Determinants

Pre-requisites from MH1810

- Determinant
  - Cofactors
  - Adjoint
  - Matrix inverse : $A^{-1} = \frac{1}{\det(A)}\mathrm{adj}(A)$

Overview and Learning Outcomes

- Properties of determinants
  - Interpret properties of determinants
- Determinants as area/volume
  - Interpret geometric properties of determinants
- Linear Transformations
  - Interpret geometry of linear transformations by determinants
  - Compute change of area/volume using determinants

3.1 Properties of determinants

1. **The determinant of the $n \times n$ identity matrix is 1.**

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \text{ and } \begin{vmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{vmatrix} = 1$$

2. **The determinant changes sign when two rows are exchanged.**

$$\begin{vmatrix} c & d \\ a & b \end{vmatrix} = - \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

If P is a permutation matrix with r row exchanges , then $|P| = 1$ for even r and $|P| = -1$ for odd r.

3. **The determinant is a linear function of each row separately.**

If 1 row of a matrix A is multiplied by t to get A', then $|A'| = t|A|$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

If one row of A is added to one row of A', then the determinants add.

$$\begin{vmatrix} a + a' & b + b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Important: *This rule applies only when the other rows do not change.*

4. **If two rows of A are equal, then $|A| = 0$.**

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0.$$

This follows from Rule 2 (Show!).

5. **Subracting a multiple of one row from another row leaves $|A|$ unchanged.**

$$\begin{vmatrix} a & b \\ c-la & d-lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

This follows from Rule 3 and Rule 4.

$|A| = |U|$ without row exchanges and $|A| = \pm|U|$ with row exchanges.

6. **A matrix with a row of zeros has $|A| = 0$.**

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0.$$

This follows from Rule 4 and Rule 5.

7. **If A is triangular, then $|A| = a_{11}a_{22}\ldots a_{nn}$ = product of diagonal entries.**

Consider the determinant of a diagonal matrix:

$$\begin{vmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{vmatrix} = D$$

Factor a_{11} from the first row. By rule 3, $D = a_{11}D'$.

Factor a_{22} from the second row. By rule 3, $D = a_{11}a_{22}D''$.

Finally, factor a_{nn} from the last row. By rule 3, $D = a_{11}a_{22}\ldots a_{nn}|I|$.

From rule 1, $|I| = 1$. So, $D = a_{11}a_{22}\ldots a_{nn}$.

Now, consider the determinants for the following triangular matrices

$$\begin{vmatrix} a & b \\ 0 & d \end{vmatrix} = D_1 \text{ and } \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} = D_2.$$

Make the off diagonal elements 0 through elimination.

$$R_1 \leftarrow R_1 - \tfrac{b}{d} R_2 : D_1^{'} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = ad$$

$$R_2 \leftarrow R_2 - \tfrac{c}{a} R_2 : D_2^{'} = \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix} = ad$$

If an $a_{ii} = 0$, elimination produces a zero row.

By Rule 5, determinant is unchanged and by Rule 6, determinant $= 0$.

Such matrices are called **singular**.

8. **If A is singular, then $|A| = 0$. If A is invertible, then $|A| \neq 0$.**

Transform A to U through elimination.
If A is singular:

- U has a zero row
- From previous rules, $|A| = |U| = 0$

If A is invertible:

- U has pivots along its diagonal
- From Rule 7, product of non-zero pivots $\Rightarrow$ non zero determinant
- $|A| = \pm|U| = \pm$(product of pivots)

[+ for even number of row exchanges and $-$ for odd number of row exchanges]

Pivots of a 2×2 matrix ($a \neq 0$):

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - (c/a)b \end{vmatrix} = ad - bc$$

(Finally, a formula for the determinant!!)

9. $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$.

Consider the ratio $D(A) = |AB|/|B|$. If $D(A)$ satisfies rules 1, 2 and 3, then it is a determinant.

