$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.0**
Lecture : **Eigen and Singular Values**
Topic : **Overview of this chapter**
Concept :

Instructor: **A/P Chng Eng Siong**
TAs: **Zhang Su, Vishal Choudhari**

Rev: 3rd July 2020

Content

Chapter 5 Eigenvalues and Eigenvectors 265

We will use Lay's (4th edition) chapter 5.1-5.4 for the slides
Ch 5.5 and 5.6 for future offerings

Lay, Linear Algebra and its Applications (4th Edition)

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m \times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.1**

Lecture : **Eigen and Singular Values**

Topic : **Introducing Eigenvectors and Eigenvalues**

Concept :

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Eigenvectors and Eigenvalues

DEFINITION

An **eigenvector** of an $n \times n$ matrix A is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an *eigenvector corresponding to* λ.[1]

[1] Note that an eigenvector must be *nonzero*, by definition, but an eigenvalue may be zero.

The requirement that an eigenvector be nonzero is imposed to avoid the unimportant case $A\mathbf{0} = \lambda\mathbf{0}$, which holds for every A and λ.

Important note:
Eigenvalues and **eigenvectors** are only for **square** matrices.

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Lay, Linear Algebra and its Applications (4th Edition)

Eigenvectors and Eigenvalues

In general, the image of a vector **x** under multiplication by a square matrix A differs from **x** in both magnitude and direction. However, in the special case where **x** is an eigenvector of A, multiplication by A leaves the direction unchanged. For example,

EXAMPLE 1 Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The images of **u** and **v** under multiplication by A are shown in Fig. 1. In fact, $A\mathbf{v}$ is just $2\mathbf{v}$. So A only "stretches," or dilates, **v**. ■

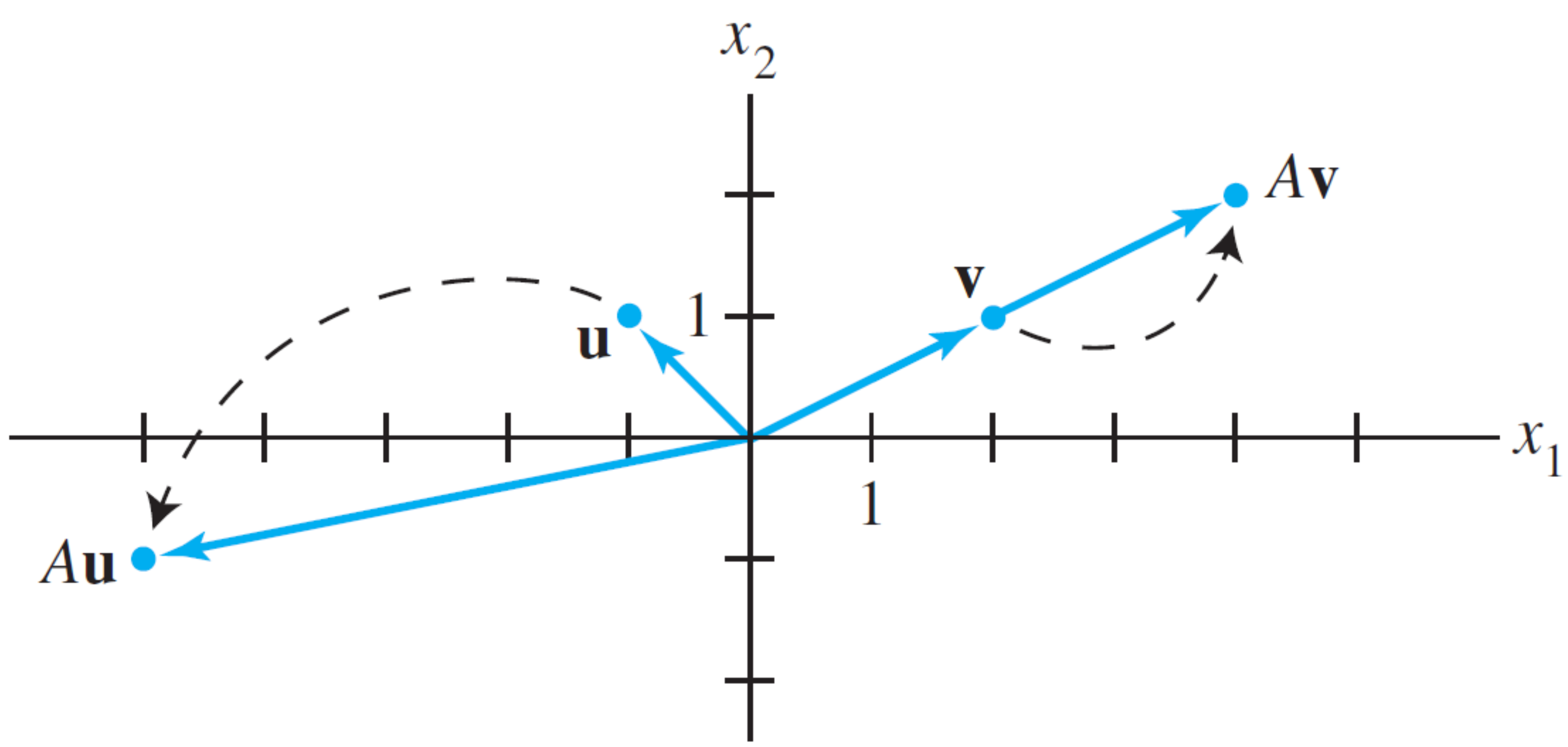


FIGURE 1 Effects of multiplication by A.

EigenVectors and Values

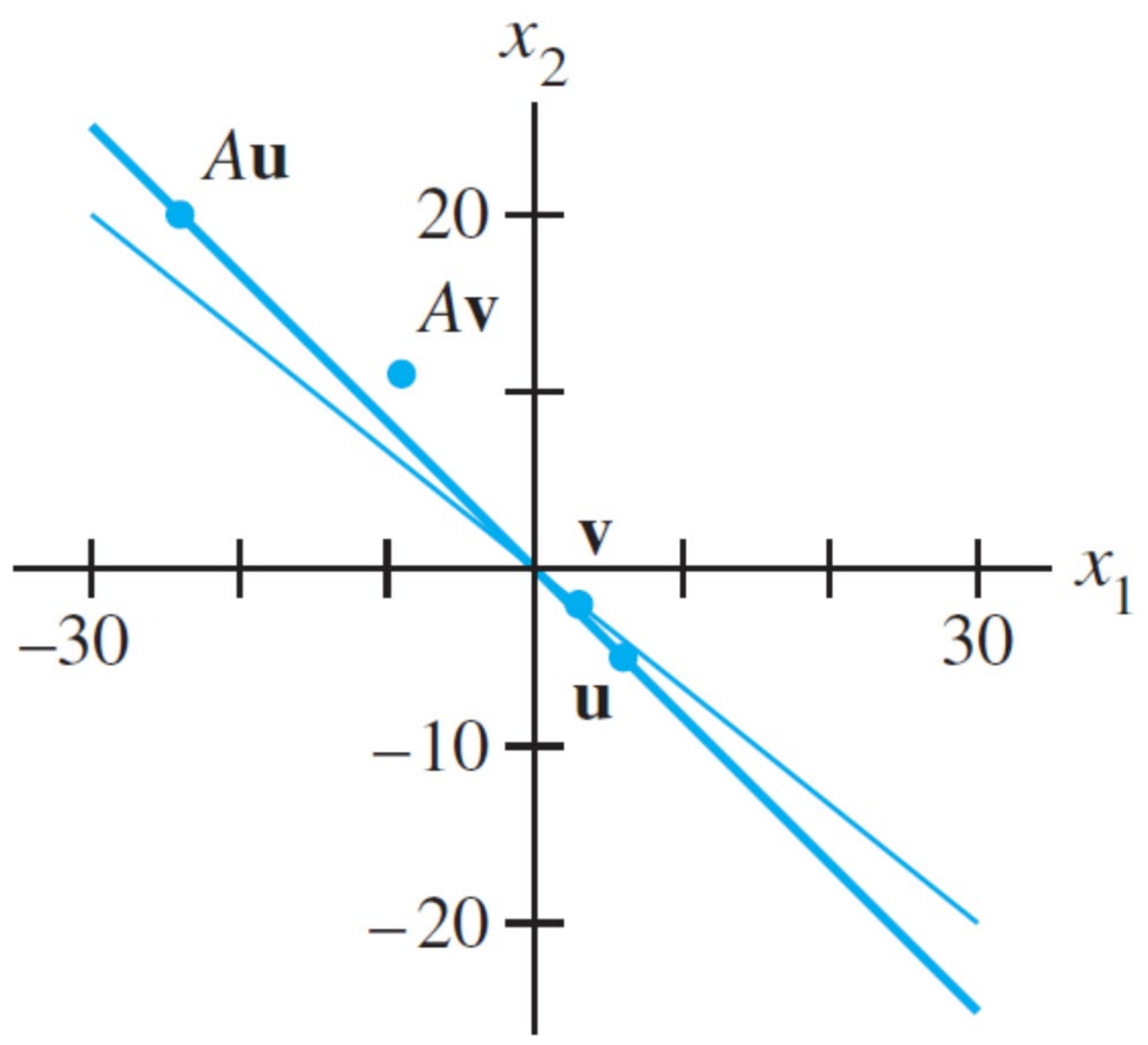
EXAMPLE 2 Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

SOLUTION

$$A\mathbf{u} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix} = -4\begin{bmatrix} 6 \\ -5 \end{bmatrix} = -4\mathbf{u}$$

$$A\mathbf{v} = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda\begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Thus $\mathbf{u}$ is an eigenvector corresponding to an eigenvalue (-4), but $\mathbf{v}$ is not an eigenvector of A, because $A\mathbf{v}$ is not a multiple of $\mathbf{v}$. ■



$A\mathbf{u} = -4\mathbf{u}$, but $A\mathbf{v} \neq \lambda\mathbf{v}$.

Example: How to find EigenVectors (V IMPT)

EXAMPLE 3 Show that 7 is an eigenvalue of matrix A in Example 2, and find the corresponding eigenvectors.

SOLUTION The scalar 7 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 7\mathbf{x} \tag{1}$$

has a nontrivial solution. But (1) is equivalent to $A\mathbf{x} - 7\mathbf{x} = \mathbf{0}$, or

$$(A - 7I)\mathbf{x} = \mathbf{0} \tag{2}$$

To solve this homogeneous equation, form the matrix

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

The columns of $A - 7I$ are obviously linearly dependent, so (2) has nontrivial solutions. Thus 7 *is* an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The general solution has the form $x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 7$. ■

Example: How to find EigenVectors (V IMPT)

EigenSpace

The equivalence of equations (1) and (2) obviously holds for any λ in place of $\lambda = 7$. Thus λ is an eigenvalue of an $n \times n$ matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0} \tag{3}$$

has a nontrivial solution. The set of *all* solutions of (3) is just the null space of the matrix $A - \lambda I$. So this set is a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** of A corresponding to λ. The eigenspace consists of the zero vector and all the eigenvectors corresponding to λ.

Note:
The $\lambda, and\ x$ that are solutions of equation (3) are the eigenvalue and vector respectively.

The eigenvector x that is a solution of the above is tied to the λ eigenvalue.

The eigenvalue could be repeated and
there could be more than 1 independent eigenvector.
(See Example 4)

Example: EigenSpace

EigenSpace

Example 3 shows that for matrix A in Example 2, the eigenspace corresponding to $\lambda = 7$ consists of *all* multiples of (1, 1), which is the line through (1, 1) and the origin. From Example 2, you can check that the eigenspace corresponding to $\lambda = -4$ is the line through $(6, -5)$. These eigenspaces are shown in Fig. 2, along with eigenvectors (1, 1) and $(3/2, -5/4)$ and the geometric action of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ on each eigenspace.

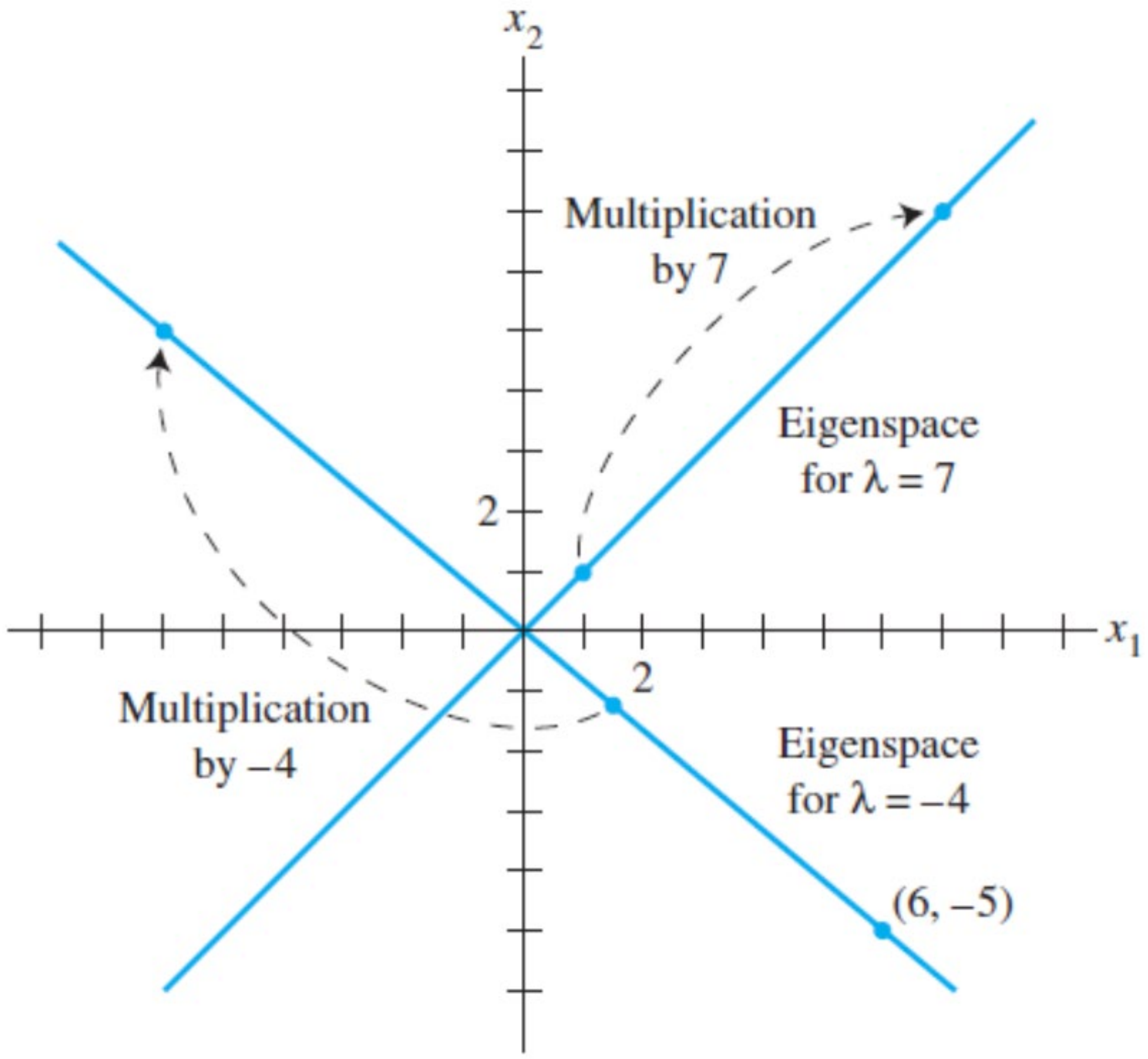


FIGURE 2 Eigenspaces for $\lambda = -4$ and $\lambda = 7$.

EigenSpace: Example two-dimensional subspace of R^3

Example:

In this example, there is **repeated eigenvalue**.

When this occur, it is possible to have more than 1 eigenvector for that eigenvalue.
I.e, the eigenspace's dimension is greater than 1.

