

MH1810 Math 1 Part 2 Chapter 6 Integration

Improper Integrals

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Improper Integrals

We know that a **continuous** function is Riemann integrable over a **closed and bounded interval** $[a, b]$ i.e., the integral $\int_a^b f(x) dx$ has a finite value. More generally, if f is not continuous at a finite number of points in $[a, b]$, then $\int_a^b f(x) dx$ exists.

However, when the integrand f is **not bounded on $[a, b]$** or the **interval $[a, b]$ is no longer bounded**, we have to consider the corresponding integrals carefully, and the integral may or may not converge to a finite value. Such integrals are known as **improper integrals**.

Improper Integrals (1): Unbounded Integrand

We shall discuss two types of improper integrals.

1 Unbounded Integrand $f(x)$.

Consider the integral

$$\int_{-1}^1 \frac{1}{x^2} dx,$$

which is not defined on $[-1, 1]$, and $\frac{1}{x^2}$ is **not** bounded on $[-1, 1]$ since $\lim_{x \rightarrow 0} \frac{1}{x^2} = \infty$. For the same reason, neither

$$\int_{-1}^0 \frac{1}{x^2} dx \quad \text{nor} \quad \int_0^1 \frac{1}{x^2} dx$$

are defined (in the Riemann sense).

Improper Integrals (2): Unbounded Interval

2 Unbounded Intervals.

Consider the integral $\int_1^{\infty} \frac{x}{1+x^2} dx$ and $\int_{-\infty}^{-1} \frac{1}{x^3} dx$.

Both intervals $[1, \infty)$ or $(-\infty, -1]$ are not bounded.

The above integrals are called **improper integrals**.

How should we assign meaning to such an **improper** integrals?

Example

Example

Evaluate $\int_{-1}^0 \frac{1}{x^2} dx$.

Since $\lim_{x \rightarrow 0^-} \frac{1}{x^2} = \infty$, 0 is known as the **singular point**. We replace the upper limit 0 by a variable $t < 0$:

$$\int_{-1}^t \frac{1}{x^2} dx.$$

and, define the improper integral $\int_{-1}^0 \frac{1}{x^2} dx$ as follows:

$$\int_{-1}^0 \frac{1}{x^2} dx = \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^2} dx.$$

Example

Therefore

$$\begin{aligned}\int_{-1}^0 \frac{1}{x^2} dx &= \lim_{t \rightarrow 0^-} \int_{-1}^t \frac{1}{x^2} dx = \lim_{t \rightarrow 0^-} \left(\frac{-1}{x} \right) \bigg|_{-1}^t \\ &= \lim_{t \rightarrow 0^-} \left(\frac{-1}{t} - 1 \right) = \infty.\end{aligned}$$

We say that the improper integral $\int_{-1}^0 \frac{1}{x^2} dx$ **diverges to infinity**.

Example

Example

Evaluate $\int_0^1 \frac{1}{\sqrt{2x-x^2}} dx$.

Solution

Note that $\lim_{x \rightarrow 0^+} \frac{1}{\sqrt{2x-x^2}} = \infty$. The integrand has a singular point at 0.

Thus, we have

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{2x-x^2}} dx &= \lim_{t \rightarrow 0^+} \left(\int_t^1 \frac{1}{\sqrt{2x-x^2}} dx \right) \\ &= \lim_{t \rightarrow 0^+} \left(\int_t^1 \frac{1}{\sqrt{1-(x-1)^2}} dx \right) \\ &= \lim_{t \rightarrow 0^+} \left(-\sin^{-1}(t-1) \right) = -\left(\frac{-\pi}{2} \right) = \frac{\pi}{2}.\end{aligned}$$

We say that the improper integral **converges to** $\frac{\pi}{2}$.

Example

Example

Evaluate $\int_0^2 \frac{1}{\sqrt{2x - x^2}} dx$.

Solution

Note that $\lim_{x \rightarrow 0^+} \frac{1}{\sqrt{2x - x^2}} = \infty$ and $\lim_{x \rightarrow 2^-} \frac{1}{\sqrt{2x - x^2}} = \infty$.

However, the function $\frac{1}{\sqrt{2x - x^2}}$ is continuous on $(0, 2)$. We evaluate the given improper integral as follows:

$$\int_0^2 \frac{1}{\sqrt{2x - x^2}} dx = \int_0^1 \frac{1}{\sqrt{2x - x^2}} dx + \int_1^2 \frac{1}{\sqrt{2x - x^2}} dx$$

The integral $\int_0^1 \frac{1}{\sqrt{2x - x^2}} dx$ is an improper integral which is evaluated in the preceding example.

Solution

Solution

We evaluate the improper integral $\int_1^2 \frac{1}{\sqrt{2x-x^2}} dx$ as follows:

$$\begin{aligned}\int_1^2 \frac{1}{\sqrt{2x-x^2}} dx &= \lim_{t \rightarrow 2^-} \left(\int_1^t \frac{1}{\sqrt{2x-x^2}} dx \right) \\&= \lim_{t \rightarrow 2^-} \left(\int_1^t \frac{1}{\sqrt{1-(x-1)^2}} dx \right) \\&= \lim_{t \rightarrow 2^-} (\sin^{-1}(t-1)) = \frac{\pi}{2}.\end{aligned}$$

Solution

Thus, we have

$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{2x-x^2}} dx &= \int_0^1 \frac{1}{\sqrt{2x-x^2}} dx + \int_1^2 \frac{1}{\sqrt{2x-x^2}} dx \\ &= \frac{\pi}{2} + \frac{\pi}{2} = \pi.\end{aligned}$$

The improper integral **converges to π** .

