NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2020–2021
MH1810 – Mathematics 1

NOVEMBER 2020 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **NINE (9)** questions and comprises **EIGHTTEEN (18)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in Appendix Pages 16-18.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.
6. This examination paper is **NOT ALLOWED** to be removed from the examination hall.

For examiners only

Questions	Marks
1 (10)	
2 (10)	
3 (10)	

Questions	Marks
4 (10)	
5 (10)	
6 (10)	

Questions	Marks
7 (15)	
8 (15)	
9 (10)	

Total (100)	

QUESTION 1. **(10 Marks)**

The points $A(1,2,3)$, $B\,(1,2,4)$ and $C\,(1,3,5)$ are shown in Figure 1.

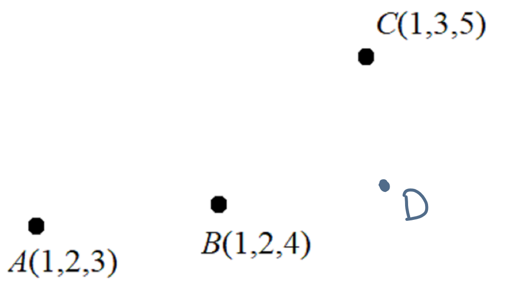


Figure 1.

(a) If D is a point such that $\overrightarrow{AD} = \left(\overrightarrow{AC}\cdot\overrightarrow{AB}\right)\overrightarrow{AB}$, find the coordinates of D. Indicate point D in Figure 1.

$$\vec{AC} = \begin{pmatrix} 1-1 \\ 3-2 \\ 5-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \qquad AC\cdot AB = 2$$

$$\vec{AB} = \begin{pmatrix} 1-1 \\ 2-2 \\ 4-3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Question 1 continues on Page 3.

(b) Find

(i) the area of triangle ABC, and

(ii) the angle $\angle CAD$.

$CA \cdot AD \Rightarrow |CA||AD|\cos\theta$

QUESTION 2. **(10 Marks)**

(a) Find all solutions of the equation $\overline{(z^4)} + 324 = 0$. Express your answers in the form $x + iy$. Indicate the solutions on the Argand diagram below.

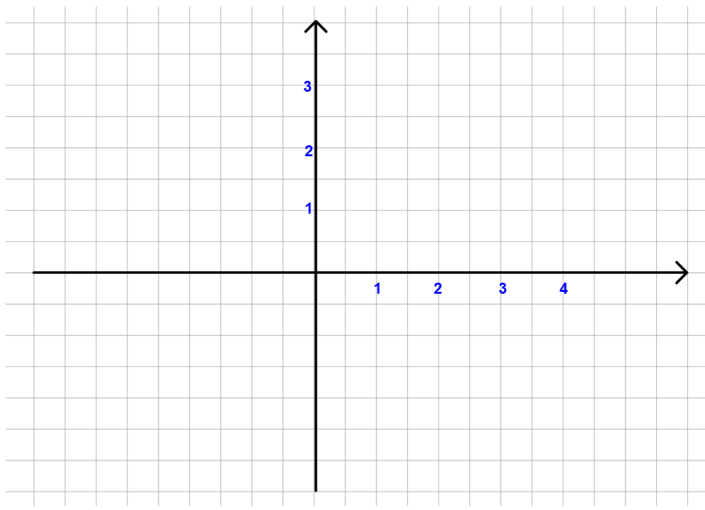


Figure 2.

$\overline{z^4} = -324$

$\overline{z^4} = 324 e^{i(\pi + 2n\pi)}$

$\overline{z} = 3\sqrt{2}\, e^{i\left(\frac{\pi + 2n\pi}{4}\right)}$

$n=0$, $3\sqrt{2}\, e^{i\frac{\pi}{4}}$, $z = 3\sqrt{2}\, e^{i-\frac{\pi}{4}}$

$n=1$, $3\sqrt{2}\, e^{i\frac{3}{4}}$, $z = 3\sqrt{2}\, e^{i-\frac{3\pi}{4}}$

$n=2$ $\frac{5}{4}$, $z = 3\sqrt{2}\, e^{i-\frac{5\pi}{4}}$ $\rightsquigarrow \frac{3}{4}$

$z = 3\sqrt{2}\, e^{i-\frac{7\pi}{4}}$

Question 2 continues on Page 5.