- Rule 1 (*Determinant of I*)
  - If $A = I$, then $D(A) = |B|/|B| = 1$
- Rule 2(*Sign reversal*)
  - Two rows of A are exchanged $\Rightarrow$ Same two rows of $|AB|$ are exchanged $\Rightarrow$ $|AB|$ changes sign $\Rightarrow$ $D(A)$ changes sign
- Rule 3 (*Linearity*)
  - When 1 row of A is multiplied by $t \Rightarrow$ so is 1 row of $AB \Rightarrow |AB|$ is multiplied by $t \Rightarrow D(A)$ is multiplied by t
  - When 1 row of A is added to 1 row of $A' \Rightarrow$ 1 row of AB is added to 1 row of $A'B \Rightarrow$ determinants add $\Rightarrow$ dividing by B, the ratios add

The ratio $|AB|/|B|$ has the same properties that define $|A|$.

Therefore, $|AB|/|B| = |A| \Rightarrow |AB| = |A||B|$

If $|B| = 0$, B is singular $\Rightarrow AB$ is singular $\Rightarrow |AB| = 0$

$|A||B| = 0$

Therefore $|AB| = |A||B|$

10. $\boldsymbol{|A^T| = |A|}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

All the above properties apply to *columns* also.

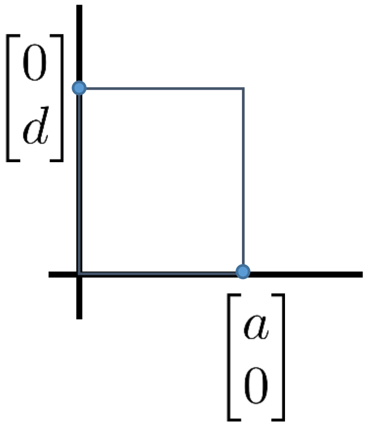
3.2 Determinants as Area or Volume

- Geometric interpretation of determinants

Theorem 3.1. *If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|A|$. If A is a 3×3 matrix, the volume of the parallelopiped determined by the columns of A is $|A|$.*

Proof. True for a 2×2 diagonal matrix:

$$\text{abs}\left(\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}\right) = \text{abs}(ad) = \text{area of rectangle}$$



$\square$

Can we transform any 2×2 matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$ into a diagonal matrix without change in area of the associated parallelogram or in $|A|$?

A can be transformed into a diagonal matrix by:

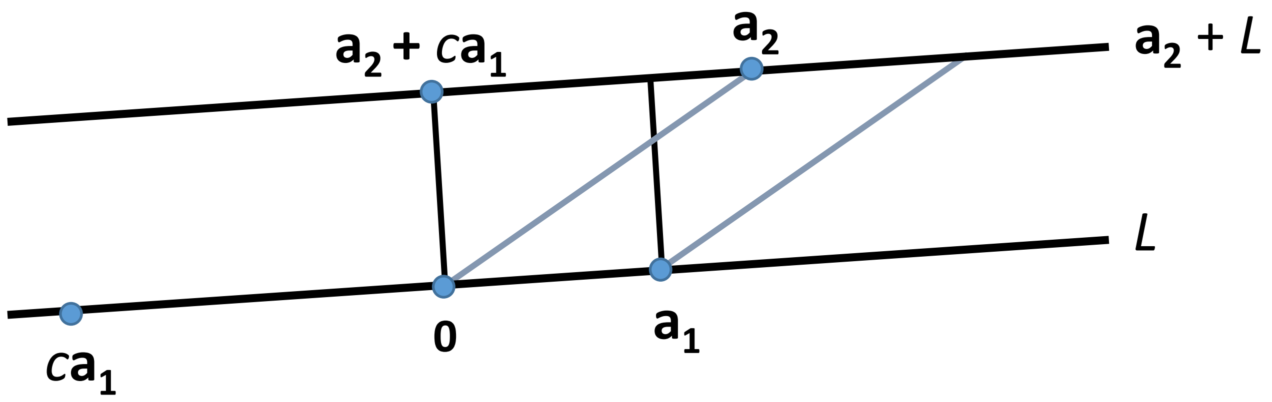
- Interchanging two columns
  - Does not change the parallelogram
  - From property 2, $|A|$ is unchanged

    Remember: properties apply to *columns* also.

