MH1810 Math 1 Part 2 Chapter 5 Differentiation

Differentiation of Implicit Functions and Inverse Functions

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Parametric Differentiation

Suppose x and y are functionally dependent but can be expressed in terms of a parameter t, i.e.,

$$\begin{cases} y & = & u(t) \\ x & = & v(t) \end{cases}$$

Then we can differentiate y with respect to x as follows, (provided derivatives $u'(t)$ and $v'(t)$ exist, and $v'(t) \neq 0$):

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{u'(t)}{v'(t)}.$$

Note that we also have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}.$$

Example

Example

Let $x = 9(t - \sin t)$ and $y = 9(1 - \cos t)$. Then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{9(\sin t)}{9(1-\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \cot\frac{t}{2}.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\cot\frac{t}{2}\right)}{9(1-\cos t)} = \frac{\frac{-1}{2}\csc^2\left(\frac{t}{2}\right)}{9\left(2\sin^2\frac{t}{2}\right)} = -\frac{1}{36\sin^4\frac{1}{2}t}.$$

Implicit Differentiation

When x and y are functionally dependent but this dependence is implicitly given by means of an equation $F(x, y) = 0$, we apply the chain rule to differentiate implicitly to obtain y' in terms of x and y.

Example

Example

Find $\frac{dy}{dx}$ if $3x^4y^2 - 7xy^3 = 4 - 8y$.

Solution

Differentiating $3x^4y^2 - 7xy^3 = 4 - 8y$ *with respect to* x*:*

$$3(4x^3)y^2 + 3x^4(2y\frac{dy}{dx}) - 7y^3 - 7x(3y^2\frac{dy}{dx}) = -8\frac{dy}{dx}.$$

By rearranging the terms, we have

$$\frac{dy}{dx} = \frac{12x^3y^2 - 7y^3}{-6x^4y + 21xy^2 - 8}.$$

Power Rule for Rational Exponents

Recall we have:

$$\frac{d(x^3)}{dx} = \qquad , \frac{d(x^{-3})}{dx} =$$

What about the following

$$\frac{d(x^{3/2})}{dx} = \qquad , \quad \frac{d(x^{-3/5})}{dx} = \qquad ?$$

Power Rule for Rational Exponents

For a rational number $r = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n\mathbb{Z}^+$, the expression $x^{\frac{m}{n}}$ is

$$x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m.$$

Theorem

Suppose $r = \dfrac{m}{n}$, where n is a positive integer, and $m \in \mathbb{Z}$. Then

$$\frac{d}{dx}\left(x^r\right) = rx^{r-1}.$$

Proof of the Power Rule for Rational Exponents

Proof.

- Let $y = x^{\frac{m}{n}}$. Then $y^n = x^m$.
- Differentiating $y^n = x^m$ (implicitly) with respect to x, we obtain
- $ny^{n-1}\dfrac{dy}{dx} = mx^{m-1}$.
- Rearranging the terms, we have
- $\dfrac{dy}{dx} = \dfrac{m}{n}\dfrac{x^{m-1}}{y^{n-1}}$
- $= \dfrac{m}{n}\dfrac{x^{m-1}}{x^{m(n-1)/n}} = \dfrac{m}{n}x^{m-1-m+m/n} = \dfrac{m}{n}x^{\frac{m}{n}-1}$.
- Replacing $\frac{m}{n}$ by r, we have the required result $\dfrac{d}{dx}\left(x^r\right) = rx^{r-1}$.

$\square$

Example

$$\frac{d}{dx}\left(x^{3/2} - \pi x^{-9/5} + x^{1/3}\right) = \frac{3}{2}x^{1/2} + \frac{9\pi}{5}x^{-14/5} + \frac{1}{3}x^{-2/3}.$$

Logarithmic Differentiation

More generally, we have the power rule for a general real number r:

Theorem

Let r be a real constant. The function $f(x) = x^r$ is defined for $x > 0$, and

$$f'(x) = rx^{r-1}.$$

To verify the above derivative, we use the technique known as logarithmic differentiation.