EXAMPLE 4 Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. An eigenvalue of A is 2. Find a basis for the corresponding eigenspace.

SOLUTION Form

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

and row reduce the augmented matrix for $(A - 2I)\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Lay,4thEd, pg 269

EigenSpace: Example two-dimensional subspace of R^3

At this point, it is clear that 2 is indeed an eigenvalue of A because the equation $(A - 2I)\mathbf{x} = \mathbf{0}$ has free variables. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 \text{ and } x_3 \text{ free}$$

The eigenspace, shown in Fig. 3, is a two-dimensional subspace of $\mathbb{R}^3$. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

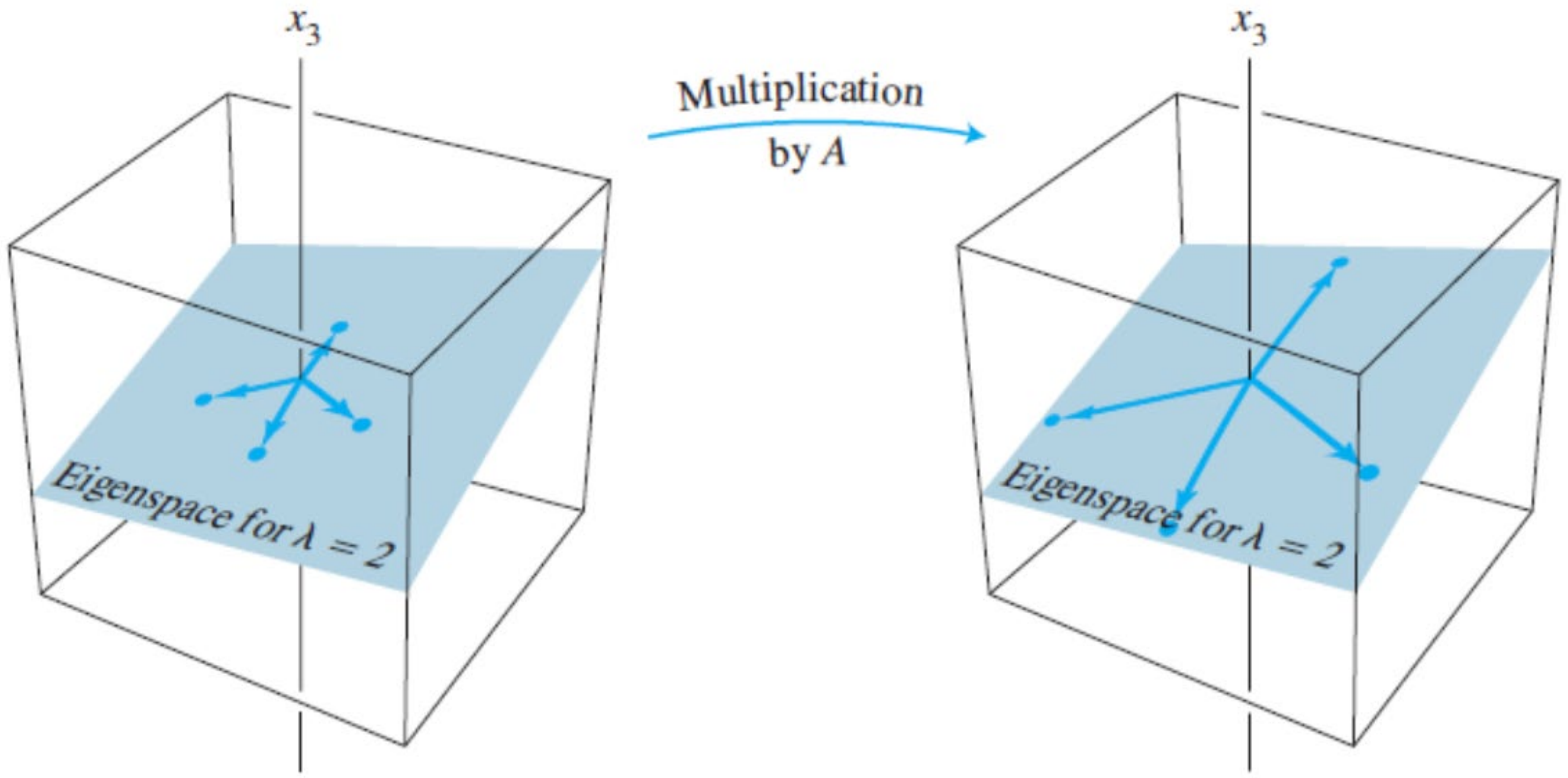


FIGURE 3 A acts as a dilation on the eigenspace.

Sanity check: Matlab(X)

```
% Lay 4th edition, pg 268, Example 4,
% a matrix with repeated roots = -2

A = [4 -1 6; 2 1 6; 2 -1 8]
[P,D] = eig(A)
A_tst = P*D*inv(P)
```

```
A =

     4    -1     6
     2     1     6
     2    -1     8
```

```
A_tst =

    4.0000   -1.0000    6.0000
    2.0000    1.0000    6.0000
    2.0000   -1.0000    8.0000
```

```
P =

   -0.5774   -0.6122    0.3205
   -0.5774   -0.7873   -0.9112
   -0.5774    0.0728   -0.2587


D =

    9.0000         0         0
         0    2.0000         0
         0         0    2.0000
```

Matlab's EIG function (X)

eig

Eigenvalues and eigenvectors

Syntax

```
e = eig(A)
[V,D] = eig(A)
[V,D,W] = eig(A)
```

Description

`e = eig(A)` returns a column vector containing the eigenvalues of square matrix `A`. example

`[V,D] = eig(A)` returns diagonal matrix `D` of eigenvalues and matrix `V` whose columns are the corresponding right eigenvectors, so that `A*V = V*D`. example

Warning:

1) D is the eigenvalue BUT they are typically not sorted.
2) V are columns of the right eigenvectors AND they (the columsn) are normalized to norm 1. Hence they may be different to Lay's example BUT their direction will be correct.

Matlab – sanity check (X)

Note: Here, $\lambda_2 = \lambda_3 = 2$. Hence, there are two eigenvectors corresponding to the eigenvalue of 2. The eigen space can be formed by any set of basis vectors that span it. Hence, the eigen space spanned by the basis on LHS is the same eigen space spanned by the 2nd and 3rd columns of matrix U.

MATLAB

```
A =

     4    -1     6
     2     1     6
     2    -1     8

>> [U,D] = eig(A)

U =

   -0.5774   -0.6122    0.3205
   -0.5774   -0.7873   -0.9112
   -0.5774    0.0728   -0.2587
```

Eigenvector corresponding to λ_1 (first column of U)

Eigenvector corresponding to λ_2 (second column of U)

Eigenvector corresponding to λ_3 (third column of U)

```
D =

    9.0000         0         0
         0    2.0000         0
         0         0    2.0000
```

λ_1 = 9.0000, λ_2 = 2.0000, λ_3 = 2.0000

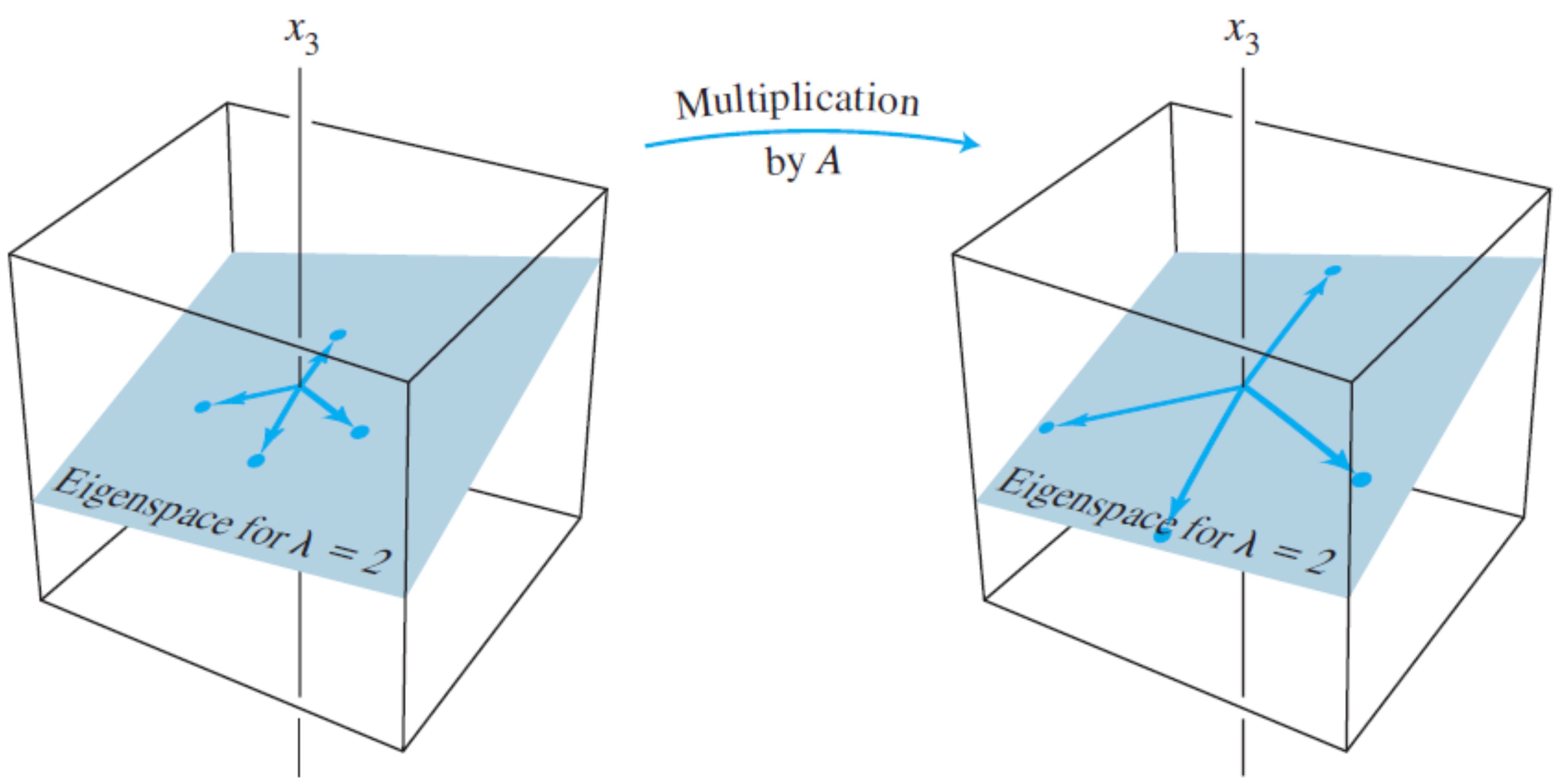


FIGURE 3 A acts as a dilation on the eigenspace.

Matlab provide solutions of eigenvectors such that the norm of the eigenvectors == 1.
When the matrix A is not symmetrical, the eigenvectors will not be orthogonal to each other.
In the special case when matrix is symmetric, eigen vectors will be orthogonal to each other.

Summary

To find the eigenvalue and vectors of a matrix, 2 steps:

a) First find the eigen value of the matrix.

- slide 8.1.2 shows how to find the eigen values.

b) Second, use Gaussian Elimination (row reduction) to find the solution of the homogenous equation

$$(A - \lambda I)x = 0$$

for each eigenvalue λ.

Example:

1) https://www.scss.tcd.ie/Rozenn.Dahyot/CS1BA1/SolutionEigen.pdf
2) https://lpsa.swarthmore.edu/MtrxVibe/EigMat/MatrixEigen.html

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.2**

Lecture : **Eigen and Singular Values**

Topic : **Characteristic Equation**

Concept : **How to find eigenvalue and vectors**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Rev: 3rd July 2020

How to find eigenValues ? (V IMPT)

Our next objective is to obtain a general procedure for finding eigenvalues and eigenvectors of an $n \times n$ matrix A. We will begin with the problem of finding the eigenvalues of A. Note first that the equation $A\mathbf{x} = \lambda\mathbf{x}$ can be rewritten as $A\mathbf{x} = \lambda I\mathbf{x}$, or equivalently, as

$$(\lambda I - A)\mathbf{x} = \mathbf{0}$$

For λ to be an eigenvalue of A this equation must have a nonzero solution for $\mathbf{x}$.

$$\begin{aligned} Ax &= \lambda x \\ Ax - \lambda x &= 0 \\ (A - \lambda I)x &= 0 \\ (\lambda I - A)x &= 0 \end{aligned}$$

For the above equation to be true, and x not a zero vector (condition for eigen vector),

then $(\lambda I - A)$ must be a singular matrix (not invertible).

See: invertible matrix theorem (corollary of (d))

How to find eigenValues ?

The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.
g. The equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.
h. The columns of A span $\mathbb{R}^n$.
i. The linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

Lay,4th, pg112 (Ch2.3)

The Invertible Matrix Theorem divides the set of all $n \times n$ matrices into two disjoint classes: the invertible (nonsingular) matrices, and the noninvertible (singular) matrices. Each statement in the theorem describes a property of every $n \times n$ invertible matrix. The *negation* of a statement in the theorem describes a property of every $n \times n$ singular matrix. For instance, an $n \times n$ singular matrix is *not* row equivalent to I_n, does *not* have n pivot positions, and has linearly *dependent* columns.

Determinant and Invertibility

THEOREM 4

A square matrix A is invertible if and only if $\det A \neq 0$.

Theorem 4 adds the statement "$\det A \neq 0$" to the Invertible Matrix Theorem. A useful corollary is that $\det A = 0$ when the columns of A are linearly dependent. Also, $\det A = 0$ when the *rows* of A are linearly dependent. (Rows of A are columns of A^T, and linearly dependent columns of A^T make A^T singular. When A^T is singular, so is A, by the Invertible Matrix Theorem.) In practice, linear dependence is obvious when two columns or two rows are the same or a column or a row is zero.

Lay, pg171 (Ch3.2)

Clever idea:
The determinant transforms the original problem of $(A - \lambda I)x = 0$,
an equation with two unknowns λ, x into a polynomial with only 1 unknown λ.
Allowing us to solve first for the eigen value λ, the roots of the polynomial.
The polynomial is called the "characteristic equation" of A.

Since $(A - \lambda I)$ is a singular matrix (NOT invertible), therefore

$$\det(A - \lambda I) = 0$$

Characteristic Eqn and Polynomial

EXAMPLE 1 Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

SOLUTION We must find all scalars λ such that the matrix equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a nontrivial solution. By the Invertible Matrix Theorem in Section 2.3, this problem is equivalent to finding all λ such that the matrix $A - \lambda I$ is *not* invertible, where

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

By Theorem 4 in Section 2.2, this matrix fails to be invertible precisely when its determinant is zero. So the eigenvalues of A are the solutions of the equation

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix} = 0$$

Determinant of 2x2 matrix

THEOREM 4

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

$$\det A = ad - bc$$

Theorem 4 says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Lay,4th,pg103,sec 2.2

Characteristic Eqn and Polynomial

Recall that

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

So

$$\begin{aligned} \boxed{\det(A - \lambda I)} &= (2-\lambda)(-6-\lambda) - (3)(3) \\ &= -12 + 6\lambda - 2\lambda + \lambda^2 - 9 \\ &\boxed{= \lambda^2 + 4\lambda - 21} \\ &= (\lambda - 3)(\lambda + 7) \end{aligned}$$

If $\det(A - \lambda I) = 0$, then $\lambda = 3$ or $\lambda = -7$. So the eigenvalues of A are 3 and -7. ■

Characteristic Eqn

Characteristic Polynomial

Equivalent statements regarding eigenvalue λ

Given $Ax = \lambda x$, the following theorem applies:

Theorem Given a square matrix A and a scalar λ, the following statements are equivalent:

- λ is an eigenvalue of A,
- $N(A - \lambda I) \neq \{\mathbf{0}\}$,
- the matrix $A - \lambda I$ is singular,
- $\det(A - \lambda I) = 0$.

