Exercises for Chapter 4

Exercise 38. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Exercise 39. Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Exercise 40. Prove using mathematical induction that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

$\forall(n)$, $P(n) =$ " $n^3 - n \bmod (3) = 0$ " for $n \in \mathbb{Z}$

Basic $\quad P(1)$: $\frac{1^3 - 1}{3} = \frac{0}{3} = 0$

Inductive $\quad$ Let $P(n)$ be true for $n \in \mathbb{Z}$

$$P(n+1) = ((n+1)^3 - (n+1)) \bmod 3$$
$$= (n^3 + 3n^2 + 3n + 1 - n - 1) \bmod 3$$
$$= (n^3 + 3n^2 + 2n) \bmod 3$$
$$= (n^3 - n + 3n^2 + 3n) \bmod 3$$
$$= (n^3 - n) \bmod 3 + (3(n^2 + n) \bmod 3)$$
$$= 0$$

Exercise 38. Let q be a positive real number. Prove or disprove the following statement: if q is irrational, then $\sqrt{q}$ is irrational.

Let $P(q)$ = " q is rational"
$Q(\sqrt{q})$ = " $\sqrt{q}$ is rational"

Prove $\neg P(q) \rightarrow \neg Q(\sqrt{q})$

If $\sqrt{q}$ is rational, $\sqrt{q} = \frac{a}{b}$

$q = \frac{a^2}{b^2}$

Exercise 39. Prove using mathematical induction that the sum of the first n odd positive integers is n^2.

Prove $n^2 = \sum_{i=0}^{n} (2n-1)$ let $P(n) = \sum_{i=0}^{n} (2n-1)$

when $P(1)$, $1^2 = 2(1) - 1$
$1 = 1$

inductive step. Let $P(n)$ be valid

$$P(n+1) = \sum_{i=0}^{n} (2n-1) + (2(n+1) - 1)$$
$$= n^2 + 2n + 1$$
$$= (n+1)^2$$

$\therefore P(n)$ is T

Exercise 41. Prove by mathematical induction that

$$1^2 + 2^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

$\frac{1}{6}n(n+1)(2n+1) = \sum_{i=1}^{n} (n^2)$ Let $P(n) = \sum_{i=1}^{n} (n^2)$

Basic $P(1)$: $\frac{1}{6}(1)(2)(3) = 1^2$
$\frac{6}{6} = 1$
$1 = 1$

Let $P(n)$ be T.

inductive step, $P(n+1)$: $\sum_{i=1}^{n+1} (n^2) = \sum_{i=1}^{n} (n^2) + (n+1)^2$

$$= \frac{1}{6} n(n+1)(2n+1) + (n+1)(n+1)$$
$$= (n+1)\left(\frac{1}{6} n(2n+1) + n + 1\right)$$
$$= (n+1)\left(\frac{1}{6}(2n^2 + n) + n + 1\right)$$
$$= (n+1)\left(\frac{1}{3}n^2 + \frac{7}{6}n + 1\right)$$
$$= (n+1)\left(\frac{1}{6}(2n^2 + 3n + 4n + 6)\right)$$
$$= (n+1)\left(\frac{1}{6}\right)(n+2)(2n+3$$
$$= \frac{1}{6}(n+1)(n+2)(2(n+1)+1)$$

Exercises for Chapter 5

Exercise 42. A set menu proposes 2 choices of starters, 3 choices of main dishes, and 2 choices of desserts. How many possible set menus are available?

Exercise 43.
- In a race with 30 runners where 8 trophies will be given to the top 8 runners (the trophies are distinct, there is a specific trophy for each place),in how many ways can this be done?
- In how many ways can you solve the above problem if a certain person, say Jackson, must be one of the top 3 winners?

Exercise 44. In how many ways can you pair up 8 boys and 8 girls?

Exercise 45. How many ternary strings of length 4 have zero ones?

Exercise 46. How many permutations are there of the word "repetition"?

42) $2 \times 3 \times 2 = 12$

43) a) $30 P 8 = 2.36 \times 10^{11}$

b) $(29 \times 28 \times 3) \times 27 P 5 = 2.36 \times 10^{10}$

44) $8! = 40320$

45) $2 \times 2 \times 2 \times 2 = 16$

46) $\frac{10!}{2!2!2!}$

Exercises for Chapter 6

Exercise 47. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

Exercise 48. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

Exercise 49. Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Exercise 50. Solve the following two recurrence relations:

$$a_n = 3a_{n-1},\ a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2},\ b_1 = 0,\ b_2 = 12.$$

Exercise 51. Solve the following linear recurrence relation:

$$b_n = 4b_{n-1} - b_{n-2},\ b_0 = 2,\ b_1 = 4.$$

$x^2 = 4x - 1$

$x = 2+\sqrt{3}, \quad x = 2-\sqrt{3}$

$b_n = U(2+\sqrt{3})^n + V(2-\sqrt{3})^n$

$b_0 = 2 = U + V$

$b_1 = 4 = U(2+\sqrt{3}) + V(2-\sqrt{3})$

$V = 1, \quad U = 1$

$\therefore \quad b_n = (2+\sqrt{3})^n + (2-\sqrt{3})^n$

Exercise 47. Consider the linear recurrence $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 3$, $a_0 = 0$.