Improper Integral (2): Unbounded Intervals

Improper integrals over an **unbounded interval**:

$$\int_a^{\infty} f(x) \, dx, \int_{-\infty}^b f(x) \, dx, \int_{-\infty}^{\infty} f(x) \, dx$$

To integrate over an **unbounded interval**

- ▶ Replace the interval by a bounded interval, and
- ▶ Pass the definite integral to limiting process: $t \rightarrow \infty$ or $t \rightarrow -\infty$.

Unbounded Intervals (a)

For the improper integral $\int_a^\infty f(x) \, dx$, we define it as

$$\int_a^\infty f(x) \, dx = \lim_{t \rightarrow \infty} \int_a^t f(x) \, dx.$$

Example

Evaluate $\int_1^\infty \frac{1}{x^2} \, dx$.

Solution

$$\begin{aligned} \int_1^\infty \frac{1}{x^2} \, dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} \, dx = \lim_{t \rightarrow \infty} \left(\left[-\frac{1}{x} \right]_1^t \right) \\ &= \lim_{t \rightarrow \infty} \left(-\frac{1}{t} + 1 \right) = 1. \end{aligned}$$

Thus, the improper integral $\int_1^\infty \frac{1}{x^2} \, dx$ converges to 1.

Unbounded Intervals (b)

For the improper integral $\int_{-\infty}^b f(x) dx$, we define it as

$$\int_{-\infty}^b f(x) dx = \lim_{t \rightarrow -\infty} \int_t^b f(x) dx.$$

Example

Evaluate $\int_{-\infty}^{-1} \frac{1}{x^3} dx$

Solution

$$\begin{aligned} \int_{-\infty}^{-1} \frac{1}{x^3} dx &= \lim_{t \rightarrow -\infty} \int_t^{-1} \frac{1}{x^3} dx \\ &= \lim_{t \rightarrow -\infty} \left(-\frac{1}{2x^2} \Big|_t^{-1} \right) \\ &= \lim_{t \rightarrow -\infty} \left(-1/2 - \left(-\frac{1}{2t^2} \right) \right) = -1/2. \end{aligned}$$

Thus, the improper integral $\int_{-\infty}^{-1} \frac{1}{x^3} dx$ converges to $-1/2$.

Comparison Test for Integrals

Often, we want to estimate whether a given improper integral is convergent or divergent. We may use comparison theorem (stated below) to compare with a known convergent or divergent improper integrals.

Theorem (Comparison Theorem for Integrals)

Suppose f and g are continuous functions such that $f(x) \geq g(x) \geq 0$, for $x \geq a$. Then

1. *If $\int_a^\infty f(x) dx$ converges, then $\int_a^\infty g(x) dx$ converges.*
2. *If $\int_a^\infty g(x) dx$ diverges, then $\int_a^\infty f(x) dx$ diverges.*

Example

Example

Determine whether $\int_2^{\infty} \frac{1}{\ln x} dx$ converges or diverges.

Solution

For $x > 2$, we have $\ln x < x$ (why?), so

$$\frac{1}{\ln x} > \frac{1}{x}.$$

However, we have

$$\int_2^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \left(\int_2^t \frac{1}{x} dx \right) = \lim_{t \rightarrow \infty} (\ln t - \ln 2) = \infty.$$

By the Comparison Theorem, we conclude that $\int_2^{\infty} \frac{1}{\ln x} dx$ also diverges.

p-Integrals

The following improper integrals known as p -integrals are useful in comparison test.

Theorem (p -integrals)

Suppose $a \in \mathbb{R}$ and $a > 0$. Then

(a)

$$\int_a^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{a^{1-p}}{p-1} & \text{for } p > 1, \\ \infty & \text{for } p \leq 1. \end{cases}$$

(b)

$$\int_0^a \frac{1}{x^p} dx = \begin{cases} \frac{a^{1-p}}{1-p} & \text{for } p < 1, \\ \infty & \text{for } p \geq 1. \end{cases}$$

Proof of (a)

Note that $\int_a^b \frac{1}{x^p} dx = \frac{b^{-p+1} - a^{-p+1}}{-p+1}$, if $p \neq 1$.

$$\begin{aligned} \int_a^\infty \frac{1}{x^p} dx &= \lim_{t \rightarrow \infty} \int_a^t \frac{1}{x^p} dx = \begin{cases} \lim_{t \rightarrow \infty} \frac{t^{-p+1} - a^{-p+1}}{-p+1}, & \text{if } p \neq 1 \\ \lim_{t \rightarrow \infty} (\ln t - \ln a), & \text{if } p = 1 \end{cases} \\ &= \begin{cases} \frac{a^{1-p}}{p-1}, & \text{if } -p+1 < 0, \text{ i.e., } p > 1; \\ \infty, & \text{if } -p+1 > 0, \text{ i.e., } p \leq 1. \end{cases} \end{aligned}$$

Proof of (b)