- Adding a multiple of one column to another

Prove the following geometric observation:

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c, the area of a parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$.

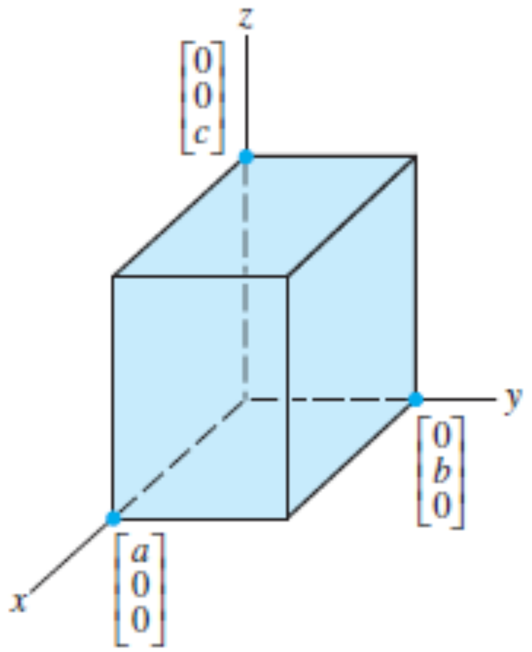


Assume $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

- L is the line through $\mathbf{0}$ and $\mathbf{a}_1 \Rightarrow \mathbf{a}_2 + L$ is the line through $\mathbf{a}_2$ and parallel to L
- Points $\mathbf{a}_2$ and $\mathbf{a}_2 + c\mathbf{a}_1$ have the same perpendicular distance to L
- Hence, two parallelograms have the same area (base X height)

Proof for $\mathbb{R}^3$ (i.e., 3×3 matrix):

True for a 3×3 diagonal matrix $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$



Can we transform any 3×3 matrix $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ into a diagonal matrix without change in volume of the associated parallelepiped or in $|A|$?

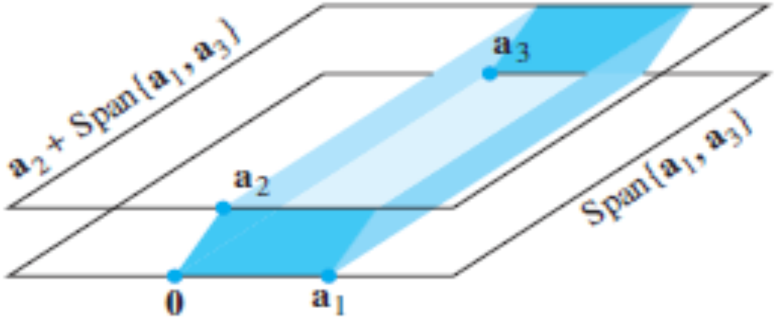
A can be transformed into a diagonal matrix by:

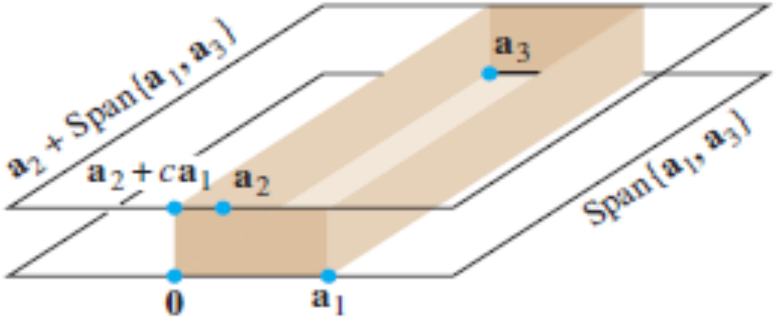
- Interchanging two columns (same as row operations on A^T)
  - Does not change the parallelepiped