(b) Solve the equation

$$|z - 3| = z - i.$$

Express your answer z in the form $x + iy$.

$$|z - 3| = z - i$$

$$\sqrt{(a-3)^2 + b^2} = a + (b-1)i \qquad b = 1$$

$$(a-3)^2 + 1 = a^2$$

$$a^2 - 6a + 10 = a^2$$

$$-6a = -10$$

$$a = \frac{5}{3}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

QUESTION 3. **(10 Marks)**

(a) Find the $\boxed{\text{determinant}}$ of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & a & 1 \end{pmatrix}$, in terms of the unknown constant a.

if $a=1$, $\det = 0$

else, $\det = 0\,[\text{un}] - a\,[2-1] + 1\,[2-1]$

$= 1-a$

(b) Use part (a) and Cramer's Rule to find the value of x that satisfies the system of linear equations

$$\frac{2}{x} + \frac{1}{y} + z = 2,$$
$$\frac{1}{x} + \frac{1}{y} + z = 1,$$
$$-\frac{1}{y} + z = 0.$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{x} \\ \frac{1}{y} \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\frac{1}{x} = \frac{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}} = \frac{2}{2}$$

$x = 1$

QUESTION 4. **(10 Marks)**

Evaluate the following limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{x}{x^2+1} e^{\sin x}$.

$-1 \leq \sin x \leq 1$

$0 \leq e^{\sin x} \leq e^{1}$

$\dfrac{x}{x^2+1} e^{-1} \leq \dfrac{x}{x^2+1} e^{\sin x} \leq \dfrac{x}{x^2+1} e^{1}$

(b) $\displaystyle\lim_{x\to 1^-} \frac{\sqrt[3]{x}-1}{x^2-1} \sin\left(\frac{\pi}{1+x}\right)$.

$\displaystyle\lim_{x\to 1^-} \frac{\sqrt[3]{x}-1}{x^2-1} \cdot \lim_{x\to 1^-} \sin\left(\frac{\pi}{1+x}\right)$

$\displaystyle\lim_{x\to 1^-} \frac{\sqrt[3]{x}-1}{x^2-1} = \frac{\frac{1}{3}x^{-\frac{2}{3}}}{2x}$

$= \frac{1}{6}$

$$\lim_{y \to x} \frac{f(x) - f(y)}{y - x}$$

QUESTION 5. **(10 Marks)**

(a) Let $f(x) > 0$ be a differentiable function on $\mathbf{R}$. Use the **definition of derivative** to prove that

$$\frac{d}{dx}\sqrt{f(x)} = \frac{1}{2}\frac{f'(x)}{\sqrt{f(x)}}.$$

$$\frac{d}{dx}\sqrt{f(x)} = \lim_{y \to x} \frac{\sqrt{f(x)} - \sqrt{f(y)}}{y - x} \cdot \frac{\sqrt{f(x)} + \sqrt{f(y)}}{\sqrt{f(x)} + \sqrt{f(y)}}$$

$$= \lim_{y \to x} \frac{f(x) - f(y)}{(y - x)} \frac{1}{\sqrt{f(x)} + \sqrt{f(y)}}$$

$$= \frac{f'(x)}{2\sqrt{f(x)}}$$

(b) Find the equation of the tangent to the graph of $y = x^x$ at $x = 2$.

$f(x) = x^x$

$f'(x) = x\left(x^{(x-1)}\right)$

when $x = 2$, $f'(x) = 4$

$\ln y = x \ln x$

$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + 1 \ln x$

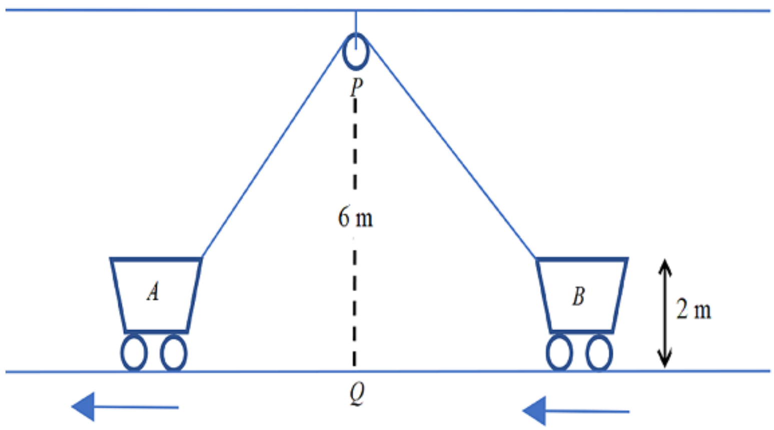
$\frac{dy}{dx} = y\left(1 + \ln x\right)$