Ref: https://textbooks.math.gatech.edu/ila/characteristic-polynomial.html

Example: EigenValues of Triangular Matrixes

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

PROOF For simplicity, consider the 3×3 case. If A is upper triangular, then $A - \lambda I$ has the form

$$A - \lambda I = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{bmatrix}$$

The scalar λ is an eigenvalue of A if and only if the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, that is, if and only if the equation has a free variable. Because of the zero entries in $A - \lambda I$, it is easy to see that $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has a free variable if and only if at least one of the entries on the diagonal of $A - \lambda I$ is zero. This happens if and only if λ equals one of the entries a_{11}, a_{22}, a_{33} in A. For the case in which A is lower triangular, see Exercise 28. ■

THEOREM 3.2 Determinant of a Triangular Matrix

If A is a triangular matrix of order n, then its determinant is the product of the entries on the main diagonal. That is,

$$\det(A) = |A| = a_{11}a_{22}a_{33} \cdot \cdot \cdot a_{nn}.$$

Fact 7. The determinant of a lower triangular matrix (or an upper triangular matrix) is the product of the diagonal entries. In particular, the determinant of a diagonal matrix is the product of the diagonal entries.

Proof: Khan's academy determinant of triangular matrix.

https://www.khanacademy.org/math/linear-algebra/matrix-transformations/determinant-depth/v/linear-algebra-upper-triangular-determinant

Example: Meaning of eigenvalue == 0 => not invertible matrix

EXAMPLE 5 Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. The eigenvalues of A are 3, 0, and 2. The eigenvalues of B are 4 and 1. ■

What does it mean for a matrix A to have an eigenvalue of 0, such as in Example 5? This happens if and only if the equation

$$A\mathbf{x} = 0\mathbf{x} \qquad (4)$$

has a nontrivial solution. But (4) is equivalent to $A\mathbf{x} = \mathbf{0}$, which has a nontrivial solution if and only if A is not invertible. Thus 0 *is an eigenvalue of* A *if and only if* A *is not invertible*. This fact will be added to the Invertible Matrix Theorem in Section 5.2.

THEOREM

The Invertible Matrix Theorem (continued)

Let A be an $n \times n$ matrix. Then A is invertible if and only if:

s. The number 0 is *not* an eigenvalue of A.

t. The determinant of A is *not* zero.

Example: eigenvalue and algebraic multiplicity

EXAMPLE 3

Find the characteristic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

SOLUTION Form $A - \lambda I$, and use Theorem 3(d):

$$\det(A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$
$$= (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

or

$$(\lambda-5)^2(\lambda-3)(\lambda-1) = 0$$

Expanding the product, we can also write

$$\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0$$

The eigenvalue 5 in Example 3 is said to have *multiplicity* 2 because $(\lambda - 5)$ occurs two times as a factor of the characteristic polynomial. In general, the (**algebraic**) **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Lay, 4th, pg276

EigenValues and Algebraic multiplicity

EXAMPLE 4 The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues and their multiplicities.

SOLUTION Factor the polynomial

$$\lambda^6 - 4\lambda^5 - 12\lambda^4 = \lambda^4(\lambda^2 - 4\lambda - 12) = \lambda^4(\lambda - 6)(\lambda + 2)$$

The eigenvalues are 0 (multiplicity 4), 6 (multiplicity 1), and -2 (multiplicity 1). ■

The **set of solutions** $(\lambda_1, \lambda_2, \dots, \lambda_{N_\lambda})$,that is, the **eigenvalues**, is called the **spectrum** of A. The charaacteristic polynomial can be factored as follows:

$$p(\lambda) = (\lambda - \lambda_1)^{n_1}(\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_{N_\lambda})^{n_{N_\lambda}} = 0.$$

The integer n_i is termed the **algebraic multiplicity** of eigenvalue λ_i .It is the number of times an eigenvalue appears as a root of the characteristic polynomial.

The algebraic multiplicities sum to N (the number of rows in A):

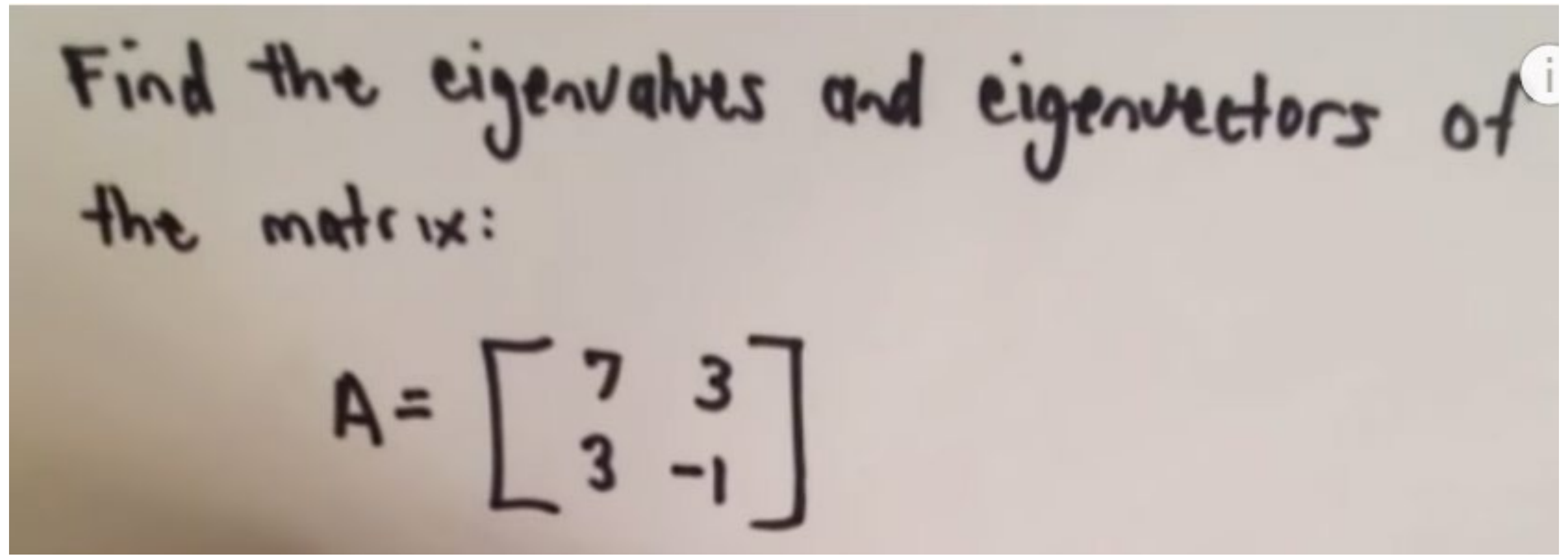
$$\sum_{i=1}^{N_\lambda} n_i = N.$$

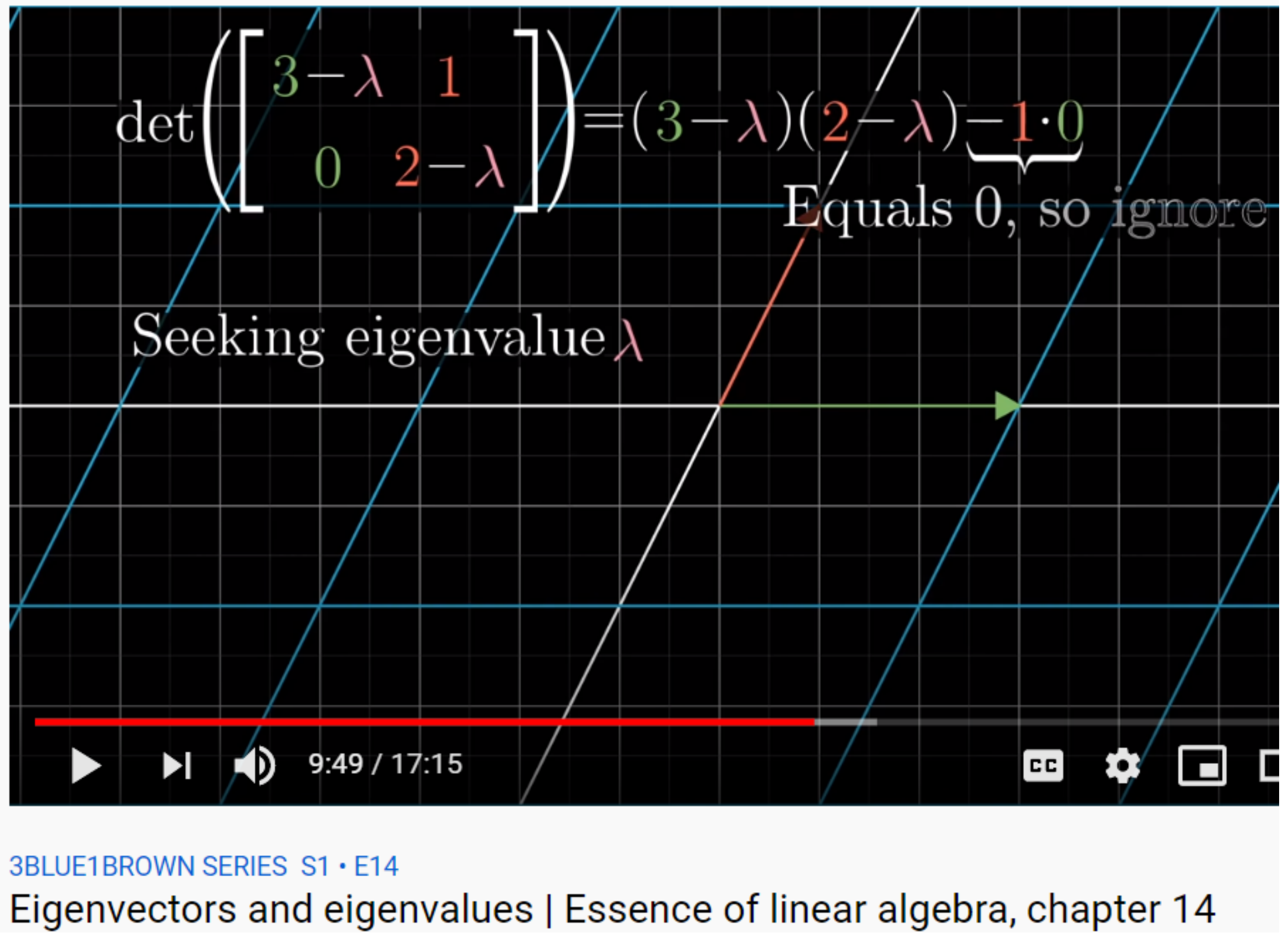
For each eigenvalue λ_i, there is a corresponding EigenSpace $E(\lambda_i)$

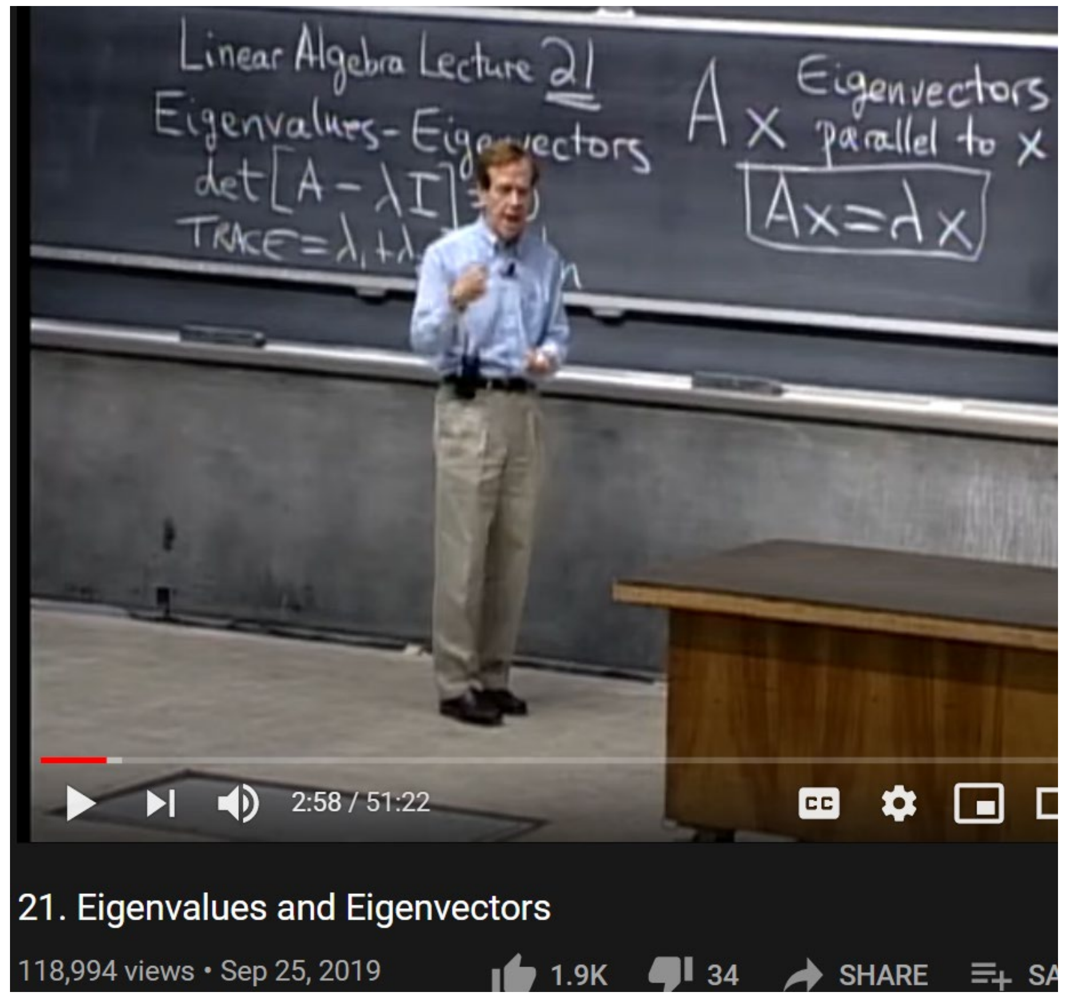
References online

1) PatrickJMT: https://www.youtube.com/watch?v=IdsV0RaC9jM

Work examples:







