Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics 1** **Tutorial 1**

[Questions with a "★" are advanced questions. Otherwise, questions are considered basic or intermediate. Please attempt the basic/intermediate questions before trying the advanced questions.]

1. Evaluate the expression and write your answer in the form $a + bi$, $a, b \in \mathbf{R}$.

   (a) $i^{179} + i^{2017} = i^3 + i^1 = -i + i = 0$

   (b) $\dfrac{1+2i}{3-4i} = \left(\dfrac{1+2i}{3-4i}\right)\left(\dfrac{3+4i}{3+4i}\right) = \left(\dfrac{3+4i+6i+8i^2}{9+12i-12i-16i^2}\right) = \dfrac{-5+10i}{25}$
   $= -\frac{1}{5} + \frac{2}{5}i$

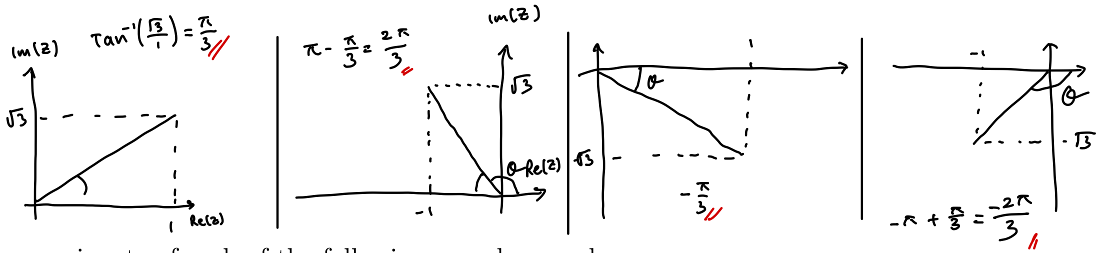
2. For each of the following, represent the complex number on the Argand diagram. Find the modulus and the principal argument of the complex number. Hence express the complex number in its polar representation.

   (a) $1 + \sqrt{3}i$

   (b) $-1 + \sqrt{3}i$

   (c) $1 - \sqrt{3}i$

   (d) $-1 - \sqrt{3}i$



3. Find the complex conjugate of each of the following complex numbers.

   (a) $(2i)' = -2i$

   (b) $(2)' = 2$

   (c) $(1+3i)' = 1-3i$

   (d) $(-3-4i)' = -3+4i$

   (e) a complex number with modulus 2 and argument $\theta = \frac{\pi}{3}$. $z = 2e^{i\frac{\pi}{3}}$
   $z' = 2e^{-i\frac{\pi}{3}}$

4. Sketch the regions defined by

   (a) $\text{Re}(z) \geq 0$

   (b) $\text{Im}(z) < 2$

   (c) $|z| \geq 2$

   (d) $\dfrac{\pi}{6} \leq \arg(z) \leq \dfrac{\pi}{3}$

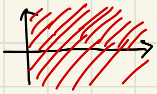
   (e) $|z - i| = |z - 1|$

   (f) $|z - (1+i)| \leq 2$

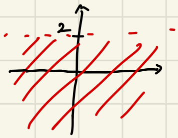
5. Solve the following equation for the real numbers, x and y.

$$\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1 + i$$

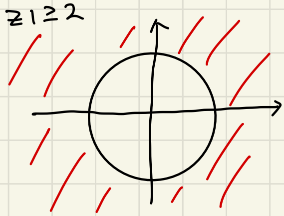
4 a $\quad Re(z) = 0$



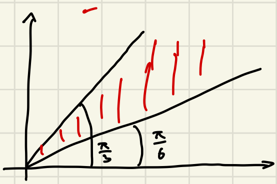
$Im(z) < 2$



$|z| \geq 2$



d . $\frac{\pi}{6} \leq \arg(z) \leq \frac{\pi}{3}$



e. $|z - i| = |z - 1|$

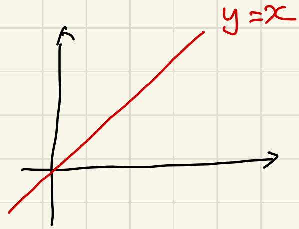
let $z = a + bi$

$|(a+bi) - i| = |(a+bi) - 1|$

$\sqrt{a^2 + (b-1)^2} = \sqrt{(a-1)^2 + b^2}$

$a^2 + (b-1)^2 = (a-1)^2 + b^2$

$a = b$



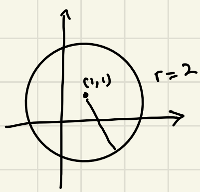
f. $|z - (1+i)| \leq 2$

$|(a+bi) - (1+i)| \leq 2$

$|a - 1 + (b-1)i| \leq 2$

$\sqrt{(a-1)^2 + (b-1)^2} \leq 2$

$(a-1)^2 + (b-1)^2 \leq 2^2$



5. $\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1+i$

$\left(\frac{(1+i)(1+i)}{(1-i)(1+i)}\right)^2 + \frac{1}{x+iy} = 1+i$

$\left(\frac{1+i+i+1}{1+1}\right)^2 + \frac{1}{x+iy} = 1+i$

$\left(\frac{2i}{2}\right)^2 + \frac{1}{x+iy} = 1+i$

$-1 + \frac{1}{x+iy} = 1+i$

$\frac{1}{x+iy} = 2+i$

$1 = (2+i)(x+iy)$

$1 = 2x + 2yi + xi - y$

$1 = (2x - y) + (x + 2y)i$

① - $2x - y = 1$ $\quad 2(-2y) - y = 1$

② - $x + 2y = 0$ $\quad y = -\frac{1}{5} \quad x = \frac{3}{5}$

6. Suppose a complex number $z = x + iy$ satisfies

$$|z - 1| = \frac{1}{2}|z - i|.$$

Show that x and y satisfy the following equation

$$\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{8}{9},$$

which represents a circle of radius $\frac{\sqrt{8}}{3}$ centred at $\left(\frac{4}{3}, \frac{-1}{3}\right)$.

let $z = x + iy$ $\quad |(x-1) + iy| = \frac{1}{2}|x + (y-1)i|$

$$\sqrt{(x-1)^2 + y^2} = \frac{1}{2}\sqrt{(x)^2 + (y-1)^2}$$
$$(x-1)^2 + y^2 = \frac{1}{4}\left(x^2 + (y-1)^2\right)$$
$$x^2 - 2x + 1 + y^2 = \frac{1}{4}x^2 + \frac{1}{4}y - \frac{1}{2}y + \frac{1}{4}$$
$$\frac{3}{4}x^2 - 2x + \frac{3}{4}y^2 + \frac{1}{2}y + \frac{3}{4} = 0$$
$$x^2 - \frac{8}{3}x + y^2 + \frac{2}{3}y + 1 = 0$$

$$\left(x - \frac{4}{3}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{8}{9}$$
$$x^2 + 2\left(\frac{4}{3}\right)x + \frac{16}{9} + y^2 + 2\left(\frac{1}{3}\right)y + \frac{1}{9} = \frac{8}{9}$$
$$x^2 - \frac{8}{3}x + y^2 + \frac{2}{3}y + 1 = 0$$

Same, ∴ shown

$(1 + \sqrt{-3})^2 = (1 + \sqrt{3}i)^2$
$= -2 + 2\sqrt{3}i$

7. Express each of the following complex numbers in the form $re^{i\theta}$, with $r \geq 0$, and $-\pi < \theta \leq \pi$.

(a) $(1 + \sqrt{-3})^2$ $\qquad$ (b) $\frac{1+i}{1-i} = \frac{(1+i)(1+i)}{(1-i)(1+i)} = \frac{2i}{2} = i$

arg θ

$\tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

$\theta = \pi - \frac{\pi}{3}$

$= \frac{2\pi}{3}$

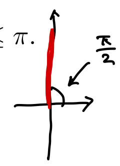
$|z|$

$\sqrt{(-2)^2 + (2\sqrt{3})^2}$

$= 4$

$|z| = 1$

$\arg\theta = \frac{\pi}{2}$ ∴ $e^{\frac{\pi}{2}i}$



8. Let $z = a + ib$, where a and b are some real numbers.

Show that $\qquad (a+ib) + (a-ib) = 2a$

(a) $z + \bar{z} = 2\text{Re}(z)$.
(b) $z = \bar{z} \iff z$ is a real number.

$z = \bar{z}$
$a + ib = a - ib$
$a = a$

z is real if $b = 0$

9. (a) If z is a root of $ax^3 + bx^2 + cx + d = 0$, where a, b, c and d are real constants, then $\bar{z}$ is also a root of $ax^3 + bx^2 + cx + d = 0$.

Remark *More generally, we have*

Suppose that $p(x) = a_0 + a_1x + ... + a_nx^n$ *is a polynomial in* x *with real coefficients* a_k*'s. If a complex number* z *is a solution of* $p(x) = 0$*, then the conjugate* $\bar{z}$ *of* z *is also a solution of* $p(x) = 0$.

(b) Given that $1 + i$ is a root of the equation $x^3 - x^2 + 2 = 0$. Find all other roots of the equation.

If $1+i$ is a root, $1-i$ is a root

$(x - (1+i))(x - (1+i))(x + a) = x^3 - x^2 + 2$

$(x^2 - x + i - x - i + 2)(x + a) =$ "

$(x^2 - 2x + 2)(x + a) = x^3 - x^2 + 2$

$a = 1$ ... ∴ root $x = -1$

10. Express each of following complex numbers in the form (i) $\cos\alpha + i\sin\alpha$ and (ii) $x + iy$.

(a) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^7$

(b) $\dfrac{1}{\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^2}$

(c) $\dfrac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^4}$

11. Without using any series expansions, prove that

$$(\sqrt{3} + i)^n + (\sqrt{3} - i)^n \quad \text{is real.}$$

Find the value of this expression when $n = 12$.

12. Find all four distinct fourth roots of $-16i$.

13. Solve the following equations.

(a) $z^4 + 4z^2 + 16 = 0$
(b) $z^4 + 1 = 0$
(c) $z^3 + z^2 + z + 1 = 0$

14. Solve the equation $\left(\frac{z - 4i}{2i}\right)^3 = i$ and represent the roots of the equation in an Argand diagram.

10. Express each of following complex numbers in the form (i) $\cos\alpha + i\sin\alpha$ and (ii) $x + iy$.

(a) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^7$

(b) $\dfrac{1}{\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^2}$

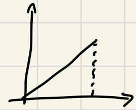
(c) $\dfrac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^4}$

a. $\left(\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\right)^7 = \cos\frac{7\pi}{3} + \sin\frac{7\pi}{3}$

$\tan\left(\frac{7\pi}{3}\right) = \frac{\sqrt{3}}{1}$

$\alpha = \frac{7\pi}{3}$

$= \frac{\pi}{3}$



ratio of $\frac{o}{a} = \frac{\sqrt{3}}{1}$

$1 = \sqrt{(a(1))^2 + (a\sqrt{3})^2}$

$1 = a^2 + 3a^2$

$a = \frac{1}{2}$

$x + iy = \frac{1}{2} + \frac{\sqrt{3}}{2}y$

b) $\dfrac{1}{\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)^2} = \dfrac{1}{\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)^2}$

$= \dfrac{1}{\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})}$

$= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

refer to above

c) $\dfrac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(-\cos\frac{\pi}{6} + i\sin\frac{\pi}{3}\right)^4} = \dfrac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left(-\cos(-\frac{\pi}{6}) - i\sin(-\frac{\pi}{6})\right)^4}$

$= \dfrac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{\left((-1)\left(\cos(-\frac{\pi}{6}) + i\sin(-\frac{\pi}{6})\right)\right)^4}$

$= \dfrac{\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}}{(-1)^4\left(\cos(-\frac{4\pi}{3}) + i\sin(-\frac{4\pi}{3})\right)}$

$= \cos\left(\frac{\pi}{6} - \left(-\frac{4\pi}{3}\right)\right) + i\sin\left(\frac{\pi}{6} - \left(-\frac{4\pi}{3}\right)\right)$

$= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)$

$= \cos\left(-\frac{1}{2}\pi\right) + i\sin\left(-\frac{1}{2}\pi\right)$

for $x + iy$: $x = 0$, $y = 1$

$\therefore -i$ //

15. ★★If α is a complex 5th root of unity with the smallest positive principal argument, determine the value of

$$(1+\alpha^4)(1+\alpha^3)(1+\alpha^2)(1+\alpha).$$

16. ★Suppose $\sin\dfrac{\theta}{2} \neq 0$. Prove that

$$\frac{1}{2}+\sum_{k=1}^{n}\cos k\theta = \frac{\sin\left[\left(n+\frac{1}{2}\right)\theta\right]}{2\sin\frac{\theta}{2}}.$$

(Hint: Let $z=\cos\theta+i\sin\theta=e^{i\theta}$. Geometric sum: $\displaystyle\sum_{k=1}^{n} z^k = z+z^2+\cdots+z^n=\frac{z(z^n-1)}{z-1}$.)

17. ★

   (a) Use De Moivre's Theorem and binomial expansion to show that

   $$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

   (b) Express $\sin 5\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

   (c) Hence, obtain an expression for $\tan 5\theta$ in terms of powers of $\tan\theta$.

18. ★

   (a) Let $z=\cos\theta+i\sin\theta$. Show that

   (i) $\dfrac{1}{z}=\cos\theta-i\sin\theta$, and hence $\cos\theta=\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$ and $\sin\theta=\dfrac{1}{2i}\left(z-\dfrac{1}{z}\right)$.

   (ii) $z^k+\dfrac{1}{z^k}=2\cos k\theta$ and $z^k-\dfrac{1}{z^k}=2i\sin k\theta$, for $k\in\mathbb{Z}^+$.

   (b) Use the results in part (a) to prove that

   $$\cos^4\theta=\frac{1}{8}\left(\cos 4\theta+4\cos 2\theta+3\right).$$

   Hence find the integral $\displaystyle\int 8\cos^4\theta\, d\theta$.

Challenging Problem (Optional. Will not be discussed in the tutorial session.)

1. Consider a regular n-sided polygon circumscribed by the unit circle. Prove that the product of the length of line segments formed by joining one vertex of the regular polygon to the rest of the vertices equals to n. (Hint: Consider the n^{th} roots of 1).

Answers

1. (a) 0 (or $0+0i$)

   (b) $\frac{-1}{5}+\frac{2}{5}i$

....

2. (a) $|z|=2, \arg(z)=\frac{\pi}{3}$

15. ★★If α is a complex 5th root of unity with the smallest positive principal argument, determine the value of

α is a root.

$$(1+\alpha^4)(1+\alpha^3)(1+\alpha^2)(1+\alpha).$$

$z^5 = 1 = e^{i(2\pi)(n)}, n \in \mathbb{I}$

$z^5 = e^{i(2\pi)(n)}$

$z = e^{i\left(\frac{2n\pi}{5}\right)}$

for $n=0$, $z = e^{i\left(\frac{0}{5}\right)} = 1$

Smallest principal argument $= \alpha$

$n=1$, $z = e^{i\left(\frac{2\pi}{5}\right)} = \alpha$

$n=2$, $z = e^{i\left(\frac{4\pi}{5}\right)} = \left(e^{i\left(\frac{2\pi}{5}\right)}\right)^2 = \alpha^2$

$n=3$, $z = e^{i\left(\frac{6\pi}{5}\right)} = \left(e^{i\left(\frac{2\pi}{5}\right)}\right)^3 = \alpha^3$

$n=4$, $z = e^{i\left(\frac{8\pi}{5}\right)} = \left(e^{i\left(\frac{2\pi}{5}\right)}\right)^4 = \alpha^4$

$n=5$ returns back to $\theta = 2\pi$.

$\therefore z^5 - 1 = (z-1)(z-\alpha)(z-\alpha^2)(z-\alpha^3)(z-\alpha^4)$

here, you wanna match with the Qn

when $z = -1$

$(-1)^5 - 1 = (-1-1)(-1-\alpha)(-1-\alpha^2)(-1-\alpha^3)(-1-\alpha^4)$

$-2 = [-1]\left[(2)(1+\alpha)(1+\alpha^2)(1+\alpha^3)(1+\alpha^4)\right]$

$\frac{-2}{-2} = (1+\alpha)(1+\alpha^2)(1+\alpha^3)(1+\alpha^4)$

(b) $|z| = 2, \arg(z) = \frac{2\pi}{3}$
(c) $|z| = 2, \arg(z) = -\frac{\pi}{3}$
(d) $|z| = 2, \arg(z) = -\frac{2\pi}{3}$

....

3. (a) $-2i$
   (b) 2
   (c) $1 - 3i$
   (d) $-3 + 4i$
   (e) $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 1 - \sqrt{3}i$

....

5. $x = \frac{2}{5}$, $y = -\frac{1}{5}$.

....

7. (a) $4e^{\frac{2\pi i}{3}}$
   (b) $e^{\frac{\pi i}{2}}$

....

9. (b) $1 - i$, -1.

10. (a) $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
    (b) $\cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$
    (c) $\cos\frac{-\pi}{2} + i\sin\frac{-\pi}{2} = -i$

....

11. 8192

....

12. $z_0 = 2\mathrm{cis}(\frac{-\pi}{8})$, $z_1 = 2\mathrm{cis}(\frac{3\pi}{8})$, $z_2 = 2\mathrm{cis}(\frac{7\pi}{8})$, $z_3 = 2\mathrm{cis}(\frac{11\pi}{8}) = 2\mathrm{cis}(\frac{-5\pi}{8})$

....

13. (a) $z_0 = 2e^{i\frac{\pi}{3}} = 1 + \sqrt{3}i, z_1 = 2e^{i\frac{4\pi}{3}} = 2e^{i\frac{-2\pi}{3}} = -1 - \sqrt{3}i,$
    $z_0' = 2e^{i\frac{-\pi}{3}} = 1 - \sqrt{3}i,$ and $z_1' = 2e^{i\frac{2\pi}{3}} = -1 + \sqrt{3}i.$
    (b) $z_0 = e^{\frac{\pi}{4}i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, z_1 = e^{\frac{3\pi}{4}i} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i,$
    $z_2 = e^{\frac{5\pi}{4}i} = e^{\frac{-3\pi}{4}i} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ or $z_3 = e^{\frac{7\pi}{4}i} = e^{\frac{-\pi}{4}i} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$
    (c) $z = -1$ or $z = e^{\frac{\pi}{2}i} = i$ or $z = e^{\frac{3\pi}{2}i} = -i$

....

14. $z = 2 + 4i,$ or $z = -1 + i(4 + \sqrt{3})$ or $, z = -1 + i(4 - \sqrt{3}).$

15. $(1 + \alpha^4)(1 + \alpha^3)(1 + \alpha^2)(1 + \alpha) = 1.$

....

17. (b) $5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$ (c) $\dfrac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$

18. (b) $\int(\cos 4\theta + 4\cos 2\theta + 3)\,d\theta = 3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta + C.$

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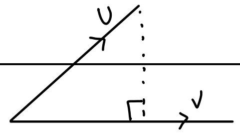
2021/22 Semester 1 **MH1810 Mathematics 1** **Tutorial 2**



$\mathrm{Proj}_V U = [U \cdot \hat{V}]\, \hat{V}$

$= \left[\begin{pmatrix}1\\1\\-5\end{pmatrix} \cdot \begin{pmatrix}\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\end{pmatrix}\right] \begin{pmatrix}\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\end{pmatrix}$

$= \left(\frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{5}{\sqrt{6}}\right) \begin{pmatrix}\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\end{pmatrix}$

$= \left(\frac{8}{\sqrt{6}}\right) \begin{pmatrix}\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ \frac{-1}{\sqrt{6}}\end{pmatrix}$

$= \begin{pmatrix}8/3\\4/3\\-4/3\end{pmatrix}$ //

Relevant sections for further reading

[S] Calculus by James Stewart Chapter 12 (Sections 12.1 -12.5) or

[T] Thomas' Calculus: Chapter 11 (Sections 11.1 -11.5).

$U \cdot V = 1(2) + 1(1) + (-5)(-1) = 8$

$\|V\| = \sqrt{6}$ //

$\|U\| = 3\sqrt{3}$ //

$U \times V = \begin{pmatrix}-1-(-5)\\-10-(-1)\\1-2\end{pmatrix} = \begin{pmatrix}4\\-9\\-1\end{pmatrix}$ //

$V \times U = \begin{pmatrix}-5-(-1)\\-1-(-10)\\2-1\end{pmatrix} = \begin{pmatrix}-4\\9\\1\end{pmatrix}$ //

1. Let $\mathbf{u} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$ $\begin{pmatrix}1\\1\\-5\end{pmatrix}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$ $\begin{pmatrix}2\\1\\-1\end{pmatrix}$.
   Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, $\mathbf{u} \cdot \mathbf{v}$, $\mathbf{u} \times \mathbf{v}$, $\mathbf{v} \times \mathbf{u}$, and $\mathrm{proj}_{\mathbf{v}}\mathbf{u}$.

