NANYANG TECHNOLOGICAL UNIVERSITY
SEMESTER I EXAMINATION 2018–2019
MH1810 – Mathematics 1

NOVEMBER 2018 TIME ALLOWED: 2 HOURS

Matriculation Number:

Seat Number:

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **SEVEN (7)** questions and comprises **NINETEEN (19)** pages, including an Appendix.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. This **IS NOT** an **OPEN BOOK** exam. However, a list of formulae is provided in the attachments.
4. Candidates may use calculators. However, they should write down systematically the steps in the workings.
5. All your solutions should be written in this booklet within the space provided after each question. If you use an additional answer book, attach it to this booklet and hand them in at the end of the examination.

For examiners only

Questions	Marks
1 (10)	
2 (10)	
3 (10)	
4 (15)	

Questions	Marks
5 (15)	
6 (15)	
7 (25)	

Total (100)	

QUESTION 1. **(10 Marks)**

The points $A(1,1,1)$, $B(2,2,1)$ and $C\left(\sqrt{2}+1,1,1\right)$ lie on a plane π.

(a) Find the vector equation and the scalar equation of the plane π.

$$AB = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad AC = \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

$$AB \times AC = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 - 0 \\ 0 - 0 \\ 0 - \sqrt{2} \end{pmatrix}$$

(b) Show that the point $D\left(\sqrt{2}+3,3,1\right)$ lies on π.

Question 1 continues on Page 3.

(c) The points A, B, C and the sides AB and AC have been drawn in Figure 1. Indicate where D should be and find the area of the quadrilateral $ACDB$.

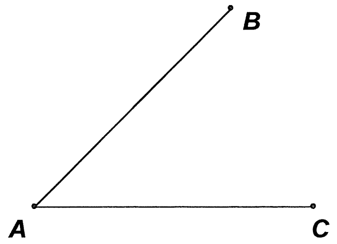
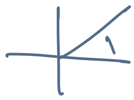


Figure 1.



QUESTION 2. **(10 Marks)**

(a) Let a be a positive real number. Express $w = \dfrac{a+ai}{a-ai}$ in the form $x + yi$. Find the modulus and argument of w.

$$W = \frac{\sqrt{2}a \ e^{i\frac{\pi}{4}}}{\sqrt{2}a \ e^{i\cdot\frac{-\pi}{4}}} =$$

(b) In Figure 2, plot the complex numbers $z_1 = 4+4i$, $z_2 = 4-4i$, $z_3 = \dfrac{4+4i}{4-4i}$ and $z_4 = \dfrac{16-16i}{4+4i}$.

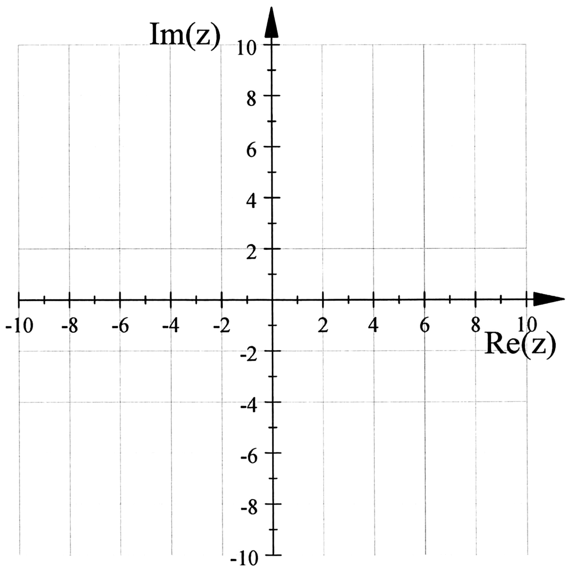


Figure 2.

Question 2 continues on Page 5.

(c) If u and v are complex numbers such that $\dfrac{u+v}{u-v}$ is purely imaginary, show that $|u| = |v|$.

$U = a + bi$ $\quad V = c + di$

$$\frac{U + V}{U - V} = Ki$$

$$U + V = Ki(U - V)$$

QUESTION 3. **(10 Marks)**

Let $f(x) = ax^2 + bx + c$. Given that $f(1) = 6$, $f(-1) = 2$ and $f'(1) = 8$.

(a) Use Cramer's Rule to find value of a. Deduce also the values of b and c.

Question 3 continues on Page 7

(b) Use the closed interval method to find the global maximum and minimum values of $f(x)$ on the interval $[-1, 1]$.

QUESTION 4. **(15 Marks)**

Let

$$f(x) = \begin{cases} \dfrac{x^3 - 1}{\sqrt{x} - 1} & \text{if} \quad x > 1, \\ a\cos(x-1) - x^2 & \text{if} \quad x \leq 1. \end{cases}$$

(a) Find the value of a that makes the function f continuous.

Question 4 continues on Page 9

(b)(i) Take $a = 1$. Show that $f(x)$ has a stationary point at $x = c$ for some $0 < c < 1$.

(ii) Use Newton's method with $x_0 = 1$, compute the second iterate x_2 to approximate the value of c. Leave your answer up to 3 decimal places.