2) 3Blue1Brown: Ch14, https://www.youtube.com/watch?v=PFDu9oVAE-g

Understanding and perspective

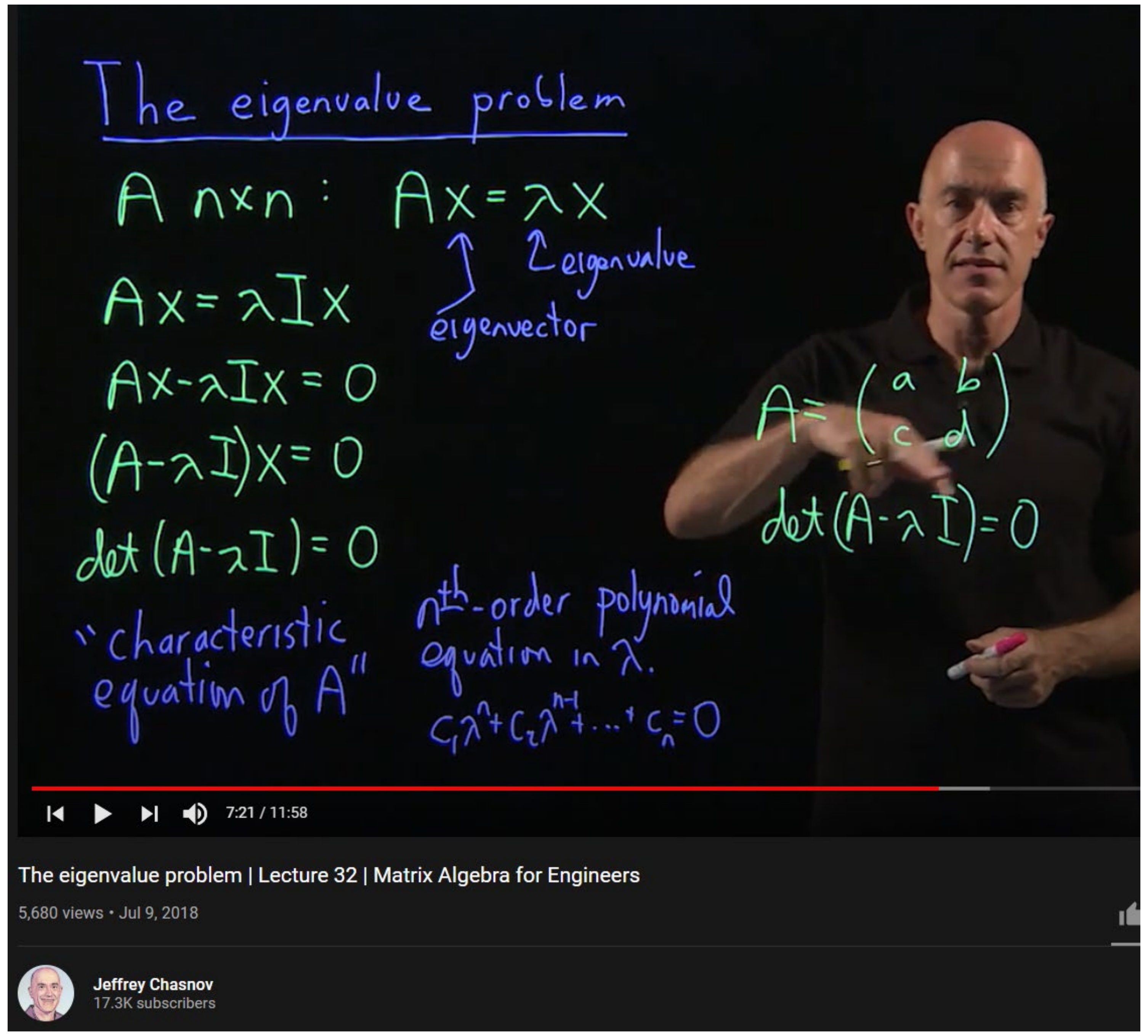
3) Strang introducing EigenValues & vectors

https://www.youtube.com/watch?v=cdZnhQjJu4I

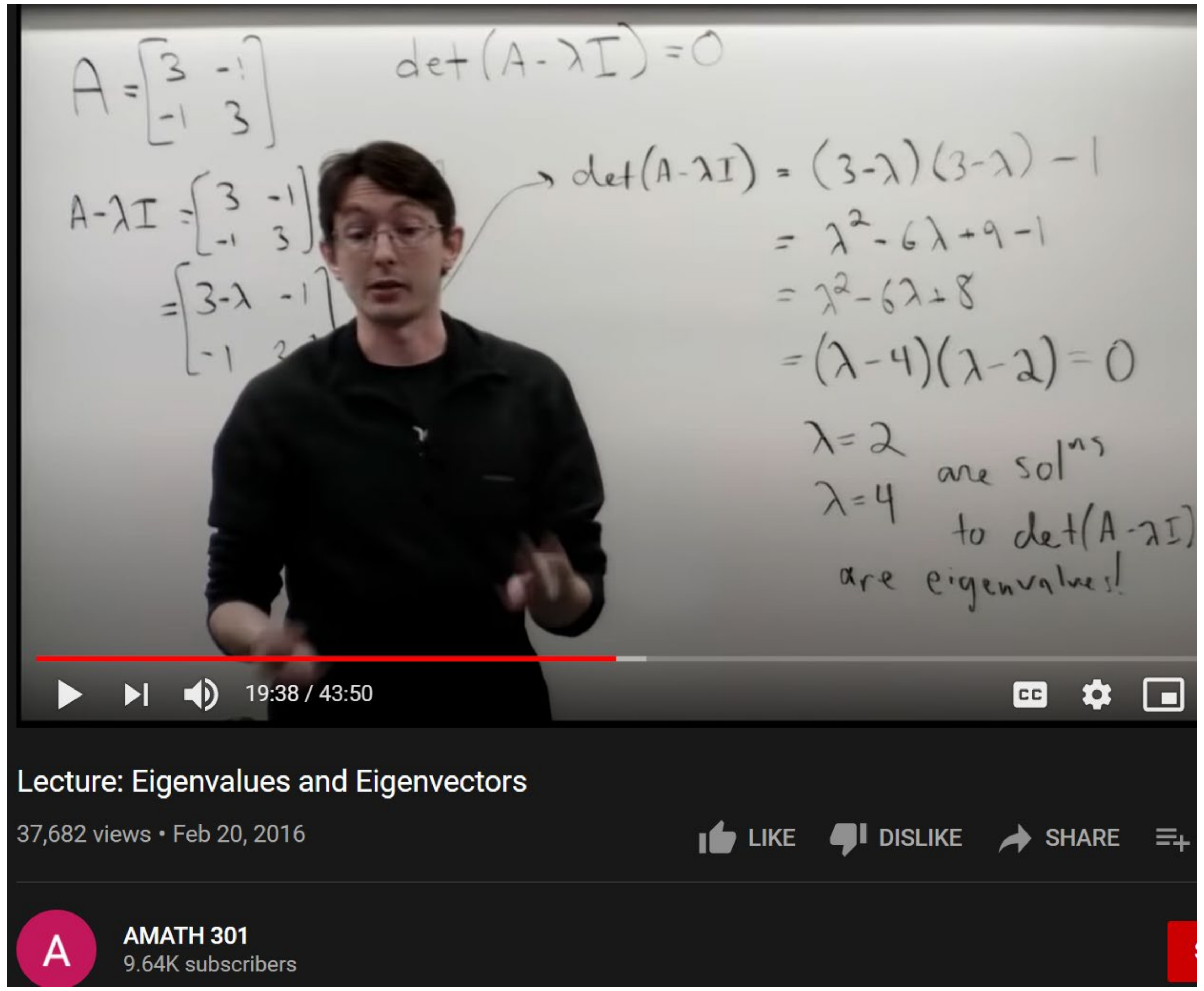
More examples

4. How to find eigenvalues/vectors (process):

a. Chasnov (L33): https://www.youtube.com/watch?v=29keVZGvqME&list=PLkZjai-2Jcxlg-Z1roB0pUwFU-P58tvOx&index=33



5. Steve Brunton's lecture for eigenvalues
https://www.youtube.com/watch?v=OELTJdaU8aA



$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.3**

Lecture : **Eigen and Singular Values**

Topic : **Similarity and Diagonalization**

Concept : **When can we represent $A = PDP^{-1}$**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Rev: 3rd July 2020

Similarity and Diagonalization

Given two nxn matrix A and P, and P is invertible (means it has n-independent column), then we can have a matrix B generated as follows:

$$P^{-1}AP = B$$
$$AP = PB$$
$$A = PBP^{-1}$$

then we say that A and B are similar matrixes,
and the transformation from A to $\mathrm{B} = P^{-1}AP$ is called
similarity transformation.

In the special case that B is a diagonal matrix, then we also say that A is diagonalizable!

Wiki: insight

Similar matrices represent the same linear map under two (possibly) different bases, with P being the change of basis matrix.[1][2]

See Lecture 8.1.5B

Similarity and Diagonalization

Similar Matrix is an important concept because Similar matrixes share certain characteristics:

In general, any property that is preserved by a similarity transformation is called a ***similarity invariant*** and is said to be ***invariant under similarity***. Table 1 lists the most important similarity invariants. The proofs of some of these are given as exercises.

Table 1 Similarity Invariants

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant.
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertible.
Rank	A and $P^{-1}AP$ have the same rank.
Nullity	A and $P^{-1}AP$ have the same nullity.
Trace	A and $P^{-1}AP$ have the same trace.
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial.
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues.
Eigenspace dimension	If λ is an eigenvalue of A (and hence of $P^{-1}AP$) then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

We like a matrix to be similar to a diagonal matrix

Why we like diagonal matrix?

Diagonal matrixes have nice properties:

1) Eigenvalues of diagonal matrixes are its diagonal element
2) Determinant == product of diagonal entries
3) Rank == number of non-zero entries in the diagonal
4) Multiplication: given A and diagonal matrix D (AD and DA):

- when we pre-multiply A by a diagonal matrix D, the rows of A are multiplied by the diagonal elements of D;
- when we post-multiply A by D, the columns of A are multiplied by the diagonal elements of D.

5) A diagonal matrix's inverse is reciprocal of diagonal elements
6) Product of diagonal matrixes are easy to compute.

Ref:
1) https://www.statlect.com/matrix-algebra/diagonal-matrix
2) http://www.robertosmathnotes.com/uploads/8/2/3/9/8239617/la10-3_diagonal_matrices.pdf

When A Diagonalizable?A must have n eigenvectors!

Special case of similarity, B is D (a diagonal matrix).

THEOREM 5

The Diagonalization Theorem

An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

Lay, pg 282, Ch 5.3

Ref:https://textbooks.math.gatech.edu/ila/similarity.html

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Proof: When is A Diagonalizable?

THEOREM 5

In other words, A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$. We call such a basis an **eigenvector basis** of $\mathbb{R}^n$.

PROOF First, observe that if P is any $n \times n$ matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, and if D is any diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, then

$$AP = A[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_n\,] = [\,A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n\,] \tag{1}$$

while

$$PD = P\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,] \tag{2}$$

Now suppose A is diagonalizable and $A = PDP^{-1}$. Then right-multiplying this relation by P, we have $AP = PD$. In this case, equations (1) and (2) imply that

$$[\,A\mathbf{v}_1 \quad A\mathbf{v}_2 \quad \cdots \quad A\mathbf{v}_n\,] = [\,\lambda_1\mathbf{v}_1 \quad \lambda_2\mathbf{v}_2 \quad \cdots \quad \lambda_n\mathbf{v}_n\,] \tag{3}$$

Equating columns, we find that

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1, \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2, \quad \ldots, \quad A\mathbf{v}_n = \lambda_n\mathbf{v}_n \tag{4}$$

Examine

$$AP = PD$$

For columns of P being eigenvectors of A, and D the eigenval of A

Practice Problems 2

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

Sol:

2. Compute $A\mathbf{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{v}_1$, and

$$A\mathbf{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \mathbf{v}_2$$

So, $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the eigenvalues 1 and 3, respectively. Thus

$$A = PDP^{-1}, \quad \text{where} \quad P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

```
% Example in Slide 8.1.3 (diagonalization)
A = [-3 12; -2 7];
[P,D] = eig(A);

P1 = [3 2; 1 1]
D1 = [ 1 0; 0 3]
P1*D1*inv(P1)
```

```
A =

    -3    12
    -2     7


P =

   -0.9487   -0.8944
   -0.3162   -0.4472


D =

    1.0000         0
         0    3.0000
```

```
P1 =

     3     2
     1     1


D1 =

     1     0
     0     3


ans =

   -3.0000   12.0000
   -2.0000    7.0000
```

Recap: Steps to diagonalize a Matrix

Given a matrix A size NxN, to diagonalize it to D, perform:

1) Find the eigenvalues of A.
2) For each eigenvalue, find the eigenvectors of corresponding λ_i
3) If there are N independent eigenvectors v_i, then the matrix A can be represented as:

$$\begin{aligned} AP &= PD \\ A &= PDP^{-1} \\ P^{-1}AP &= D \end{aligned}$$

Where D = diagonal matrix with eigenvalues λ_i
And P is a matrix with columns that are corresponding eigenvectors v_i.

When is A diagonalizable? Sufficient condition:
If A has n Distinct EigenValues -> Diagonalizable

THEOREM 6 Lay, 4thEd, pg 284, Ch 5.3

An $n \times n$ matrix with n distinct eigenvalues is diagonalizable.

PROOF Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be eigenvectors corresponding to the n distinct eigenvalues of a matrix A. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent, by Theorem 2 in Section 5.1. Hence A is diagonalizable, by Theorem 5. ∎

It is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable. The 3×3 matrix in Example 3 is diagonalizable even though it has only two distinct eigenvalues.

Theorem & proof: distinct eigenvalues means distinct eigenvectors

THEOREM 2 If $\mathbf{v}_1, \dots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly independent.

Lay 4th, pg 270, Ch 5.1

PROOF Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ is linearly dependent. Since $\mathbf{v}_1$ is nonzero, Theorem 7 in Section 1.7 says that one of the vectors in the set is a linear combination of the preceding vectors. Let p be the least index such that $\mathbf{v}_{p+1}$ is a linear combination of the preceding (linearly independent) vectors. Then there exist scalars $c_1, \dots, c_p$ such that

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{v}_{p+1} \tag{5}$$

Multiplying both sides of (5) by A and using the fact that $A\mathbf{v}_k = \lambda_k\mathbf{v}_k$ for each k, we obtain

$$\begin{aligned} c_1A\mathbf{v}_1 + \cdots + c_pA\mathbf{v}_p &= A\mathbf{v}_{p+1} \\ c_1\lambda_1\mathbf{v}_1 + \cdots + c_p\lambda_p\mathbf{v}_p &= \lambda_{p+1}\mathbf{v}_{p+1} \end{aligned} \tag{6}$$

Multiplying both sides of (5) by λ_{p+1} and subtracting the result from (6), we have

$$c_1(\lambda_1 - \lambda_{p+1})\mathbf{v}_1 + \cdots + c_p(\lambda_p - \lambda_{p+1})\mathbf{v}_p = \mathbf{0} \tag{7}$$

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is linearly independent, the weights in (7) are all zero. But none of the factors $\lambda_i - \lambda_{p+1}$ are zero, because the eigenvalues are distinct. Hence $c_i = 0$ for $i = 1, \dots, p$. But then (5) says that $\mathbf{v}_{p+1} = \mathbf{0}$, which is impossible. Hence $\{\mathbf{v}_1, \dots, \mathbf{v}_r\}$ cannot be linearly dependent and therefore must be linearly independent. ■

Recap: linear independence

Lay 4th Ed, ch1.7) Revision linear independence

THEOREM 7 **Characterization of Linearly Dependent Sets**

An indexed set $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$.

A set of two vectors $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if at least one of the vectors is a multiple of the other. The set is linearly independent if and only if neither of the vectors is a multiple of the other.