2. For which values of k are $\mathbf{x} = (k, k, 1)$ and $\mathbf{y} = (k, 5, 6)$ in $\mathbb{R}^3$ perpendicular to each other (i.e., orthogonal)?

3. Consider the parallelogram $ABPC$ with adjacent sides AB and AC and vertices $A\,(1, 0, 0)$, $B\,(0, 1, 0)$ and $C\,(0, 0, 1)$.

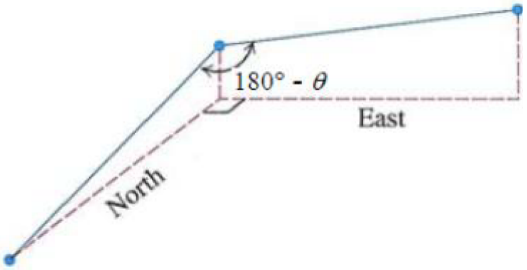
   (a) Find the area of the parallelogram.

   (b) Find the coordinates of the vertex P.

   (c) Find the angle between the diagonals of the parallelogram.

4. Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude 5 N) along the line from the origin to the point $(1, 1)$. (Distance measured in metres).

5. A water main is to be constructed with at 20% grade (i.e., slope $= \dfrac{\text{height}}{\text{horizontal distance}} = 0.2$) in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east (i.e., the angle θ you need to bend the water main).



6. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$.

   (a) Using $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ and some properties of dot products, prove that

   $$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2.$$

   Hence prove that

   $$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2.$$

   (b) Use part (a) to prove that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$. Also, interpret this geometrically in $\mathbf{R}^2$.

2. For which values of k are $\mathbf{x} = (k, k, 1)$ and $\mathbf{y} = (k, 5, 6)$ in $\mathbb{R}^3$ perpendicular to each other (i.e., orthogonal)?

If $x \perp y$, $x \cdot y = 0 = kk + 5k + 6$
$= (k+2)(k+3)$

$\therefore k = -2, \; k = -3$

3. Consider the parallelogram $ABPC$ with adjacent sides AB and AC and vertices $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$.

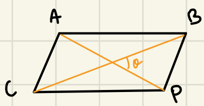
(a) Find the area of the parallelogram.

(b) Find the coordinates of the vertex P.

(c) Find the angle between the diagonals of the parallelogram.

$AB = AO + OB = OB - OA = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$AC = OC - OA = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$



a) Area $= \| AB \times AC \|$

$= \left\| \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\|$

$= \left\| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\|$

$= \sqrt{3}$

b) $P = A + AB + AC$

$= \begin{pmatrix} 1 - 1 - 1 \\ 0 + 1 + 0 \\ 0 + 0 + 1 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

c) $AP = OP - OA = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$BC = OC - OB = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

$AP \cdot BC = \|AP\| \|BC\| \cos\theta$

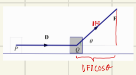
$-1 + 1 = \|AP\| \|BC\| \cos\theta$

$0 = \cos\theta$

$\theta = \frac{\pi}{2}$

4. Find the work done by a force $\mathbf{F} = 5\mathbf{i}$ (magnitude $5\ N$) along the line from the origin to the point $(1,1)$. (Distance measured in metres).

Work done $=$ force $\times$ distance



$= (\|F\| \cos\theta)(\|d\|)$

$= F \cdot d$

$= \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$= 5J$

5. A water main is to be constructed with at 20% grade (i.e., slope $= \dfrac{\text{height}}{\text{horizontal distance}} = 0.2$) in the north direction and a 10% grade in the east direction. Determine the angle θ required in the water main for the turn from north to east (i.e., the angle θ you need to bend the water main).

Let North be the x axis. The pipe$_1$ is traveling positively.

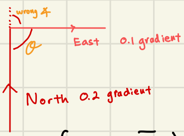
let pipe$_1$ intersect $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. at $x = 1$, $z = 0.2$.

hence, point $\begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix}$

Pipe$_1$ lies on vector $\begin{pmatrix} 1 \\ 0 \\ 0.2 \end{pmatrix}$

Let east be y axis. let pipe$_2$ intersect $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 1 \\ 0.1 \end{pmatrix}$

* pipe$_2$ so $\measuredangle$ can be found, $\begin{pmatrix} 0 \\ -1 \\ -0.1 \end{pmatrix}$



$\therefore \text{abs}(\text{Pipe}_1 \cdot \text{Pipe}_2) = \|\text{Pipe}_1\| \|\text{Pipe}_2\| \cos\theta$

$\dfrac{0.02}{(\sqrt{1+0.2^2})(\sqrt{1+0.1^2})} = \cos\theta$

$\theta = 88.88°$

6. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^3$.

(a) Using $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ and some properties of dot products, prove that

$$\|\mathbf{u} \pm \mathbf{v}\|^2 = \|\mathbf{u}\|^2 \pm 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2.$$

Hence prove that

$$\mathbf{u} \cdot \mathbf{v} = \frac{1}{4}\|\mathbf{u} + \mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u} - \mathbf{v}\|^2.$$

(b) Use part (a) to prove that two vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular if and only if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\|$. Also, interpret this geometrically in $\mathbf{R}^2$.

a). $\|w\|^2 = w \cdot w$

$\|u \pm v\|^2 = (u \pm v) \cdot (u \pm v)$

$= u \cdot u \pm u \cdot v \pm v \cdot u + v \cdot v$

$= \|u\|^2 \pm 2 u \cdot v + \|v\|^2$

$\frac{1}{4}(\cancel{\|u\|^2} + 2 u \cdot v + \cancel{\|v\|^2}) - \frac{1}{4}(\cancel{\|u\|^2} - 2 u \cdot v + \cancel{\|v\|^2})$

$= \frac{1}{4}(4\, u \cdot v)$

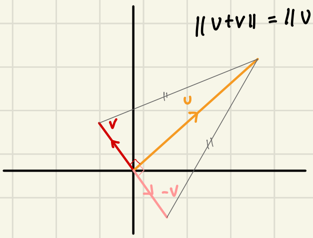
$= u \cdot v$

b) $u \cdot v = 0$ means $u \perp v$.

$\frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 = 0$

$\frac{1}{4}\|u+v\|^2 = \frac{1}{4}\|u-v\|^2$

$\|u+v\| = \|u-v\|$



7. (a) Find the vector equation of the line through $A(1, 0, 1)$ and $B(1, -1, 1)$.
   (b) Find the parametric equation of the line through $P(1, 2, -1)$ and $Q(-1, 0, 1)$.
   (c) Find the parametric equation of the line through the point $R(2, 4, 5)$ and perpendicular to the plane $3x + 7y - 5z = 21$.

8. Consider vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
   (a) Find a unit vector that is perpendicular to vectors $\mathbf{u}$ and $\mathbf{v}$.
   (b) Determine the scalar equation of the plane Π which passes through the point $(1, 1, 0)$ and is parallel to $\mathbf{u}$ and $\mathbf{v}$. What is the distance between planes Π and the plane containing the origin and parallel to $\mathbf{u}$ and $\mathbf{v}$?

9. (a) Find the vector equation and scalar equation of the plane through the point $P(1, -1, 3)$ parallel to the plane $3x + y + z = 7$.
   (b) Find the vector equation of the plane through $A(1, -2, 1)$ perpendicular to OA.

10. (a) Find the distance from $S(3, -1, 4)$ to the line $\ell : x = 4 - t, y = 3 + 2t, z = -5 + 3t$.
    (b) Find the distance from $S(2, -3, 4)$ to the plane $x + 2y + 2z = 13$.
    (c) Find the distance between the two planes $x + 2y + 6z = 1$ and $x + 2y + 6z = 10$.

11. Consider four distinct points $A(0, 0, 0)$, $B(1, 2, 0)$, $C(0, -3, 2)$ and $D(3, -4, 5)$ where AB, AC and AD are three edges of a parallelepiped.
    (a) Find the volume of the parallelepiped via scalar triple product.
    (b) If A, B and C are three vertices on the base of the parallelepiped, compute the height of the parallelepiped.
    (c) ★Let ℓ_1 be the line through A and B and ℓ_2 the line through D and parallel to AC. What is the distance between the skew lines ℓ_1 and ℓ_2?

Challenging Questions(will not be discussed)

1. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be three non zero, non coplanar vectors.
   (a) Let $\mathbf{v}_1 = \mathbf{x}_1$ and $\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1$. Show that $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$.
   (b) Find a vector $\mathbf{v}_3$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are perpendicular to each other.

2. Given any three non collinear points A, B, C. Let O be the circumcenter of ΔABC (i.e., A, B, C are point on the circle with center O and radius $= ||\overrightarrow{OA}|| = ||\overrightarrow{OB}|| = ||\overrightarrow{OC}||$). Let $\overrightarrow{OT} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Show that T is the orthocenter of ΔABC. (The orthocenter of a triangle ΔABC is a point H such that $AH \perp BC, BH \perp AC$, and $CH \perp AB$.)

7. (a) Find the vector equation of the line through $A(1,0,1)$ and $B(1,-1,1)$.

(b) Find the parametric equation of the line through $P(1,2,-1)$ and $Q(-1,0,1)$.

(c) Find the parametric equation of the line through the point $R(2,4,5)$ and perpendicular to the plane $3x+7y-5z=21$.

a) $l : A + \lambda \vec{AB},\ \lambda \in \mathbb{R}$

$\begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda \begin{pmatrix}0\\-1\\0\end{pmatrix}$

b) $PQ : \begin{pmatrix}-1-1\\0-2\\1+1\end{pmatrix} = \begin{pmatrix}-2\\-2\\2\end{pmatrix}$

$l : r = \begin{pmatrix}-1\\0\\1\end{pmatrix} + t\begin{pmatrix}-1\\-1\\1\end{pmatrix},\ t \in \mathbb{R}$

$x = -1-t,\ y = -t,\ z = 1+t$

c) Plane : $r \cdot \begin{pmatrix}3\\7\\-5\end{pmatrix} = 21$

$\therefore l : r = \begin{pmatrix}2\\4\\5\end{pmatrix} + \mu \begin{pmatrix}3\\7\\-5\end{pmatrix},\ \mu \in \mathbb{R}$

$x = 2+3\mu,\ y = 4+7\mu,\ z = 5-5\mu$

8. Consider vectors $\mathbf{u} = \mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i}+\mathbf{j}-2\mathbf{k}$.

(a) Find a unit vector that is perpendicular to vectors $\mathbf{u}$ and $\mathbf{v}$.

(b) Determine the scalar equation of the plane Π which passes through the point $(1,1,0)$ and is parallel to $\mathbf{u}$ and $\mathbf{v}$. What is the distance between planes Π and the plane containing the origin and parallel to $\mathbf{u}$ and $\mathbf{v}$?

a) $\begin{pmatrix}1\\1\\2\end{pmatrix} \times \begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}-2-2\\4+2\\1-2\end{pmatrix} = \begin{pmatrix}-4\\6\\-1\end{pmatrix}$

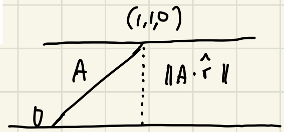
$\sqrt{4^2+6^2+1} = \sqrt{53}$

$\hat{r} = \frac{1}{\sqrt{53}}\begin{pmatrix}-4\\6\\-1\end{pmatrix}$ //

b) $r \cdot \begin{pmatrix}-4\\6\\-1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}-4\\6\\1\end{pmatrix}$

$r \cdot \begin{pmatrix}-4\\6\\-1\end{pmatrix} = 2$

$\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \frac{1}{\sqrt{53}}\begin{pmatrix}-4\\6\\-1\end{pmatrix} = \frac{1}{\sqrt{53}}(-4+6) = \frac{2}{\sqrt{53}}$ //



9. (a) Find the vector equation and scalar equation of the plane through the point $P(1,-1,3)$ parallel to the plane $3x+y+z=7$.

(b) Find the vector equation of the plane through $A(1,-2,1)$ perpendicular to OA.

a) $3x+y+z=7$

$r \cdot \begin{pmatrix}3\\1\\1\end{pmatrix} = 7$

$r \cdot \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}3\\1\\1\end{pmatrix}$

$r \cdot \begin{pmatrix}3\\1\\1\end{pmatrix} = 5$ //

$3x+y+z = 5$ //

b) $r \cdot \begin{pmatrix}1\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\1\end{pmatrix} \begin{pmatrix}1\\-2\\1\end{pmatrix}$

$r \cdot \begin{pmatrix}1\\-2\\1\end{pmatrix} = 6$

8.33

10. (a) Find the distance from $S(3,-1,4)$ to the line $\ell : x = 4-t, y = 3+2t, z = -5+3t$.

(b) Find the distance from $S(2,-3,4)$ to the plane $x+2y+2z=13$.

(c) Find the distance between the two planes $x+2y+6z=1$ and $x+2y+6z=10$.

a) $l : r = \begin{pmatrix}4\\3\\-5\end{pmatrix} + \lambda\begin{pmatrix}-1\\2\\3\end{pmatrix},\ \lambda \in \mathbb{R}$

let $\begin{pmatrix}4\\3\\-5\end{pmatrix}$ be U

$\vec{US} = \begin{pmatrix}3\\-1\\4\end{pmatrix} - \begin{pmatrix}4\\3\\-5\end{pmatrix} = \begin{pmatrix}-1\\-4\\9\end{pmatrix}$

$\sqrt{1^2+2^2+3^2} = \sqrt{14}$

distance : $|\vec{US} \times \hat{l}| = \left|\begin{pmatrix}-1\\-4\\9\end{pmatrix} \times \begin{pmatrix}\frac{-1}{\sqrt{14}}\\ \frac{2}{\sqrt{14}}\\ \frac{3}{\sqrt{14}}\end{pmatrix}\right| = \left|\begin{pmatrix}\frac{-12}{\sqrt{14}} - \frac{18}{\sqrt{14}}\\ \frac{-9}{\sqrt{14}} + \frac{3}{\sqrt{14}}\\ \frac{-2}{\sqrt{14}} - \frac{4}{\sqrt{14}}\end{pmatrix}\right|$

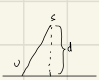
$= \left|\begin{pmatrix}\frac{-30}{\sqrt{14}}\\ \frac{-6}{\sqrt{14}}\\ \frac{-6}{\sqrt{14}}\end{pmatrix}\right|$

$= \sqrt{\frac{900}{14} + \frac{36}{14} + \frac{36}{14}} = \sqrt{\frac{486}{7}} = \frac{9\sqrt{42}}{7}$ //

b) $l : r \cdot \begin{pmatrix}1\\2\\2\end{pmatrix} = 13$

$\hat{n} = \frac{1}{\sqrt{9}}\begin{pmatrix}1\\2\\2\end{pmatrix}$

$\therefore$ point $\begin{pmatrix}13\\0\\0\end{pmatrix}$ exists. let this be U



$\vec{SU} = \begin{pmatrix}13\\0\\0\end{pmatrix} - \begin{pmatrix}2\\-3\\4\end{pmatrix} = \begin{pmatrix}11\\3\\-4\end{pmatrix}$

distance : $|\vec{SU} \cdot \hat{n}| = \left|\begin{pmatrix}11\\3\\-4\end{pmatrix} \cdot \frac{1}{\sqrt{9}}\begin{pmatrix}1\\2\\2\end{pmatrix}\right|$

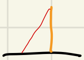
$= \frac{1}{\sqrt{9}}(11+6-8)$

$= \frac{9}{\sqrt{9}} = 3$

c) $l_1 : r \cdot \begin{pmatrix}1\\2\\6\end{pmatrix} = 1 \quad l_2 : r \cdot \begin{pmatrix}1\\2\\6\end{pmatrix} = 10$ $\qquad \sqrt{1^2+2^2+6^2} = \sqrt{41}$

Point $\begin{pmatrix}1\\0\\0\end{pmatrix}$ exists on l_1, point $\begin{pmatrix}10\\0\\0\end{pmatrix}$ exist on l_2

The vector connecting these two points, $\begin{pmatrix}9\\0\\0\end{pmatrix}$



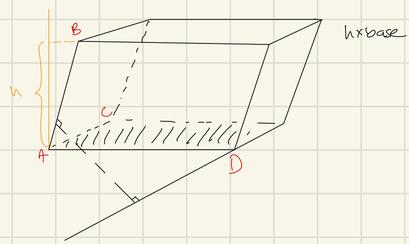
$$\left| \begin{pmatrix}9\\0\\0\end{pmatrix} \cdot \frac{1}{\sqrt{41}} \begin{pmatrix}1\\2\\6\end{pmatrix} \right| = \left| \frac{1}{\sqrt{41}} (9) \right| = \frac{9}{\sqrt{41}} \text{ //}$$

11. Consider four distinct points $A\,(0,0,0)$, $B\,(1,2,0)$, $C\,(0,-3,2)$ and $D\,(3,-4,5)$ where AB, AC and AD are three edges of a parallelepiped.

(a) Find the volume of the parallelepiped via scalar triple product.

(b) If A, B and C are three vertices on the base of the parallelepiped, compute the height of the parallelepiped.

(c) ★Let ℓ_1 be the line through A and B and ℓ_2 the line through D and parallel to AC. What is the distance between the skew lines ℓ_1 and ℓ_2?



a) $AC \times AD = \begin{pmatrix}0\\-3\\2\end{pmatrix} \times \begin{pmatrix}3\\-4\\5\end{pmatrix} = \begin{pmatrix}-15+8\\6-0\\0+9\end{pmatrix}$

$= \begin{pmatrix}-7\\6\\9\end{pmatrix}$

$h = |AB \cdot \hat{n}| = \left| AB \cdot \frac{AC \times AD}{|AC \times AD|} \right|$

Area $= h\left(|AC \times AD|\right) = \left| AB \cdot \frac{AC \times AD}{\cancel{|AC \times AD|}} \right| \left(\cancel{|AC \times AD|}\right)$

$= |AB \cdot (AC \times AD)|$

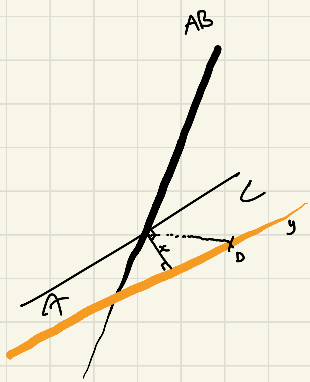
$= \left| \begin{pmatrix}1\\2\\0\end{pmatrix} \cdot \begin{pmatrix}-7\\6\\9\end{pmatrix} \right| = 5$ //

$AC = \begin{pmatrix}0\\-3\\2\end{pmatrix} \quad AD = \begin{pmatrix}3\\-4\\5\end{pmatrix}$

$AB = \begin{pmatrix}1\\2\\0\end{pmatrix}$

b) $|AC \times AD| = \sqrt{166}$

$h = \frac{5}{\sqrt{166}}$ /



c) $x = AB \times AC = \begin{pmatrix}0\\-3\\2\end{pmatrix} \times \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}-4\\2\\3\end{pmatrix}$ $\qquad |x| = \sqrt{29}$

$l_{AB} : r = \lambda \begin{pmatrix}1\\2\\0\end{pmatrix}, \qquad \lambda \in \mathbb{R}$

$l_y : r = \begin{pmatrix}3\\-4\\5\end{pmatrix} + \mu \begin{pmatrix}0\\-3\\2\end{pmatrix} = \begin{pmatrix}3\\-4-3\mu\\5+2\mu\end{pmatrix} \qquad \mu \in \mathbb{R}$

$l_x : r = t \begin{pmatrix}-4\\2\\3\end{pmatrix}, \qquad t \in \mathbb{R}$

Project$_x$ OD : $\left| \begin{pmatrix}3\\-4\\5\end{pmatrix} \cdot \frac{1}{\sqrt{29}} \begin{pmatrix}-4\\2\\3\end{pmatrix} \right| = \left| \frac{1}{\sqrt{29}} (-5) \right|$

$= \frac{5}{\sqrt{29}}$ //

Challenging Questions(will not be discussed)

1. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be three non zero, non coplanar vectors.
   (a) Let $\mathbf{v}_1 = \mathbf{x}_1$ and $\mathbf{v}_2 = \mathbf{x}_2 - \left(\frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\right)\mathbf{v}_1$. Show that $\mathbf{v}_1$ is perpendicular to $\mathbf{v}_2$.
   (b) Find a vector $\mathbf{v}_3$ such that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are perpendicular to each other.
2. Given any three non collinear points A, B, C. Let O be the circumcenter of ΔABC (i.e., A, B, C are point on the circle with center O and radius $= ||\overrightarrow{OA}|| = ||\overrightarrow{OB}|| = ||\overrightarrow{OC}||$). Let $\overrightarrow{OT} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$. Show that T is the orthocenter of ΔABC. (The orthocenter of a triangle ΔABC is a point H such that $AH \perp BC, BH \perp AC$, and $CH \perp AB$.)