- Solve it using the backtracking method.
- Solve it using the characteristic equation.

a) $a_n = 2a_{n-1} - a_{n-2}$

$= 2(2a_{n-2} - a_{n-3}) - a_{n-2} = 3a_{n-2} - 2a_{n-3}$

$= 3(2a_{n-3} - a_{n-4}) - 2a_{n-3} = 4a_{n-3} - 3a_{n-4}$

$= \ldots$

General term is $(i+1)a_{n-i} - i\,a_{n-(i+1)}$

Last term $=$ $n-i-1=0$ at $n a_1 - (n-1)a_0 = 3n$

$\therefore a_n = 3n$

b) $a_n = 2a_{n-1} - a_{n-2}$

Degree 2. $x^2 = 2x - 1$ $\qquad Us^n + n \cdot Vs^n$

$x^2 - 2x + 1 = 0$

$x = 1$, single root

$\therefore a_n = U(1)^n + (n)(V)(1)^n$

$= U + Vn$

for $a_1 = U + V$ $\qquad 3 = U + V$ $\qquad \therefore V = 3$

$a_0 = U$ $\qquad \therefore$ $0 = U$ $\qquad \therefore a_n = 3n$ //

Exercise 48. What is the solution of the recurrence relation

$$a_n = a_{n-1} + 2a_{n-2}$$

with $a_0 = 2$ and $a_1 = 7$?

2 degree $x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$, $x = 2$, $x = -1$

$\therefore a_n = U(2)^n + V(-1)^n$

$a_0 = U + V = 2$ $\qquad 2 = U + 2U - 7$

$a_1 = 2U - V = 7 \longrightarrow V = 2U - 7$ $\qquad U = 3$

$V = -1$

$\therefore a_n = 3(2)^n - (-1)^n$

Exercise 49. Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ be a linear homogeneous recurrence. Assume both sequences a_n, a'_n satisfy this linear homogeneous recurrence. Show that $a_n + a'_n$ and αa_n also satisfy it, for α some constant.

Exercise 50. Solve the following two recurrence relations:

$$a_n = 3a_{n-1},\ a_1 = 4$$

and

$$b_n = 4b_{n-1} - 3b_{n-2},\ b_1 = 0,\ b_2 = 12.$$

$a_n = 3a_{n-1}$

$= 3(3a_{n-2})$

$= 3^3(a_{n-3})$

$= \ldots$

$= 3^{(n-1)}(a_1)$

$= 3^{(n-1)}(4)$

$b_n = 4b_{n-1} - 3b_{n-2}$

$x^2 = 4x - 3$ $\qquad x = 3$, $x = 1$

$b_n = U(3)^n + V$

$b_1 = 0 = U(3) + V$ $\Rightarrow V = -3U$

$b_2 = 12 = U(9) + V$

$\hookrightarrow 12 = 9U - 3U$, $U = 2$, $V = -6$ $\qquad \therefore b_n = 2(3)^n - 6$

Exercises for Chapter 7

Exercise 52. 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \; n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

You will need 1. for this!

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Exercise 53. Consider the set $A = \{1, 2, 3\}$, $P(A)$ =power set of A.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.
- Compute the cardinality of $P(A)$ using the counting approach.

Exercise 54. Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B), \; P(A), \; P(A \cup B), \; P(A \times B).$$

Exercise 55. Prove by contradiction that for two sets A and B

$$(A - B) \cap (B - A) = \varnothing.$$

Exercise 56. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B.$$

Exercise 52. 1. Show that

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

for $1 \leq k \leq l$, where by definition

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1.$$

2. Prove by mathematical induction that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

You will need 1. for this!

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}$$

$$= \frac{n!(n-k+1)}{k!(n-k+1)!} + \frac{n!(k)}{k!(n-k+1)!}$$

$$= \frac{n!(n-k+1+k)}{k!(n-k+1)!}$$

$$= \frac{(n+1)!}{k!(n+1-k)!}$$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

Let $P(n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

Base $P(1)$: $\binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1 = x+y$

Let $P(n)$ be T

$P(n+1) = (x+y) \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$

$=$

3. Deduce that the cardinality of the power set $P(S)$ of a finite set S with n elements is 2^n.

Power set is set of all subsets.