QUESTION 5 **(15 Marks)**

(a) (i) State, without proof, the **Mean Value Theorem**.

$$\frac{f(b) - f(a)}{b - a}$$

(ii) If $f'(x) = 0$ for all $x \in \mathbb{R}$, **use the Mean Value Theorem** to show that f is a constant function.

Question 5 continues on Page 11.

$\tan = \frac{\sin}{\cos}$

(b) Find the following limits.

(i) $\lim_{x\to 0} \frac{\tan(3x^2)}{x\,(\sin x)}$ $= \frac{\cos(3x^2)}{\sin(3x^2)} \cdot \frac{1}{x(\sin x)}$

$\frac{\sin(x)}{x} \to 1$

(ii) $\lim_{x\to -\infty} \sqrt{x^2+x+1}\,\sin\frac{1}{x}$

QUESTION 6. **(15 Marks)**

(a) Let

$$f(x)=\begin{cases} x^3\cos\dfrac{1}{|x|} & \text{if} \quad x\neq 0, \\ 0 & \text{if} \quad x=0. \end{cases}$$

Use the **definition of derivative,** determine the value of $f'(0)$.

$$f'(x)=\lim_{y\to x}\frac{f(x)-f(y)}{y-x}$$

$$=\lim_{y\to x}\frac{x^3\cos\frac{1}{|x|}-y^3\cos\frac{1}{|y|}}{y-x}$$

Question 6 continues on Page 13.

(b) Show that the volume of the largest possible circular cone that is contained in a sphere is $\dfrac{8}{27}$ of the volume of the sphere.
[Volume of sphere of radius $r = \frac{4}{3}\pi r^3$; Volume of cone of radius r and height $h = \frac{1}{3}\pi r^2 h$].

QUESTION 7. **(25 Marks)**

(a) Evaluate

(i) $\lim_{n\to\infty} \frac{1}{n}\left[\left(\frac{1}{n}\right)^{1.5} + \left(\frac{2}{n}\right)^{1.5} + \left(\frac{3}{n}\right)^{1.5} + \ldots + \left(\frac{n}{n}\right)^{1.5}\right]$

(b) (i) Find $\int \frac{x}{(1+x^2)^2}dx$.

let $u = 1+x^2$

$\frac{du}{dx} = 2x$

$\int \frac{x}{(1+x^2)^2}dx = \int \frac{x}{u^2} \frac{1}{2\cancel{x}} = \int \frac{1}{2u^2}$

Question 7 continues on Page 15.

(b) (ii) By using the substitution $x = \tan\theta$ or otherwise, prove that

$$\int \frac{1}{(1+x^2)^2}dx = \frac{1}{2}\left(\frac{x}{1+x^2} + \tan^{-1} x\right) + C,$$

where C is an arbitrary constant.

(iii) Hence find $\displaystyle\int_0^1 \frac{x^2+3x+2}{(1+x^2)^2}dx$, express the answer in terms of π.

Question 7 continues on Page 16.

(c) Let R be the region bounded by the curve of $y = x^4$, x-axis and $x = 1$. Find the volume of the solid obtained by rotating R about the line $x = 2$.

END OF PAPER

Appendix

Numerical Methods.

- Linearization Formula:

$$L(x) = f(a) + f'(a)\,(x-a)$$

- Newton's Method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

- Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left[y_0 + 2\,(y_1 + y_2 + \cdots + y_{n-1}) + y_n\right]$$

- Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right]$$

Derivatives.

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \text{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\text{csch}^2 x$$

$$\frac{d}{dx}(\text{sech } x) = -\text{sech} x \tanh x \qquad \frac{d}{dx}(\text{csch } x) = -\text{csch} x \coth x$$

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$$

Antiderivatives.

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \; n \neq -1 \qquad \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C \qquad \int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C \qquad \int \csc^2 x \, dx = -\cot x + C$$

$$\int \tan x \sec x \, dx = \sec x + C \qquad \int \cot x \csc x \, dx = -\csc x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C \qquad \int \cot x \, dx = \ln|\sin x| + C$$

$$\int e^x \, dx = e^x + C \qquad \int a^x \, dx = \frac{a^x}{\ln a} + C, \; a > 0$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C \qquad \int \frac{1}{1+x^2} \, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}} \, dx = \sin^{-1}\left(\frac{x}{a}\right) + C, \; |x| < |a| \qquad \int \frac{1}{x^2+a^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{x^2+1}} \, dx = \sinh^{-1} x + C \qquad \int \frac{1}{\sqrt{x^2+a^2}} \, dx = \sinh^{-1}\left(\frac{x}{a}\right) + C$$

MH1810 MATHEMATICS 1

Please read the following instructions carefully:

1. **Please do not turn over the question paper until you are told to do so. Disciplinary action may be taken against you if you do so.**

2. You are not allowed to leave the examination hall unless accompanied by an invigilator. You may raise your hand if you need to communicate with the invigilator.

3. Please write your Matriculation Number on the front of the answer book.

4. Please indicate clearly in the answer book (at the appropriate place) if you are continuing the answer to a question elsewhere in the book.