Logarithmic Differentiation

Let $y = x^r$. Since $\ln x$ is an injective function, we apply the function $\ln x$ to $y = x^r$. This gives

$$\ln y = r \ln x.$$

Differentiate implicitly with respect to x, we have

$$\frac{1}{y}\frac{dy}{dx} = r\frac{1}{x}.$$

Thus, we have

$$\frac{dy}{dx} = r\left(\frac{y}{x}\right) = r\left(\frac{x^r}{x}\right) = rx^{r-1}.$$

Example

Example

Find the derivative $\frac{d}{dx}\left(x^{\pi}-\pi^{x}\right)$.

Example

Example

Find the derivative of $y = x^x$ for $x > 0$.

Derivative of Inverse Function

We state without proof the result on the derivative of inverse function.

Theorem (Derivative of inverse)

If f is increasing (respectively decreasing) and continuous on an interval (a, b) and $f'(x_0) > 0$ (respectively $f'(x_0) < 0$) for some $x_0 \in (a, b)$, then f^{-1} is differentiable at the point $y_0 = f(x_0)$, and

$$\begin{aligned}(f^{-1})'(y_0) &= \frac{1}{f'(f^{-1}(y_0))} \\ &= \frac{1}{f'(x_0)}.\end{aligned}$$

Derivative of Inverse Function

Note that the condition that f is increasing and continuous on (a, b) tells us that the function f is injective and the inverse f^{-1} exists.

In the next example, we demonstrate the formula

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}.$$

Example

Example

Let $f(x) = \cos x$, where $x \in (0, \pi)$. Find $(f^{-1})'(0)$.

Solution

Note that

- $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))}$

Example

Example

Let $f(x) = \cos x$, where $x \in (0, \pi)$. Find $(f^{-1})'(0)$.

Solution

Note that

- $(f^{-1})'(0) = \frac{1}{f'(f^{-1}(0))}$
- $= \dfrac{1}{-\sin(\cos^{-1}(0))} = \dfrac{1}{-\sin(\pi/2)} = -1.$

Inverse Trigonometric Functions

Theorem

1. $\dfrac{d}{dx}\left(\sin^{-1} x\right) = \dfrac{1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.
2. $\dfrac{d}{dx}\left(\cos^{-1} x\right) = \dfrac{-1}{\sqrt{1-x^2}}$, *for* $-1 < x < 1$.
3. $\dfrac{d}{dx}\left(\tan^{-1} x\right) = \dfrac{1}{1+x^2}$, *for* $x \in \mathbb{R}$.
4. $\dfrac{d}{dx}\left(\cot^{-1} x\right) = \dfrac{-1}{1+x^2}$, *for* $x \in \mathbb{R}$.
5. $\dfrac{d}{dx}\left(\sec^{-1} x\right) = \dfrac{1}{|x|\sqrt{x^2-1}}$, *for* $x < -1$ *or* $x > 1$.
6. $\dfrac{d}{dx}\left(\csc^{-1} x\right) = \dfrac{-1}{|x|\sqrt{x^2-1}}$, , *for* $x < -1$ *or* $x > 1$.

Proof of (1).

We use implicit differentiation to obtain the derivative of the inverse function.
Let $y = f(x) = \sin^{-1}(x)$, where $x \in (-1, 1)$ and $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$.
Note that

$$y = f(x) = \sin^{-1}(x) \Longleftrightarrow \sin y = x.$$

Differentiate $\sin y = x$ with respect to x implicitly, we have

$$(\cos y)\,\frac{dy}{dx} = 1, \text{ which gives } \frac{dy}{dx} = \frac{1}{\cos y}.$$

$\square$

Inverse Trigonometric Functions

Proof of (1) (Cont'd).

We have: $\frac{dy}{dx} = \frac{1}{\cos y}$.

By the trigonometric identity $\cos^2 y + \sin^2 y = 1$, we have

$$\cos^2 y = 1 - \sin^2 y = 1 - x^2.$$

Since $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we have $\cos y > 0$. Therefore, $\cos y = \sqrt{1 - x^2}$.Thus, we have

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}.$$

$\square$

Inverse Trigonometric Functions

The derivatives of the other inverse trigonometric functions can be proved similarly. You should at least verify (via implicit differentiation)

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$