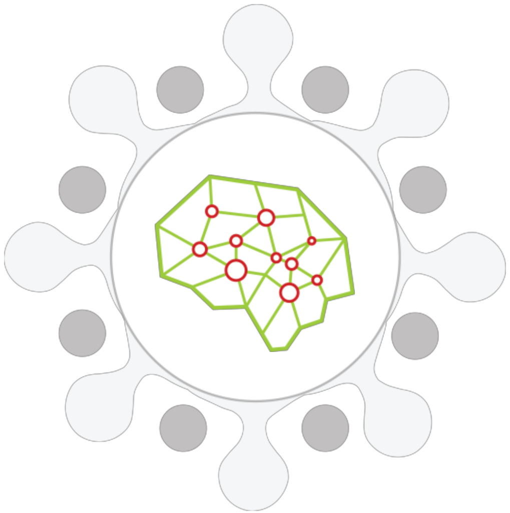
Logical Operators

Logical Operators: Symbolic Logic

- Use **symbols** to represent statements (both have the **same truth values**)

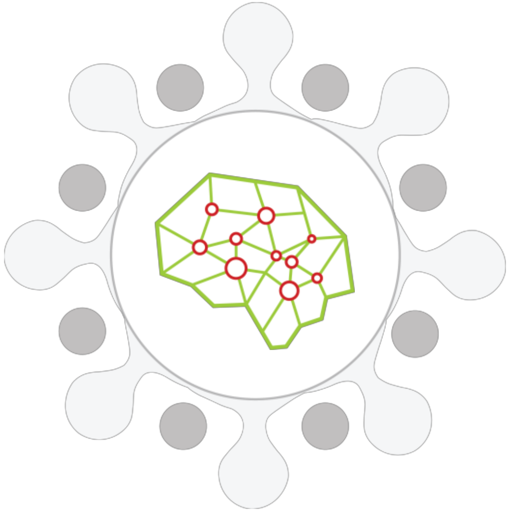
- Use **logical operators** to combine statements:

Compound Propositions	=	Propositions Combined with Logical Operator(s)



Logical Operators: Three Basic Operators

- $\wedge$ Conjunction (and)
- $\vee$ Disjunction (or)
- $\neg$ Negation (not, alternatively $\sim$)

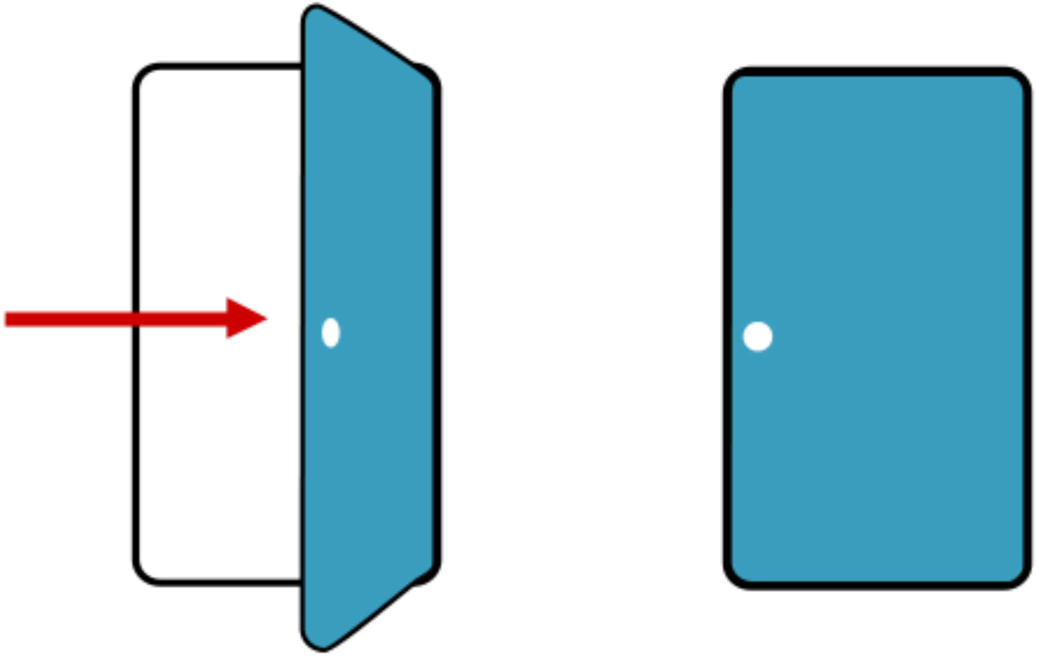


Logical Operators: Negation

- Negation (not) of p: $\neg p$ ($\sim p$ is also used)

p	$\neg p$
T	F
F	T

Truth Table



p: You may enter

$\neg p$: You may not enter

Logical Operators: Disjunction

$\vee$

- **Disjunction (or) of p with q: $p \vee q$**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$q \vee p$
T
T
T
F