1)

$$V_2 = x_2 - \left(\frac{x_2\cdot V_1}{V_1\cdot V_1}\right)V_1$$

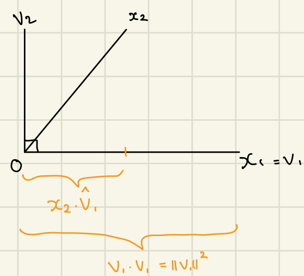
$$V_2 = x_2 - \left(\frac{x_2\cdot V_1}{\|V_1\|^2}\right)V_1$$

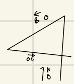
$$= x_2 - \left(\frac{x_2\cdot \hat{V}_1}{\|V_1\|}\right)V_1$$

$$V_2 = x_2 - (x_2\cdot\hat{V}_1)\hat{V}_1$$

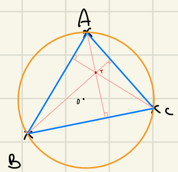
$$V_1\cdot V_2 =$$

$$V_1\cdot V_2 = 0$$





2)



Answers

1. $\sqrt{27}$; $\sqrt{6}$; 8; $4\mathbf{i}-9\mathbf{j}-\mathbf{k}$; $-4\mathbf{i}+9\mathbf{j}+\mathbf{k}$; $\frac{8}{6}\,(2\mathbf{i}+\mathbf{j}-\mathbf{k})=\frac{8}{3}\mathbf{i}+\frac{4}{3}\mathbf{j}-\frac{4}{3}\mathbf{k}$
2. $k=-2$ or $k=-3$
   (a) The area is $\sqrt{3}$.
   (b) $(-1,1,1)$
   (c) $\pi/2$
3. $5J$.
4. $\theta=\arccos(\frac{2}{\sqrt{104}\sqrt{101}})$ or $\theta\approx 1.55$ rad, or $88.88°$.
7. (a) $\mathbf{r}=(1,0,1)+t(0,-1,0), t\in\mathbb{R}$.
   (b) $x=1-2t, y=2-2t, z=-1+2t$
   (c) $x=2+3t, y=4+7t, z=5-5t, t\in\mathbb{R}$.
8. (a) $\frac{1}{\sqrt{53}}(-4,6,-1)$
   (b) $-4x+6y-z=2$, and the distance is $\dfrac{2}{\sqrt{53}}$.
9. (a) Vector equation: $\mathbf{r}\cdot(3,1,1)=5$, Scalar equation: $3x+y+z=5$
   (b) Vector equation: $\mathbf{r}\cdot(1,-2,1)=6$.
10. (a) $\frac{9\sqrt{42}}{7}$, (b) 3, (c) $\frac{9}{\sqrt{41}}$
11. (a) 5, (b), Height of the parallelepiped $=\frac{5}{\sqrt{29}}$, (c) $\frac{5}{\sqrt{29}}$.

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics 1** **Tutorial 3**

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1. Find matrices $A = [a_{ij}]$, $B = [b_{ij}]$ and $C = [c_{ij}]$ whose sizes and entries satisfy the stated conditions.

   (a) A is 3×4 and $a_{ij} = \begin{cases} 1 & \text{if} \quad i < j, \\ 7 & \text{if} \quad i = j, \\ 9 & \text{if} \quad i > j. \end{cases}$

   $$A = \begin{bmatrix} 7 & 1 & 1 & 1 \\ 9 & 7 & 1 & 1 \\ 9 & 9 & 7 & 1 \end{bmatrix}$$

   (b) B is 4×4 and $b_{ij} = \begin{cases} 1 & \text{if} \quad |i - j| > 1, \\ -1 & \text{if} \quad |i - j| \leq 1. \end{cases}$

   $$B = \begin{bmatrix} -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$$

   (c) C is 3×2 and $C_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 2^i 3^j & \text{if} \quad i \neq j. \end{cases}$

   $$C = \begin{bmatrix} 1 & 18 \\ 12 & 1 \\ 24 & 72 \end{bmatrix}$$

2. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

   Compute the following (where possible). Note M^T means the transpose of M. It is obtained from M by interchanging the rows and columns. Example: For $M = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, we have $M^T = \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$.

   (a) BA (b) BC (c) $D^T - E^T$ (d) $(D - E)^T$ (e) DE (f) $(DA)^T$

3. Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

   (a) Find the third column of AA.

   (b) Find the second row of AB.

   (c) Find the first row of $(AB)^T$.

4. ★Indicate whether the statement is always true or sometimes false. Justify your answer with a logical statement or a counterexample.

   (a) If A is a square matrix with two identical rows, then AA has two identical rows.

   (b) If A is a square matrix and AA has a column of zeros, then A must have a column of zeros.

   (c) If the matrix sum $AB + BA$ is defined, then A and B must be square.

5. Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

   (a) Verify that $A^2 - 6A + 5I = 0$.

   (b) Use part (a) to find a matrix B such that $AB = I$ and $BA = I$. Explain why A is invertible and find its inverse.

2. Consider the matrices

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

Compute the following (where possible). Note M^T means the transpose of M. It is obtained from M by interchanging the rows and columns. Example: For $M = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$, we have $M^T = \begin{bmatrix} 1 & 3 \\ 4 & 1 \\ 2 & 5 \end{bmatrix}$.

(a) BA (b) BC (c) $D^T - E^T$ (d) $(D - E)^T$ (e) DE (f) $(DA)^T$

a) $BA = 2\times 2 \quad 3\times 2$

$= $ non-defined

b) $BC = 2\times2,\ 2\times3$

$= \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}$

$= \begin{bmatrix} 1 & 15 & 3 \\ 6 & 1 & 10 \end{bmatrix}$

c) $D^T - E^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix} - \begin{bmatrix} 6 & -1 & 4 \\ 1 & 1 & 1 \\ 3 & 2 & 3 \end{bmatrix}$

$= \begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

d) Same as c

e) $DE = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix} \times \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 24 & 25 \end{bmatrix}$

3. Let

$$A = \begin{bmatrix} 3 & -2 & 7 \\ 6 & 5 & 4 \\ 0 & 4 & 9 \end{bmatrix} \text{ and } B = \begin{bmatrix} 6 & -2 & 4 \\ 0 & 1 & 3 \\ 7 & 7 & 5 \end{bmatrix}$$

(a) Find the third column of AA.

(b) Find the second row of AB.

(c) Find the first row of $(AB)^T$.

a) 3rd col ō $AA = \begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$

b) 2nd row ō $AB = [64\ 21\ 64]$

c) 1 row ō $(AB)^T = [67\ 64\ 63]$

4. ★Indicate whether the statement is always true or sometimes false. Justify your answer with a logical statement or a counterexample.

(a) If A is a square matrix with two identical rows, then AA has two identical rows.

(b) If A is a square matrix and AA has a column of zeros, then A must have a column of zeros.

(c) If the matrix sum $AB + BA$ is defined, then A and B must be square.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 9 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & 0 & 0 \end{bmatrix}$ True

b) ~~True~~ $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ False

c) True

5. Let $A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$.

(a) Verify that $A^2 - 6A + 5I = 0$.

(b) Use part (a) to find a matrix B such that $AB = I$ and $BA = I$. Explain why A is invertible and find its inverse.

a) $AA = \begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix}$

$\begin{bmatrix} 13 & 12 \\ 12 & 13 \end{bmatrix} - \begin{bmatrix} 18 & 12 \\ 12 & 18 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

$\begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$

b) $A^{-1} = B$

$B = \frac{1}{3^2 - 2^2} \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$

$= \begin{bmatrix} 0.6 & -0.4 \\ -0.4 & 0.6 \end{bmatrix}$

$AA^{-1} = I$

$AB = I$

$AA^{-1} = AB$

$A^{-1} = B$

$A^2 - 6A + 5I = 0$

$I = \frac{1}{5}(6A - A^2)$

$I = \underbrace{\frac{A}{5}(6I - A)}_{\text{Solve}}$

6. Find A where

(a) $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$ (b) $(I+2A)^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

7. Consider the matrix A where $A = \begin{bmatrix} 2 & k-2 \\ 3 & k \end{bmatrix}$.

   (a) State all values of k for A to be invertible.

   (b) Suppose A is invertible. Find the inverse A^{-1} and use it to find the solution of the simultaneous equations

$$\begin{array}{rcrcl} 2x & + & (k-2)y & = & 4 \\ 3x & + & ky & = & 5 \end{array}$$

   (c) State the value of k for which there are no solutions to the simultaneous solutions. Justify your answer.

8. Suppose the first row of an $n \times n$ matrix A is identical to the second row of A. Is there an $n \times n$ matrix B such that $AB = I$? Is A invertible?

9. The trace of a square matrix is defined to be the sum of its diagonal entries. We denote by $\mathrm{tr}(A)$ the trace of a square matrix A.

   (a) Find the trace of each of the following matrices.

$$X = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 6 & 1 & 3 & 9 \\ -1 & 1 & 2 & 3 \\ 0 & 1 & -7 & 3 \\ 4 & 1 & 3 & 0 \end{bmatrix}$$

   (b) Let $A = [a_{ij}]$. Express $\mathrm{tr}(A)$ in terms of a_{ij}.

   (c) Prove: If A and B are $n \times n$ matrices, then $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.

   (d) What can you say about $\mathrm{tr}(\alpha A)$? (You may continue to explore other properties of the trace function. eg, Is $\mathrm{tr}(AB) = \mathrm{tr}(A) \cdot \mathrm{tr}(B)$?)

10. Campus Yogurt sells three types of yogurt: nonfat, regular, and super creamy at three locations. Location N sells 50 gallons of nonfat, 100 gallons of regular, 50 gallons of super creamy each day. Location C sells 10 gallons of nonfat and Location S sells 60 gallon of nonfat each day. Daily sales of regular yogurt are 90 gallons at Location C and 120 gallons at Location S. At Location C, 50 gallons of super creamy are sold each day, and 40 gallons of super creamy are sold each day at Location S.

    The income per gallon for nonfat, regular, and super creamy is \$ 12, \$ 10, and \$ 15, respectively.

    Use matrix product to find the daily income at each of the three locations.

11. A new mass transit system has just gone into operation. The transit authority has made studies that predict the percentage of commuters who will change to mass transit or continue driving their automobile. Based on the following information:

		This	year
		Mass transit	Automobile
Next	Mass transit	0.7	0.2
year	Automobile	0.3	0.8

    For example, 30% of commuters taking mass transit this year will change to driving automobile next year.

    Suppose the population of the area remains constant, and that initially 30 percent of the commutes use mass transit and 70 percent use their automobiles.

    What percentage of commuters will be using the mass transit system after 1 year? After 2 years?

6. Find A where

(a) $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$ (b) $(I + 2A)^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

a) $(7A)^{-1} = \begin{bmatrix} -3 & 7 \\ 1 & -2 \end{bmatrix}$

det: -1

$7A = -\begin{bmatrix} -2 & -7 \\ -1 & -3 \end{bmatrix}$

$7A = \begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$

$A = \frac{1}{7}\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$

b) $(I + 2A)^{-1} = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$ det = 7

$I + 2A = \frac{1}{7}\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

$2A = \begin{bmatrix} (\frac{2}{7} - 1) & -\frac{1}{7} \\ \frac{1}{7} & (\frac{3}{7} - 1) \end{bmatrix}$

$A = \begin{bmatrix} -\frac{5}{14} & -\frac{1}{14} \\ \frac{1}{14} & -\frac{2}{7} \end{bmatrix}$

7. Consider the matrix A where $A = \begin{bmatrix} 2 & k-2 \\ 3 & k \end{bmatrix}$.

(a) State all values of k for A to be invertible.

(b) Suppose A is invertible. Find the inverse A^{-1} and use it to find the solution of the simultaneous equations

$$\begin{aligned} 2x + (k-2)y &= 4 \\ 3x + ky &= 5 \end{aligned}$$

(c) State the value of k for which there are no solutions to the simultaneous solutions. Justify your answer.

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6-k}\begin{pmatrix} 10-k \\ -2 \end{pmatrix}$ (c) $k = 6$

a) for A to be invertible,

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

$\frac{3}{2}(k-2) \neq k$

$\frac{3}{2}k - 3 \neq k$

$\frac{1}{2}k \neq 3$

$k \neq 6$

b) $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6-k}\begin{pmatrix} k & -(k-2) \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$= \frac{1}{6-k}\begin{pmatrix} 4k - 5(k-2) \\ 4(-3) + 5(2) \end{pmatrix} = \frac{1}{6-k}\begin{pmatrix} 10-k \\ -2 \end{pmatrix}$

Basically, show parallel, K=6 sub in

c) $2x + 4y = 4$

$3x + 6y = 5$

$\Downarrow$

$x + 2y = 2$

$x + 2y = \frac{5}{3}$ > grad same!

8. Suppose the first row of an $n \times n$ matrix A is identical to the second row of A. Is there an $n \times n$ matrix B such that $AB = I$? Is A invertible?

A is non invertible if row ≠ 0,0,0,0,0

9. The trace of a square matrix is defined to be the sum of its diagonal entries. We denote by tr(A) the trace of a square matrix A.

(a) Find the trace of each of the following matrices.

$$X = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, \quad Z = \begin{bmatrix} 6 & 1 & 3 & 9 \\ -1 & 1 & 2 & 3 \\ 0 & 1 & -7 & 3 \\ 4 & 1 & 3 & 0 \end{bmatrix}$$

(b) Let $A = [a_{ij}]$. Express tr(A) in terms of a_{ij}.

(c) Prove: If A and B are $n \times n$ matrices, then $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$.

(d) What can you say about tr(αA)? (You may continue to explore other properties of the trace function. eg, Is $\text{tr}(AB) = \text{tr}(A) \cdot \text{tr}(B)$?)

a) $Tr(X) = 3 + 2 = 5$

$Tr(Y) = 5$

$Tr(Z) = 0$

b) $Tr(A) = a_{11} + a_{22} + a_{33} + a_{44} + \ldots a_{i-1\,j-1} + a_{ij}$

c) $Tr(B) = b_{11} + b_{22} + \ldots + b_{ij}$

$Tr(A+B) = a_{11} + b_{11} + a_{22} + b_{22} + \ldots a_{ij} + b_{ij}$

$= [a_{11} + a_{22} + \ldots + a_{ij}] + [b_{11} + b_{22} + \ldots b_{ij}]$

10. Campus Yogurt sells three types of yogurt: nonfat, regular, and super creamy at three locations. Location N sells 50 gallons of nonfat, 100 gallons of regular, 50 gallons of super creamy each day. Location C sells 10 gallons of nonfat and Location S sells 60 gallon of nonfat each day. Daily sales of regular yogurt are 90 gallons at Location C and 120 gallons at Location S. At Location C, 50 gallons of super creamy are sold each day, and 40 gallons of super creamy are sold each day at Location S.

The income per gallon for nonfat, regular, and super creamy is \$ 12, \$ 10, and \$ 15, respectively.

Use matrix product to find the daily income at each of the three locations.

1X3 3X3 → 1X3

	N	C	S
non	50	10	60
reg	100	90	120
cream	50	50	40

non = 12

reg = 10

cream = 15

$\begin{bmatrix} 50 & 100 & 50 \\ 10 & 90 & 50 \\ 60 & 120 & 40 \end{bmatrix} \times \begin{bmatrix} 12 \\ 10 \\ 15 \end{bmatrix} = \begin{bmatrix} 2350 \\ 1770 \\ 2520 \end{bmatrix}$

11. A new mass transit system has just gone into operation. The transit authority has made studies that predict the percentage of commuters who will change to mass transit or continue driving their automobile. Based on the following information:

		This year Mass transit	Automobile
Next year	Mass transit	0.7	0.2
	Automobile	0.3	0.8

For example, 30% of commuters taking mass transit this year will change to driving automobile next year.

Suppose the population of the area remains constant, and that initially 30 percent of the commutes use mass transit and 70 percent use their automobiles.

What percentage of commuters will be using the mass transit system after 1 year? After 2 years?

$\begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix} \times \begin{bmatrix} 30 & 70 \end{bmatrix} = \begin{bmatrix} 0.21 + 0.14 \\ 0.09 + 0.56 \end{bmatrix}$

Next year, 35%

★ a certain proportion, no change

→ can continue to use.

→ Don't need Δ trans matrix

$\begin{pmatrix} M_n \\ A_n \end{pmatrix} = P^n\begin{pmatrix} 0.35 \\ 0.65 \end{pmatrix}$

$P = \begin{bmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{bmatrix}$

$\begin{pmatrix} M_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$ $\begin{pmatrix} M_1 \\ A_1 \end{pmatrix} = P\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 0.35 \\ 0.65 \end{pmatrix}$

$\therefore \begin{pmatrix} M_2 \\ A_2 \end{pmatrix} = P\begin{pmatrix} M_1 \\ A_1 \end{pmatrix} = P\begin{pmatrix} 0.35 \\ 0.65 \end{pmatrix}$

Challenging Questions

1. Suppose A and B are $n \times n$ matrices.
   (a) Prove that $(AB)^T = B^T A^T$.
   (b) Prove that if A is invertible, then A^T is invertible and its inverse is $(A^{-1})^T$.
2. A square matrix A is said to be *nilpotent* if $A^n = 0$ for some integer n. Show that if A is nilpotent, then
   (a) $(I - A)^{-1} = \sum_{k=0}^{\infty} A^k$.
   (b) $(I + A)^{-1} = \sum_{k=0}^{\infty} (-1)^k A^k$. (Here we take $A^0 = I$).
3. If the product of n square matrices $A_1, A_2, ..., A_n$ is invertible, show, from the definition of inverse, that each matrix A_i is invertible. (Hint: show that the statement is true for $n = 2$, then use induction.)

Answers

1. $A = \begin{bmatrix} 7 & 1 & 1 & 1 \\ 9 & 7 & 1 & 1 \\ 9 & 9 & 7 & 1 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 18 \\ 12 & 1 \\ 24 & 72 \end{bmatrix}$

2. (a) Not defined;
(b) $\begin{bmatrix} 1 & 15 & 3 \\ 6 & 2 & 10 \end{bmatrix}$; (c) & (d) $\begin{bmatrix} -5 & 0 & -1 \\ 4 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$; (e) $\begin{bmatrix} 9 & 8 & 19 \\ -2 & 0 & 0 \\ 32 & 9 & 25 \end{bmatrix}$; (f) $\begin{bmatrix} 0 & -2 & 11 \\ 12 & 1 & 8 \end{bmatrix}$

3. (a) $\begin{bmatrix} 76 \\ 98 \\ 97 \end{bmatrix}$; (b) $(64 \quad 21 \quad 59)$; (c) $[67 \quad 64 \quad 63]$.

5. Answers: (b) $A^{-1} = \left(\frac{6}{5}I - \frac{1}{5}A\right)$.

6. (a) $A = \frac{1}{7}\begin{bmatrix} 2 & 7 \\ 1 & 3 \end{bmatrix}$; (b) $A = \begin{bmatrix} -\frac{5}{14} & -\frac{1}{14} \\ \frac{1}{14} & -\frac{4}{14} \end{bmatrix}$.

7. (a) $k \neq 6$, (b) $A^{-1} = \frac{1}{6-k}\begin{pmatrix} k & -(k-2) \\ -3 & 2 \end{pmatrix}$, $\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{6-k}\begin{pmatrix} 10-k \\ -2 \end{pmatrix}$ (c) $k = 6$

9. (a) $\operatorname{tr}(X) = 3 + 2 = 5, \operatorname{tr}(Y) = 5, \operatorname{tr}(Z) = 0$.

10. $2350, 1770, 2520$.

11. 35% and 37.5%

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 | **MH1810 Mathematics I** | **Tutorial 4**

Reference for Limits: [S] Chapter 2, Section 2.1 - 2.3, 2.5 - 2.6. OR [T] Chapter 2, Section 2.1 - 2.2, 2.4 - 2.6.