$|P(S)| = 1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} \cdots + \binom{n}{n-1} + 1$

$= 2^n$

Exercise 53. Consider the set $A = \{1, 2, 3\}$, $P(A)$ =power set of A.

- Compute the cardinality of $P(A)$ using the binomial theorem approach.
- Compute the cardinality of $P(A)$ using the counting approach.

a) $|P(A)| = 1 + \binom{3}{1} + \binom{3}{2} + 1 = 8$

Exercise 54. Let $P(C)$ denote the power set of C. Given $A = \{1, 2\}$ and $B = \{2, 3\}$, determine:

$$P(A \cap B),\ P(A),\ P(A \cup B),\ P(A \times B).$$

Exercise 55. Prove by contradiction that for two sets A and B

$$(A - B) \cap (B - A) = \varnothing.$$

Let $(A-B) \cap (B-A) \neq \emptyset$

there is $x \in A-B$ that belongs to $x \in B-A$,

which is a contradiction.

Exercise 56. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) = P(B) \iff A = B.$$

① $P(A) = P(B) \rightarrow A = B$

Exercise 57. Let $P(C)$ denote the power set of C. Prove that for two sets A and B

$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Exercise 58. Show that the empty set is a subset of all non-null sets.

$\emptyset \subseteq A$

$\therefore\ \forall x\,(x \in \emptyset \rightarrow x \in A)$

$x \in \emptyset$ is F, no such x

$\therefore\ \emptyset \subseteq A$ always T

Exercise 57. Let $P(C)$ denote the power set of C. Prove that for two sets A and B
$$P(A) \subseteq P(B) \iff A \subseteq B.$$

Exercise 58. Show that the empty set is a subset of all non-null sets.

Exercise 59. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

Exercise 60. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

Exercise 61. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

Exercise 62. How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.

Exercise 63. Prove the following set equality:

$$\{12a + 25b, a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

Exercise 64. Let A, B, C be sets. Prove or disprove the following set equality:
$$A - (B \cup C) = (A - B) \cap (A - C).$$

Exercise 65. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

Exercise 66. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Exercise 59. Show that for two sets A and B

$$A \neq B \equiv \exists x[(x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)].$$

$A \neq B \equiv \neg \forall x (x \in A \leftrightarrow x \in B)$

$\equiv \neg \forall x (x \in A \rightarrow x \in B \wedge x \in B \rightarrow x \in A)$

$\equiv \neg \forall x ((x \notin A \vee x \in B) \wedge (x \notin B \vee x \in A))$

$\equiv \exists x (\neg(x \notin A \vee x \in B) \vee \neg(x \notin B \vee x \in A))$

$\equiv \exists x ((x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A))$

Exercise 60. Prove that for the sets A, B, C, D

$$(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D).$$

Does equality hold?

Exercise 61. Does the equality

$$(A_1 \cup A_2) \times (B_1 \cup B_2) = (A_1 \times B_1) \cup (A_2 \times B_2)$$

hold?

Exercise 62. How many subsets of $\{1, \ldots, n\}$ are there with an even number of elements? Justify your answer.

Exercise 63. Prove the following set equality:

$$\{12a + 25b, a, b \in \mathbb{Z}\} = \mathbb{Z}.$$

Exercise 64. Let A, B, C be sets. Prove or disprove the following set equality:

$$A - (B \cup C) = (A - B) \cap (A - C).$$

$(A - B) \cap (A - C) = (A \cup \bar{B}) \cap (A \cup \bar{C})$

$= A \cup (\bar{B} \cap \bar{C})$

$= A \cup (\overline{B \cup C})$

$= A - (B \cup C)$

Exercise 65. For all sets A, B, C, prove that

$$\overline{(A - B) - (B - C)} = \bar{A} \cup B.$$

using set identities.

Exercise 66. This exercise is more difficult. For all sets A and B, prove $(A \cup B) \cap \overline{A \cap B} = (A - B) \cup (B - A)$ by showing that each side of the equation is a subset of the other.

Exercise 67. The symmetric difference of A and B, denoted by $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B.

1. Prove that $(A \oplus B) \oplus B = A$ by showing that each side of the equation is a subset of the other.

2. Prove that $(A \oplus B) \oplus B = A$ using a membership table.

Exercise 68. In a fruit feast among 200 students, 88 chose to eat durians, 73 ate mangoes, and 46 ate litchis. 34 of them had eaten both durians and mangoes, 16 had eaten durians and litchis, and 12 had eaten mangoes and litchis, while 5 had eaten all 3 fruits. Determine, how many of the 200 students ate none of the 3 fruits, and how many ate only mangoes?

Exercise 69. Let A, B, C be sets. Prove or disprove the following set equality:

$$A \times (B - C) = (A \times B) - (A \times C).$$