True when "at least one" of them is true

Truth Table

$$p \vee q \equiv q \vee p$$

i.e., operator $\vee$ commutes

Means "equivalent"

```
gap>
gap> (5>3) or (1>5);
true
gap>
```

Logical Operators: Conjunction

$\wedge$

- Conjunction (and) of p with q: $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$q \wedge p$
T
F
F
F

Truth Table

True only when "both" of them are true

$\wedge$ is also commutative: $p \wedge q \equiv q \wedge p$

```
gap> (5>3) and (7>5);
true
gap>
gap>
gap> (5>3) and (1>5);
false
```

De Morgan’s Laws

De Morgan's Laws: Definition

$$\neg (p \land q) \equiv \neg p \lor \neg q$$
$$\neg (p \lor q) \equiv \neg p \land \neg q$$

p	q	$\neg p$	$\neg q$	$p \land q$	$\neg(p \land q)$	$\neg p \lor \neg q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Augustus De Morgan
(1806 - 1871)

Augustus De Morgan by Sophia Elizabeth De Morgan under WikiCommons (PD-US)

Contradiction and Tautology

Contradiction and Tautology: Definition

A compound proposition that is always false is called a **contradiction**.

Example

This course is easy "and" this course is not easy.

$$p \wedge (\neg p) \equiv F$$

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Contradiction and Tautology: Definition

A compound proposition that always gives a true value is called a **tautology**.

Example

$$p \lor (\neg p) \equiv \mathrm{T}$$

p	$\neg p$	$p \lor \neg p$
T	F	T
F	T	T

Always true!

Equivalent Expressions

Equivalent Expressions: Bob and Alice

1. Alice is not married but Bob is not single.

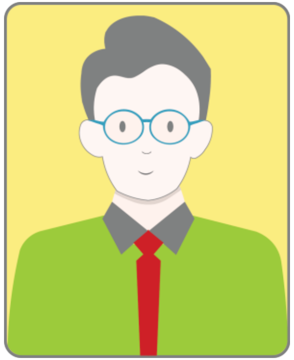
$$\neg h \land \neg b$$

2. Bob is not single and Alice is not married.

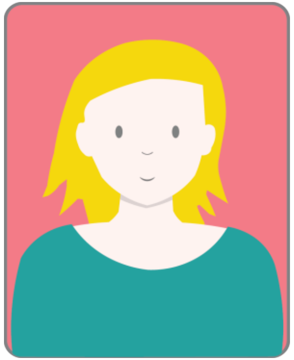
$$\neg b \land \neg h$$

3. Neither Bob is single nor Alice is married.

$$\neg (b \lor h)$$



Bob



Alice

These three statements are equivalent.

$$\neg h \land \neg b \equiv \neg b \land \neg h \equiv \neg (b \lor h)$$

Equivalent Expressions: The Statements

These three statements are equivalent:

$$\neg h \wedge \neg b \equiv \neg b \wedge \neg h \equiv \neg(b \vee h)$$

b h	$\neg b$	$\neg h$	$b \vee h$	$\neg h \wedge \neg b$	$\neg b \wedge \neg h$	$\neg(b \vee h)$
T T	F	F	T	F	F	F
T F	F	T	T	F	F	F
F T	T	F	T	F	F	F
F F	T	T	F	T	T	T

Topic Summary

Let’s recap…

- We have covered:
  - Proposition (Compound Propositions)
  - Paradox
  - Contradiction
  - Tautology
  - Equivalent Expressions

- Basic logical operators (and De Morgan’s laws):
  - Negation
  - Conjunction
  - Disjunction

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Topic 2.2 - Propositional Logic II