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

1. For each of the following matrices, find its matrix of cofactors $C = (C_{ij})$.[C_{ij} is the (i, j)-cofactor of A].

   (a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ (b) $A = \begin{pmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$

   a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ $\boxed{C = \begin{bmatrix} 2 & -3 \\ -2 & 1 \end{bmatrix}}$

   $=2(1)$, $=-1(3)$, $=-1(2)$, $=1(1)$

   b) $B = \begin{bmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ $\boxed{D = \begin{bmatrix} 0 & 4 & -8 \\ -5 & 1 & 3 \\ 5 & -1 & 17 \end{bmatrix}}$

   $=1(2-2)$, $=-1(-3-1)$, $=1(-6-2)$, $=-1(5)$, $=1(1)$, $=-1(2-5)$, $=1(5)$, $=-1(1)$, $=1(2-(-15))$

2. Evaluate the following determinant without using cofactor expansion.

   (a) $\begin{vmatrix} 3 & -17 & -3 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{vmatrix}$ (b) $\begin{vmatrix} \sqrt{2} & 0 & 0 & 0 \\ -8 & \sqrt{2} & 0 & 0 \\ 7 & 0 & -1 & 0 \\ 9 & 5 & 1 & 6 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & -4 & 8 & 5 \\ 0 & 0 & 0 & 0 \\ 9 & 0 & -7 & 0 \\ -11 & 3 & 0 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & 7 & 9 \\ \sqrt{2} & \pi & e \\ 1 & 7 & 9 \end{vmatrix}$

   a) Triangle Matrix. $3 \times 5 \times -2 = -30$

   b) Triangle matrix $\sqrt{2} \times \sqrt{2} \times -1 \times 6 = -12$

   c) Determinant $= 0$, Row 0 zero

   d) Determinant $= 0$, Doubles

3. Let $A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$.

   (a) Find

   (i) C_{21} (ii) C_{23} (iii) C_{44} (iv) C_{13}

   (b) Evaluate the determinant of A by cofactor expansion along

   (i) the first column, (ii) the third row.

   $C_{21} = (-1)(1)\begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 6$

   $\begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix}$ $= (1)(2)(3) + 3(2)(2) - (3)(3) - (3)(2)(1) = -6$

   $C_{23} = -1(1)\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = -12$

   $C_{44} = (1)(3)\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{vmatrix} = 6$

   $C_{13} = 0$

   i) $1(2)\begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} + 6 = -4 + 6 = 2$

4. Solve for all real numbers x which satisfies the following equation.

$$\begin{vmatrix} x & -1 \\ 3 & 1-x \end{vmatrix} = \begin{vmatrix} 1 & 0 & -3 \\ 2 & x & -6 \\ 1 & 3 & x-5 \end{vmatrix}$$

   $x(1-x) + 3 = 1\begin{vmatrix} x & -6 \\ 3 & x-5 \end{vmatrix} + (-3)\begin{vmatrix} 2 & x \\ 1 & 3 \end{vmatrix}$

   $-x^2 - x + 3 = (x)(x-5) + 18 - 3(6-x)$

   $-x^2 - x + 3 = x^2 - 5x + 18 - 18 + 3x$

   $-2x^2 + x + 3 = 0$

   $x = \frac{3}{2}, \quad x = -1$

5. The matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is the matrix of rotation of points in $\mathbb{R}^3$, it rotates points about the z-axis by θ radians in counter-clockwise direction.

   Show that the matrix R is invertible for all values of θ and find the inverse R^{-1} of R.

6. Solve the linear system by Cramer's rule, if it applies.

$$\begin{aligned} 4x + 5y \quad &= 2 \\ 11x + y + 2z &= 3 \\ x + 5y + 2z &= 1 \end{aligned}$$

   det $= -132$

   $\begin{bmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$

   $x = \dfrac{\begin{vmatrix} 2 & 5 & 0 \\ 3 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}} = \dfrac{3}{11}$ $\quad y = \dfrac{\begin{vmatrix} 4 & 2 & 0 \\ 11 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}} = \dfrac{2}{11}$ $\quad z = \dfrac{\begin{vmatrix} 4 & 5 & 2 \\ 11 & 1 & 3 \\ 1 & 5 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 5 & 0 \\ 11 & 1 & 2 \\ 1 & 5 & 2 \end{vmatrix}} = -\dfrac{1}{11}$

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 8 & 6 & 7 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ \frac{1}{z} \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

$$\frac{1}{x} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 5 & 4 & 1 \\ 0 & 6 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 8 & 6 & 7 \end{vmatrix}} = 1$$

$$-18$$

$$\frac{1}{y} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 3 & 5 & 1 \\ 8 & 0 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 8 & 6 & 7 \end{vmatrix}} = 1$$

$$\frac{1}{z} = \frac{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 5 \\ 8 & 6 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 2 & 1 \\ 3 & 4 & 1 \\ 8 & 6 & 7 \end{vmatrix}} = -\frac{36}{18}$$

$$z = -\frac{1}{2}$$

7. Solve for x, y and z.

$$\begin{aligned} \frac{1}{x} + \frac{2}{y} + \frac{1}{z} &= 1 \\ \frac{3}{x} + \frac{4}{y} + \frac{1}{z} &= 5 \\ \frac{8}{x} + \frac{6}{y} + \frac{7}{z} &= 0 \end{aligned}$$

8. (AY 2012/13 Semester 1) Consider the following system of linear equations

$$\begin{aligned} 2a + 3b - c &= 1 \\ -a + 4b + 2c &= 0 \\ a + rb - c &= -1 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 1 & r & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

has to be valid. $\begin{vmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 1 & r & -1 \end{vmatrix} \neq 0$

$(1)\begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} - r\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - 1\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} \neq 0$

$(1)(10) - r(3) - 1(11) \neq 0$

$3r \neq -1$

$r \neq -\frac{1}{3}$

(i) Find the values of r at which Cramer's rule is applicable.

(ii) For $r = 1$, use Cramer's Rule to determine the unknown b.

$$\begin{bmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad b = \frac{\begin{vmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & -1 & -1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & -1 \\ -1 & 4 & 2 \\ 1 & 1 & -1 \end{vmatrix}} = \frac{4}{-4} = -1$$

9. Consider the function $f : [-3, 5] \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} 2 - x & \text{if } -3 \le x < 1 \\ 0 & \text{if } x = 1 \\ \sqrt{x} & \text{if } 1 < x < 3 \\ (x-1)^2 & \text{if } 3 \le x \le 5. \end{cases}$$

(a) Sketch the graph $y = f(x)$ for $-3 \le x \le 5$. From your sketch, write down the range of f, i.e., the set of values where $f(x)$ assumes for $-3 \le x \le 5$.

(b) From your graph, determine each of the following limits if it exists:
(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to 2} f(x)$ (iii) $\lim_{x\to 4} f(x)$ (iv) $\lim_{x\to 1^-} f(x)$ (v) $\lim_{x\to 1^+} f(x)$
(vi) $\lim_{x\to 1} f(x)$ (vii) $\lim_{x\to 3} f(x)$

10. Does the following limit exist? If it does, what is its value? If it is an infinite limit, determine whether it is $+\infty$ and $-\infty$.

(a) $\displaystyle\lim_{x\to 5^+} \frac{6}{x-5}$ (b) $\displaystyle\lim_{x\to \pi^-} \csc x$

$[\csc\theta = \frac{1}{\cos\theta}]$

11. (a) Sketch graphs of exponential functions $y = a^x$, where $0 < a < 1$ and $a > 1$.

(b) Use the graphs in part (a) to write down each of the following limits.
(i) $\lim_{x\to\infty} (1.001)^x$ (ii) $\lim_{x\to-\infty} \pi^x$ (iii) $\lim_{x\to\infty} 0.37^x$ (iv) $\lim_{x\to-\infty} 181^x$

12. Sketch the graph of $y = \ln(2-x)$ and use it to determine each of the following limits.

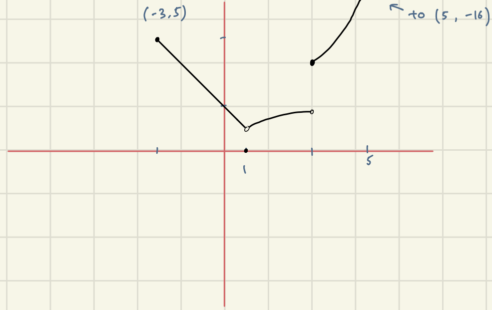
(a) $\lim_{x\to 2^-} \ln(2-x)$ (b) $\lim_{x\to 1^-} \ln(2-x)$ (c) $\lim_{x\to 3^+} \ln(2-x)$ (d) $\lim_{x\to -3} \ln(2-x)$ (e) $\lim_{x\to-\infty} \ln(2-x)$.

9. Consider the function $f : [-3, 5] \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} 2 - x & \text{if } -3 \le x < 1 \\ 0 & \text{if } x = 1 \\ \sqrt{x} & \text{if } 1 < x < 3 \\ (x-1)^2 & \text{if } 3 \le x \le 5. \end{cases}$$

(a) Sketch the graph $y = f(x)$ for $-3 \le x \le 5$. From your sketch, write down the range of f, i.e., the set of values where $f(x)$ assumes for $-3 \le x \le 5$.

(b) From your graph, determine each of the following limits if it exists:
(i) $\lim_{x\to 0} f(x)$ (ii) $\lim_{x\to 2} f(x)$ (iii) $\lim_{x\to 4} f(x)$ (iv) $\lim_{x\to 1^-} f(x)$ (v) $\lim_{x\to 1^+} f(x)$
(vi) $\lim_{x\to 1} f(x)$ (vii) $\lim_{x\to 3} f(x)$



a) range of $f(x)$ is $\{0\} \cup (0, 16]$

number

not including

including 16

and

b) i) $\lim_{x\to 0} f(x) = 2$

ii) $\lim_{x\to 2} f(x) = \sqrt{2}$

iii) $\lim_{x\to 4} f(x) = 9$

iv) $\lim_{x\to 1^-} f(x) = 1$

v) $\lim_{x\to 1^+} f(x) = 1$

vi) $\lim_{x\to 1} f(x) = 1$

vii) $\lim_{x\to 3} f(x) = \text{DNE}$

the range of f is $\{0\} \cup (1, 16]$
(i) $\lim_{x\to 0} f(x) = 2$
(ii) $\lim_{x\to 2} f(x) = \sqrt{2}$
(iii) $\lim_{x\to 4} f(x) = 9$
(iv) $\lim_{x\to 1^-} f(x) = 1$
(v) $\lim_{x\to 1^+} f(x) = 1$
(vi) $\lim_{x\to 1} f(x) = 1$.
(vii) $\lim_{x\to 3} f(x)$ does not exist .

Does the following limit exist? If it does, what is its value? If it is an infinite limit, determine whether it is $+\infty$ and $-\infty$.

(a) $\displaystyle\lim_{x\to 5^+} \frac{6}{x-5}$ (b) $\displaystyle\lim_{x\to \pi^-} \csc x$

$[\csc\theta = \frac{1}{\cos\theta}]$ $\to$ $\csc\theta = \frac{1}{\sin\theta}$?

a) $\displaystyle\lim_{x\to 5^+} \frac{6}{x-5} = \lim_{x\to 5^+} 6\left(\frac{1}{x-5}\right)$

let $f(x) = x - 5$
$f(5) = 0$
$x > 5$
$x - 5 > 0$
$f(x) > 0$ when $x > 5$

$\therefore \displaystyle\lim_{x\to 5^+} \frac{1}{f(x)} = +\infty$

b) $\displaystyle\lim_{x\to \pi^-} \csc x = \lim_{x\to \pi^-} \frac{1}{\sin x}$

let $g(x) = \sin x$
$g(\pi) = 0$
$x < \pi$
$x - \pi < 0$

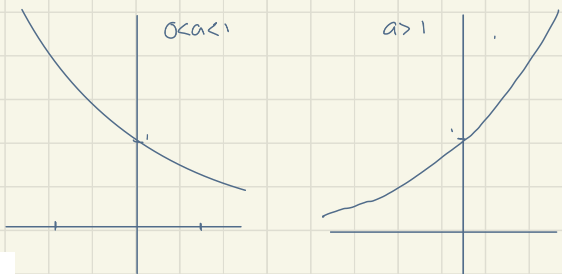
$\sin x > 0$
$g(x) > 0$

$\therefore \displaystyle\lim_{x\to \pi^-} \frac{1}{\sin x} = \infty$

$\displaystyle\lim_{x\to \pi^-} \frac{1}{\sin x} \to f(x) = 0$

$\frac{1}{0} = -\infty$

(a) Sketch graphs of exponential functions $y = a^x$, where $0 < a < 1$ and $a > 1$.
(b) Use the graphs in part (a) to write down each of the following limits.
(i) $\lim_{x\to\infty} (1.001)^x$ (ii) $\lim_{x\to-\infty} \pi^x$ (iii) $\lim_{x\to\infty} 0.37^x$ (iv) $\lim_{x\to-\infty} 181^x$



b) i) $\lim_{x\to\infty} (1.001)^x = \infty$

ii) $\lim_{x\to-\infty} \pi^x = 0$

iii) $\lim_{x\to\infty} 0.37^x = 0$

iv) $\lim_{x\to-\infty} 181^x = 0$

Sketch the graph of $y = \ln(2-x)$ and use it to determine each of the following limits.
(a) $\lim_{x\to 2^-} \ln(2-x)$ (b) $\lim_{x\to 1^-} \ln(2-x)$ (c) $\lim_{x\to 3^+} \ln(2-x)$ (d) $\lim_{x\to -3} \ln(2-x)$ (e) $\lim_{x\to-\infty} \ln(2-x)$.

a) $\lim_{x\to 2^-} \ln(2-x) = 0$?

b) $\lim_{x\to 1^-} \ln(2-x) = 0$

c) $\lim_{x\to 3^+} \ln(2-x) = \text{DNE}$

d) $\lim_{x\to -3} \ln(2-x) = \ln(5)$

e) $\lim_{x\to-\infty} \ln(2-x) = \infty$

Challenging Problems (will not be discussed in tutorial).

1. The *adjoint* of a matrix A is the transpose of the cofactor matrix C of A . It is denoted by adj A, i.e.,

$$\text{adj}\, A = C^T,$$

where $C = (C_{ij})$ is the cofactor matrix. Use the cofactors matrices found in Question 1 of the tutorial to find adj A for

$$A = \begin{pmatrix} 1 & 5 & 0 \\ -3 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Verify that the matrix product $A\,\text{adj}\, A$ is a diagonal matrix and hence find the inverse of A. Based on the above observation, propose a way to find the inverse of a nonsingluar matrix.

2. Determine whether there is a nonsingular matrix A such that

$$A^3 = ABA + A^2,$$

where

$$B = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

3. A matrix A is *skew symmetric* if $A^T = -A$. Prove that an $n \times n$ skew symmetric matrix, where n is odd, is singular.

4. A matrix A is *othogonal* if $A^T = A^{-1}$.

   (a) Prove that if A is orthogonal, then $\det A = \pm 1$.

   (b) Give a characterization of 2x2 othogonal matrices.

Answers

1. (a) $\begin{pmatrix} 2 & -3 \\ -2 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 0 & 4 & -8 \\ -5 & 1 & 3 \\ 5 & -1 & 17 \end{pmatrix}$

....

2. (a) -30 (upper triangular matrix)

   (b) -12 (lower triangular matrix)

   (c) 0 (zero row)

   (d) 0 (Identical rows)

....

(a) (i) $C_{21} = -\begin{vmatrix} 1 & 3 & 3 \\ 2 & 2 & 0 \\ 1 & 2 & 3 \end{vmatrix} = 6$ (ii) $C_{23} = -\begin{vmatrix} 2 & 1 & 3 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = -12$

(iii) $C_{44} = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 2 \end{vmatrix} = 0$ (iv) $C_{13} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = 6$

....

3. $x = \dfrac{3 \pm \sqrt{33}}{4}$.

....

6. $x = \frac{3}{11}, y = \frac{2}{11}, z = -\frac{1}{11}$.

....

7. $x = y = 1, z = -1/2$.

....

8. (i) $r \neq -\frac{1}{3}$
(ii) $b = -1$.

....

9. (a) the range of f is $\{0\} \cup (1, 16]$
(b) (i) $\lim_{x\to 0} f(x) = 2$
(ii) $\lim_{x\to 2} f(x) = \sqrt{2}$
(iii) $\lim_{x\to 4} f(x) = 9$
(iv) $\lim_{x\to 1^-} f(x) = 1$
(v) $\lim_{x\to 1^+} f(x) = 1$
(vi) $\lim_{x\to 1} f(x) = 1$.
(vii) $\lim_{x\to 3} f(x)$ does not exist .

....

10. (a) $+\infty$ (b) $+\infty$.

....

11. (b) (i) $\lim_{x\to\infty} (1.001)^x = +\infty$
(ii) $\lim_{x\to -\infty} \pi^x = 0$
(iii) $\lim_{x\to\infty} 0.37^x = 0$
(iv) $\lim_{x\to -\infty} 181^x = 0$

....

12. (a) $\lim_{x\to 2^-} \ln(2-x) = -\infty$
(b) $\lim_{x\to 1^-} \ln(2-x) = 0$
(c) $\lim_{x\to 3^+} \ln(2-x)$ is not defined
(d) $\lim_{x\to -3} \ln(2-x) = \ln 5$
(e) $\lim_{x\to -\infty} \ln(2-x) = \infty$

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics I** **Tutorial 5**

Reference

[S] Chapter 2, Section 2.1 - 2.3, 2.5 - 2.6.

[T] Chapter 2, Section 2.1 - 2.2, 2.4 - 2.6.

1. Suppose that $\lim_{x\to 1} p(x) = 4$, $\lim_{x\to 1} q(x) = \pi$ and $\lim_{x\to 1} r(x) = 3$. Determine each of the following limits and justify each step by indicating the appropriate Limit Law(s).

   (a) $\lim_{x\to 1} [\pi p(x) + q(x) - (qr)(x)]$ (b) $\lim_{x\to 1} \dfrac{p(x)+q(x)}{r(x)}$

2. Find the limit.

   (a) $\lim_{x\to\pi/2} \cos x$ (b) $\lim_{x\to\infty} 179$ (c) $\lim_{x\to 3^-} (x^2+\pi x+\sqrt{2})$ (d) $\lim_{y\to 3} 4^y$ (e) $\lim_{t\to 125} \sqrt[3]{t}$

3. Use continuity to determine the following limits.

   (a) $\lim_{x\to 1} \sqrt{\dfrac{x}{1+3x}}$

   (b) $\lim_{x\to 1} \sin(x-1)^2$

   (c) $\lim_{x\to 1} \tan\left(\dfrac{(2-x^2)\pi}{3}\right)$

   (d) $\lim_{x\to 3} \ln|x-2|$

   (e) $\lim_{x\to\sqrt{2}} \tan^{-1}\left(\dfrac{x^2}{2}\right)$

4. Use appropriate techniques to find the following limits.

   (a) $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1}$

   (b) $\lim_{x\to\sqrt{2}} \dfrac{x^2-2}{x-\sqrt{2}}$

   (c) $\lim_{x\to 0^-} \dfrac{x}{\sqrt{x+1}-1}$

   (d) $\lim_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3}$

   (e) $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

   (f) $\lim_{t\to\frac{\pi}{4}} \dfrac{\cos 2t}{\cos t-\sin t}$

   (g) $\lim_{h\to 0} \dfrac{(2+h)^3-8}{h}$.

   (h) $\lim_{x\to 7^+} \dfrac{\sqrt{x+2}-3}{x-7}$.

(i) $\lim_{t \to 0^+} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$. $\left(\frac{1}{t} - \frac{1}{t(t+1)} \right) = \frac{(t+1) - 1}{t(t+1)} = \frac{\cancel{t}}{\cancel{t}(t+1)}$

(j) $\lim_{x \to 0} \left(x^4 \cos \frac{1}{x} \right)$

5. Determine whether $\lim_{x \to 2} f(x)$ exists where

$$f(x) = \begin{cases} \dfrac{3x - 6}{x^2 - 4} & \text{if } 0 < x < 2, \\ 0 & \text{if } x = 2, \\ \dfrac{x - 2}{\sqrt{3 - x} - 1} & \text{if } 2 < x < 3. \end{cases}$$

6. ★If the product $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, is it always true that $f(x)$ and $g(x)$ must be continuous at $x = 0$? Give reasons to your answer.

