NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM I (CA1)

MH1812 – Discrete Mathematics

February 2017 TIME ALLOWED: 40 minutes

Name:

Matric. no.: Tutor group:

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. This **IS NOT** an **OPEN BOOK** exam.
6. Candidates should clearly explain their reasoning when answering each question.

QUESTION 1. **(40 marks)**

(a) Which integer $a \in \{0, 1, 2\}$ is congruent to $2017 + 2020 + 2023$ modulo 3? (10 marks)

(b) Write down each integer $a \in \{0, 1, 2, 3\}$ for which there exists an integer n such that $a \equiv n^2 \pmod 4$? (10 marks)

(c) Decide whether the set S is closed under the operation Δ when

- $S = \{\text{even integers}\}$ and Δ is subtraction. (10 marks)
- $S = \{\text{irrational numbers}\}$ and Δ is multiplication. (10 marks)

Briefly justify your answers.

Solution:

(a) We have

$$\begin{aligned} 2017 + 2020 + 2023 &= 2017 + (2017 + 3) + (2017 + 2 \cdot 3) \\ &= 3 \cdot 2017 + 3 \cdot 3. \end{aligned}$$

So $2017 + 2020 + 2023$ is divisible by 3, hence $2017 + 2020 + 2023 \equiv 0 \pmod 3$.

(b) Let n be an integer. Then n is congruent to either 0, 1, 2, or 3 modulo 4.

- If $n \equiv 0 \pmod 4$ then $n^2 \equiv 0^2 \equiv 0 \pmod 4$;
- if $n \equiv 1 \pmod 4$ then $n^2 \equiv 1^2 \equiv 1 \pmod 4$;
- if $n \equiv 2 \pmod 4$ then $n^2 \equiv 2^2 \equiv 4 \equiv 0 \pmod 4$;
- and if $n \equiv 3 \pmod 4$ then $n^2 \equiv 3^2 \equiv 9 \equiv 1 \pmod 4$.

So the only $a \in \{0, 1, 2, 3\}$ for which there exists an integer n such that $a \equiv n^2 \pmod 4$ are 0 and 1.

(c)
- $S = \{\text{even integers}\}$ and Δ is subtraction. Here S is closed under Δ: Take two elements $a, b \in S$; both a and b are even integers so $a = 2a'$ and $b = 2b'$ for some integers a' and b'. Now $a - b = 2a' - 2b' = 2(a' - b')$ is also an even integer.

- $S = \{\text{irrational numbers}\}$ and Δ is multiplication. Here S is *not* closed under Δ. Indeed, $\sqrt{2}$ is irrational and $\sqrt{2} \cdot \sqrt{2} = 2$ is not irrational. In other words, $\sqrt{2} \in S$ and $\sqrt{2} \cdot \sqrt{2} \notin S$.

QUESTION 2. **(40 marks)**

(a) Prove or disprove the following statement (20 marks):

$$(p \wedge \neg q) \rightarrow r \equiv (p \wedge \neg r) \rightarrow q.$$

(b) Decide whether or not the following argument is valid (20 marks):

$$\begin{array}{l} p \rightarrow q; \\ \neg p \rightarrow r; \\ \neg(r \wedge q); \\ \therefore \neg p \end{array}$$

Briefly justify your answer.

Solution:

(a)

$$\begin{aligned} (p \wedge \neg q) \rightarrow r &\equiv \neg(p \wedge \neg q) \vee r && \text{conversion theorem} \\ &\equiv (\neg p \vee \neg(\neg q)) \vee r && \text{De Morgan} \\ &\equiv (\neg p \vee q) \vee r && \text{double negation} \\ &\equiv \neg p \vee (q \vee r) && \text{associativity} \\ &\equiv \neg p \vee (r \vee q) && \text{commutativity} \\ &\equiv (\neg p \vee r) \vee q && \text{associativity} \\ &\equiv \neg(p \wedge \neg r) \vee q && \text{De Morgan} \\ &\equiv (p \wedge \neg r) \rightarrow q && \text{conversion theorem} \end{aligned}$$

Alternatively, one can prove the statement using a truth table.

(b) The argument is invalid. We want to find when the conclusion is false and all premises are true. The conclusion being false implies p is true.

Now with p true, we want to make all the premises true.

(i) With p true and $p \rightarrow q$ true we must have that q is true (modus ponens).

(ii) Since $\neg p$ is false $\neg p \rightarrow r$ is true regardless of the truth value of r.

(iii) To make $\neg(r \wedge q)$ true, we need r false, since q is true.

So we have a counterexample: p true, q true, and r false.

QUESTION 3. **(20 marks)**

Consider the domains $X = \mathbb{Z} = \{\text{integers}\}$ and $Y = \{0, 1, 2\}$, and the predicate $P(x, y) =$ "3 divides $x - y$".

Determine the truth values of the following statements:

(a) $\forall x \in X,\ \exists y \in Y,\ P(x, y)$; (10 marks)

(b) $\neg\,(\forall y \in Y,\ \exists x \in X,\ \neg P(x, y))$. (10 marks)

Briefly justify your answers.

Solution:

(a) For all $x \in X$ we can write $x = 3k + 0$, $x = 3k + 1$, or $x = 3k + 2$ for some integers k.

- If $x = 3k + 0$ then 3 divides $x - y$ for $y = 0$;
- if $x = 3k + 1$ then 3 divides $x - y$ for $y = 1$;
- if $x = 3k + 2$ then 3 divides $x - y$ for $y = 2$.

Hence the statement is true.

(b) Let us check the statement $\forall y \in Y,\ \exists x \in X,\ \neg P(x, y)$.

- If $y = 0$ then 3 does not divide $x - y$ for $x = 1$;
- if $y = 1$ then 3 does not divide $x - y$ for $x = 2$;
- if $y = 2$ then 3 does not divide $x - y$ for $x = 1$.

Therefore the statement $\forall y \in Y,\ \exists x \in X,\ \neg P(x, y)$ is true and its negation is false. So $\neg\,(\forall y \in Y,\ \exists x \in X,\ \neg P(x, y))$ is false.