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Topic Overview

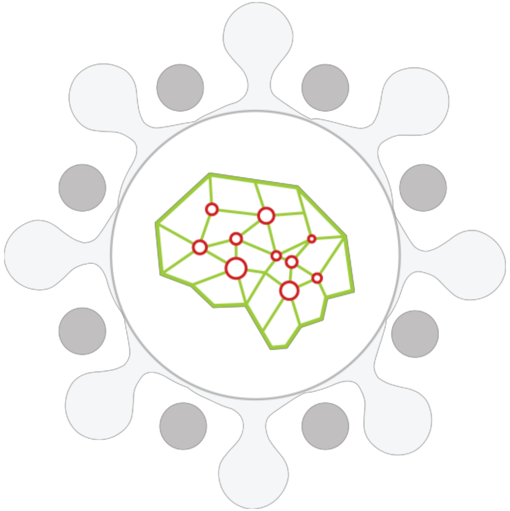
What’s in store…

Logical Equivalence Laws

Conditional Operator

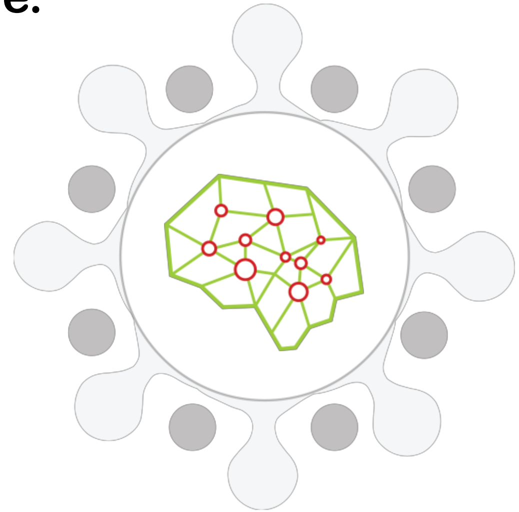
Biconditional Operator

Operator Precedence



By the end of this lesson, you should be able to…

- Prove equivalence using logical equivalence laws.
- Use the conditional operator to combine propositions.
- Use the biconditional operator to combine propositions.
- Evaluate logical expressions using operator precedence.



Logical Equivalence Laws

Logical Equivalence Laws: Already Seen

- Useful laws to **transform** one logical expression to an equivalent one

Axioms

T ≡ Tautology

C ≡ Contradiction

$\neg T \equiv F$

$\neg F \equiv T$

$\neg T \equiv C \equiv F$

$\neg C \equiv T \equiv T$

De Morgan

$\neg(p \wedge q) \equiv \neg p \vee \neg q$

$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Commutativity

$p \wedge q \equiv q \wedge p$

$p \vee q \equiv q \vee p$

Logical Equivalence Laws: More Laws

Double Negation

$$\neg(\neg p) \equiv p$$

Absorption

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

Idempotent

$$p \wedge p \equiv p$$

$$p \vee p \equiv p$$

Logical Equivalence Laws: Distributive Law

Distributivity

$$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$$

$$p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$$

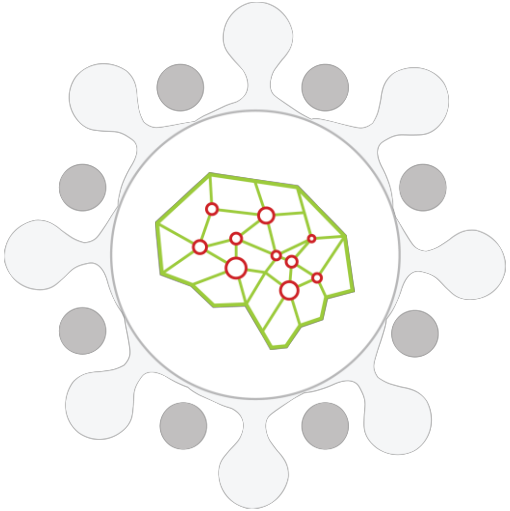
Conditional Operator

Conditional Operator: Known Operators

$\wedge$ Conjunction (and)

Disjunction (or) $\vee$

$\neg$ Negation (not, alternatively $\sim$)



Conditional Operator: If Then

If p then q: $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

By definition, when p is false, $p \to q$ is true.

This is called vacuously true or true by default.

```
gap>
gap> a:=10;; if (a>5) then Print("yes"); fi;
yes
gap> a:=1;; if (a>5) then Print("yes"); fi;
gap>
gap>
```

Not really the same!