7. Find real constants c and d that makes g continuous at $x = 4$.

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4, \\ d & \text{if } x = 4, \\ cx + 20 & \text{if } x > 4. \end{cases}$$

8. Suppose that $3x \leq f(x) \leq x^3 + 2$ for $0 \leq x \leq 2$.

   (a) What is $f(1)$?

   (b) Use Squeeze Theorem to evaluate $\lim_{x \to 1} f(x)$.

   (c) Is f continuous at $x = 1$?

9. Under certain circumstances a rumor spreads according the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

   where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants.

   (a) Find $\lim_{t \to \infty} p(t)$.

   (b) For $a = 10$, $k = 0.5$ and t being measured in hours, how long will it take for 80% of the population to hear the rumor?

10. Determine the following infinite limits.

   (a) $\lim_{x \to 1^-} \frac{1}{1 - x^2} = \frac{1}{(x-1)(x+1)}$

   (b) $\lim_{x \to 1^+} \frac{x}{1 - \sqrt{x}}$

Challenging Problems

1. If $\lim_{x \to a} [f(x) + g(x)] = 1$ and $\lim_{x \to a} [f(x) - g(x)] = 2$, find $\lim_{x \to a} f(x) g(x)$.

2. Let $\lfloor x \rfloor$ denote the greatest integer $\leq x$, e.g., $\lfloor 3.4 \rfloor = 3$, $\lfloor 3 \rfloor = 3$, $\lfloor -1.1 \rfloor = -2$.

   (a) Sketch the region in the plane defined by each of the following equations

1. Suppose that $\lim_{x\to 1} p(x) = 4$, $\lim_{x\to 1} q(x) = \pi$ and $\lim_{x\to 1} r(x) = 3$. Determine each of the following limits and justify each step by indicating the appropriate Limit Law(s).

(a) $\lim_{x\to 1} [\pi p(x) + q(x) - (qr)(x)]$ (b) $\lim_{x\to 1} \dfrac{p(x)+q(x)}{r(x)}$

a) $\lim_{x\to 1} (\pi p(x) + q(x) - (qr)(x)) = \pi \lim_{x\to 1} p(x) + \lim_{x\to 1} q(x) - \lim_{x\to 1} qr(x) = 2\pi$

b) $\lim_{x\to 1} \dfrac{p(x)+q(x)}{r(x)} = \dfrac{\lim_{x\to 1} p(x) + \lim_{x\to 1} q(x)}{\lim_{x\to 1} r(x)}$ as $\lim_{x\to 1} r(x) \neq 0$

$= \dfrac{4+\pi}{3} = \dfrac{4}{3} + \dfrac{\pi}{3}$

2. Find the limit.

(a) $\lim_{x\to\pi/2} \cos x$ (b) $\lim_{x\to\infty} 179$ (c) $\lim_{x\to 3^-} (x^2+\pi x+\sqrt{2})$ (d) $\lim_{y\to 3} 4^y$ (e) $\lim_{t\to 125} \sqrt[3]{t}$

a) $\lim_{x\to\frac{\pi}{2}} \cos x = 0$ b) $\lim_{x\to\infty} 179 = 179$ c) $\lim_{x\to 3^-} (x^2+\pi x+\sqrt{2}) = 9+3\pi+\sqrt{2}$ d) $\lim_{y\to 3} 4^y = 64$ e) $\lim_{t\to 125} \sqrt[3]{t} = 5$

Use continuity to determine the following limits.

(a) $\lim_{x\to 1} \sqrt{\dfrac{x}{1+3x}}$

(b) $\lim_{x\to 1} \sin(x-1)^2$

(c) $\lim_{x\to 1} \tan\left(\dfrac{(2-x^2)\pi}{3}\right)$

(d) $\lim_{x\to 3} \ln|x-2|$

(e) $\lim_{x\to\sqrt{2}} \tan^{-1}\left(\dfrac{x^2}{2}\right)$

a) $\lim_{x\to 1} \sqrt{\dfrac{x}{1+3x}} =$

4. Use appropriate techniques to find the following limits.

(a) $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1}$

(b) $\lim_{x\to\sqrt{2}} \dfrac{x^2-2}{x-\sqrt{2}}$

(c) $\lim_{x\to 0^-} \dfrac{x}{\sqrt{x+1}-1}$

(d) $\lim_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3}$

(e) $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

(f) $\lim_{t\to\frac{\pi}{4}} \dfrac{\cos 2t}{\cos t - \sin t}$

(g) $\lim_{h\to 0} \dfrac{(2+h)^3-8}{h}$.

(h) $\lim_{x\to 7^+} \dfrac{\sqrt{x+2}-3}{x-7}$.

a) $\lim_{x\to 1} \dfrac{x^3-1}{x^2-1} = \lim_{x\to 1} \dfrac{\cancel{(x-1)}(x^2+x+1)}{\cancel{(x-1)}(x+1)} = \dfrac{3}{2}$

$$\begin{array}{r} x^2+x+1 \\ x-1 \enclose{longdiv}{x^3-1} \\ -(x^3-x^2) \\ x^2-1 \\ -(x^2-x) \\ x-1 \end{array}$$

b) $\lim_{x\to\sqrt{2}} \dfrac{x^2-2}{x-\sqrt{2}} \cdot \dfrac{x+\sqrt{2}}{x+\sqrt{2}} = \lim_{x\to\sqrt{2}} \dfrac{\cancel{(x^2-2)}(x+\sqrt{2})}{\cancel{(x^2-2)}(1)} = 2\sqrt{2}$

c) $\lim_{x\to 0^-} \dfrac{x}{\sqrt{x+1}-1} \cdot \dfrac{\sqrt{x+1}+1}{\sqrt{x+1}+1} = \lim_{x\to 0^-} \dfrac{x(\sqrt{x+1}+1)}{x+1-1} = \lim_{x\to 0^-} \dfrac{\sqrt{x+1}+1}{1} = 2$

d) $\lim_{t\to -3} \dfrac{t^2-9}{2t^2+7t+3} = \dfrac{(x-3)\cancel{(x+3)}}{2(x+\frac{1}{2})\cancel{(x+3)}} = \dfrac{6}{5}$

e) $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1} \cdot \dfrac{x}{x} = \lim_{x\to 1} \dfrac{1-x}{x(x-1)} = \lim_{x\to 1} \dfrac{-1\cancel{(x-1)}}{x\cancel{(x-1)}} = -1$

f) $\lim_{t\to\frac{\pi}{4}} \dfrac{\cos 2t}{\cos t - \sin t} = \lim_{t\to\frac{\pi}{4}} \dfrac{\cos^2 t - \sin^2 t}{\cos t - \sin t} = \lim_{t\to\frac{\pi}{4}} \dfrac{\cancel{(\cos t - \sin t)}(\cos t + \sin t)}{\cancel{(\cos t - \sin t)}(1)} = \sqrt{2}$

g) $\lim_{h\to 0} \dfrac{(2+h)^3-8}{h} = \lim_{h\to 0} \dfrac{h^3+6h^2+12h}{h} = \dfrac{0+0+12}{1} = 12$

h) $\lim_{x\to 7^+} \dfrac{\sqrt{x+2}-3}{x-7} = \lim_{x\to 7^+} \dfrac{x+2-9}{(x-7)(\sqrt{x+2}+3)} = \lim_{x\to 7^+} \dfrac{1}{\sqrt{x+2}+3} = \dfrac{1}{6}$

(i) $\lim_{t\to 0^+} \left(\dfrac{1}{t} - \dfrac{1}{t^2+t}\right)$.

(j) $\lim_{x\to 0} \left(x^4 \cos\dfrac{1}{x}\right)$

i) $\lim_{t\to 0^+} \left(\dfrac{1}{t} - \dfrac{1}{t^2+t}\right) = \lim_{t\to 0^+} \left(\dfrac{t^2+t}{t^3+t^2} - \dfrac{t}{t^3+t^2}\right) = \lim_{t\to 0^+} \left(\dfrac{t^2}{t^3+t^2}\right) = \lim_{t\to 0^+} \left(\dfrac{\cancel{t^2}}{(t+1)\cancel{(t)(t)}}\right) = 1$

j) $\lim_{x\to 0} (x^4 \cos\frac{1}{x})$

$-1 \le \cos\theta \le 1$

$-1 \le \cos\frac{1}{x} \le 1$ multiply by x^4

$-x^4 \le x^4 \cos\frac{1}{x} \le x^4$

when $x\to 0$, $-x^4$ & $x^4 = 0$

$\therefore$ $x^4 \cos(\frac{1}{x}) = 0$

Determine whether $\lim_{x\to 2} f(x)$ exists where

5)

$$f(x) = \begin{cases} \dfrac{3x-6}{x^2-4} & \text{if } 0 < x < 2, \\ 0 & \text{if } x = 2, \\ \dfrac{x-2}{\sqrt{3-x}-1} & \text{if } 2 < x < 3. \end{cases}$$

For limit to exist, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

$$\lim_{x\to 2^-} \frac{3x-6}{x^2-4} = \lim_{x\to 2^-} \frac{3\cancel{(x-2)}}{\cancel{(x-2)}(x+2)} = \frac{3}{4} \qquad \lim_{x\to 2^+} \frac{x-2}{\sqrt{3-x}-1} \cdot \frac{\sqrt{3-x}+1}{\sqrt{3-x}+1} = \lim_{x\to 2^+} \frac{\cancel{x-2}(\sqrt{3-x}+1)}{-1\cancel{(x-2)}} = -2$$

$\therefore$ DNE

. ★If the product $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, is it always true that $f(x)$ and $g(x)$ must be continuous at $x = 0$? Give reasons to your answer.

if $\lim_{x\to 0} h(x)$ is Valid, $\lim_{x\to 0} (f(x) \cdot g(x)) = \lim_{x\to 0} f(x) \cdot \lim_{x\to 0} g(x)$

Find real constants c and d that makes g continuous at $x = 4$.

7)

$$g(x) = \begin{cases} x^2 - c^2 & \text{if } x < 4, \\ d & \text{if } x = 4, \\ cx + 20 & \text{if } x > 4. \end{cases}$$

$\lim_{x\to 4^-} (x^2 - c^2) = d = \lim_{x\to 4^+} (cx + 20)$

Assuming continuity,...

$4^2 - c^2 = 4c + 20 = d$

$-c^2 - 4c - 4 = 0$

$-1(c+2)(c+2) = 0$

$\therefore$ $c = -2$, $d = 12$

Suppose that $3x \le f(x) \le x^3 + 2$ for $0 \le x \le 2$.

(a) What is $f(1)$?

(b) Use Squeeze Theorem to evaluate $\lim_{x\to 1} f(x)$.

(c) Is f continuous at $x = 1$?

a) $3 \le f(x) \le 1+2$

$f(1) = 3$

b) $\lim_{x\to 1} 3x = 3$ $\quad \lim_{x\to 1} x^3 + 2 = 3$

c) lim exist f(x) exist and they equal

Under certain circumstances a rumor spreads according the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants.

(a) Find $\lim_{t\to\infty} p(t)$.

(b) For $a = 10$, $k = 0.5$ and t being measured in hours, how long will it take for 80% of the population to hear the rumor?

$$\lim_{t\to\infty} \frac{1}{1 + ae^{-kt}} = \lim_{t\to\infty} \frac{1}{1+0} = 1$$

b) $0.8 = \dfrac{1}{1 + 10(e^{-0.5t})}$

$e^{-0.5t} = \dfrac{1 - 0.8}{8}$

$-0.5t = \ln\left(\frac{1}{40}\right)$

10. Determine the following infinite limits.

(a) $\lim_{x\to 1^-} \dfrac{1}{1-x^2}$

(b) $\lim_{x\to 1^+} \dfrac{x}{1-\sqrt{x}}$

a) when $f(x) = 1 - x^2$,

$f(1) = 0$.

$x < 1$

$x^2 < 1$

$-x^2 > -1$

$1 - x^2 > 0$, $f(x) > 0$

$\therefore \lim_{x\to 1^-} \dfrac{1}{1-x^2} = \infty$

b) when $f(x) = 1 - \sqrt{x}$

$f(1) = 1$

$x > 1$

$\sqrt{x} > 1$

$-\sqrt{x} < -1$

$1 - \sqrt{x} < 0$, $f(x) < 0$

$\lim_{x\to 1^+} \dfrac{x}{1-\sqrt{x}} = -\infty$

i. $\lfloor x \rfloor + \lfloor y \rfloor = 1$
ii. $\lfloor x \rfloor - \lfloor y \rfloor = 1$

(b) Find $\lim_{x\to\infty} \frac{\lfloor x \rfloor}{x}$.

Answers

1. (a) 2π
   (b) $\frac{4+\pi}{3}$

....

2. (a) 0
   (b) 179
   (c) $9 + 3\pi + \sqrt{2}$
   (d) 64
   (e) 5

....

3. (a) $\frac{1}{2}$
   (b) 0
   (c) $\sqrt{3}$
   (d) 0
   (e) $\frac{\pi}{4}$

....

4.

(a)	$\frac{3}{2}$	(e)	-1	(i)	1
(b)	$2\sqrt{2}$	(f)	$\sqrt{2}$	(j)	0
(c)	2	(g)	12		
(d)	$\frac{6}{5}$	(h)	$\frac{1}{6}$		

....

7. $c = -2, d = 12$

....

8. (a) 3
   (c) Yes.

....

9. (a) 1
   (b) 7.3778 hours

....

10. (a) $+\infty$
    (b) $-\infty$

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics I** **Tutorial 6**

Reference

[S] Chapter 2, Section 2.5, 2.6 or [T] Chapter 2, Section 2.5, 2.6

1. Determine the following limits at infinity.

   (a) $\lim_{x\to\infty} \dfrac{3x+5}{x-4} = \lim_{x\to\infty} \dfrac{3+\frac{5}{x}}{1-\frac{4}{x}} = 3$

   (b) $\lim_{x\to\infty} \dfrac{x^3-2x+3}{5-2x^2} = \lim_{x\to\infty} \dfrac{x-\frac{2}{x}+\frac{3}{x^2}}{\frac{5}{x^2}-2} = \dfrac{\infty}{-2} = -\infty$

   (c) $\lim_{x\to\infty} \dfrac{x+2}{\sqrt{9x^2+1}} = \lim_{x\to\infty} \dfrac{1+\frac{2}{x}}{\frac{\sqrt{9x^2+1}}{\sqrt{x^2}}} = \lim_{x\to\infty} \dfrac{1+\frac{2}{x}}{\sqrt{9+\frac{1}{x^2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

   (d) $\lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+4}} = \lim_{x\to-\infty} \dfrac{x}{\sqrt{x^2+4}} \cdot \dfrac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to-\infty} \dfrac{1}{\frac{\sqrt{x^2+4}}{-\sqrt{x^2}}} = \lim_{x\to-\infty} \dfrac{1}{-\sqrt{1+\frac{4}{x^2}}} = -1$

   (e) $\lim_{x\to\infty} \left(\sqrt{x^4+6x^2}-x^2\right) = \lim_{x\to\infty} \dfrac{(\sqrt{x^4+6x^2}-x^2)(\sqrt{x^4+6x^2}+x^2)}{\sqrt{x^4+6x^2}+x^2} = \lim_{x\to\infty} \dfrac{x^4+6x^2-x^4}{x^2+\sqrt{x^4+6x^2}} = \lim_{x\to\infty} \dfrac{6x^2}{x^2+\sqrt{x^4+6x^2}} \cdot \dfrac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{6}{1+\frac{\sqrt{x^4+6x^2}}{\sqrt{x^4}}}$
   $= \lim_{x\to\infty} \dfrac{6}{1+\sqrt{1+\frac{6}{x^2}}} = 3$

   (f) $\lim_{x\to\infty} \dfrac{e^{3x}-e^{-3x}}{e^{3x}+e^{-3x}} = \lim_{x\to\infty} \dfrac{\frac{e^{3x}-e^{-3x}}{e^{3x}}}{\frac{e^{3x}+e^{-3x}}{e^{3x}}} = \dfrac{1-e^{-6x}}{1+e^{-6x}} = \dfrac{1-0}{1+0} = 1$

   (g) $\lim_{x\to\infty} [\ln(2+x)-\ln(1+x)] = \lim_{x\to\infty} \ln\left(\dfrac{2+x}{1+x}\right) = \ln\left(\lim_{x\to\infty}\left(\dfrac{2+x}{1+x}\right)\right) = \ln(1) = 0$

2. Find the domain of $g(x) = \dfrac{\sqrt{x+3}}{x^2-3x-10}$.

3. Use the Intermediate Value Theorem to show that there is a root of the equation $(1-x)^3 = \sin x$ in the interval $(0,1)$.

4. Explain why the equation $x^3 - 15x + 1 = 0$ has at least three solutions in the interval $[-4, 4]$.

5. Use the Intermediate Value Theorem to show that the two graphs $y = e^x$ and $y = -x$ intersect.

6. ★Suppose that a function f is continuous on the closed interval $[0,1]$ and $0 \le f(x) \le 1$ for every $x \in [0,1]$. Is it true that $f(c) = c$ for some $c \in [0,1]$? Justify your answer.

Definition The *derivative* of a function f at a number c, denoted by $f'(c)$, is defined as follows:

$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c},$$

$$\left(\text{ or equivalently, } f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h},\right)$$

if the limit exists, and we say that f is *differentiable* at c. Otherwise, f is not differentiable at c. If f is differentiable at each $c \in (a,b)$, f is said to be differentiable *on* (a,b), and the function f', where

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}, \; x \in (a,b)$$

is called the drivative of f.

7. Let $f(x) = 1 - x^3$.
   (a) Write down the definition of $f'(0)$ and use it to find $f'(0)$.
   (b) Use the value $f'(0)$ you have found in part (a) to write down an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.
8. Use the *definition of derivative* to find the first derivative of the following functions:
   (a) $f(x) = \dfrac{1}{5 - 3x}$
   (b) $g(x) = \sqrt{x^2 + 3}$.
9. Consider the function f where $f(x) = x|x|$. Is f differentiable at $x = 0$? If it is, determine $f'(0)$.
10. Consider f which is defined as follows:

$$f(x) = \begin{cases} \dfrac{e^x}{2+x} & \text{if } x \geq 0, \\ \cos(1 - e^{\pi x}) & \text{if } x < 0. \end{cases}$$

    Is f differentiable at $x = 0$?

Challenging Problems (will not be discussed in tutorial)

1. Show that if $f : \mathbf{R} \to \mathbf{R}$ is continuous and one-to-one, then f must be (strictly) increasing or decreasing.
2. If a map $f : [0, 1] \to [0, 1]$ is continuous, $f(0) = 0$, $f(1) = 1$ and $f \circ f(x) = x$ for all $x \in [0, 1]$, show that $f(x) = x$ for all $x \in [0, 1]$.
3. True or false: If f is a differentiable function, then its derivative f' is continuous.

Answers

1. (a) 3
   (b) $-\infty$
   (c) $\frac{1}{3}$
   (d) -1
   (e) 3
   (f) 1
   (g) 0

   ....

2. The domain is $[-3, \infty) \backslash \{-2, 5\}$.

   ....

6. TRUE

   ....

7. (a) $f'(0) = 0$
   (b) $y = 1$

   ....

2. Find the domain of $g(x) = \dfrac{\sqrt{x+3}}{x^2 - 3x - 10}$.

$g(x)$ exists when $x+3 \geq 0$ & $x^2 - 3x - 10 \neq 0$ $\quad [-3, \infty) / \{5, -2\}$

$x \geq -3 \quad (x-5)(x+2) \neq 0$

$x \neq 5, x \neq -2$

3. Use the Intermediate Value Theorem to show that there is a root of the equation $(1-x)^3 = \sin x$ in the interval $(0, 1)$.

let $f(x) = \sin x - (1-x)^3$

$f(0) = -1 \quad f(1) = \sin(1)$

$f(0) < 0 < f(1)$

By IVT, there exists a $c \in (0,1)$ such that $f(c) = 0$

4. Explain why the equation $x^3 - 15x + 1 = 0$ has at least three solutions in the interval $[-4, 4]$.

just use IVT for all fnx

5. Use the Intermediate Value Theorem to show that the two graphs $y = e^x$ and $y = -x$ intersect.

$f(x) = e^x + x$ $\quad$ when $f(-\infty) = -\infty$

when $f(\infty) = \infty + \infty = \infty$

$\therefore f(-\infty) \leq 0 \leq f(\infty)$

7. Let $f(x) = 1 - x^3$.

(a) Write down the definition of $f'(0)$ and use it to find $f'(0)$.