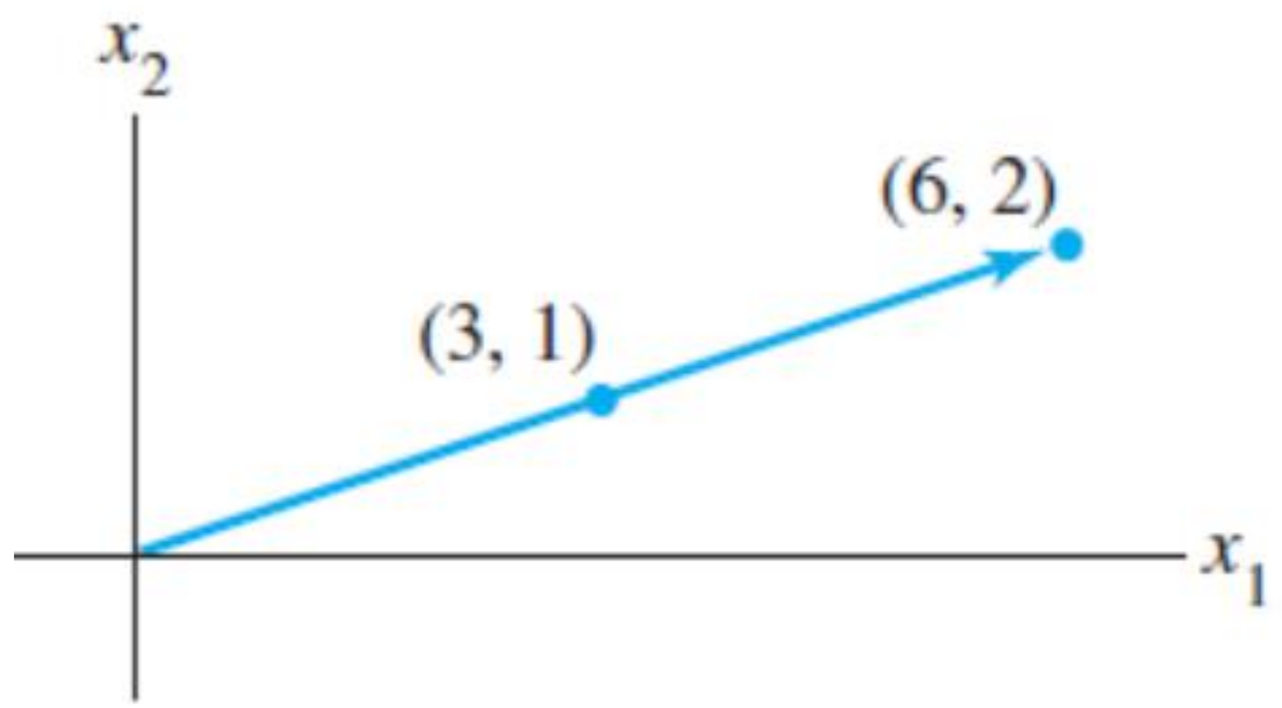
Linear Independence of Matrix Columns

Suppose that we begin with a matrix $A = [\,\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n\,]$ instead of a set of vectors. The matrix equation $A\mathbf{x} = \mathbf{0}$ can be written as

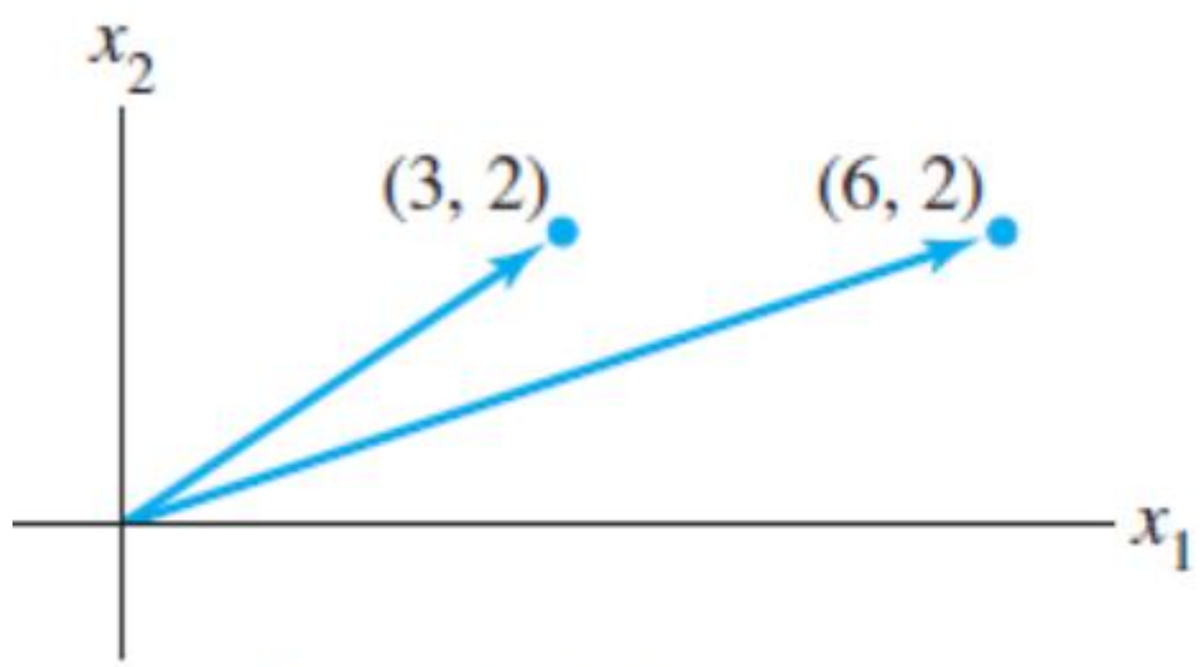
$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{0}$$

Each linear dependence relation among the columns of A corresponds to a nontrivial solution of $A\mathbf{x} = \mathbf{0}$. Thus we have the following important fact.

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution. (3)



Linearly dependent



Linearly independent

Example: Distinct EigenValues -> Diagonalizable

EXAMPLE 5 Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

SOLUTION This is easy! Since the matrix is triangular, its eigenvalues are obviously 5, 0, and -2. Since A is a 3×3 matrix with three distinct eigenvalues, A is diagonalizable. ■

See Slide 8.1.2 (pg 8) To see slide "EigenValues of Triangular Matrixes"

THEOREM 1 The eigenvalues of a triangular matrix are the entries on its main diagonal.

Distinct eigenvalue is a sufficient BUT
not necessary condition to have linearly independent eigenvectors.

Example 3 (later) shows that eigenvalues are repeated, but it is still diagonalizable.
And Example 4 shows counter-example.

When is a matrix with repeated roots diagonalizable? Introducing algebraic and geometrix multiplicity.

Algebraic
multiplicity = multiplicity of eigenvalues λ_k

Geometric
multiplicity = Dimension of eigenspace corresponding to eigenvalue λ_k

Ref: https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

Introducing terminology: Algebraic and Geometric Multiplicity

Eigenspaces

Let λ be an eigenvalue of A. Recall that the eigenvectors of A for λ are the nonzero vectors in the nullspace of $A - \lambda I$. We call the nullspace $A - \lambda I$ the **eigenspace** of A for λ denoted by $\mathcal{E}_A(\lambda)$. In other words, $\mathcal{E}_A(\lambda)$ consists of all the eigenvectors of A for λ and the zero vector.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$. Note that -1 is an eigenvalue of A. Then $A - (-1)I_2 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$. The nullspace of this matrix is spanned by the single vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Hence, $\mathcal{E}_A(-1)$ is the span of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Sanity check:

```
% example in sli
A = [1 2; 1 0]
[P,D] = eig(A)
```

```
A =

     1     2
     1     0


P =

    0.8944   -0.7071
    0.4472    0.7071


D =

     2     0
     0    -1
```

Ref:https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

Geometric Multiplicity (is the dimension of Eigen Space)
Algebraic Multiplicity (is the number of repeated roots)

Algebraic multiplicity vs geometric multiplicity

The **geometric multiplicity** of an eigenvalue λ of A is the dimension of $\mathcal{E}_A(\lambda)$.

In the example above, the geometric multiplicity of -1 is 1 as the eigenspace is spanned by one nonzero vector.

In general, determining the geometric multiplicity of an eigenvalue requires no new technique because one is simply looking for the dimension of the nullspace of $A - \lambda I$.

The **algebraic multiplicity** of an eigenvalue λ of A is the number of times λ appears as a root of p_A. For the example above, one can check that -1 appears only once as a root. Let us now look at an example in which an eigenvalue has multiplicity higher than 1.

In mathematics, the **dimension** of a vector space *V* is the cardinality (i.e. the number of vectors) of a basis of *V* over its base field.[1]

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$.

```
% Example:
% repeated roots lambda=1
% BUT eigenspace ==1
A = [1 2; 0 1]
[P,D] = eig(A)
```

```
A =

     1     2
     0     1


P =

    1.0000   -1.0000
         0    0.0000


D =

     1     0
     0     1
```

Ref:https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

When is a matrix with repeated roots diagonalizable? Introducing algebraic and geometrix multiplicity.

In general, **the algebraic multiplicity and geometric multiplicity of an eigenvalue can differ. However, the geometric multiplicity can never exceed the algebraic multiplicity**.

It is a fact that summing up the algebraic multiplicities of all the eigenvalues of an $n \times n$ matrix A gives exactly n. **If for every eigenvalue of A, the geometric multiplicity equals the algebraic multiplicity, then A is said to be *diagonalizable***.

See also (Theorem 7) in Lay, 4thEd, pg 285, Ch 5.3

Ref: https://people.math.carleton.ca/~kcheung/math/notes/MATH1107/wk10/10_algebraic_and_geometric_multiplicities.html

Example 3: Diagonalizable A with repeated eigenValue

EXAMPLE 3 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

That is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

SOLUTION There are four steps to implement the description in Theorem 5.

Step 1. ***Find the eigenvalues of A.*** As mentioned in Section 5.2, the mechanics of this step are appropriate for a computer when the matrix is larger than 2×2. To avoid unnecessary distractions, the text will usually supply information needed for this step. In the present case, the characteristic equation turns out to involve a cubic polynomial that can be factored:

$$\begin{aligned} 0 = \det(A - \lambda I) &= -\lambda^3 - 3\lambda^2 + 4 \\ &= -(\lambda - 1)(\lambda + 2)^2 \end{aligned}$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$.

Note: In this example, the eigenvalues are NOT distinct ($\lambda = -2$), i.e repeated, But this matrix is diagonalizable.

Algebraic multiplicity of eigenvalue = -2 is 2
AND
Geometric multiplicity of eigenvalue = -2 is ALSO 2.

HENCE there is a complete set of linearly independent eigenvectors for A, allowing A to be diagonalizable.

Lay4th, pg 283, Ch 5.2

Example 3: Diagonalizable A with repeated eigenValue

Step 2. ***Find three linearly independent eigenvectors of A.*** *Three* vectors are needed because A is a 3×3 matrix. This is the critical step. If it fails, then Theorem 5 says that A cannot be diagonalized. The method in Section 5.1 produces a basis for each eigenspace:

$$\text{Basis for } \lambda = 1: \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

You can check that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

Step 3. ***Construct P from the vectors in step 2.*** The order of the vectors is unimportant. Using the order chosen in step 2, form

$$P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Sanity check:

```
% Example: slide 5.1.3, pg 17
A = [1 3 3; -3 -5 -3; 3 3 1]
[P,D] = eig(A)
```

```
A =

     1     3     3
    -3    -5    -3
     3     3     1


P =

   -0.5774   -0.7876    0.4206
    0.5774    0.2074   -0.8164
   -0.5774    0.5802    0.3957


D =

    1.0000         0         0
         0   -2.0000         0
         0         0   -2.0000
```

```
P23 = P(:,2:3);
v2  = [-1 1 0]'
c2 = pinv(P23)*v2
v2_est = P23*c2
```

```
c2 =

    0.7121
   -1.0440


v2_est =

   -1.0000
    1.0000
   -0.0000
```

Note that matlab produces col 2,3 of P that does not resemble v_2 *and* v_3.
But you can check that v_2 *and* v_3 can be formed by appropriate linear combinations of col2,3, of P!

Example 3: Diagonalizable A with repeated eigenValue

Step 4. Construct D from the corresponding eigenvalues. In this step, it is essential that the order of the eigenvalues matches the order chosen for the columns of P. Use the eigenvalue $\lambda = -2$ twice, once for each of the eigenvectors corresponding to $\lambda = -2$:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

It is a good idea to check that P and D really work. To avoid computing P^{-1}, simply verify that $AP = PD$. This is equivalent to $A = PDP^{-1}$ when P is invertible. (However, be sure that P is invertible!) Compute

$$AP = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ -1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \qquad \blacksquare$$

Example 4: NOT Diagonalizable A with repeated eigenValue

EXAMPLE 4 Diagonalize the following matrix, if possible.

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$$

SOLUTION The characteristic equation of A turns out to be exactly the same as that in Example 3:

$$0 = \det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$$

The eigenvalues are $\lambda = 1$ and $\lambda = -2$. However, it is easy to verify that each eigenspace is only one-dimensional:

$$\text{Basis for } \lambda = 1: \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis for } \lambda = -2: \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

There are no other eigenvalues, and every eigenvector of A is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. Hence it is impossible to construct a basis of $\mathbb{R}^3$ using eigenvectors of A. By Theorem 5, A is *not* diagonalizable. ■

Note: In this example, the eigenvalues are NOT distinct ($\lambda = -2$), i.e repeated,
But this matrix is NOT diagonalizable.

Algebraic multiplicity of eigenvalue = -2 is 2
BUT
Geometric multiplicity of eigenvalue = -2 is ONLY 1.

Hence incomplete basis of eigenvectors for A =>
A is NOT diagonalizable.

Example 4: NOT Diagonalizable A with repeated eigenValue

It is NOT possible to diagonalize as A does
NOT have a full set of independent eigenvectors.
Sanity check: what does Matlab produce?

```
A =

     2     4     3
    -4    -6    -3
     3     3     1

>> [P,D] = eig(A)

P =

   0.5774 + 0.0000i   0.7071 + 0.0000i   0.7071 - 0.0000i
  -0.5774 + 0.0000i  -0.7071 + 0.0000i  -0.7071 + 0.0000i
   0.5774 + 0.0000i   0.0000 - 0.0000i   0.0000 + 0.0000i


D =

   1.0000 + 0.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i  -2.0000 + 0.0000i   0.0000 + 0.0000i
   0.0000 + 0.0000i   0.0000 + 0.0000i  -2.0000 - 0.0000i
```

Note that columns of P are
the eigen vectors.
Column 2 == Column 3.

Compare Column 2 to v_2 .
We see Direction is the same,
BUT scaled differently.
Matlab gives eigenVectors
with norm == 1

References (optional)

Similar Matrixes
A) Trefor Bazett: Similar matrices have similar properties
Link: https://www.youtube.com/watch?v=jNtiENbAcFM

B) MIT Strang: "Similar Matrixes"
https://www.youtube.com/watch?v=LKMGo8G7-vk

Diagonalization:
A) MIT Strang: "Diagonalizing a matrix"
https://www.youtube.com/watch?v=U8R54zOTVLw

B) MIT Strang: "Diagonalization and Powers of A"
https://www.youtube.com/watch?v=13r9QY6cmjc
Example: Time == 22:10 (triangular matrix and eigenvalues)
Time == 24:00 (algebraic multiplicity==2, geometric multiplicity=1) -> not diagonalizable.

Prof. Strang shows similar matrices A and B have the same eigenvalues in the first 3 minutes and works out some examples of similar matrices later!

CX1104: **Linear Algebra for Computing**

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

Chap. No : **8.1.4**

Lecture : **Eigen and Singular Values**

Topic : **Powers of A**

Concept : **Efficiently compute** A^K

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Rev: 3rd July 2020

Power of A: Example 1

In many cases, the eigenvalue–eigenvector information contained within a matrix A can be displayed in a useful factorization of the form $A = PDP^{-1}$ where D is a diagonal matrix. In this section, the factorization enables us to compute A^k quickly for large values of k, a fundamental idea in several applications of linear algebra. Later, in Sections 5.6 and 5.7, the factorization will be used to analyze (and *decouple*) dynamical systems.

The following example illustrates that powers of a diagonal matrix are easy to compute.