Conditional Operator: Conversion Theorem

Theorem: $p \to q \equiv \neg p \vee q$

Proof

p q	$p \to q$	$\neg p$	$\neg p \vee q$
T T	T	F	T
T F	F	F	F
F T	T	T	T
F F	T	T	T

Conditional Operator: Converse, Inverse, Contrapositive

Statement	$p \to q$
Converse	$q \to p$
Inverse	$\neg p \to \neg q$
Contrapositive	$\neg q \to \neg p$

Theorem: $\neg q \to \neg p \equiv p \to q$

Proof

$\neg q \to \neg p$

$\equiv \neg(\neg q) \lor \neg p$

$\equiv q \lor \neg p$

$\equiv \neg p \lor q$

$\equiv p \to q$

Conditional Operator: Only If

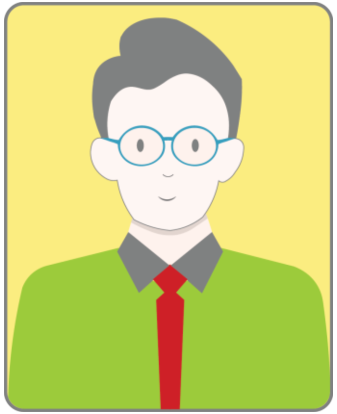
- p **only if** q $\triangleq$ $\neg q \rightarrow \neg p$
- $\neg q \rightarrow \neg p$ is the **contrapositive** of $p \rightarrow q$
- (If not q then not p) $\equiv$ ($p \rightarrow q$) (why?)

Example

"Bob pays taxes **only if** his income $\geq \$1000$"

$\triangleq$ "if Bob's income $< \$1000$ then he does not pay taxes"

$\equiv$ "if Bob pays tax then his income $\geq \$1000$"



Bob

Conditional Operator: Sufficient and Necessary Conditions

When $p \rightarrow q$, p is called a **sufficient condition** for q, q is a **necessary condition** for p.

- Being an apple is a **sufficient condition** for being a fruit.

  ≡ If it is an apple then it must be a fruit.

- Being a fruit is a **necessary condition** for being an apple.

  ≡ If it is not a fruit then it cannot be an apple.

Conditional Operator: Example

Let f: "you fix my ceiling", p: "I will pay my rent"

"You fix my ceiling **or** I won't pay my rent!"

$$f \vee \neg p \equiv p \rightarrow f$$

Landlord

Tenant

"**If** you do not fix my ceiling, **then** I won't pay my rent."

$$\neg f \rightarrow \neg p \equiv p \rightarrow f$$

"I will pay my rent **only if** you fix my ceiling."

$$\neg f \rightarrow \neg p \equiv p \rightarrow f$$

Biconditional Operator

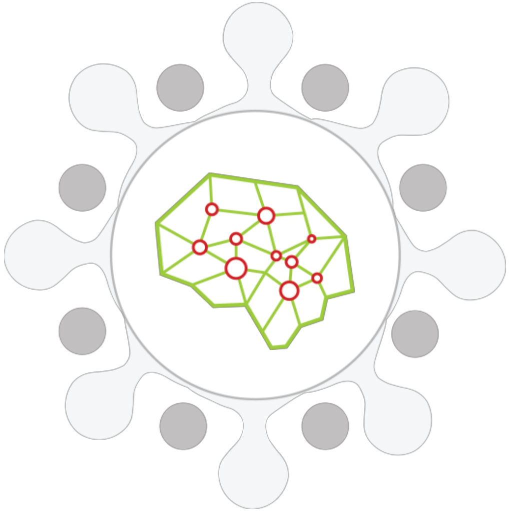
Biconditional Operator: Known Operators

$\neg$ Negation (not)

Conjunction (and) $\wedge$

$\vee$ Disjunction (or)

Conditional (if then) $\rightarrow$

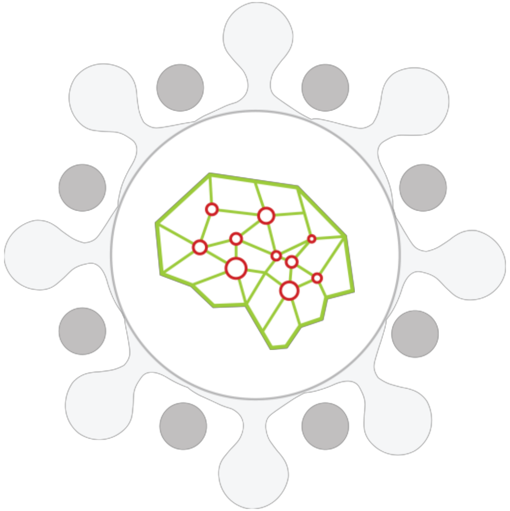


Biconditional Operator: If and Only If

- The **biconditional** of p and q: $p \leftrightarrow q \triangleq (p \rightarrow q) \wedge (q \rightarrow p)$
  – True only when p and q have identical truth value