(b) Use the value $f'(0)$ you have found in part (a) to write down an equation of the tangent line to the curve $y = 1 - x^3$ at the point $(0, 1)$.

a) $f'(x) = \lim\limits_{y \to x} \dfrac{f(y) - f(x)}{y - x} = \lim\limits_{h \to 0} \dfrac{f(x-h) - f(x)}{h}$

$f'(0) = \lim\limits_{h \to 0} \dfrac{f(0-h) - f(0)}{h} = \lim\limits_{h \to 0} \dfrac{1 - h^3 - 1}{h} = \lim\limits_{h \to 0} h^2 = 0$ //

b) $\lim\limits_{y \to x} \dfrac{x^3 - y^3}{y - x} = \dfrac{-1\,\cancel{(y-x)}(x^2 + y^2 + xy)}{\cancel{y-x}} = -x^2 - x^2 - x^2$

$= -3x^2$

Use the *definition of derivative* to find the first derivative of the following functions:

(a) $f(x) = \dfrac{1}{5 - 3x}$

(b) $g(x) = \sqrt{x^2 + 3}$.

a) $f'(x) = \lim\limits_{y \to x} \dfrac{f(y) - f(x)}{y - x} = \lim\limits_{y \to x} \dfrac{\frac{1}{5-3y} - \frac{1}{5-3x}}{y - x} = \lim\limits_{y \to x} \dfrac{\frac{(5-3x)-(5-3y)}{(5-3y)(5-3x)}}{y - x} = \lim\limits_{y \to x} \dfrac{3y - 3x}{(5-3y)(5-3x)(y-x)} = \lim\limits_{y \to x} \dfrac{3\,\cancel{(y-x)}}{(5-3y)(5-3x)\,\cancel{(y-x)}}$

$= \dfrac{3}{(5-3x)^2}$

b) $g'(x) = \lim\limits_{y \to x} \dfrac{\sqrt{y^2+3} - \sqrt{x^2+3}}{y - x} \cdot \dfrac{\sqrt{y^2+3} + \sqrt{x^2+3}}{\sqrt{y^2+3} + \sqrt{x^2+3}} = \lim\limits_{y \to x} \dfrac{y^2 - x^2}{\cancel{(y-x)}(\sqrt{y^2+3} + \sqrt{x^2+3})}$

$= \lim\limits_{y \to x} \dfrac{y + x}{\sqrt{y^2+3} + \sqrt{x^2+3}}$

$= \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}}$

8. (a) $\frac{3}{(5-3x)^2}$
   (b) $\frac{x}{\sqrt{x^2+3}}$

....

9. f is differentiable at $x=0$ and $f'(x)=0$

....

10. Not diffrentiable at $x=0$.

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2020/21 Semester 1 **MH1810 Mathematics I** **Tutorial 7**

Reference [S] Chapter 3 or [T] Chapter 3.

1. Suppose f is differentiable and $f(x) > 0$.

   Use the following definition of derivative, $g'(x) = \lim_{t \to x} \dfrac{g(t) - g(x)}{t - x}$, to prove that

   (a) $\dfrac{d}{dx}(179 f(x)) = 179 f'(x)$.

   (b) $\dfrac{d}{dx}\sqrt{f(x)} = \dfrac{f'(x)}{2\sqrt{f(x)}}$.

   (c) $\dfrac{d}{dx}\left(\dfrac{1}{f(x)}\right) = \dfrac{-f'(x)}{(f(x))^2}$.

2. If $r(t) = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is $\dfrac{dr}{dt}$ at $t = 0$?

3. Calculate y'.

   (a) $y = \cos(\tan x)$

   (b) $y = \left(x + \dfrac{1}{x^2}\right)^{\sqrt{7}}$

   (c) $y = \dfrac{1}{\sin(x - \sin x)}$

   (d) $x^2 \cos y + \sin 2y = xy$

   (e) $x \tan y = y - 1$

   (f) $y = \ln(\sec x)$

   (g) $y = \ln(\sec x + \tan x)$

   (h) $y = \sin^{-1}(1 - x)$

4. Find the second derivative $f''(x)$ of $f(x) = \dfrac{x}{1 + x^2}$.

5. Find $f'(x)$.

   (a) $f(x) = \log_{10}\left(\dfrac{x}{x - 1}\right)$

   (b) $f(x) = \left(\dfrac{1 + \ln x}{1 - \ln x}\right)$

   (c) $f(x) = x \ln(1 + e^x)$

   (d) $f(x) = (\ln(1 + e^x))^2$

6. Find an equation of the tangent line to the curve $y = \dfrac{e^x}{x}$ at the point (i) $(1, e)$, (ii) where $x = -1$.

Suppose f is differentiable and $f(x) > 0$.

Use the following definition of derivative, $g'(x) = \lim_{t \to x} \frac{g(t) - g(x)}{t - x}$, to prove that

(a) $\frac{d}{dx}(179 f(x)) = 179 f'(x)$.

(b) $\frac{d}{dx}\sqrt{f(x)} = \frac{f'(x)}{2\sqrt{f(x)}}$.

(c) $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-f'(x)}{(f(x))^2}$.

a) Let $g(x) = 179 f(x)$

$$g'(x) = \lim_{t \to x} \frac{179f(t) - 179f(x)}{t-x} = 179 \lim_{t \to x} \frac{f(t) - f(x)}{t-x}$$

$$= 179 \, f'(x)$$

b) Let $h(x) = \sqrt{f(x)}$

$$h'(x) = \lim_{t \to x} \frac{\sqrt{f(t)} - \sqrt{f(x)}}{t-x} \cdot \frac{\sqrt{f(t)} + \sqrt{f(x)}}{\sqrt{f(t)} + \sqrt{f(x)}} = \frac{f(t) - f(x)}{(t-x)(\sqrt{f(t)} + \sqrt{f(x)})}$$

$$= f'(x) \cdot \frac{1}{\sqrt{f(t)} + \sqrt{f(x)}}$$

$$= \frac{f'(x)}{2\sqrt{f(x)}}$$

c) Let $g(x) = \frac{1}{f(x)}$

$$g'(x) = \frac{\frac{1}{f(t)} - \frac{1}{f(x)}}{t-x} = \frac{\frac{f(x) - f(t)}{f(t) f(x)}}{t-x} = \frac{-(f(t) - f(x))}{(t-x)(f(t)f(x))}$$

$$= \frac{-f'(x)}{(f(x))^2}$$

If $r(t) = \sin(f(t))$, $f(0) = \pi/3$, and $f'(0) = 4$, then what is $\frac{dr}{dt}$ at $t = 0$?

2) when $f(0)$, $r(t) = \frac{\sqrt{3}}{2}$

$$\frac{dr}{dt} = \frac{dr}{du} \times \frac{du}{dt} = 4\cos\left(\frac{\pi}{3}\right) = 2$$

Calculate y'.

(a) $y = \cos(\tan x)$

(b) $y = \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$

(c) $y = \frac{1}{\sin(x - \sin x)}$

(d) $x^2 \cos y + \sin 2y = xy$

(e) $x \tan y = y - 1$

(f) $y = \ln(\sec x)$

(g) $y = \ln(\sec x + \tan x)$

(h) $y = \sin^{-1}(1 - x)$

a) $y = \cos(\tan x)$

$y' = \sec^2 x(-\sin(\tan x))$

b) $y = \left(x + \frac{1}{x^2}\right)^{\sqrt{7}}$

$y' = \sqrt{7}\left(x + \frac{1}{x^2}\right)^{\sqrt{7}-1}\left(1 - 2x^{-3}\right)$

c) $y = \frac{1}{\sin(x - \sin x)}$

$y' = \frac{-(1 - \cos x)\cos(x - \sin x)}{\sin(x - \sin x)^2}$

d) $x^2 \cos y + \sin 2y = xy$

$2x\cos y - x^2 \frac{dy}{dx} \sin y + 2\frac{dy}{dx}\cos 2y = y + x\frac{dy}{dx}$

$-x^2 \sin y \frac{dy}{dx} + 2\frac{dy}{dx}\cos 2y - x\frac{dy}{dx} = y - 2x\cos y$

$\frac{dy}{dx} = \frac{y - 2x\cos y}{2\cos 2y - x - x^2 \sin y}$

e) $x \tan y = y - 1$

$\tan y + x \frac{dy}{dx} \sec^2 y = \frac{dy}{dx}$

$\frac{dy}{dx}(1 - x\sec^2 y) = \tan y$

$\frac{dy}{dx} = \frac{\tan y}{1 - x\sec^2 y}$

f) $y = \ln(\sec x)$

$\frac{dy}{dx} = \frac{1}{\sec x}(\sec x \tan x)$

$= \tan x$

g) $y = \ln(\sec x + \tan x)$

$\frac{dy}{dx} = \frac{1}{\sec x + \tan x}(\sec x \tan x + \sec^2 x)$

$= \frac{\sec x(\cancel{\tan x + \sec x})}{\cancel{\sec x + \tan x}} = \sec x$

h) $y = \sin^{-1}(1 - x)$

$1 - x = \sin y$

$-1 = \frac{dy}{dx}\cos y$

$\frac{dy}{dx} = \frac{-1}{\cos y}$

$= \frac{-1}{\sqrt{1 - \sin^2 y}}$

$= \frac{-1}{\sqrt{1 - \sin^2(\sin^{-1}(1-x))}}$

$= \frac{-1}{\sqrt{1 - (1-x)^2}}$

Find the second derivative $f''(x)$ of $f(x) = \frac{x}{1 + x^2}$.

4

$f(x) = x(1 + x^2)^{-1}$

$f'(x) = (1 + x^2)^{-1} - x(1 + x^2)^{-2}(2x) = \frac{1 - x^2}{(1 + x^2)^2}$

$f''(x) = \frac{(-2x)(1 + x^2)^{\cancel{2}} - (1 - x^2)(2)\cancel{(1 + x^2)}(2x)}{(1 + x^2)^{\cancel{4}\,3}}$

$= \frac{(-2x - 2x^3) - (4x - 4x^3)}{(1 + x^2)^3} = \frac{6x - 2x^3}{(1 + x^2)^3}$

Find $f'(x)$.

(a) $f(x) = \log_{10}\left(\frac{x}{x-1}\right)$

(b) $f(x) = \left(\frac{1+\ln x}{1-\ln x}\right)$

(c) $f(x) = x\ln(1+e^x)$

(d) $f(x) = (\ln(1+e^x))^2$

a) $f(x) = \log_{10}\left(\frac{x}{x-1}\right)$

$= \frac{\ln\left(\frac{x}{x-1}\right)}{\ln 10}$

$= \frac{1}{\ln 10}\left(\ln x - \ln(x-1)\right)$

$f'(x) = \frac{1}{\ln 10}\left(\frac{1}{x} - \frac{1}{x-1}\right)$

b) $f(x) = \frac{1+\ln x}{1-\ln x}$

$f'(x) = \frac{\frac{1}{x}\left[(1-\ln x)+(1+\ln x)\right]}{(1-\ln x)^2}$

$= \frac{2}{x(1-\ln x)^2}$

c) $f(x) = x\ln(1+e^x)$

$f'(x) = \ln(1+e^x) + x\left(\frac{e^x}{1+e^x}\right)$

d) $f(x) = (\ln(1+e^x))^2$

$f'(x) = 2(\ln(1+e^x))\left(\frac{e^x}{1+e^x}\right)$

6. Find an equation of the tangent line to the curve $y = \frac{e^x}{x}$ at the point (i) $(1, e)$, (ii) where $x = -1$.

$\frac{dy}{dx} = \frac{xe^x - e^x}{x^2}$

when $x = 1$, $m = 0$.

$y - e = 0$

$y = e$

$x = -1$, $y = -e^{-1}$

. If n is a positive number, prove that

$$\frac{d}{dx}(\sin^n x \cos nx) = n\sin^{n-1} x \cos(n+1)x$$

$n\sin^{n-1} x \cos nx \cos x - \sin^n x \; n \cos nx \; \sin nx$

(a) Use implicit differentiation to prove that

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}.$$

(b) Use the formula established in part (a) to find $\frac{dy}{dx}$ for

(i) $y = x\tan^{-1}\left(\frac{x}{2}\right)$, (ii) $y = \tan^{-1}(\ln x)$ and (iii) $\tan^{-1}(xy) = 1 + x^2y$.

a) $y = \tan^{-1} x$

$\tan y = x$

$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y + 1} = \frac{1}{x^2+1}$

b) $y = x\tan^{-1}\left(\frac{x}{2}\right)$

$\frac{dy}{dx} = \tan^{-1}\left(\frac{x}{2}\right) + \frac{\frac{1}{2}x}{1+\frac{1}{4}x^2}$

b) $y = \tan^{-1}(\ln x)$

$\frac{dy}{dx} = \frac{1}{1+(\ln x)^2}\left(\frac{1}{x}\right)$

$\tan^{-1}(xy) = 1 + x^2y$

$\frac{y + \frac{dy}{dx}x}{1+(xy)^2} = 2xy + x^2\frac{dy}{dx}$

$\frac{dy}{dx}\left(\frac{1}{1+(xy^2)} - x^2\right) = 2xy - \frac{y}{1+(xy)^2}$

$x^2\cos y = (x^2)(\cos y) \longrightarrow \frac{d}{dx}\left[f(x)\,g(x)\right] = f'(x)g(x) + f(x)g'(x)$

$\frac{d}{dx}x^2\cos y = \left[\frac{d}{dx}x^2\right](\cos y) + (x^2)\left[\frac{d}{dx}\cos y\right]$

$\frac{d}{dx}x^2\cos y = 2x(\cos y) + x^2\left[-\sin y\right]\left[\frac{dy}{dx}\right]$

$\frac{d}{dx}f(g(x)) = (f'(g(x)))(g'(x))$

$\frac{d}{dx}\cos y = \left[-\sin(y)\right]\left[\frac{dy}{dx}\right]$

7. If n is a positive number, prove that

$$\frac{d}{dx}(\sin^n x \cos nx) = n \sin^{n-1} x \cos(n+1)x$$

8. (a) Use implicit differentiation to prove that

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$

(b) Use the formula established in part (a) to find $\frac{dy}{dx}$ for
(i) $y = x\tan^{-1}\left(\frac{x}{2}\right)$, (ii) $y = \tan^{-1}(\ln x)$ and (iii) $\tan^{-1}(xy) = 1 + x^2 y$.

9. Find the derivative of the following function

$$f(x) = (\ln x)^{\cos x}, x > 1.$$

10. The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

(a) How are $\frac{dP}{dt}$, $\frac{dR}{dt}$ and $\frac{dI}{dt}$ related if P, R and I are functions of t?

(b) How is $\frac{dR}{dt}$ related to $\frac{dI}{dt}$ if $P = P_0$ is constant?

11. (**Accelerations whose magnitudes are proportional to displacement**) Suppose that the position of a body moving along a coordinate line at time t is $s = a\cos kt + b\sin kt$. Show that the acceleration $\frac{d^2 s}{dt^2}$ is proportional to s and it is directed to the origin.

12. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

13. A spotlight on the ground shines on a wall 12 m away. If a 2 m tall man walks from the spotlight straight towards the building at a speed of 1.6 m/s, how fast is the length of his shadow on the building decreasing, at the moment when he is 4 m from the building?

14. A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$m?

Challenging Problems

1. **Leibniz's Formula** Let f and g be n times differentiable functions. Prove that

$$\frac{d^n}{dx^n}(fg) = \sum_{m=0}^{n} \binom{n}{m} \frac{d^{n-m} f}{dx^{n-m}} \frac{d^m g}{dx^m}.$$

(Hint: Use induction and the following fact: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$. Proof of "fact": How many different ways can you choose a commitee of k people from $n+1$ people? Of which, how many ways if a particular person is chosen? If that person is not chosen?)

2. Let $f(x) = \tan^{-1} x$ (or $\arctan x$). Find $f^{(n)}(0)$ for $n = 0, 1, 2, 3...$, where $f^{(n)}(x) = \frac{d^n f(x)}{dx^n}$. (Hint: Leibniz formula).

Find the derivative of the following function

$$f(x) = (\ln x)^{\cos x}, x > 1.$$

$y = (\ln x)^{\cos x}$

$\ln y = \ln(\ln x^{\cos x})$

$\ln y = \cos x \ln(\ln x)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = -\sin x \ln(\ln x) + \cos x \frac{1}{\ln x}\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = \cos x \frac{y}{x \ln x} - y \sin x \ln(\ln x)$

The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.

(a) How are $\frac{dP}{dt}$, $\frac{dR}{dt}$ and $\frac{dI}{dt}$ related if P, R and I are functions of t?

(b) How is $\frac{dR}{dt}$ related to $\frac{dI}{dt}$ if $P = P_0$ is constant?

(**Accelerations whose magnitudes are proportional to displacement**) Suppose that the position of a body moving along a coordinate line at time t is $s = a\cos kt + b\sin kt$. Show that the acceleration $\frac{d^2s}{dt^2}$ is proportional to s and it is directed to the origin.

$s = a\cos kt + b\sin kt$

3. A function is even if $f(x) = f(-x)$ for all $x \in \mathbf{R}$. A function is odd if $f(x) = -f(-x)$ for all $x \in \mathbf{R}$. Let f be a differentiable function on $\mathbf{R}$. Prove that

   (a) if f is even, then f' is odd;

   (b) if f is odd, then f' is even;

   (c) $(f' \text{ is odd}) \Leftrightarrow (f \text{ is even})$.

Answers

2. 2

3. (a) $-(\sec^2 x)\sin(\tan x)$

   (b) $\sqrt{7}(1-\frac{2}{x^3})(x+\frac{1}{x^2})^{\sqrt{7}-1}$

   (c) $\frac{-(\cos(x-\sin x))(1-\cos x)}{\sin^2(x-\sin x)}$

   (d) $\frac{2x\cos y - y}{x^2\sin y - 2\cos(2y) + x}$

   (e) $\frac{\tan y}{1-x\sec^2 y}$

   (f) $\tan x$

   (g) $\sec x$

   (h) $\frac{-1}{\sqrt{1-(1-x)^2}}$

....

4. $\frac{-2x(3-x^2)}{(1+x^2)^3}$

....

5. (a) $\frac{-1}{(\ln\ 10)x(x-1)}$

   (b) $\frac{2}{x(1-\ln x)^2}$

   (c) $\ln(1+e^x) + \frac{xe^x}{1+e^x}$

   (d) $\frac{2e^x\ln(1+e^x)}{1+e^x}$

....

6. (i) $y = e$

   (ii) $y = (-2/e)x - 3/e$

....

8. (a) $\tan^{-1}(\frac{x}{2}) + \frac{2x}{4+x^2}$

   (b) $\frac{1}{x(1+(\ln x)^2)}$

   (c) $\frac{2xy+2(xy)^3-y}{(x-x^2(1+(xy)^2))}$

....

9. $(\ln x)^{\cos x}\left((-\sin x)\ln(\ln x) + \frac{\cos x}{x\ln x}\right)$

....

10. (a) $\frac{dP}{dt} = I^2\frac{dR}{dt} + 2RI\frac{dI}{dt}$

(b) $\frac{dR}{dt} = -\frac{2R}{I}\frac{dI}{dt} = -\frac{2P_0}{I^3}\frac{dI}{dt}$

. . . .

12. $-\frac{1}{20\pi} \approx -0.0159$

. . . .

13. The shadow is getting shorter at a rate of 0.6 m/s.

. . . .

14. 1 rad/sec

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics I** **Tutorial 9**

Topics: L'Hospital's Rule, Mean Value Theorem, First and second derivatives, Increasing/Decreasing, Concavity, Local extrema, Antiderivatives.

References: [S]4.2 - 4.7; [T] 4.2 - 4.5

1. Find each of the following limits.
   Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule does not apply, explain why.