- Adding a multiple of one column to another

  In the figure below:

  - Volume of parallelepiped = area of base × height
  - Base is in Span$\{\mathbf{a}_1, \mathbf{a}_3\}$ Height = $\mathbf{a}_2 + \text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$
  - $\mathbf{a}_2+c\mathbf{a}_1$ lies in the plane $\mathbf{a}_2+\text{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$, which is parallel to Span$\{\mathbf{a}_1, \mathbf{a}_3\}$
  - Hence, any vector $\mathbf{a}_2 + c\mathbf{a}_1$ has the same height as $\mathbf{a}_2$
  - Therefore, the volume of the parallelepiped is unchanged when $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3\end{bmatrix}$ is changed to $\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 + c\mathbf{a}_1 & \mathbf{a}_3\end{bmatrix}$



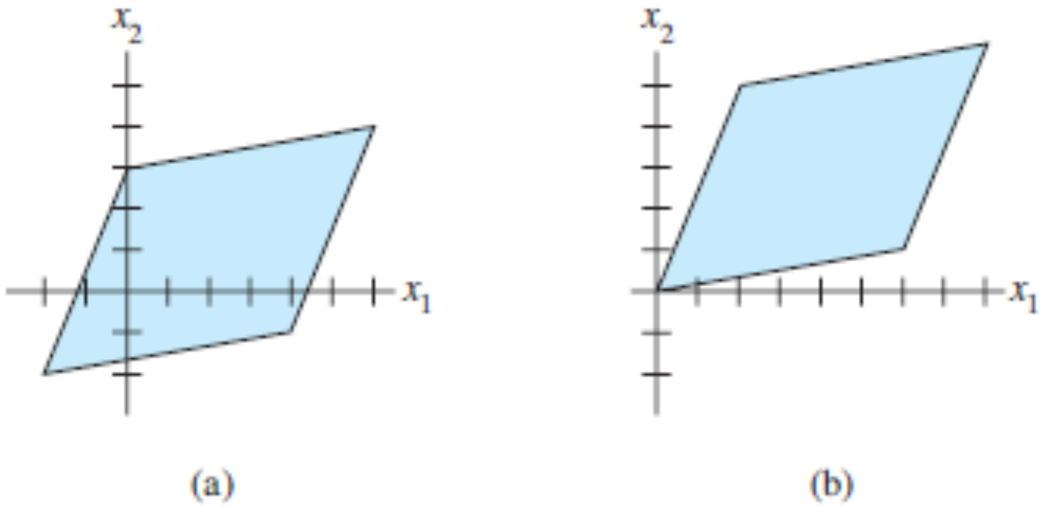


Exercise 3.2.1

Calculate the area of the parallelogram determined by the points $(-2,-2),(0,3),(4,-1),(6,4)$.

Solution

Translate the parallelogram to one having the origin as a vertex, e.g., subtract (-2, -2) from each of the four vertices.



Translating a parallelogram does not change its area

New vertices are at $(0,0),(2,5),(6,1),(8,6)$.

This parallelogram is determined by the columns of $A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$

$\text{abs}(|A|) = |-28|$

Therefore, area of the parallelogram is 28.

3.3 Linear Transformations

- How does the area (or volume) of a transformed set compare with the area (or volume) of the original

Theorem 3.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then area of $T(S) = abs(|A|)\times$ area of S.*
If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then volume of $T(S) = abs(|A|)\times$ volume of S.

Proof.
Consider the 2×2 case, $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$
A parallelogram at the origin in $\mathbb{R}^2$ determined by the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ has the form
$S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2 : 0 \leq s_1 \leq 1, 0 \leq s_2 \leq 1\}$

The image of S under T consists of the points of the form

$$T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) = s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2) = s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2,$$

where $0 \le s_1 \le 1, 0 \le s_2 \le 1$.