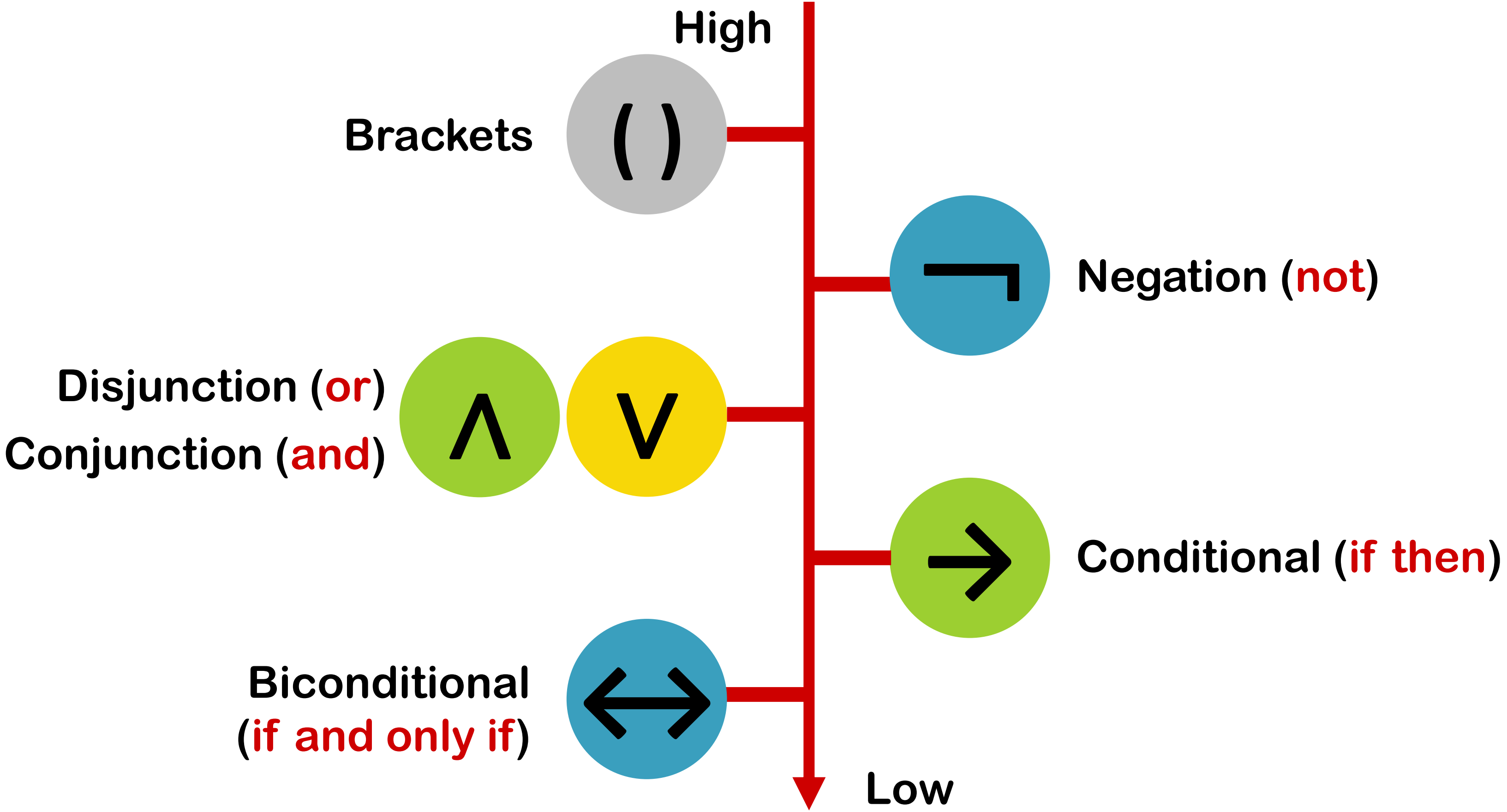
- If and only if (**iff**)

p q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T T	T	T	T
T F	F	T	F
F T	T	F	F
F F	T	T	T



Operator Precedence

Operator Precedence: High to Low



Operator Precedence: Leftmost and Rightmost

Leftmost Precedence

When equal priority instances of **binary connectives** are not separated by (), the **leftmost** one has precedence.

E.g., $p \rightarrow q \rightarrow r \equiv (p \rightarrow q) \rightarrow r$

High → Low

Rightmost Precedence

When instances of $\neg$ are not separated by (), the **rightmost** one has precedence.