   (a) $\lim\limits_{x\to 0} \dfrac{x+\tan x}{\sin x}$ $= \lim\limits_{x\to 0} \dfrac{1+\sec^2 x}{\cos x} = \lim\limits_{x\to 0} \dfrac{1+(1+\tan^2 x)}{\cos x} = \dfrac{2}{1} = 2$

   (b) $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x - (x^2/2)}{x^3}$ $= \lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{3x^2} = \lim\limits_{x\to 0} \dfrac{e^x - 1}{6x} = \lim\limits_{x\to 0} \dfrac{e^x}{6} = \dfrac{1}{6}$

   (c) $\lim\limits_{t\to\infty} \dfrac{\pi t^5 - 9t^3 + 5}{t^5 + 7t^4 + 3t^2 - 1}$ $= \pi$

   (d) $\lim\limits_{x\to -\infty} x^2 e^x$ $= \lim\limits_{x\to -\infty} \dfrac{x^2}{e^{-x}} = \lim\limits_{x\to -\infty} \dfrac{2x}{-e^{-x}} = 0$

   (e) $\lim\limits_{x\to 1} \left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right)$ $= \lim\limits_{x\to 1} \left(\dfrac{(x-1)-\ln x}{\ln x(x-1)}\right) = \lim\limits_{x\to 1}\left(\dfrac{1-\frac{1}{x}}{\ln x + \frac{x-1}{x}}\right) = \lim\limits_{x\to 1}\left(\dfrac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x}+-\frac{2x-2}{x^2}}\right) = \dfrac{1}{2}$

   (f) $\lim\limits_{x\to\infty} \dfrac{3x + e^{2x}}{x^2 + e^{3x}}$ $=$

   (g) $\lim\limits_{x\to\infty} x \tan^{-1}\left(\dfrac{1}{x}\right)$

   (h) $\lim\limits_{x\to\infty} (e^x + x)^{1/x}$

2. Prove, by mathematical induction, that, for every positive integer n,
$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

3. Suppose that $3 \le f'(x) \le 7$ for all real numbers x. Use the Mean value Theorem to show that
$$15 \le f(9) - f(4) \le 35.$$

4. Consider the equation
$$x^3 + 3x^2 + 4x + 1 = 0.$$

   (a) Use the Intermediate Value Theorem to show that the above equation has at least one real root.

   (b) Use Mean value Theorem (or Rolle's Theorem) to show that the equation $x^3 + 3x^2 + 4x + 1 = 0$ has at most one real root.

   (c) Conclude from Parts (a) and (b) that the above equation has exactly one real solution.

5. At 2:00 PM a car was travelling at a velocity of 50 km/h. At 2:10 PM, it's velocity increased to 65 km/h. Show that at some time between 2:00 and 2:10 PM, the acceleration is exactly 90 km/h^2.

6. Two runners start the race at the same time and finish in a tie. Prove that at some time during the race, they have the same speed.

$$\lim_{x\to\infty} \frac{\tan^{-1}\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{-\frac{1}{x^2}}{1+\left(\frac{1}{x}\right)^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{-\frac{1}{\left(1+\left(\frac{1}{x}\right)^2\right)x^2}}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x^2+1}}{\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \frac{x^2}{x^2+1} = \lim_{x\to\infty} \frac{2x}{2x} = \lim_{x\to\infty} \frac{2}{2} = 1$$

$$\lim_{x\to\infty} (e^x + x)^{1/x} = \lim_{x\to\infty} e^{(1/x)\ln(e^x+x)}$$

$$= e^{\lim_{x\to\infty} (1/x)\ln(e^x+x)}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(e^x+x)}{x}} = e^{\lim_{x\to\infty} \frac{\frac{e^x+1}{e^x+x}}{1}} = e^{\lim_{x\to\infty} \frac{e^x}{e^x+1}} = e^{\lim_{x\to\infty} \frac{e^x}{e^x}} = e^{\lim_{x\to\infty} 1} = e$$

2. Prove, by mathematical induction, that, for every positive integer n,

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0.$$

② ? ? ? ? ?

3. Suppose that $3 \le f'(x) \le 7$ for all real numbers x. Use the Mean value Theorem to show that

$$15 \le f(9) - f(4) \le 35.$$

$3 \le f'(x) \le 7$ for $x \in \mathbb{R}$ Show $15 \le f(9) - f(4) \le 35$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$3 \le \frac{f(9)-f(4)}{9-4} \le 7$$

$$15 \le f(9) - f(4) \le 35 \,/\!/$$

4. Consider the equation

$$x^3 + 3x^2 + 4x + 1 = 0.$$

(a) Use the Intermediate Value Theorem to show that the above equation has at least one real root.
(b) Use Mean value Theorem (or Rolle's Theorem) to show that the equation $x^3 + 3x^2 + 4x + 1 = 0$ has at most one real root.
(c) Conclude from Parts (a) and (b) that the above equation has exactly one real solution.

a) $x^3 + 3x^2 + 4x + 1 = 0$

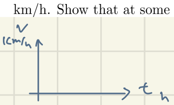
$f(1) = 9$ $f(-1) = -1$

b) Suppose 2 roots, a & b

$f(a) = f(b) = 0$

$$f'(x_0) = \frac{f(a)-f(b)}{a-b}$$ but $f'(x) = 3x^2 + 6x + 4$, never zero

5. At 2:00 PM a car was travelling at a velocity of 50 km/h. At 2:10 PM, it's velocity increased to 65 km/h. Show that at some time between 2:00 and 2:10 PM, the acceleration is exactly 90 km/h^2.



$f(0) = 50$ km/h $f\left(\frac{1}{6}\right) = 65$ km/h

$$f'(x_0) = \frac{f\left(\frac{1}{6}\right) - f(0)}{\frac{1}{6} - 0} = \frac{65-50}{\frac{1}{6}}$$

$$f'(x_0) = 90 \,/\!/$$

6. Two runners start the race at the same time and finish in a tie. Prove that at some time during the race, they have the same speed.

7. Consider the function $f(x) = 2\sqrt{x} - (3 - \frac{1}{x})$ on $[1, \infty)$.

   (a) Explain why f increasing on $[1, \infty)$.

   (b) Use part(a) to prove that for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

8. Determine the global maximum value of $f(x) = \dfrac{e^x}{1 + e^{2x}}$, $x \in \mathbf{R}$. Justify your answer.

9. Classify all critical points of the following functions.

   (a) $f(x) = \sqrt{3 + 2x - x^2}$, for $x \in (-1, 3)$.

   (b) $f(x) = \dfrac{x}{2} - 2\sin\dfrac{x}{2}$, for $x \in (0, 2\pi)$.

   (c) $f(x) = x^3 - 2x + 4$ for $x \in \mathbb{R}$.

10. A function F is said to be an *antiderivative* of f on an interval (a, b) if $F'(x) = f(x)$ for all $x \in (a, b)$. If F is an antiderivate of f, then $F + c$, where c is a constant, is also an antiderivative of f. In fact, every antiderivative of f is of the form $F + c$. The class of all antiderivatives of f is called the indefinite integral of f, denoted by $\int f(x)\ dx$.

    Find the general antiderivative for each of the following functions. Check your answers by differentiation.

    (a) $\sec^2 2x - \sin(3x + 5)$

    (b) $\left(1 - x^2\right)^2 + \dfrac{1}{1 + 3x}$

    (c) $e^{2x} + \dfrac{1}{\sqrt{1 - x^2}} - \dfrac{1}{x^2 + 1}$

11. Find the following indefinite integrals. Check your answers by differentiation.

    (a) $\displaystyle\int (\cos 2x + 2\cos x)\ dx$

    (b) $\displaystyle\int \left(1 + \tan^2\theta\right)\ d\theta$

    (c) $\displaystyle\int \cot^2 x + 3\sec^2(3x)\ dx$

12. Find the curve $y = f(x)$ that passes through the point $(9, 4)$ and whose gradient at each point (x, y) is $3\sqrt{x}$.

7. Consider the function $f(x) = 2\sqrt{x} - (3 - \frac{1}{x})$ on $[1, \infty)$.

(a) Explain why f increasing on $[1, \infty)$.

(b) Use part(a) to prove that for all $x > 1$,

$$2\sqrt{x} > 3 - \frac{1}{x}.$$

$x > 1$

$f'(x) = x^{-\frac{1}{2}} - x^{-2}$

$= \frac{x^2 - \sqrt{x}}{\sqrt{x}\, x^2} = \frac{x^{\frac{3}{2}} - 1}{x^2}$ +ve / +ve

$f'(x) > 0$ for all x more than 1

b)

8. Determine the global maximum value of $f(x) = \frac{e^x}{1 + e^{2x}}$, $x \in \mathbf{R}$. Justify your answer.

$f'(x) = \frac{e^x(1+e^{2x}) - e^x(2e^{2x})}{(1+e^{2x})^2} = \frac{e^x(1 - e^{2x})}{(1+e^{2x})^2}$

$f'(x) = 0$ when $e^x(1 - e^{2x}) = 0$

$1 - e^{2x} = 0$

$e^{2x} = 1$

$x = 0$

9. Classify all critical points of the following functions.

(a) $f(x) = \sqrt{3 + 2x - x^2}$, for $x \in (-1, 3)$.

(b) $f(x) = \frac{x}{2} - 2\sin\frac{x}{2}$, for $x \in (0, 2\pi)$.

(c) $f(x) = x^3 - 2x + 4$ for $x \in \mathbb{R}$.

a) $f'(x) = \frac{-2x + 2}{2\sqrt{3 + 2x - x^2}}$

$-2x + 2 = 0$

$x = 1$ //

$2\sqrt{3 + 2x - x^2} = 0$

$2\sqrt{(x-3)(x+1)(-1)} = 0$

$x = 3, \quad x = -1$

b) $f(x) = \frac{x}{2} - 2\sin\frac{x}{2}$

$f'(x) = \frac{1}{2} - \cancel{2}\cos(\frac{x}{2})(\cancel{\tfrac{1}{2}})$

$\frac{1}{2} - \cos(\frac{x}{2}) = 0$

$\cos(\frac{x}{2}) = \frac{1}{2}$

$x = \frac{1}{6}\pi$ /

(a) $\sec^2 2x - \sin(3x + 5)$

(b) $(1 - x^2)^2 + \frac{1}{1 + 3x}$

(c) $e^{2x} + \frac{1}{\sqrt{1 - x^2}} - \frac{1}{x^2 + 1}$

a) $\int (\sec^2 2x - \sin(3x + 5))\,dx$

$= \frac{\tan(2x)}{2} + \frac{\cos(3x + 5)}{3} + C$

b) $\int [(1 - x^2)^2 + \frac{1}{1 + 3x}]\,dx$

$= \qquad + \frac{\ln|1 + 3x|}{3}$

c) $\int [e^{2x} + \frac{1}{\sqrt{1 - x^2}} - \frac{1}{x^2 + 1}]\,dx$

$= \frac{1}{2}e^{2x} + \sin^{-1}x - \tan^{-1}x + C$

(a) $\int (\cos 2x + 2\cos x)\; dx$ $= \frac{1}{2}\sin(2x) + 2\sin x + C$

(b) $\int (1 + \tan^2\theta)\; d\theta$ $= \int(\sec^2\theta) = \tan\theta + C$

(c) $\int \cot^2 x + 3\sec^2(3x)\; dx$

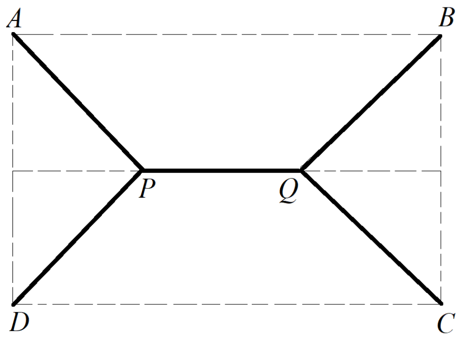
sin cos = 1

1 + tan = sec

1 + cot = csc

Challenging Problems

1. Four towns A, B, C, D are located at four corners of a huge rectangle with $AB \geq AD$. Two interchanges P and Q along the line of symmetry of the rectangle (see daigram below) and roads connected the towns via P and Q are to be built. (See diagram below). P and Q must be built equidistant from the center of the rectangle.



Show that if the total length of the roads in the network is minimal, then the angles at the intersections of the roads (e.g., $\angle APQ$) must be $120°$.

2. Suppose that f is a differentiable function such that $f'(x) \neq 0$ for all x, prove that f is one-to-one.

3. Suppose that $f : [-1, 1] \to \mathbf{R}$ is continuous, $f'(x)$ exists and $f'(x) \leq 1$ for all $x \in (-1, 1)$. If $f(-1) = -1$ and $f(1) = 1$, then $f(x) = x$ for all $x \in (-1, 1)$.

Answer

1. (a) 2
   (b) 1/6
   (c) π
   (d) 0
   (e) 1/2
   (f) 0
   (g) 1
   (h) e

....

8. The global maximum is $f(0) = 0.5$.

....

9. (a) $f(1)$ is a local maximum.
   (b) $f(\frac{2\pi}{3})$ is a local minimum.
   (c) $f\left(\sqrt{\frac{2}{3}}\right)$ is a local minimum whereas $f\left(-\sqrt{\frac{2}{3}}\right)$ is a local maximum.

....

10. (a) $\frac{1}{2}\tan(2x) + \frac{1}{3}\cos(3x+5) + C$
    (b) $x - \frac{2x^3}{3} + \frac{x^5}{5} + \frac{1}{3}\ln|1+3x| + C$
    (c) $\frac{e^{2x}}{2} + \sin^{-1}(x) - \tan^{-1}x + C$

....

11. (a) $\frac{\sin 2x}{2} + 2\sin x + C$
    (b) $\tan\theta + C$
    (c) $-\cot x - x + \tan 3x + C$

....

12. The curve is $y = 2x^{3/2} - 50$.

....

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/2022 Semester 1 **MH1810 Mathematics I** **Tutorial 10**

Topics Maximum and Minimum, Concavity, Indefinite Integrals, Riemann Sum, Fundamental Theorem of Calculus

1. Find the local maximum and minimum of f using both the First and Second Derivative Tests. Which method do you prefer?

   (a) $f(x) = x^5 - 5x + 3$,

   (b) $f(x) = \dfrac{x}{x^2+4}$.

2. For what values of c does the polynomial $f(x) = x^4 + cx^3 + x^2$ have

   (a) No inflection point?

   (b) 1 inflection point?

   (c) 2 inflection points?

3. Show that if $f(x) = x^4$, then $f''(0) = 0$ but $(0,0)$ is not a point of inflection.

4. Show that if $g(x) = x\,|x|$, then $(0,0)$ is a point of inflection but $g''(0)$ does not exist.

5. Express each of the following limits as a definite integral $\displaystyle\int_0^1 f(x)\,dx$ and use it to evaluate the limit.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n}\left\{\sin(\frac{\pi}{n}) + \sin(\frac{2\pi}{n}) + \cdots + \sin(\frac{k\pi}{n}) + \cdots + \sin(\frac{(n-1)\pi}{n}) + \sin(\frac{n\pi}{n})\right\}$.

   (b) $\displaystyle\lim_{n\to\infty} \left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k} + \cdots + \frac{1}{n+n}\right\}$

   (c) $\displaystyle\lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3}$

6. Find each of the following derivatives.

   (a) $\displaystyle\frac{d}{dx}\left(\int_1^x (2+t^4)^5\,dt\right)$

   (b) $\displaystyle\frac{d}{dx}\left(\int_{1/x^2}^0 \sin^3 t\,dt\right)$

   (c) $\displaystyle\frac{d}{dx}\left(\int_{\cos x}^{5x} \cos(u^2)\,du\right)$

7. Evaluate the following definite integrals.

   (a) $\displaystyle\int_1^3 \left(5 - \frac{x}{2} + \frac{3}{x^2} - \frac{1}{x}\right)\,dx$

   (b) $\displaystyle\int_{-1}^1 \left(x^2 - 2x + 3\right)\,dx$

1. Find the local maximum and minimum of f using both the First and Second Derivative Tests. Which method do you prefer?

(a) $f(x) = x^5 - 5x + 3$,

(b) $f(x) = \dfrac{x}{x^2+4}$.

a) $f(x) = x^5 - 5x + 3$

$f'(x) = 5x^4 - 5$

$f'(x) = 0$ when $x = 1, -1$

$f''(x) = 20x^3$

at $f''(1) = 20$ ∪

$f''(-1) = -20$ ∩

b) $f(x) = \dfrac{x}{x^2+4}$

$f'(x) = \dfrac{1(x^2+4) - (2x)(x)}{(x^2+4)^2} = \dfrac{-x^2+4}{(x^2+4)^2}$

$f'(x) = 0$ when $x = -2$, $x = 2$

2. For what values of c does the polynomial $f(x) = x^4 + cx^3 + x^2$ have

(a) No inflection point?

(b) 1 inflection point?

(c) 2 inflection points?

$f(x) = x^4 + cx^3 + x^2$

$f'(x) = 4x^3 + 3cx^2 + 2x$

$f''(x) = 12x^2 + 6cx + 2$

no inflection point means $f''(x) \neq 0$ $\quad b^2 - 4ac < 0$

$(6c)^2 - 4(12)(2) < 0$

$36c^2 < 96$

$c^2 < \frac{8}{3}$

$-\sqrt{\frac{8}{3}} < c < \sqrt{\frac{8}{3}}$

$-\frac{2\sqrt{6}}{3} < c < \frac{2\sqrt{6}}{3}$

$f''(x) = 0$, $c = -\frac{2\sqrt{6}}{3}$ or $\frac{3\sqrt{6}}{3}$

$f''(x) = 0$ twice $\quad -\frac{2\sqrt{6}}{3} > c > \frac{2\sqrt{6}}{3}$

3. Show that if $f(x) = x^4$, then $f''(0) = 0$ but $(0,0)$ is not a point of inflection.

$f(x) = x^4$

$f'(x) = 4x^3$

$f''(x) = 12x^2$

$f''(x) = 0$ when $x = 0$

$f''(0.01) = \frac{3}{2500}$ $\quad f''(-0.01) = \frac{3}{2500}$

both +ve.

4. Show that if $g(x) = x|x|$, then $(0,0)$ is a point of inflection but $g''(0)$ does not exist.

$g(x) = x|x|$

$g'(x) =$

5. Express each of the following limits as a definite integral $\int_0^1 f(x)\,dx$ and use it to evaluate the limit.

(a) $\lim_{n\to\infty} \frac{1}{n}\left\{\sin(\frac{\pi}{n}) + \sin(\frac{2\pi}{n}) + \cdots + \sin(\frac{k\pi}{n}) + \cdots + \sin(\frac{(n-1)\pi}{n}) + \sin(\frac{n\pi}{n})\right\}$.