EXAMPLE 1 If $D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$, then $D^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}$

and

$$D^3 = DD^2 = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix} = \begin{bmatrix} 5^3 & 0 \\ 0 & 3^3 \end{bmatrix}$$

In general,

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix} \quad \text{for } k \geq 1$$

Lay, pg 281

Example 2: Finding A^k from $A = PDP^{-1}$

EXAMPLE 2 Let $A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^k, given that $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$$

SOLUTION The standard formula for the inverse of a 2×2 matrix yields

$$P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Then, by associativity of matrix multiplication,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Example 2: Finding A^k from $A = PDP^{-1}$

Again,

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{I}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

In general, for $k \geq 1$,

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 3^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \cdot 5^k - 3^k & 5^k - 3^k \\ 2 \cdot 3^k - 2 \cdot 5^k & 2 \cdot 3^k - 5^k \end{bmatrix} \quad \blacksquare$$

A square matrix A is said to be **diagonalizable** if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

Impt: when A is diagonalizable to $A = PDP^{-1}, then$

$$A^k = PD^kP^{-1}$$

Practice Problems 1

PRACTICE PROBLEMS

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

SOLUTIONS TO PRACTICE PROBLEMS

1. $\det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$. The eigenvalues are 2 and 1, and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Next, form

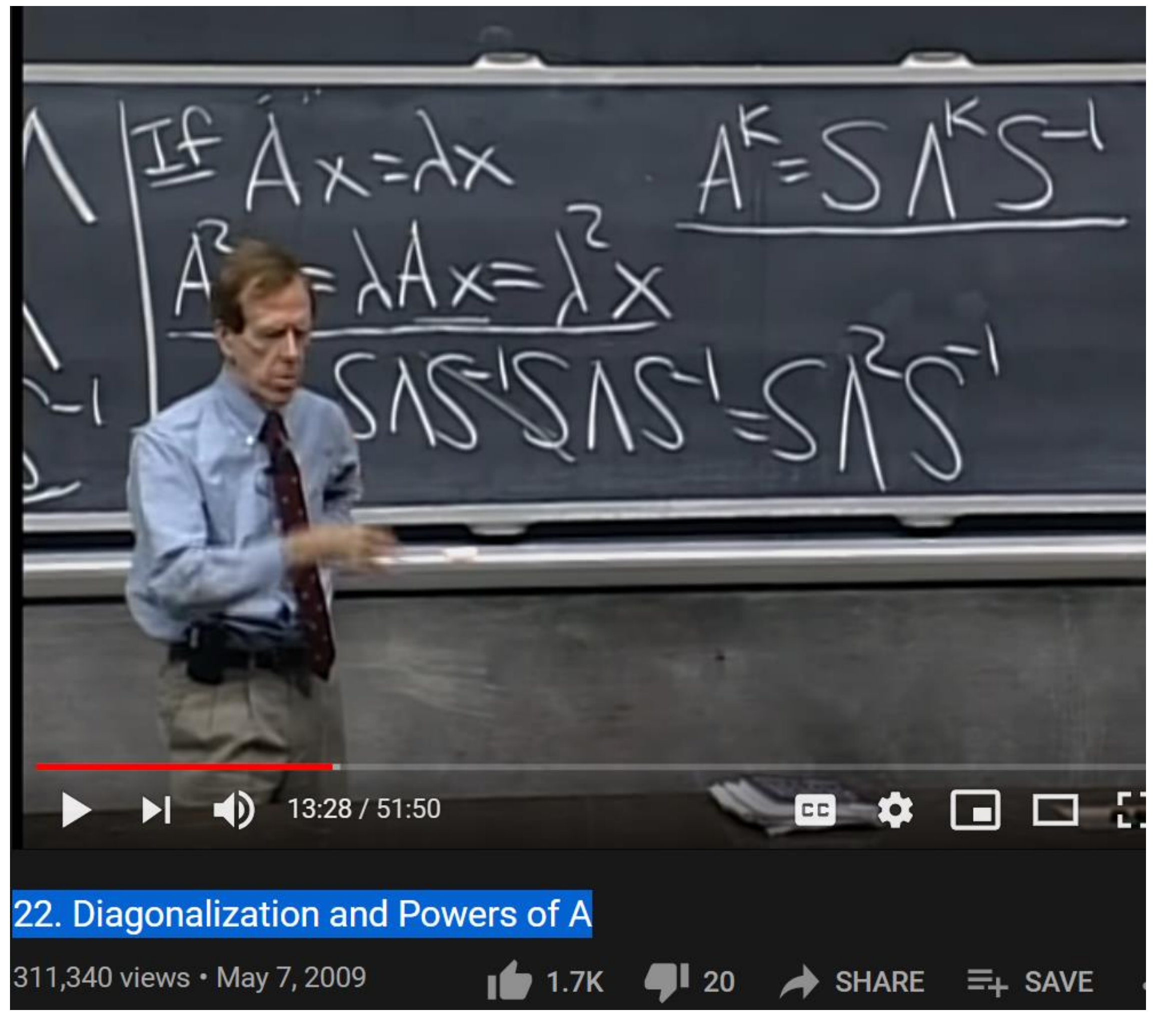
$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Since $A = PDP^{-1}$,

$$\begin{aligned} A^8 = PD^8P^{-1} &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix} \end{aligned}$$

References (optional)

1) MIT Strang : Lect 22. Diagonalization and Powers of A
https://www.youtube.com/watch?v=13r9QY6cmjc



$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}}_{b}$$

CX1104: **Linear Algebra for Computing**

Chap. No : **8.1.5**

Lecture : **Eigen and Singular Values**

Topic : **Dynamical System**

Concept : **x[k+1] = Ax[k]**

Instructor: **A/P Chng Eng Siong**

TAs: **Zhang Su, Vishal Choudhari**

Rev: 3rd July 2020

Trajectory of Dynamical System

The equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ determines an infinite collection of equations.
Beginning with an initial vector $\mathbf{x}_0$, we have

$$\begin{aligned} \mathbf{x}_1 &= A\mathbf{x}_0 \\ \mathbf{x}_2 &= A\mathbf{x}_1 \\ \mathbf{x}_3 &= A\mathbf{x}_2 \\ &\vdots \end{aligned}$$

The set $\{\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots\}$ is called a trajectory of the system.
Note that $\mathbf{x}_k = A^k\mathbf{x}_0$.

Eigenvalues and eigenvectors provide the key to understanding the long-term behavior, or *evolution*, of a dynamical system described by a difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

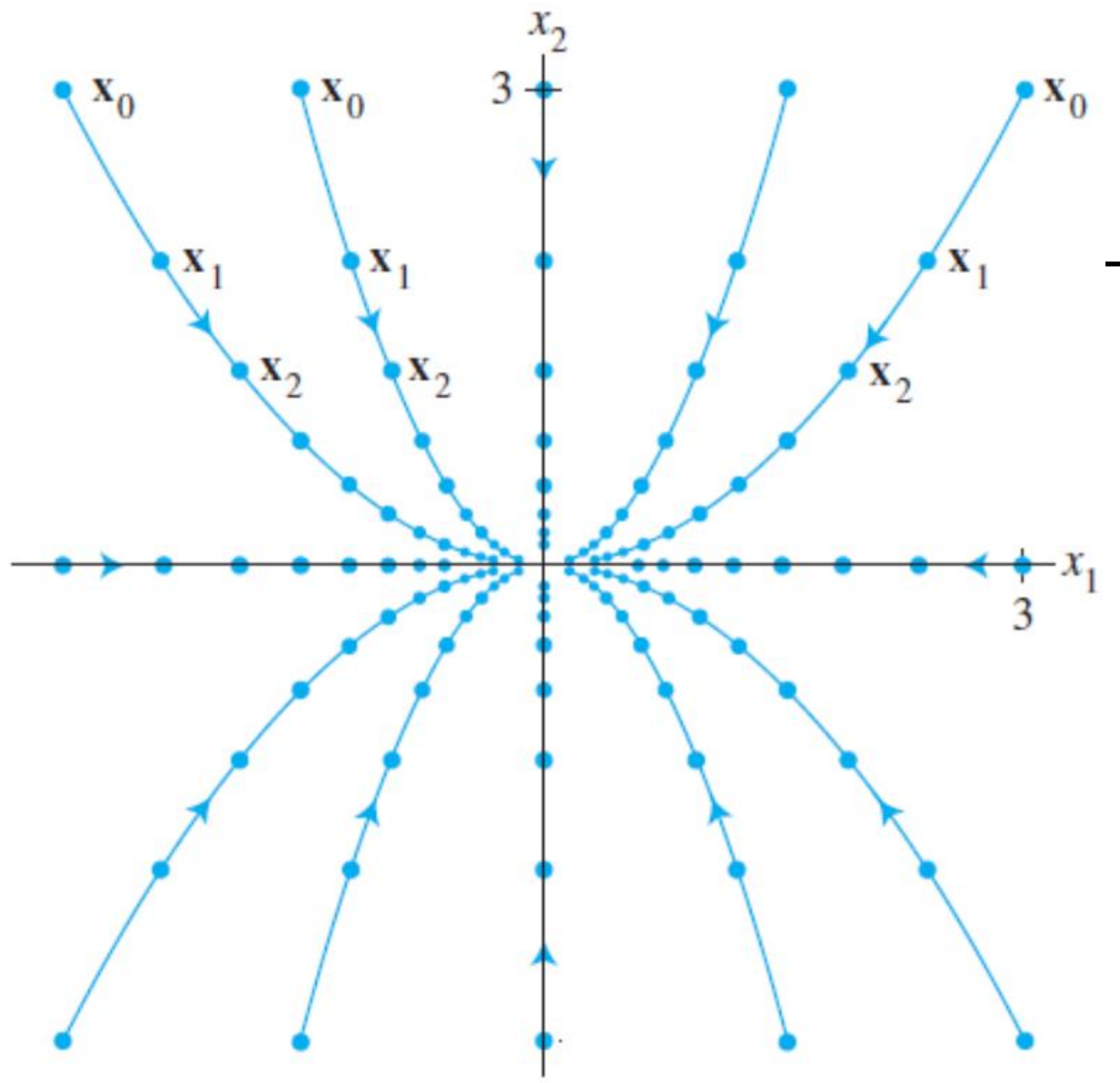


FIGURE 1 The origin as an attractor.

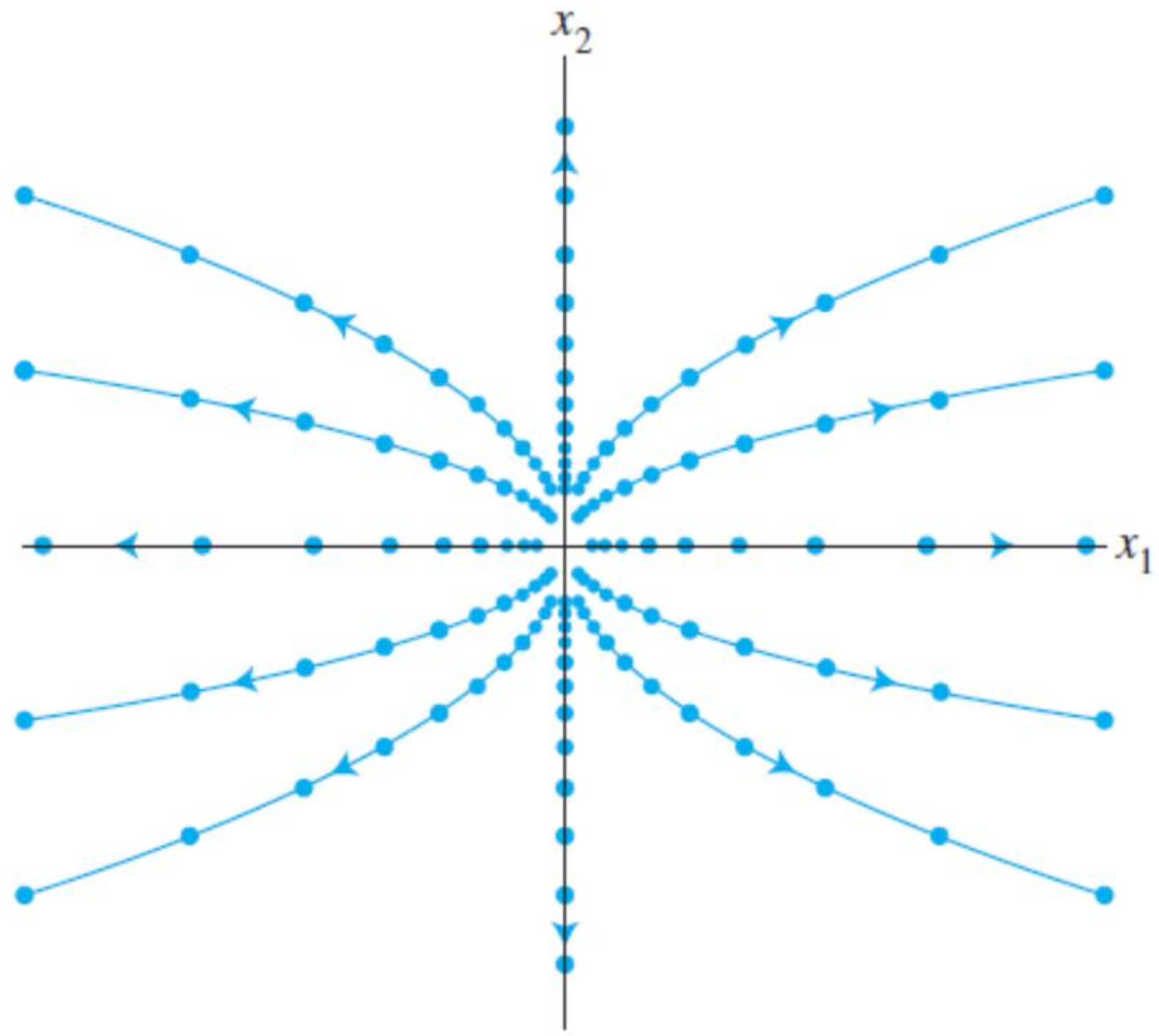


FIGURE 2 The origin as a repeller.

Example 2: when A is a diagonal matrix

Graphical Description of Solutions

When A is 2×2, algebraic calculations can be supplemented by a geometric description of a system's evolution. We can view the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$ as a description of what happens to an initial point $\mathbf{x}_0$ in $\mathbb{R}^2$ as it is transformed repeatedly by the mapping $\mathbf{x} \mapsto A\mathbf{x}$. The graph of $\mathbf{x}_0, \mathbf{x}_1, \ldots$ is called a **trajectory** of the dynamical system.