E.g., $\neg\neg\neg p \equiv \neg(\neg(\neg p))$

Low ← High

Operator Precedence: Example

Show that $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

$p \vee q \rightarrow r$

$\equiv (p \vee q) \rightarrow r$ **Operator precedence**

$\equiv \neg(p \vee q) \vee r$ **Why?**

$\equiv (\neg p \wedge \neg q) \vee r$ **De Morgan's**

$\equiv (\neg p \vee r) \wedge (\neg q \vee r)$ **Why?**

$\equiv (p \rightarrow r) \wedge (q \rightarrow r)$ **Why?**

Topic Summary

Let’s recap…

- Useful logical equivalence laws:
  - Proving equivalence using these laws
- Conditional and biconditional operators:
  - Sufficient and necessary conditions
- Operator precedence

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Topic 2.3 - Propositional Logic III

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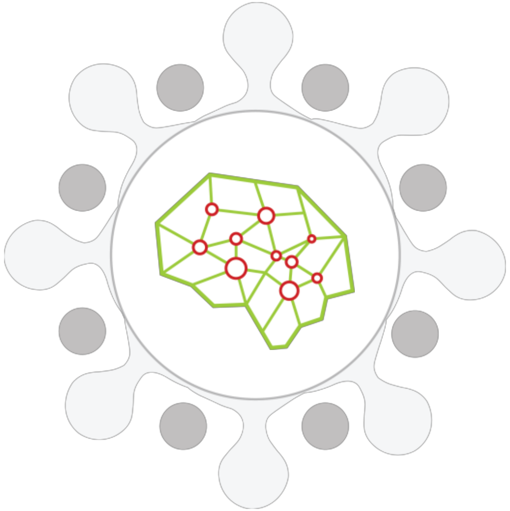
Topic Overview

What’s in store…

Arguments

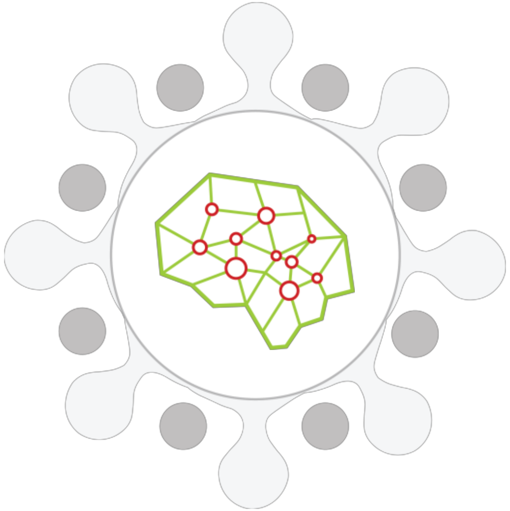
Inference Rules



By the end of this lesson, you should be able to…

- Determine whether or not an argument is valid.

- Apply basic inference rules.



Arguments

Arguments: Valid Argument

An **argument** is a sequence of statements. The last statement is called the **conclusion**. All the previous statements are called **premises** (or **assumptions**/**hypotheses**).

A **valid argument** is an argument where the conclusion is true if the premises are all true.

Arguments: Valid Argument

A **valid argument** is an argument where the conclusion is true if the premises are all true.

Example

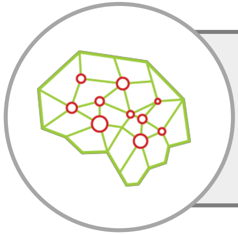
- "If you pay up in full then I will deliver it"
- "You pay up in full"

Premises

- "I will deliver it"

Conclusion

Arguments: Valid Argument



A **valid argument** is an argument where the conclusion is true if the premises are all true.

A series of statements form a valid argument if and only if "the conjunction of premises implying the conclusion" is a tautology.

