NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2021-2022

MH1812 - DISCRETE MATHEMATICS

December 2021 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SIX (6)** questions and comprises **FOUR (4)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the end of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This **IS NOT** an **OPEN BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

Throughout this examination paper, we denote by $\mathbb{R}$ the set of all real numbers, $\mathbb{R}^+$ the set of all positive real numbers, $\mathbb{Q}$ the set of all rational numbers, $\mathbb{Z}$ the set of all integers and $\mathbb{Z}^+$ the set of all positive integers.

QUESTION 1. **(15 marks)**

Determine the truth value of each of the following statements. You need not state the reason.

(a) $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \forall c \in \mathbb{R}\ ((b^2 - 4ac \geq 0) \rightarrow (\exists x \in \mathbb{R}\ (ax^2 + bx + c = 0)))$ **(3 marks)**

(b) $\exists x \in \mathbb{R}\ \forall n \in \mathbb{Z}^+\ ((x+1)^2 < 1/n)$ **(3 marks)**

(c) $(\exists x \in \mathbb{R}^+\ (x > 1812)) \rightarrow (\forall y \in \mathbb{R}^+\ (y > 2021))$ **(3 marks)**

(d) $\forall n \in \mathbb{Z}^+\ \exists x \in \mathbb{Q}\ (|x - \sqrt{6}| < 1/n)$ **(3 marks)**

(e) $\exists x \in \mathbb{R}\ ((\exists y \in \mathbb{R}\ (x = (1-y)^2)) \wedge (\exists y \in \mathbb{R}\ (x = -y^2)))$ **(3 marks)**

QUESTION 2. **(15 marks)**

For functions $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) = 2x + 3$ and $g(x) = x^3 + 1$,

(a) find $f \circ g$ and its inverse function $(f \circ g)^{-1}$; **(6 marks)**

(b) find f^{-1}, g^{-1}, and $g^{-1} \circ f^{-1}$; **(7 marks)**

(c) are $(f \circ g)^{-1}$ and $g^{-1} \circ f^{-1}$ identical ? **(2 marks)**

Determine the truth value of each of the following statements. You need not state the reason.

(a) $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ \forall c \in \mathbb{R}\ ((b^2 - 4ac \geq 0) \rightarrow (\exists x \in \mathbb{R}\ (ax^2 + bx + c = 0)))$ **(3 marks)**

(b) $\exists x \in \mathbb{R}\ \forall n \in \mathbb{Z}^+\ ((x+1)^2 < 1/n)$ **(3 marks)**

(c) $(\exists x \in \mathbb{R}^+\ (x > 1812)) \rightarrow (\forall y \in \mathbb{R}^+\ (y > 2021))$ **(3 marks)**

(d) $\forall n \in \mathbb{Z}^+\ \exists x \in \mathbb{Q}\ (|x - \sqrt{6}| < 1/n)$ **(3 marks)**

(e) $\exists x \in \mathbb{R}\ ((\exists y \in \mathbb{R}\ (x = (1-y)^2)) \wedge (\exists y \in \mathbb{R}\ (x = -y^2)))$ **(3 marks)**

a) F, $a=0$, $b=0$, $c=-1$

b) T, for some x, $x=-1$

$(-1+1)^2 = 0$

$0 < 1/n$ for any n

c)

e) separate

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(a) find $f \circ g$ and its inverse function $(f \circ g)^{-1}$; **(6 marks)**

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(c) are $(f \circ g)^{-1}$ and $g^{-1} \circ f^{-1}$ identical ? **(2 marks)**

$f \circ g = 2(x^3+1)+3$,

$y = 2(x^3+1)+3$

$\frac{y-3}{2} = x^3 + 1$

$\frac{y-3}{2} - 1 = x^3$

$x = \sqrt[3]{\frac{y-3}{2} - 1}$,

QUESTION 3. **(20 marks)**

A ternary string is a string whose characters are 0, 1 or 2. For each $n \in \mathbb{Z}^+$, let a_n denote the number of strings of length n without a substring "201". It is clear that there are 3^n ternary strings of length n and thus $a_n \leq 3^n$.

(a) Write down a recurrence relation for a_n and the initial value(s). You do not need to solve the recurrence relation. **(10 marks)**

(b) Show that $a_n \geq 3 \cdot 2^n$ for all $n \in \mathbb{Z} \cap [3, \infty)$. **(10 marks)**

QUESTION 4. **(15 marks)**

In the 100-yard dash with 8 runners, the runner or runners who finish with the fastest time receive gold medals, the runner or runners who finish with exactly one runner ahead receive silver medals, and the runner or runners who finish with exactly two runners ahead receive bronze medals. How many ways are there to award the medals to the 8 runners, if exactly 3 of them win medals and ties are possible? (Your answer must be an explicit integer and should not be an unevaluated or partially evaluated expression.)

QUESTION 5. **(20 marks)**

On a set $S = \{a, b, c, d\}$ we define a relation $R = \{(a, a), (a, b), (b, b), (b, c), (c, c), (c, d), (d, d)\}$.

(a) Find the transitive closure R^t. **(4 marks)**

(b) Is R^t reflexive? Symmetric? Anti-symmetric? An equivalence relation? A partial order? You need not state the reason. **(8 marks)**

(c) Find R^2, R^3, and $R \cup R^2 \cup R^3$. **(6 marks)**

(d) Do R, R^2, and R^3 partition R^t? **(2 marks)**

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(a) Write down a recurrence relation for a_n and the initial value(s). You do not need to solve the recurrence relation. **(10 marks)**

(b) Show that $a_n \geq 3 \cdot 2^n$ for all $n \in \mathbb{Z} \cap [3, \infty)$. **(10 marks)**

QUESTION 6. **(15 marks)**

Let $M = \mathbb{Z}^+ \cup \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. Define $S \subseteq \mathbb{R} \times \mathbb{R}$ to be

$$S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = yz \text{ for some } z \in M\}.$$

For each $x \in \mathbb{R}$, define a set H_x to be

$$H_x = \{y \in \mathbb{R} : (x, y) \in S\}.$$

(a) Determine H_0. **(5 marks)**

(b) Show that $\mathbb{Z}^+ \subseteq H_{1/2}$. **(5 marks)**

(c) Find all $x \in \mathbb{R}$ such that $H_x = M$. If such an x does not exist, state "does not exist" and give your reason. **(5 marks)**

END OF PAPER

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(c) Find all $x \in \mathbb{R}$ such that $H_x = M$. If such an x does not exist, state "does not exist" and give your reason. **(5 marks)**

$H_0 = \{y \in \mathbb{R} : (0, y) \in S\}$

$0 = yz$, $z \in M$

$z \neq 0$

$y = 0$

$\therefore H_0 = \{0\}$

b) $H_{1/2} = \{y \in \mathbb{R} : (1/2, y) \in S\}$

$1/2 = yz$ $\quad z \in M$

let $z = 1/2$

$1/2 = y \, 1/2$

$y = 1$

let $z = 1/4$

$1/2 = y \, 1/4$

$y = 2$

$\therefore z = \frac{1}{2 \times n}$, $n \in \mathbb{Z}^+$ for $y \in \mathbb{Z}^+$