$= 4\left(1 + \ln 2\right)$

$y = 4$

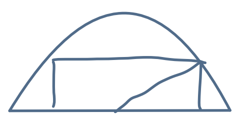
QUESTION 6 **(10 Marks)**

A rope of length 12 m that passes over a pulley at P is connected to the top corners of two identical carts A and B of height 2 m. The point Q is on the floor 6 m directly below P. Cart A is being pulled away from Q at a speed of 0.5 m/s. How fast is Cart B moving towards Q when Cart A is 3 m from Q?



QUESTION 7. **(15 Marks)**

(a) Determine the dimensions of the rectangle of largest possible area that can be inscribed in a semicircle of radius 3 cm.



$3^2 = h^2 + (\frac{1}{2} w)^2$

$9 = h^2 + \frac{1}{4} w^2$

$\frac{1}{4} w^2 = 9 - h^2$

$w^2 = 36 - 4h^2$

Area $= h \times w = h \times \sqrt{36 - 4h^2}$

Question 7 continues on Page 11.

(b) Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

$x^3 + 3x + 1 = 0$

QUESTION 8. **(15 Marks)**

(a) Find the values of p such that the improper integral $\int_0^1 x^p \ln x \, dx$ converges. For those values of p such that the integral converges, evaluate the value of the integral.

$U' = x^p \quad V = \ln x$

$U = \frac{x^{p+1}}{p+1} \quad V' = \frac{1}{x}$

$\int x^p \ln x \, dx = px^{p-1} \ln x - \int (px^{p-1})(\frac{1}{x}) \, dx$

$= px^{p-1} \ln x - \int px^{p-2} \, dx$

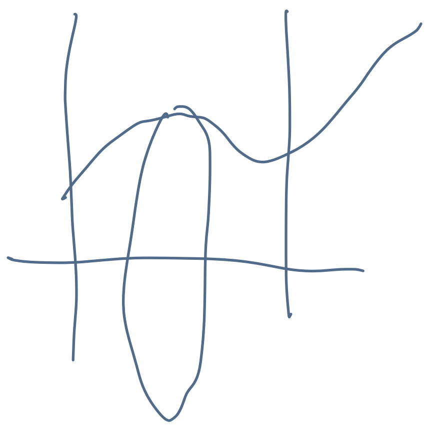
$= px^{p-1} \ln x - \frac{px^{p-1}}{p-1}$

Question 8 continues on Page 13.

(b) Let

$$f(x) = \begin{cases} x\left(|\ln x|\right)^{1/2}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

and R be the region bounded by the curve of $y = f(x)$, x-axis, y-axis and $x = 1$. Find the volume of the solid obtained by rotating R about the x-axis.



$$\pi\left(x\left(|\ln x|\right)^{\frac{1}{2}}\right)^2$$

QUESTION 9. **(10 Marks)**

Evaluate the following integrals.

(a) $\displaystyle\int \frac{1}{(x+2)(x+3)} dx.$

a) $\displaystyle\int \frac{1}{(x+2)(x+3)} dx = \int \left[\frac{1}{x+2} - \frac{1}{x+3}\right] dx$

$1 = A(x+3) + B(x+2)$

$A = 1$
$B = -1$

Question 9 continues on Page 15.

(b) $\displaystyle\int \frac{1}{x^2+x+1}\,dx.$

$\left(x+\frac{1}{2}\right)^2 + 1 - \frac{1}{2}^2$

$= \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x - a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech} x \tanh x \qquad \frac{d}{dx}(\text{csch } x) = -\text{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1 \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \tan x \sec x\,dx = \sec x + C \qquad \int \cot x \csc x\,dx = -\csc x + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \cot x\,dx = \ln|\sin x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int a^x\,dx = \frac{a^x}{\ln a} + C,\ a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2}\,dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C,\ |x| < |a| \qquad \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$