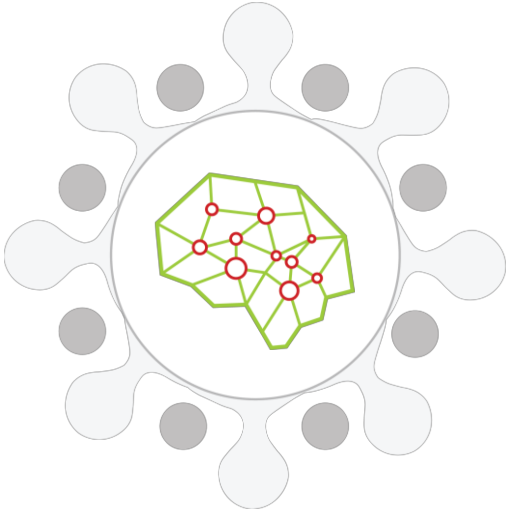
((Premise) ∧ (Premise)) → Conclusion

Arguments: A Valid Argument Template

$$\left.\begin{array}{r} p \rightarrow q; \\ p; \end{array}\right\} \text{Premises}$$

$$\left.\begin{array}{r} \therefore\ q \end{array}\right\} \text{Conclusion}$$

- By definition, a valid argument satisfies: “If the premises are true, then the conclusion is true”.
- To check if the above argument is valid, we need to check that $((p \rightarrow q) \land p) \rightarrow q$ is a tautology.



Arguments: Valid Argument Template

$$\begin{array}{r} p \to q; \\ p; \\ \therefore\ q \end{array}$$

$\left.\begin{array}{r} p \to q; \\ p; \end{array}\right\}$ **Premises**

$\left.\therefore\ q\right\}$ **Conclusion**

- **Critical rows** are rows with all premises true.
- If in all critical rows the conclusion is true, then the **argument is valid** (otherwise it is invalid).

p	q	$p \to q$	$(p \to q) \wedge p$	$((p \to q) \wedge p) \to q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

No need to calculate

Arguments: Counterexample

If in all critical rows the conclusion is true, then the argument is valid (otherwise it is invalid).

A critical row with a false conclusion is a counterexample.

A counterexample:

- Invalidates the argument (i.e., makes the argument not valid)
- Indicates a situation where the conclusion does not follow from the premises

Arguments: Invalid Argument Example

- "If it is falling and it is directly above me then I'll run"
- "It is falling"
- "It is not directly above me"

Premises

- "I will not run"

Conclusion

$$S = (f \land a \rightarrow r);$$
$$f;$$
$$\neg a;$$
$$\therefore \neg r$$

Premises

Conclusion

Arguments: Invalid Argument Example

$S = (f \wedge a \rightarrow r)$;
f;
$\neg a$;
$\therefore \neg r$

Premises

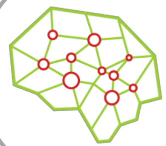
Conclusion

a r	f	$\neg a$	$f \wedge a$	S	$\neg r$
T T	T	F	T	T	F
T T	F	F	F	T	F
T F	T	F	T	F	T
T F	F	F	F	T	T
F T	**T**	**T**	F	**T**	**F**
F T	F	T	F	T	F
F F	T	T	F	T	T
F F	F	T	F	T	T

Counterexample

Critical rows

Arguments: Fallacy



A fallacy is an error in reasoning that results in an invalid argument.

Arguments: Fallacy 1 (Converse Error)

Example

p	q	$p \to q$
T	F	F
T	T	T
F	F	T
F	T	T

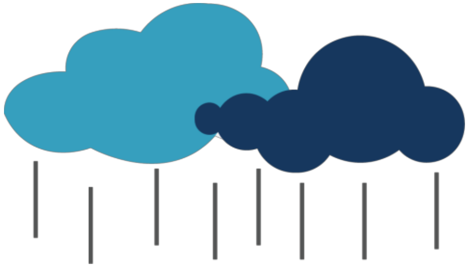
- If it is Christmas then it is a holiday.
- It is a holiday. Therefore, it is Christmas!

$$\begin{array}{r} p \to q; \\ q; \\ \therefore\ p \end{array}$$

Arguments: Fallacy 2 (Inverse Error)

Example

q	p	$p \to q$	$\neg p$	$\neg q$
T	F	T	T	F
~~T~~	~~T~~	~~T~~	~~F~~	~~F~~
F	F	T	T	T
~~F~~	~~T~~	~~F~~	~~F~~	~~T~~



- If it is raining then I will stay at home.
- It is not raining. Therefore, I will not stay at home!

$$p \to q;$$
$$\neg p;$$
$$\therefore \neg q$$

Arguments: Invalid Argument, Correct Conclusion

An argument may be invalid, but it may still draw a correct conclusion (e.g., by coincidence).

Example

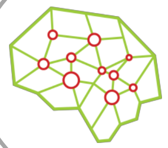
- If New York is a big city then New York has tall buildings.
- New York has tall buildings.
  - So, New York is a big city.

So, what happened?

- We have just made an invalid argument, i.e., converse error!
- But the conclusion is true (a fact true by itself).