(b) $\lim_{n\to\infty} \left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+k} + \cdots + \frac{1}{n+n}\right\}$

(c) $\lim_{n\to\infty} \frac{1^2 + 2^2 + \cdots + n^2}{n^3}$

a) $\Delta x = \frac{1}{n}$

$\lim_{n\to\infty} \sum_{k=1}^{n} f(\frac{k}{n}) \cdot \frac{1}{n} = \int_0^1 f(x)\,dx$

Wite in summation

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \sin(\frac{k}{n}\pi) = \lim_{n\to\infty} \sum_{k=1}^{n} f(\frac{k}{n}) \cdot \frac{1}{n}$

$f(x) = \sin(\pi x)$

$= \int_0^1 \sin \pi x\,dx$

$= \left[\frac{-\cos \pi x}{\pi}\right]_0^1$

b) $\lim_{n\to\infty} \left\{\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+k} + \cdots + \frac{1}{n+n}\right\}$

$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n+k}\right) = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{(n+k)(\frac{n}{1})(\frac{1}{n})}\right)$

$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{(n+k)(\frac{1}{n})}\right)\left(\frac{1}{n}\right)$

$= \lim_{n\to\infty} \left(\frac{1}{n}\right) \sum_{k=1}^{n} \left(\frac{1}{(1+\frac{k}{n})}\right)$

$f(x) = \frac{1}{1+x}$

$\int_0^1 \left(\frac{1}{1+x}\right) dx = \left[\ln|1+x|\right]_0^1$

$= \ln 2$

c) $\lim_{x\to\infty} \frac{1^2+2^2+\cdots\cdots n^2}{n^3} = \lim_{x\to\infty} \sum_{k=1}^{n} \left(\frac{k^2}{n^3}\right)$

$= \lim_{x\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n}\left(\frac{k}{n}\right)^2\right)$

$f(x) = x^2$

$\int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$

6. Find each of the following derivatives.

(a) $\frac{d}{dx}\left(\int_1^x (2+t^4)^5\,dt\right)$

(b) $\frac{d}{dx}\left(\int_{1/x^2}^0 \sin^3 t\,dt\right)$

(c) $\frac{d}{dx}\left(\int_{\cos x}^{5x} \cos(u^2)\,du\right)$

a) $\frac{d}{dx}\left(\int_1^x (2+t^4)^5 dt\right) = (2+t^4)^5$

b) $\frac{d}{dx}\left(\int_{1/x^2}^0 \sin^3 t\, dt\right) = -\frac{d}{dx}\left(\int_0^{1/x^2} \sin^3 t\, dt\right)$

let $u = \frac{1}{x^2}$, $\frac{du}{dx} = \frac{-2}{x^3}$ $\qquad \frac{d}{dx} = \frac{d}{du} \times \frac{du}{dx}$

$= -\left(\frac{-2}{x^3}\right)\frac{d}{du}\left(\int_0^u \sin^3 t\, dt\right)$

$= \frac{2}{x^3} \sin^3 u$

$= \frac{2}{x^3} \sin^3\left(\frac{1}{x^2}\right)$

c) $\frac{d}{dx}\left(\int_{\cos x}^{5x} \cos(u^2)\, du\right)$

$= \frac{d}{dx}\left(\int_0^{5x} \cos(u^2) du - \int_0^{\cos x} \cos(u^2) du\right)$

let $i = 5x$ $\qquad \frac{di}{dx} = 5 \qquad \frac{d}{dx} = \frac{di}{dx} \times \frac{d}{di}$

let $j = \cos x$ $\qquad \frac{dj}{dx} = -\sin x \qquad \frac{d}{dx} = \frac{dj}{dx} \times \frac{d}{dj}$

$= 5\frac{d}{di}\int_0^i \cos(u^2) du + \sin x \frac{d}{dj}\int_0^j \cos(u^2) du$

$= 5\cos(i^2) + \sin x \cos(j^2)$

$= 5\cos(25x) + \sin x \cos(\cos^2 x)$

7. Evaluate the following definite integrals.

(a) $\int_1^3 \left(5 - \frac{x}{2} + \frac{3}{x^2} - \frac{1}{x}\right)\, dx$

(b) $\int_{-1}^1 \left(x^2 - 2x + 3\right)\, dx$

(c) $\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\, dx$

(d) $\int_0^{\pi/4} (1 + \cos x - \tan^2 x)\, dx$

(e) $\int_0^{\pi/4} (\sin(2x) - \cos(5x))\, dx$

(f) $\int_{\pi/2}^{\pi} \frac{\sin 2x}{2\sin x} dx$

(g) $\int_{-1}^0 (2^u + e^u)\, du$

(h) $\int_{-\pi}^{\pi/2} f(x)\,dx$ where $f(x) = \begin{cases} e^x & \text{if } -\pi \le x \le 0, \\ \cos x & \text{if } 0 < x \le \pi. \end{cases}$

(i) $\int_0^{0.5} \frac{1}{\sqrt{1-x^2}}\, dx$

(j) $\int_1^{\sqrt{3}} \frac{1}{1+x^2}\, dx$

a) $\int_1^3 \left(5 - \frac{x}{2} + \frac{3}{x^2} - \frac{1}{x}\right) dx = \left[5x - \frac{x^2}{4} - \frac{3}{x} - \ln x\right]_1^3$

$= \frac{47}{4} - \frac{7}{4} - \ln 3$

$= 10 - \ln 3$

b) $\int_{-1}^1 (x^2 - 2x + 3)\, dx = \left[\frac{x^3}{3} - x^2 + 3x\right]_{-1}^1$

$= \frac{7}{3} + \frac{13}{3} = \frac{20}{3}$

c) $\int_1^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \left[\frac{2x^{\frac{3}{2}}}{3} + \frac{\sqrt{x}}{\frac{1}{2}}\right]_1^9$

$= 24 - \frac{8}{3} = 21\frac{1}{3} = \frac{64}{3}$

d) $\int_0^{\pi/4} (1 + \cos x - \tan^2 x)\, dx = \int_0^{\pi/4} (1 + \cos x - (\tan^2 x + 1) + 1)\, dx$ $\qquad \tan^2 + 1 = \sec^2 \theta$

$= \int_0^{\pi/4} (2 + \cos x - \sec^2 x)\, dx$

$= \left[2x + \sin x - \tan x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\pi + \frac{\sqrt{2}}{2} - 1 - 0 - 0 + 0$

e) $\int_0^{\pi/4} (\sin 2x - \cos 5x) dx = \left[-\frac{\cos(2x)}{2} - \frac{\sin(5x)}{5}\right]_0^{\frac{\pi}{4}}$

$= 0 + \frac{1}{2} + \frac{1}{5\sqrt{2}}$

f) $\int_{\frac{\pi}{2}}^{\pi}$

(c) $\displaystyle\int_{1}^{9}\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\, dx$

(d) $\displaystyle\int_{0}^{\pi/4}\left(1+\cos x-\tan^2 x\right)\, dx$

(e) $\displaystyle\int_{0}^{\pi/4}(\sin(2x)-\cos(5x))\, dx$

(f) $\displaystyle\int_{\pi/2}^{\pi}\frac{\sin 2x}{2\sin x}dx$

(g) $\displaystyle\int_{-1}^{0}(2^u+e^u)\, du$

(h) $\displaystyle\int_{-\pi}^{\pi/2} f(x)\,dx$ where $f(x)=\begin{cases} e^x & \text{if } -\pi\le x\le 0, \\ \cos x & \text{if } 0<x\le\pi. \end{cases}$

(i) $\displaystyle\int_{0}^{0.5}\frac{1}{\sqrt{1-x^2}}\, dx$

(j) $\displaystyle\int_{1}^{\sqrt{3}}\frac{1}{1+x^2}\, dx$

Challenging Questions (Will not be discussed)

1. Show that if f is continuous on an interval $[a,b]$ and $f'(x)=0$ for all $x\in(a,b)$, then f is a constant function. Hence show that if G_1 and G_2 are antiderivatives of a function g, then G_1-G_2 is a constant function.

2. Evaluate the integrals:

   (a) $\int\frac{1}{(x+1)(x+2)(x+3)(x+4)}dx$

   (b) $\int\frac{1}{x^3-1}dx$

   (c) $\int\frac{1}{x^6-1}dx$

   (d) $\int\frac{1}{x^7-x}dx$

Answers

1. (a) Local maximum at $x = -1$,local minimum at $x = 1$.
   (b) Local minimum at $x = -2$,local maximum at $x = 2$.

....

2. (a) $|c| \leq \sqrt{\frac{8}{3}}$
   (b) No such c exists.
   (c) $|c| > \sqrt{\frac{8}{3}}$

....

5. (a) $\frac{2}{\pi}$
   (b) $\ln 2$
   (c) $\frac{1}{3}$

....

6. (a) $(2 + x^4)^5$
   (b) $\frac{2}{x^3} \sin^3(\frac{1}{x^2})$
   (c) $5 \cos 25x^2 + (\sin x) \cos (\cos^2 x)$

....

7. (a) $10 - \ln 3$
   (b) $\frac{20}{3}$
   (c) $\frac{64}{3}$
   (d) $\frac{\pi}{2} + \frac{1}{\sqrt{2}} - 1$
   (e) $\frac{1}{2} + \frac{1}{5\sqrt{2}}$
   (f) -1
   (g) $\frac{1}{2 \ln 2} + 1 - \frac{1}{e}$
   (h) $2 - e^{-\pi}$
   (i) $\frac{\pi}{6}$.
   (j) $\frac{\pi}{12}$.

Challenging Question

2.

a. $\frac{1}{6} \ln |x+1| - \frac{1}{2} \ln |x+2| + \frac{1}{2} \ln |x+3| - \frac{1}{6} \ln |x+4| + C$

b. $\frac{1}{3} \ln |x-1| - \frac{1}{6} \ln \left(x^2 + x + 1\right) - \frac{1}{3}\sqrt{3} \tan^{-1} \sqrt{3} \left|\frac{2}{3}x + \frac{1}{3}\right| + C$

c. $\frac{1}{12} \ln \left(x^2 - x + 1\right) - \frac{1}{12} \ln \left(x^2 + x + 1\right) + \frac{1}{6} \ln |x-1| - \frac{1}{6} \ln |x+1| - \frac{1}{6}\sqrt{3} \tan^{-1} \sqrt{3} \left(\frac{2}{3}x - \frac{1}{3}\right)$
$-\frac{1}{6}\sqrt{3} \tan^{-1} \sqrt{3} \left(\frac{2}{3}x + \frac{1}{3}\right) + C$

d. $\frac{1}{6} \ln \left|x^6 - 1\right| - \ln x + C$

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics I** **Tutorial 11**

Topics Techniques of integration

1. Evaluate each of the following integrals by an appropriate substitution.

   (a) $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx$

   (b) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

   (c) $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

   (d) $\displaystyle\int 7\sqrt{7x-1}\,dx$

   (e) $\displaystyle\int \frac{4x^3}{(x^4+1)^2}\,dx$

   (f) $\displaystyle\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx$

Log
inverse
algebra
trigo
exponent

2. Evaluate the following integrals.

   $V' = x$ $U = \ln(x)$
   $V = \frac{x^2}{2}$ $U' = \frac{1}{x}$

   (a) $\displaystyle\int x\ln x\,dx$ $= x\ln(x) - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx = \frac{x^2\ln(x)}{2} - \int\left(\frac{x}{2}\right)dx = \frac{x^2\ln(x)}{2} - \frac{x^2}{4} + c$

   $V' = 1$ $U = \sin^{-1}(y)$
   $V = x$ $U' = \frac{1}{\sqrt{1-y^2}}$

   (b) $\displaystyle\int \sin^{-1} y\,dy$ $= y\sin^{-1}y - \int\frac{y}{\sqrt{1-y^2}}dy = y\sin^{-1}y - \int\frac{\cancel{y}}{\sqrt{u}}\left(\frac{1}{-2\cancel{y}}\right)dy$ $u = 1-y^2$, $\frac{du}{dy} = -2y$, $dy = du\left(\frac{1}{-2y}\right)$

   $= y\sin^{-1}y - \int\frac{1}{-2\sqrt{u}}\,du$

   $= y\sin^{-1}y + \frac{1}{2}\left(2u^{\frac{1}{2}}\right)$

   $= y\sin^{-1}y + \sqrt{1-y^2}$

   $V' = e^{-x}$ $U = x$
   $V = -e^{-x}$ $U' = 1$

   (c) $\displaystyle\int xe^{-x}\,dx$ $= -e^{-x}(x) - \int(-e^{-x})\,dx = -e^{-x}(x) - e^{-x} + c$

   (d) $\displaystyle\int x\sin(\pi x)\,dx$

   (e) $\displaystyle\int \sin(\ln x)\,dx.$

3. Evaluate $\displaystyle\int x^3\tan^{-1} x\,dx.$ $V' = 1$ $U = \sin(\ln x)$ $\int\sin(\ln x)\,dx$
   $V = x$ $U' = \cos(\ln x)\left(\frac{1}{x}\right)$

4. Evaluate the integrals.

   (a) $\displaystyle\int \frac{x-1}{x^2+3x+2}\,dx$

   (b) $\displaystyle\int \frac{x+4}{x^2+5x+6}\,dx$

   (c) $\displaystyle\int \frac{x^2}{(x-3)(x+2)^2}\,dx$

   (d) $\displaystyle\int \frac{x^3}{(x+1)^3}\,dx$

1. Evaluate each of the following integrals by an appropriate substitution.

(a) $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx$

(b) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

(c) $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx$

(d) $\displaystyle\int 7\sqrt{7x-1}\,dx$

(e) $\displaystyle\int \frac{4x^3}{(x^4+1)^2}\,dx$

(f) $\displaystyle\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx$

a) $\displaystyle\int \frac{\tan^{-1} x}{1+x^2}\,dx$

$u = \tan^{-1} x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$dx = du(1+x^2)$

$= \displaystyle\int \frac{u}{\cancel{1+x^2}}\,\cancel{(1+x^2)}\,du$

$= \displaystyle\int u\,du$

$= \frac{u^2}{2} = \frac{(\tan^{-1}(x))^2}{2}$

change

$u = \frac{1}{\sqrt{1-x^2}}$

c) $\displaystyle\int_0^{1/2} \frac{\sin^{-1} x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{u}\,du$

$\frac{du}{dx} = \sin^{-1}(x)$

b) $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

$u = e^x$

$\frac{du}{dx} = e^x$

$dx = du\left(\frac{1}{e^x}\right)$

$\displaystyle\int \frac{u}{u+1}\left(\frac{1}{u}\right)du$

$\displaystyle\int \frac{\cancel{u}}{u+1}\,\frac{1}{\cancel{u}}\,du = \ln|u+1|$

$= \ln|e^x+1|$

d) $\displaystyle\int 7\sqrt{7x-1}\,dx = 7\int \sqrt{7x-1}\,dx$

$= \cancel{7}\displaystyle\int \sqrt{u}\left(\frac{1}{\cancel{7}}\right)du$

$= \displaystyle\int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{2}{3}(7x-1)^{\frac{3}{2}} + C$

$u = 7x-1$

$\frac{du}{dx} = 7$

$dx = du\left(\frac{1}{7}\right)$

e) $\displaystyle\int \frac{4x^3}{(x^4+1)^2}\,dx = \int \frac{1}{u^2}\,du$

$u = x^4+1$

$\frac{du}{dx} = 4x^3$

$dx = du\frac{1}{4x^3}$

(e) $\displaystyle\int \frac{1}{x^2+16}\,dx$

(f) $\displaystyle\int \frac{1}{x^2+2x+5}\,dx$

(g) $\displaystyle\int \frac{x}{x^2+4x+13}dx$

5. Let $I_n = \displaystyle\int \cos^n x\,dx$ for $n = 0, 1, 2, 3, \cdots$.

   (a) Prove the reduction formula

   $$I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2} \text{ for } n \geq 2.$$

   (b) Use part(a) to evaluate

   (i) $\displaystyle\int \cos^3 x\ dx.$

   (ii) $\displaystyle\int_0^{\pi/2} \cos^4 x\ dx.$

Challenging Questions (Will not be discussed)

1. If n is a positive integer, prove that

   $$\int_0^1 (\ln x)^n\,dx = (-1)^n\,n!.$$

2. Let

   $$\Gamma(z) = \int_0^\infty \frac{x^{z-1}}{e^x}dx.$$

   Prove, by induction, that $\Gamma(n) = (n-1)!$ for any positive integer.

Answers

1. (a) $\frac{(\tan^{-1}x)^2}{2} + C$
   (b) $\ln(1+e^x) + C$
   (c) $\frac{\pi^2}{72}$
   (d) $\frac{2}{3}(7x-1)^{3/2} + C$
   (e) $\frac{-1}{x^4+1} + C$
   (f) $\frac{3(1+\sqrt{x})^{4/3}}{2} + C$

. . . .

2. (a) $\frac{x^2\ln x}{2} - \frac{x^2}{4} + C$
   (b) $y\sin^{-1}y + \sqrt{1-y^2} + C$
   (c) $-xe^{-x} - \int(-e^{-x})dx = -xe^{-x} - e^{-x} + C$
   (d) $\frac{-x\cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2} + C$
   (e) $\frac{1}{2}\left(x\sin(\ln x) - x\cos(\ln x)\right) + C$

. . . .

3. $\frac{1}{4}\Big((x^4-1)\tan^{-1}x - (\frac{x^3}{3} - x)\Big) + C$

. . . .

4. (a) $-2\ln|x+1| + 3\ln|x+2| + C$
   (b) $-\ln|x+3| + 2\ln|x+2| + C$
   (c) $\frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$
   (d) $x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C$
   (e) $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$
   (f) $\frac{1}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$
   (g) $\frac{1}{2}\ln(x^2+4x+13) - \frac{2}{3}\tan^{-1}(\frac{x+2}{3}) + C$

5. (b) (i) $\frac{1}{3}\cos^2 x\sin x + \frac{2}{3}\sin x + C.$
   (ii) $\frac{3\pi}{16}$

Nanyang Technological University

SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Mathematics I** **Tutorial 12**

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1. Evaluate each of the following improper integrals.

   (a) $\displaystyle\int_0^1 x \ln x \, dx$

   (b) $\displaystyle\int_0^1 \frac{4r}{\sqrt{1-r^4}} dr$

   (c) $\displaystyle\int_1^\infty \frac{1}{x^3} \, dx$

   (d) $\displaystyle\int_5^\infty \frac{1}{\sqrt{x-1}} \, dx$

   (e) $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^2} \, dx$

2. Find the values of p for which the integral converges

   (a) $\displaystyle\int_1^2 \frac{1}{x\,(\ln x)^p} \, dx$

   (b) $\displaystyle\int_2^\infty \frac{1}{x\,(\ln x)^p} \, dx$

3. Sketch each of the region enclosed by the given lines and curves. Find the area of the enclosed region.

   (a) $y = 2x - x^2$ and $y = -3$

   (b) $y = x^2 - 2x$ and $y = x$

   (c) $x = y^2$ and $x = y + 2$

   (d) $x = y^3 - y^2$ and $x = 2y$

4. (a) Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.

   (b) Sketch the region bounded by the given curves and find the area of the region:

   $$y = \sin x, y = e^x, x = 0, x = \pi/2$$

5. The base of a solid S is circular disk with radius r. Parallel cross-sections perpendicular to the base are squares. Show that the volume of the solid S is $\dfrac{16}{3}r^3$.

   (Note that the equation of a circle with radius r and centered at $(0, 0)$ is $x^2 + y^2 = r^2$.)

6. (a) Find the volume of the solid obtained by revolving the region bounded by the curves $x = y - y^2$ and $x = 0$ about the y-axis.

   (b) Find the volume of the solid generated by revolving the regions bounded by the curve $y = x^3$ and lines $y = 0$ and $x = 2$ about the x-axis.

   (c) Find the volume of the solid obtained by revolving the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the line $x = 2$.

7. Find the volume of the solid generated by revolving the regions bounded by the lines and curves about the y-axis.

   (a) $x = y^{3/2}$, $x = 0$, $y = 2$

   (b) $x = \sqrt{2\sin 2y}$, $0 \leq y \leq \pi/2$, $x = 0$

8. Consider the region bounded by the graphs of $y = \tan^{-1} x$, $y = 0$ and $x = 1$.

   (a) Find the area of the region.

   (b) Find the volume of the solid formed by revolving this region about the y-axis.

9. Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the curves $x + y = 3$ and $x = 4 - (y-1)^2$ about the x-axis.

10. Estimate each of the following definite integrals using the Trapezoidal Rule with $n = 4$.

    (a) $\displaystyle\int_1^2 x \; dx$ (b) $\displaystyle\int_1^3 (2x-1) \; dx$

11. Estimate each of the following definite integrals using Simpson's Rule with $n = 4$.

    (a) $\displaystyle\int_{-1}^1 \left(x^2+1\right) \; dx$ (b) $\displaystyle\int_{-2}^0 \left(x^2-1\right) \; dx$

12. Prove that the volume of the cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.

13. (a) The equation of a circle with center at the origin and radius r is described by the equation $x^2 + y^2 = r^2$. Use integration to prove that the area of the circle is πr^2.

    (b) When the region bounded by the x-axis and the curve $y = \sqrt{r^2 - x^2}$ for $-r \leq x \leq r$ is rotated about the x-axis, a sphere with radius r is obtained. Use integration to prove that the volume of the sphere is given by $\dfrac{4}{3}\pi r^3$.

14. Use integration by substitution to prove the following.

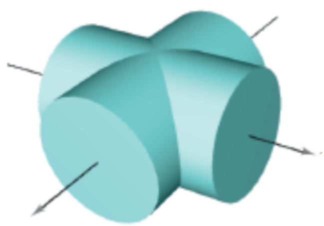
    (a) $\displaystyle\int \tan x dx = \ln|\sec x| + C$

    (b) $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + C$

    (c) $\displaystyle\int \sin^3 x \cos^8 x dx = -\frac{\cos^9 x}{9} + \frac{\cos^{11}}{11} + C$

Challenging Problems (Will not be discussed.)

1. Show that the volume common to two circular cyclinders, each with radius r, if the axes of the cyclinders intersect at right angle, is $\frac{16}{3}r^3$.



Two intersecting cylinders

Answers

1. (a) $-\frac{1}{4}$.
   (b) π.
   (c) $\frac{1}{2}$.
   (d) diverges
   (e) $\frac{1}{\ln 2}$.

. . . .

2. (a) $p < 1$, (b) $p > 1$

. . . .

   (a) $\frac{32}{3}$
   (b) $\frac{9}{2}$
   (c) $\frac{9}{2}$
   (d) $\frac{37}{12}$

. . . .

3. (a) $\frac{1}{12}$.
   (b) $e^{\frac{\pi}{2}} - 2$.

. . . .

6. (a) $\frac{\pi}{30}$
   (b) $\frac{128\pi}{7}$
   (c) $\frac{8}{15}\pi$

. . . .

7. (a) 4π
   (b) 2π

. . . .

8. (a) $\frac{\pi}{4} - \frac{1}{2}\ln 2$
   (b) $\pi(\frac{\pi}{2} - 1)$.

9. $27\pi/2$

10. (a) $\frac{3}{2}$, (b) 6

11. (a) $\frac{8}{3}$, (b) $\frac{2}{3}$