EXAMPLE 2 Plot several trajectories of the dynamical system $\mathbf{x}_{k+1} = A\mathbf{x}_k$, when

$$A = \begin{bmatrix} .80 & 0 \\ 0 & .64 \end{bmatrix}$$

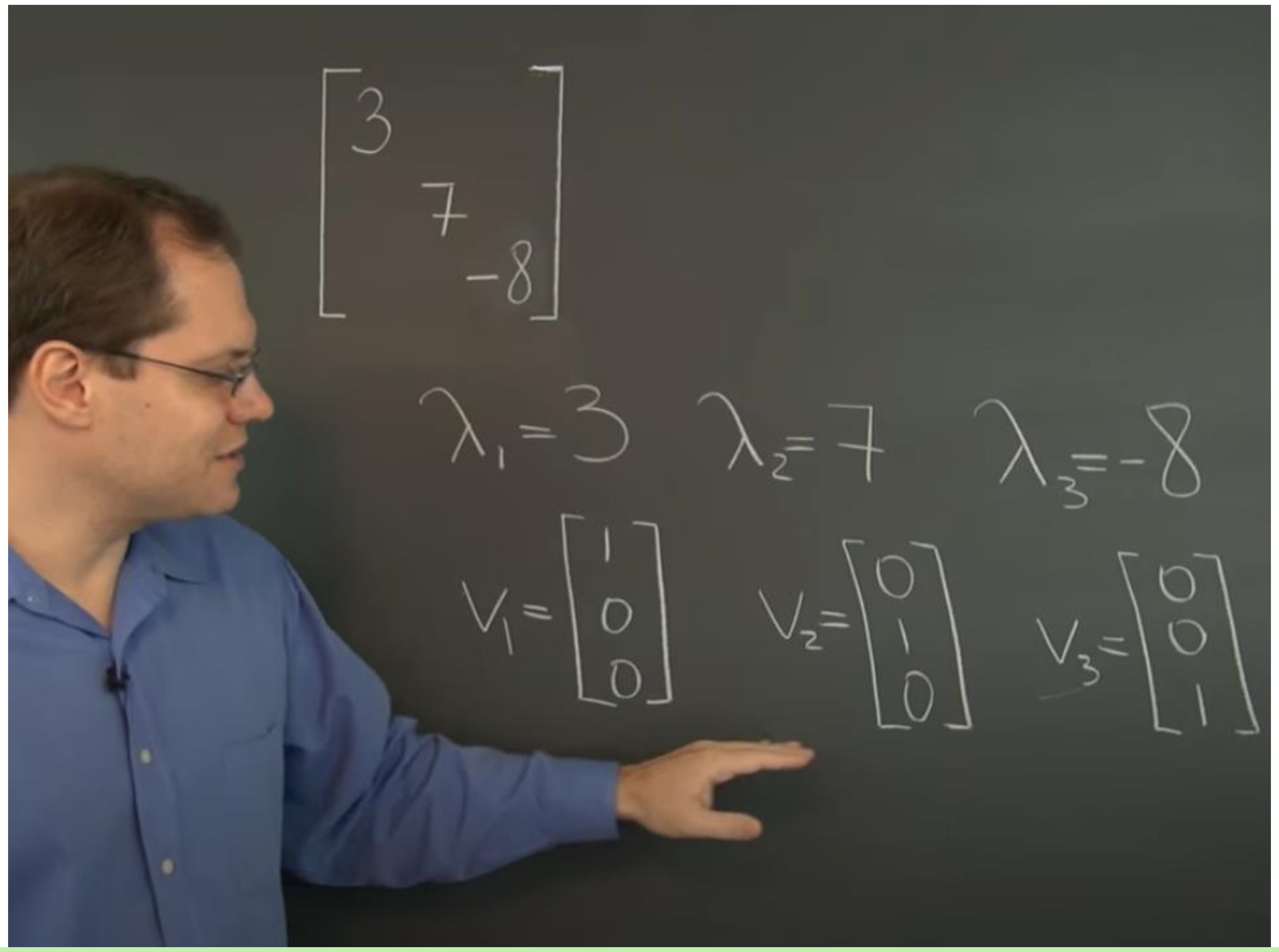
SOLUTION The eigenvalues of A are .8 and .64, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then

$$\mathbf{x}_k = c_1(.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Of course, $\mathbf{x}_k$ tends to $\mathbf{0}$ because $(.8)^k$ and $(.64)^k$ both approach 0 as $k \to \infty$. But *the way* $\mathbf{x}_k$ goes toward $\mathbf{0}$ is interesting. Figure 1 (on page 304) shows the first few terms of several trajectories that begin at points on the boundary of the box with corners at $(\pm 3, \pm 3)$. The points on each trajectory are connected by a thin curve, to make the trajectory easier to see. ■

Lay (4th), p303

https://www.youtube.com/watch?v=2mPl3qKMFL4



Why the standard basis is the eigen vector of a diagonal matrix is easily seen when you pluck these std basis as x into Ax to observe the output.

Why

If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then

$$\mathbf{x}_k = c_1(.8)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(.64)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Proof: Lay 4th, pg 278, example 5

If $x_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then $x_0 = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_0 \quad = c_1 v_1 + c_2 v_2$

$x_1 \quad = \quad Ax_0 \quad = A(c_1 v_1 + c_2 v_2)$

$\quad = \quad c_1\lambda_1 v_1 + c_2\lambda_2 v_2$

$x_2 \quad = \quad Ax_1 \quad = A(c_1\lambda_1 v_1 + c_2\lambda_2 v_2)$

$\quad = \quad c_1(\lambda_1)^2\, v_1 + c_2(\lambda_2)^2 v_2$

Therefore, x_k

$x_k \quad = c_1(\lambda_1)^k\, v_1 + c_2(\lambda_2)^k v_2$

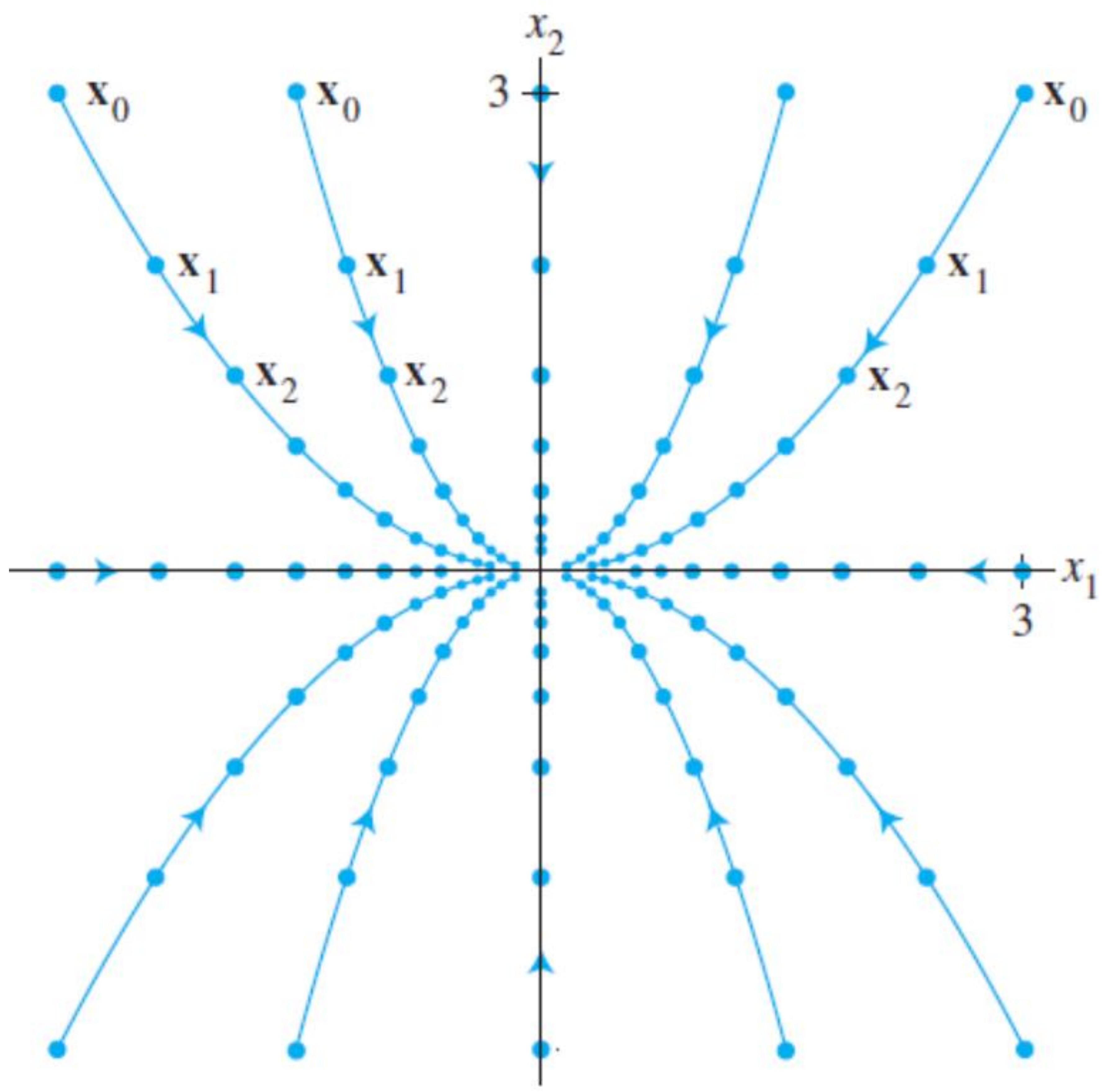


FIGURE 1 The origin as an attractor.

In Example 2, the origin is called an **attractor** of the dynamical system because all trajectories tend toward **0**. This occurs whenever both eigenvalues are less than 1 in magnitude. The direction of greatest attraction is along the line through **0** and the eigenvector $\mathbf{v}_2$ for the eigenvalue of smaller magnitude.

Example 3: repeller at origin

In the next example, both eigenvalues of A are larger than 1 in magnitude, and $\mathbf{0}$ is called a **repeller** of the dynamical system. All solutions of $\mathbf{x}_{k+1} = A\mathbf{x}_k$ except the (constant) zero solution are unbounded and tend away from the origin.[2]

EXAMPLE 3 Plot several typical solutions of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$, where

$$A = \begin{bmatrix} 1.44 & 0 \\ 0 & 1.2 \end{bmatrix}$$

SOLUTION The eigenvalues of A are 1.44 and 1.2. If $\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$\mathbf{x}_k = c_1(1.44)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(1.2)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Both terms grow in size, but the first term grows faster. So the direction of greatest repulsion is the line through $\mathbf{0}$ and the eigenvector for the eigenvalue of larger magnitude. Figure 2 shows several trajectories that begin at points quite close to $\mathbf{0}$. ■

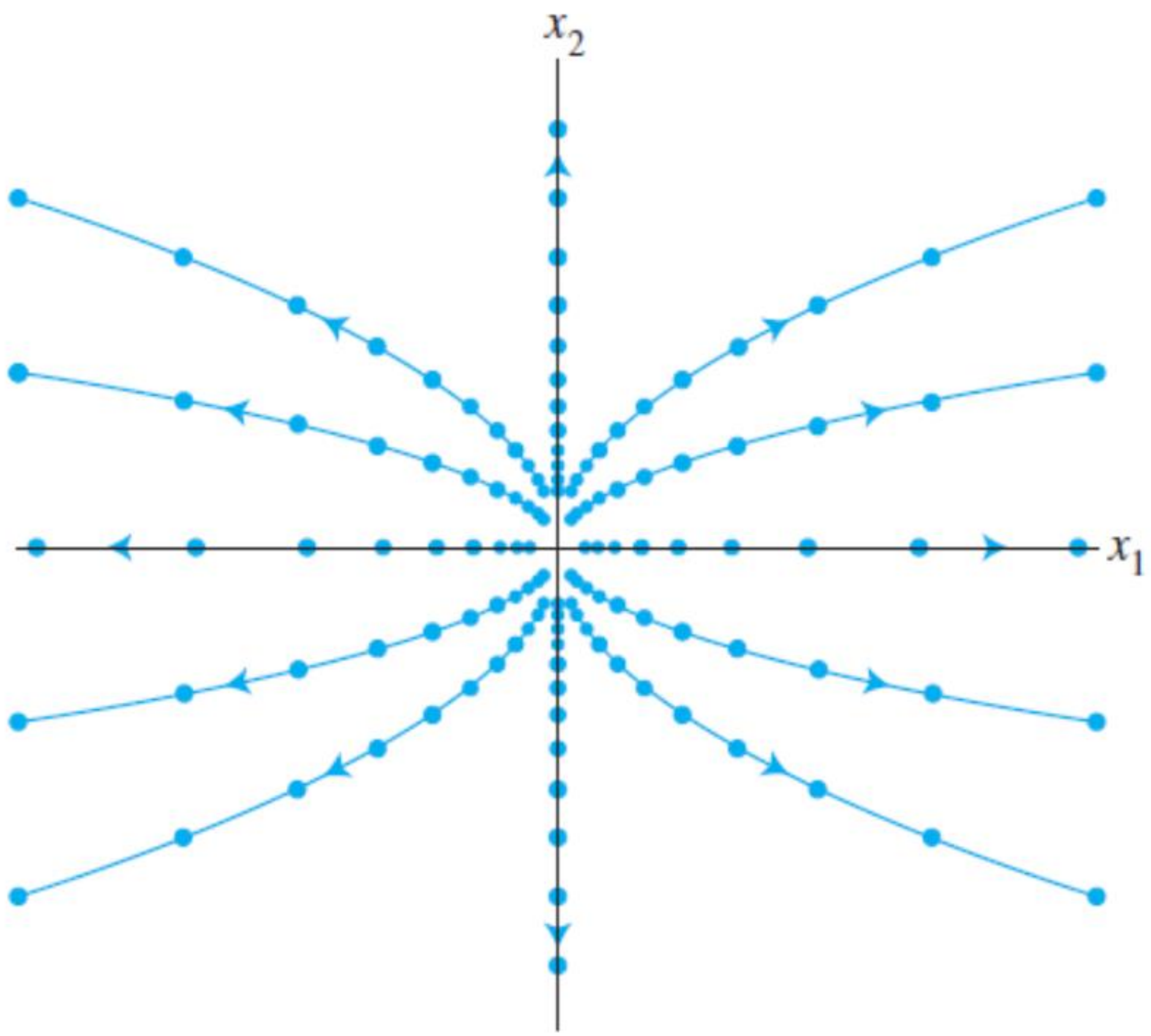


FIGURE 2 The origin as a repeller.

Lay (4th), p305

Example 4: saddle point

EXAMPLE 4 Plot several typical solutions of the equation $\mathbf{y}_{k+1} = D\mathbf{y}_k$, where

$$D = \begin{bmatrix} 2.0 & 0 \\ 0 & 0.5 \end{bmatrix}$$

(We write D and $\mathbf{y}$ here instead of A and $\mathbf{x}$ because this example will be used later.) Show that a solution $\{\mathbf{y}_k\}$ is unbounded if its initial point is not on the x_2-axis.

SOLUTION The eigenvalues of D are 2 and .5. If $\mathbf{y}_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, then

$$\mathbf{y}_k = c_1 2^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(.5)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{8}$$

If $\mathbf{y}_0$ is on the x_2-axis, then $c_1 = 0$ and $\mathbf{y}_k \to \mathbf{0}$ as $k \to \infty$. But if $\mathbf{y}_0$ is not on the x_2-axis, then the first term in the sum for $\mathbf{y}_k$ becomes arbitrarily large, and so $\{\mathbf{y}_k\}$ is unbounded. Figure 3 shows ten trajectories that begin near or on the x_2-axis. ■

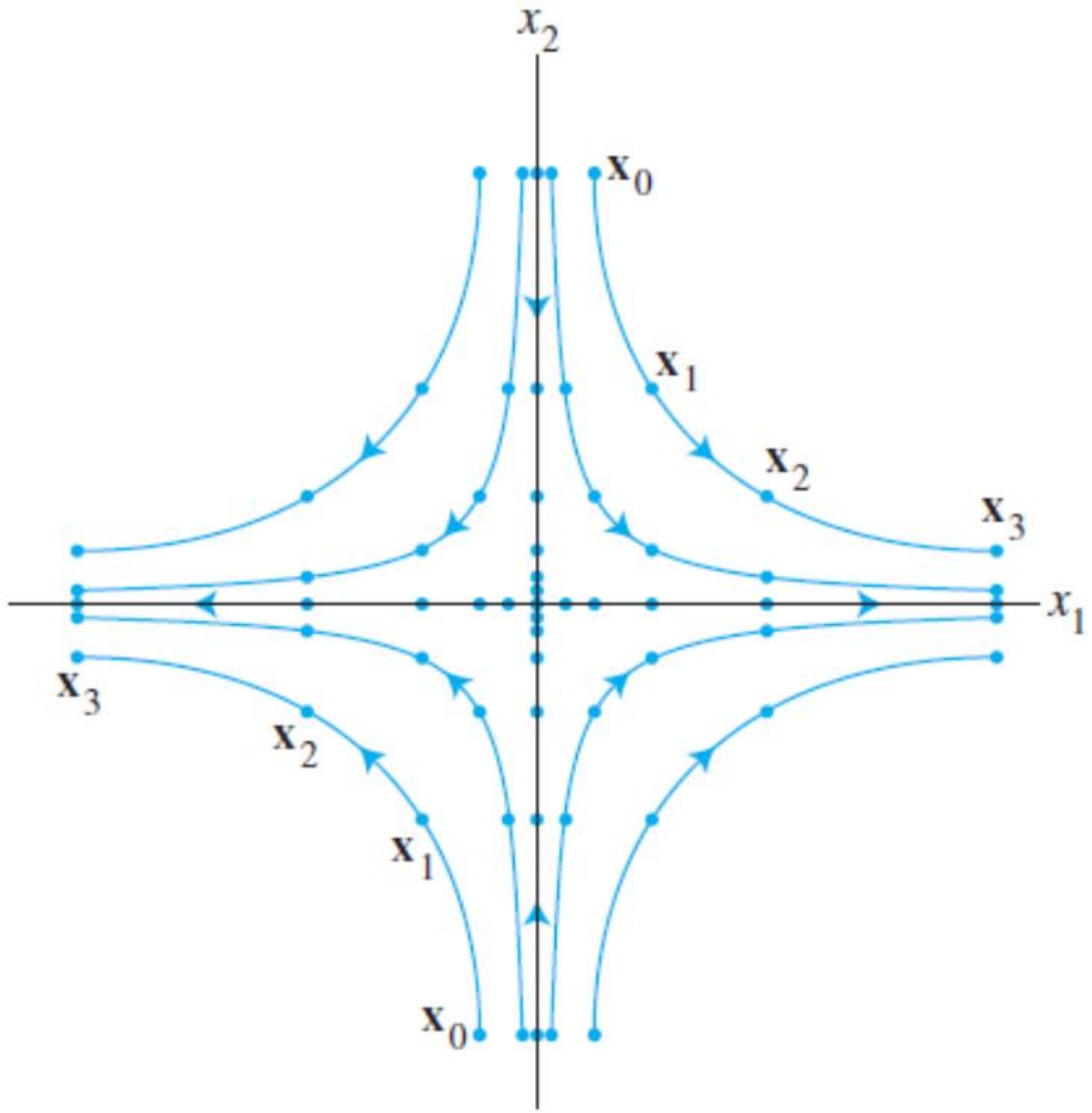


FIGURE 3 The origin as a saddle point.