Inference Rules

Inference Rules: Definition



A **rule of inference** is a logical construct which takes premises, analyses their syntax and returns a conclusion.

We already saw…

$$\begin{array}{r} p \rightarrow q; \\ p; \\ \therefore \quad q \end{array}$$

Modus Ponens
(Method of Affirming)

$$\begin{array}{r} p \rightarrow q; \\ \neg\, q; \\ \therefore \neg\, p \end{array}$$

Modus Tollens
(Method of Denying)

Inference Rules: More Inference Rules

Conjunctive Simplification (Particularising)

$p \wedge q;$

$\therefore\ p$

Disjunctive Syllogism (Case Elimination)

$p \vee q;$

$\neg p;$

$\therefore\ q$

Conjunctive Addition (Specialising)

$p;$

$q;$

$\therefore\ p \wedge q$

Rule of Contradiction

$\neg p \rightarrow C;$

$\therefore\ p$

Disjunctive Addition (Generalisation)

$p;$

$\therefore\ p \vee q$

Alternative Rule of Contradiction

$\neg p \rightarrow F;$

$\therefore\ p$

Inference Rules: Dilemma

Dilemma (case by case discussions)

$p \vee q$;
$p \rightarrow r$;
$q \rightarrow r$;
$\therefore r$

$p \vee q$

$\neg p \vee r$

$\neg q \vee r$

either p or q or both T

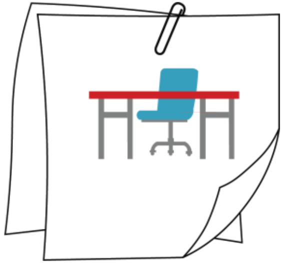
if p is T
$\neg p$ is F
r is T

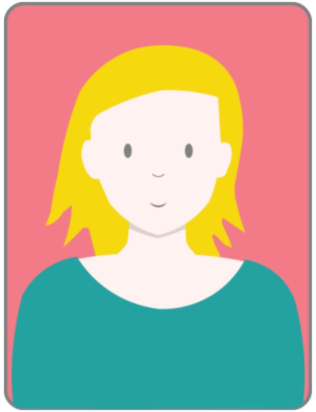
if q is T
$\neg q$ is F
r is T

Inference Rules: Hypothetical Syllogism

Hypothetical Syllogism

$$p \to q;$$
$$q \to r;$$
$$\therefore p \to r$$





Alice

Example

- If I do not wake up, then I cannot go to work.
- If I cannot go to work, then I will not get paid.
- Therefore, if I do not wake up, then I will not get paid.

Inference Rules: Proof Hypothetical Syllogism

$(p \to q) \wedge (q \to r)$	(Hypotheses; Assumed True)
$\equiv (p \to q) \wedge (\neg q \vee r)$	(Conversion Theorem)
$\equiv [(p \to q) \wedge \neg q] \vee [(p \to q) \wedge r]$	(Distributive)
$\equiv [((p \to q) \wedge \neg q) \vee (p \to q)] \wedge [((p \to q) \wedge \neg q) \vee r]$	(Distributive)
$\equiv (p \to q) \wedge [((p \to q) \wedge \neg q) \vee r]$	(Recall absorption law: a ∨ (a ∧ b) ≡ a, hence $[((p \to q) \wedge \neg q) \vee (p \to q)] \equiv p \to q$)
$\equiv (p \to q) \wedge [((\neg p \vee q) \wedge \neg q) \vee r]$	(Conversion)
$\equiv (p \to q) \wedge [((\neg p \wedge \neg q) \vee (q \wedge \neg q)) \vee r]$	(Distributive)
$\equiv (p \to q) \wedge [((\neg p \wedge \neg q) \vee F) \vee r]$	(Contradiction)
$\equiv (p \to q) \wedge [(\neg p \wedge \neg q) \vee r]$	(Unity)
$\equiv (p \to q) \wedge [(\neg p \vee r) \wedge (\neg q \vee r)]$	(Distributive)
$\equiv [(p \to q) \wedge (\neg q \vee r)] \wedge (\neg p \vee r)$	(Commutative; Associative)
$\therefore (\neg p \vee r) \equiv p \to r$	(Conjunctive Simplification; Conversion)

Topic Summary

Let’s recap…

- Arguments:
  - Valid arguments
  - Invalid arguments
  - Counterexample
  - Fallacy

- Inference rules:
  - Derive conclusions from a bunch of information
  - Some basic inference rules