$T(S)$ is the parallelogram determined by columns of $\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = AB$ where $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$.

$$\text{area of } T(S) = abs(|AB|) = (abs|A|)(abs|B|) = (abs|A|)(\text{area of } S)$$

Now for the general case:

An arbitrary parallelogram has the form $\mathbf{p} + S$
where $\mathbf{p}$ is a vector and S is a parallelogram at the origin.

$T(\mathbf{p} + S) = T(\mathbf{p}) + T(S)$

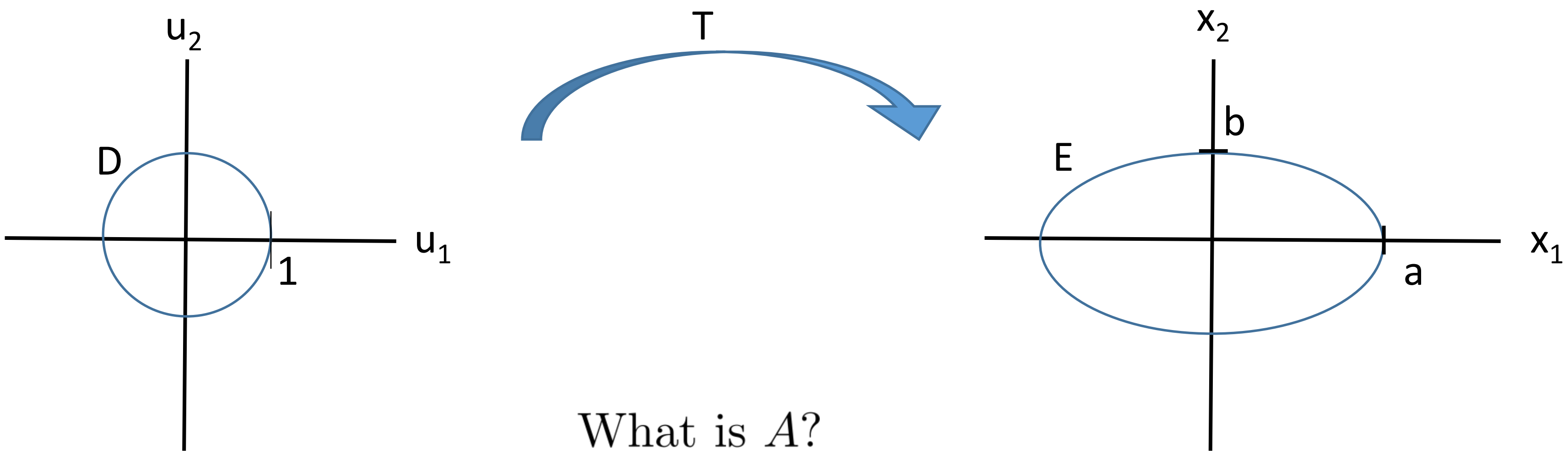
Translation does not affect the area of a set

$$\begin{aligned}
\text{area of } T(\mathbf{p} + S) &= \text{area of } (T(\mathbf{p}) + T(S)) \\
&= \text{area of } T(S) \\
&= abs(|A|) \times \text{area of } S \\
&= abs(|A|) \times \text{area of } \mathbf{p} + S
\end{aligned}$$

Proof for 3×3 is analogous.

Thereom 3.2 is applicable for arbitrary shapes also.

Example



What is A?

$$\begin{aligned} \text{area of ellipse} &= \text{area of } T(D) \\ &= abs(|A|) \times \text{area of } D \\ &= ab \times \pi 1^2 = \pi ab \end{aligned}$$

If $\mathbf{x} = A\mathbf{u}$ with $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, equation of ellipse given by $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, what is $\mathbf{u}$?

***** END OF CHAPTER *****