Lay Example: Dynamical System

EXAMPLE 5 Let $A = \begin{bmatrix} .95 & .03 \\ .05 & .97 \end{bmatrix}$. Analyze the long-term behavior of the dynamical system defined by $\mathbf{x}_{k+1} = A\mathbf{x}_k$ $(k = 0, 1, 2, \dots)$, with $\mathbf{x}_0 = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$.

SOLUTION The first step is to find the eigenvalues of A and a basis for each eigenspace. The characteristic equation for A is

$$\begin{aligned} 0 &= \det \begin{bmatrix} .95 - \lambda & .03 \\ .05 & .97 - \lambda \end{bmatrix} = (.95 - \lambda)(.97 - \lambda) - (.03)(.05) \\ &= \lambda^2 - 1.92\lambda + .92 \end{aligned}$$

By the quadratic formula

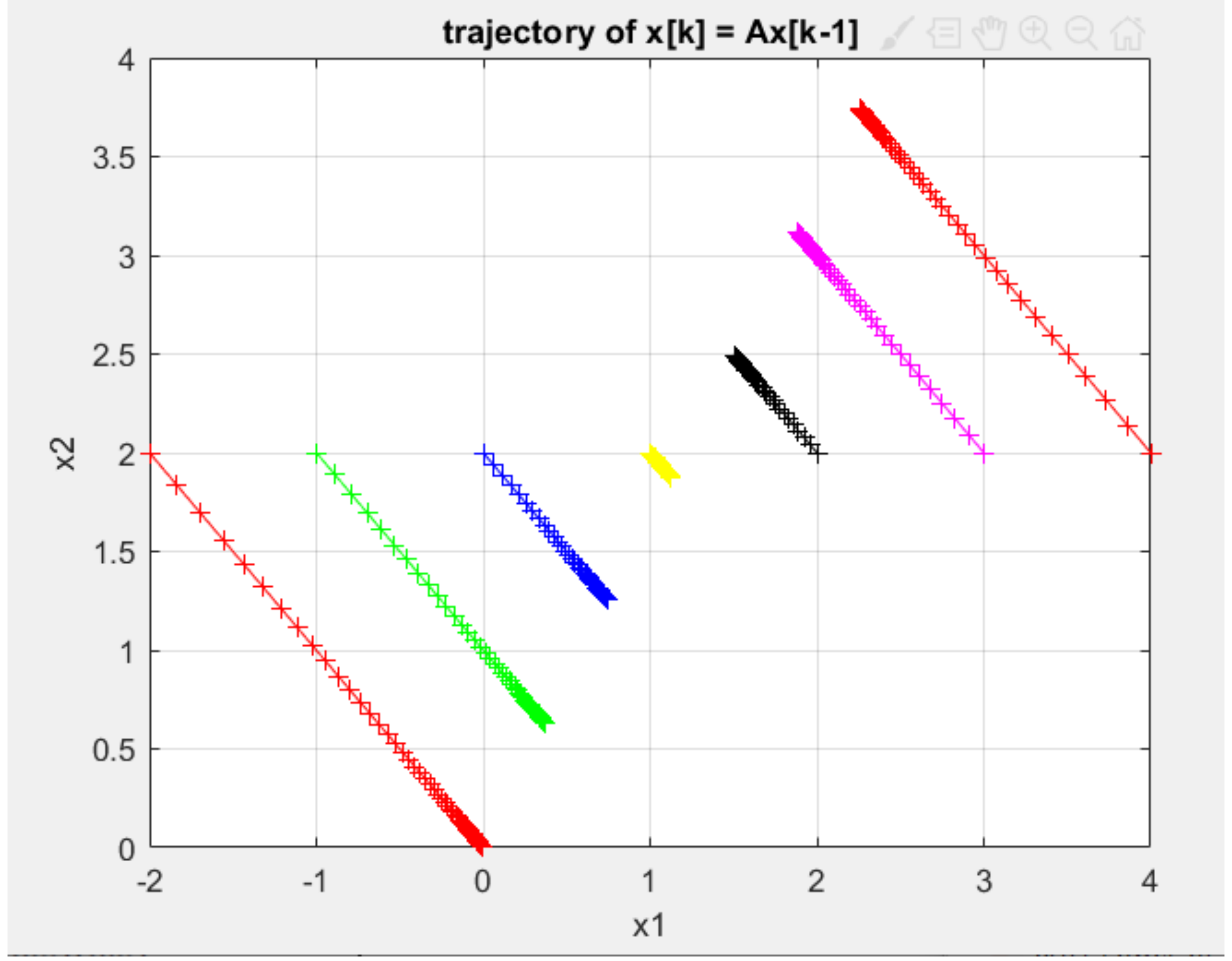
$$\begin{aligned} \lambda &= \frac{1.92 \pm \sqrt{(1.92)^2 - 4(.92)}}{2} = \frac{1.92 \pm \sqrt{.0064}}{2} \\ &= \frac{1.92 \pm .08}{2} = 1 \quad \text{or} \quad .92 \end{aligned}$$

It is readily checked that eigenvectors corresponding to $\lambda = 1$ and $\lambda = .92$ are multiples of

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

respectively.

Lay4thEdition: sec5.2,pg278




```
A  = [0.95 0.03; 0.05 0.97];
N  = 10000

close all;
figure(idxFig);
figID =1;
idxFig = 0;
for  (y = 2:-1:2)
    for (x = -2:1:4)
     x0 = [x y];
     bruteForce_computeAx(A,x0,N,figID,idxFig);
     idxFig=idxFig+1
    end
end
```

```
function bruteForce_computeAx(A,x0,N, figID, idxFig)
colormapStr = 'rgbykm';
x  = x0(:);
hist_x = [];
% brute force way to compute x[k] = Ax[k-1]
for k=0:N
   hist_x = [hist_x; x'];
   x = A*x;
end
tt = mod(idxFig,length(colormapStr))+1;
colorStr = [colormapStr(tt),'-+'];
plot(hist_x(:,1),hist_x(:,2),colorStr)
xlabel('x1'); ylabel('x2');
title('trajectory of x[k] = Ax[k-1]');
grid on; hold on;
end
```

Lay Example: Dynamical System

The next step is to write the given $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. This can be done because $\{\mathbf{v}_1, \mathbf{v}_2\}$ is obviously a basis for $\mathbb{R}^2$. (Why?) So there exist weights c_1 and c_2 such that

$$\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \tag{3}$$

In fact,

$$\begin{aligned}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]^{-1}\mathbf{x}_0 = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}^{-1}\begin{bmatrix} .60 \\ .40 \end{bmatrix} \\ &= \frac{1}{-8}\begin{bmatrix} -1 & -1 \\ -5 & 3 \end{bmatrix}\begin{bmatrix} .60 \\ .40 \end{bmatrix} = \begin{bmatrix} .125 \\ .225 \end{bmatrix}\end{aligned} \tag{4}$$

Very important concept!!!
We decompose x into
the eigen basis

Because $\mathbf{v}_1$ and $\mathbf{v}_2$ in (3) are eigenvectors of A, with $A\mathbf{v}_1 = \mathbf{v}_1$ and $A\mathbf{v}_2 = .92\mathbf{v}_2$, we easily compute each $\mathbf{x}_k$:

$$\begin{aligned}\mathbf{x}_1 = A\mathbf{x}_0 &= c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 && \text{Using linearity of } \mathbf{x} \mapsto A\mathbf{x} \\ &= c_1\mathbf{v}_1 + c_2(.92)\mathbf{v}_2 && \mathbf{v}_1 \text{ and } \mathbf{v}_2 \text{ are eigenvectors.} \\ \mathbf{x}_2 = A\mathbf{x}_1 &= c_1A\mathbf{v}_1 + c_2(.92)A\mathbf{v}_2 \\ &= c_1\mathbf{v}_1 + c_2(.92)^2\mathbf{v}_2\end{aligned}$$

and so on. In general,

$$\mathbf{x}_k = c_1\mathbf{v}_1 + c_2(.92)^k\mathbf{v}_2 \quad (k = 0, 1, 2, \dots)$$

Using c_1 and c_2 from (4),

$$\mathbf{x}_k = .125\begin{bmatrix} 3 \\ 5 \end{bmatrix} + .225(.92)^k\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad (k = 0, 1, 2, \dots) \tag{5}$$

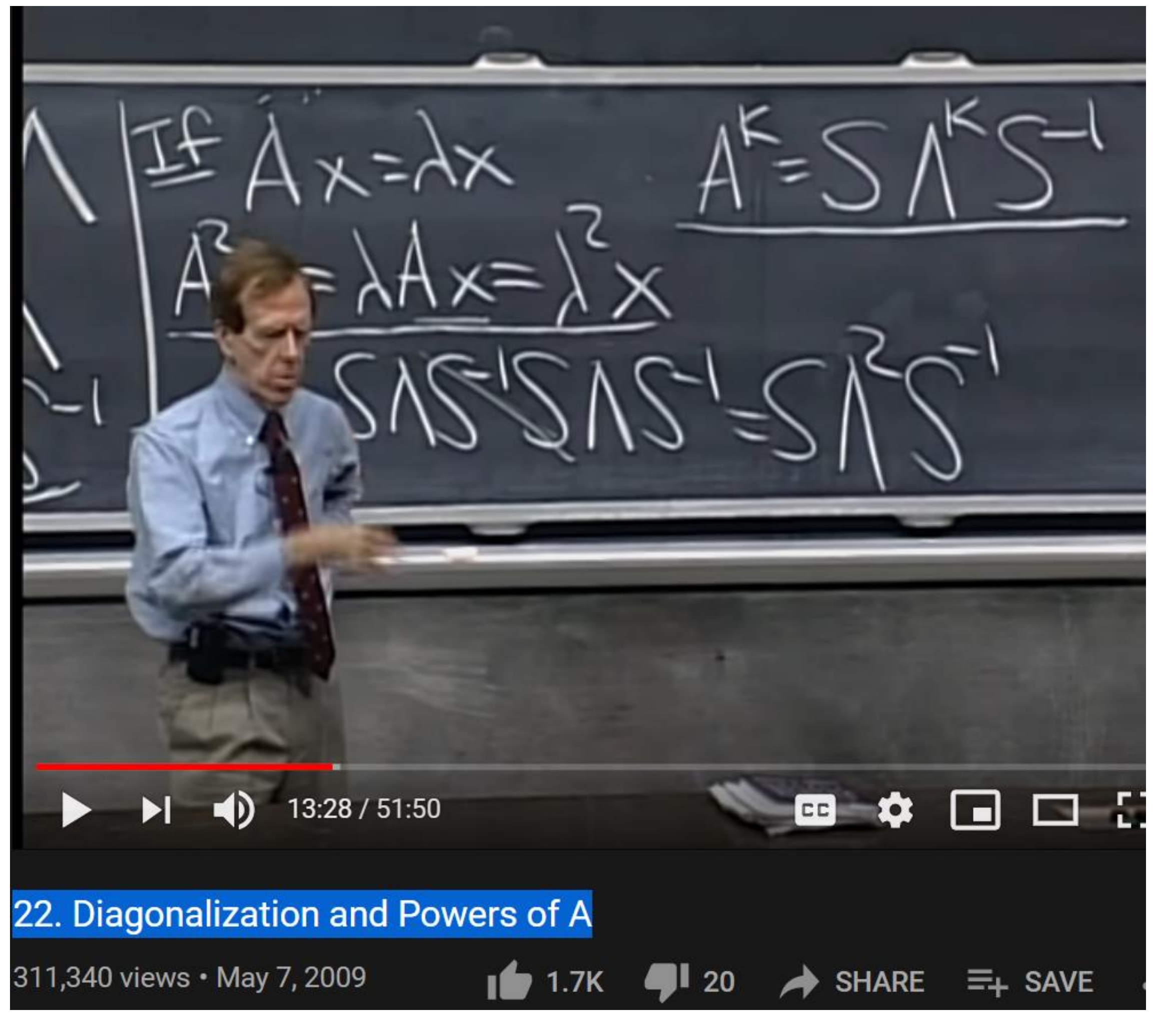
This explicit formula for $\mathbf{x}_k$ gives the solution of the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

As $k \to \infty$, $(.92)^k$ tends to zero and $\mathbf{x}_k$ tends to $\begin{bmatrix} .375 \\ .625 \end{bmatrix} = .125\mathbf{v}_1$. ■

References

1) Lay sec 5.6 (4^{th} Ed), pg 301

2) MIT Strang : Lect 22. Diagonalization and Powers of A
https://www.youtube.com/watch?v=13r9QY6cmjc



Closing Comments on Diagonalisation of Matrices

TKB

1004

Diagonalisable if & only if n eigenvectors

- Some matrices have no eigenvalues at all.
- Example 2x2 rotation matrices
- $a_{11} = 0$, $a_{12} = -1$
- $a_{21} = 1$, $a_{22} = 0$
- Char poly is $\lambda^2 + 1 = 0$.
- So no real eigenvalues
- Therefore no real eigenvectors
- Thus rotation matrices cannot be diagonalised by real matrices!

Diagonalising rotation with complex numbers

- However if we allow complex eigenvalues then rotation matrices can be diagonalised!
- Eigenvalues i, -i
- Can proceed in same way to get complex eigenvectors: just solve
- $(A - iI)x = 0$ & $(A + iI)x = 0$!
- Solving, you will get (i,1) & ((-i,1).
- Hence can form eigenvector matrix P with these 2 vectors.
- Then we can diagonalised the rotation matrix and the diagonals values are i and –i !

Diagonalising rotation with complex numbers

- So we say rotation matrices cant be diagonalised over R butthey can be diagonalised over C, the complex numbers

- With the diagonalised form, we can still compute powers of A when A has FULL set of n complex eigenvectors!

- This is the case with rotation matrices & many other matrices!

Some matrix not even diagonalisable over C

- Eg A has entries 1st row (1,1) and 2nd row (0,1).
- Sine matrix is upper triangle, eigenvalues are 1 (repeated twice).
- Can check A has only 1 eigenvector (1,0)!
- So this A cant be diagonalised at all, over both R and C.

Assignment n Some Good news

- Diagonalisation over complex numbers not in your take home assignment syllabus
- Abt 80% content from eigenvalues onwards.
- Remaining 20% from the rest
- Approximately 2 hour paper
- Constitutes 10% of final grade
- Date: 23rd Saturday 9am