NANYANG TECHNOLOGICAL UNIVERSITY

MIDTERM I (CA1)

MH1812 – Discrete Mathematics

February 2018 TIME ALLOWED: 40 minutes

Name:

Matric. no.: Tutor group:

INSTRUCTIONS TO CANDIDATES

1. **DO NOT TURN OVER PAPER UNTIL INSTRUCTED.**
2. This midterm paper contains **THREE (3)** questions.
3. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
4. Candidates can write anywhere on this midterm paper.
5. This **IS NOT** an **OPEN BOOK** exam.
6. Candidates should clearly explain their reasoning when answering each question.

QUESTION 1. (40 marks)

(a) Which integer $a \in \{0, 1, 2, 3\}$ satisfies $a \equiv 2^{2018} \pmod 4$? (10 marks)

(b) Wednesday is two days after Monday. What day of the week is it 500 days after Tuesday? (10 marks)

(c) Decide whether the set S is closed under the operation Δ when

- $S = \{\text{odd integers}\}$ and Δ is addition. (10 marks)
- $S = \{\text{even integers}\}$ and Δ is division. (10 marks)

Briefly justify your answers.

Solution:

(a) We have

$$\begin{aligned} 2^{2018} &= 4^{1009} \\ &= 4 \cdot 4^{1008}. \end{aligned}$$

So 2^{2018} is divisible by 4, hence $2^{2018} \equiv 0 \pmod 4$.

(b) The days of the week repeat every 7 days and $500 \equiv 3 \pmod 7$. Friday is three days after Tuesday and so Friday is also 500 days after Tuesday.

(c)
- $S = \{\text{odd integers}\}$ and Δ is addition. Here S is *not* closed under Δ: Indeed, $1 \in S$ but $1 + 1 = 2$ is even, and hence $1 + 1$ is not in S.
- $S = \{\text{even integers}\}$ and Δ is division. Here S is *not* closed under Δ. Indeed, $2 \in S$ but $2/2 = 1$ is odd, and hence $2/2$ is not in S.

QUESTION 2. **(40 marks)**

(a) Prove or disprove the following statement (20 marks):

$$(p \vee r) \to (p \wedge q) \equiv (p \to q) \wedge (r \to q).$$

(b) Decide whether or not the following argument is valid (20 marks):

$$\begin{aligned}
&p \vee q; \\
&\neg p \to r; \\
&\neg q \to r; \\
&r \vee p; \\
&\therefore r
\end{aligned}$$

Briefly justify your answers.

Solution:

(a) Here we want to disprove the statement. It suffices to find truth values for p, q, and r such that the truth value of the left hand side is different from that of the right hand side. For $p = F$, $q = T$, and $r = T$, the left hand side is false, but the right hand side is true. Hence the left hand side is not equivalent to the right hand side.

Alternatively, one can disprove the statement using a truth table.

(b) The argument is invalid. We want to find when the conclusion is false and all premises are true. The conclusion being false implies r is false.

Now with r false, we want to make all the premises true.

(i) With r false and $\neg p \to r$ true we must have that p is true.

(ii) With r false and $\neg q \to r$ true we must have that q is true.

So we have a counterexample: p true, q true, and r false.

Alternatively, one can show that the argument is invalid using a truth table.

QUESTION 3. (20 marks)

(a) Consider the domain $\mathbb{Q} = \{\text{rational numbers}\}$ and the predicate $P(x, y) =$ "xy is an integer".

Determine the truth value of the statement: (10 marks)

$\forall x \in \mathbb{Q},\ \exists y \in \mathbb{Q},\ P(x, y)$.

(b) Let X and Y be domains, and let $P(x)$ and $Q(y)$ be predicates. Which of the following statements is the *negation* of the statement

$\forall y \in Y,\ \exists x \in X,\ P(x) \to Q(y)$? (10 marks)

(i) $\forall y \in Y,\ \exists x \in X,\ \neg P(x) \wedge Q(y)$;
(ii) $\exists y \in Y,\ \exists x \in X,\ \neg P(x) \vee Q(y)$;
(iii) $\exists y \in Y,\ \forall x \in X,\ P(x) \wedge \neg Q(y)$;
(iv) $\exists y \in Y,\ \forall x \in X,\ \neg P(x) \wedge \neg Q(y)$.

Briefly justify your answers.

Solution:

(a) For all $x \in \mathbb{Q}$ we can write $x = a/b$ for some integers a and b with $b \neq 0$. Take $y = b \in \mathbb{Q}$. Then

$$xy = \frac{a}{b} \cdot b = a,$$

which is an integer. Hence the statement is true.

(b) We can write

$$\forall y \in Y,\ \exists x \in X,\ P(x) \to Q(y) \equiv \forall y \in Y,\ R(y),$$

where $R(y)$ is the predicate $R(y) = \exists x \in X,\ P(x) \to Q(y)$. The negation of "$\forall y \in Y,\ R(y)''$ is $\exists y \in Y,\ \neg R(y)$. Next we see that the negation of $R(y)$ is just $\forall x \in X,\ \neg(P(x) \to Q(y))$. Then

$$\begin{aligned}
\neg(P(x) \to Q(y)) &\equiv \neg(\neg P(x) \vee Q(y)) && \text{(conversion theorem)}\\
&\equiv \neg\neg P(x) \wedge \neg Q(y)) && \text{(De Morgan's law)}\\
&\equiv P(x) \wedge \neg Q(y)) && \text{(double negation)}
\end{aligned}$$

Hence the answer is (iii).