MA1513 TUTORIAL 2

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Link: http://tinyurl.com/MA1513Solutions

KEY CONCEPTS – CHAPTER 2 VECTORS IN *N*-SPACE

Determinants (only for square matrix)

- For $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det\mathbf{A}=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$.

$$\mathbf{A}^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- For $\mathbf{A}=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, then

$$\det\mathbf{A}=a\begin{vmatrix} e & f \\ h & i \end{vmatrix}-b\begin{vmatrix} d & f \\ g & i \end{vmatrix}+c\begin{vmatrix} d & e \\ g & h \end{vmatrix}.$$

<u>Properties</u>

1. For **diagonal** or **triangular** matrices, its determinant is the product of diagonal entries.
2. A square matrix with **identical rows** or **columns** have zero determinant.
3. Effect of E.R.O. on determinants

E.R.O.	Determinant
$\mathbf{A}\xrightarrow{kR_i}\mathbf{B}$	$\det\mathbf{B}=k\det\mathbf{A}$
$\mathbf{A}\xrightarrow{R_i\leftrightarrow R_j}\mathbf{B}$	$\det\mathbf{B}=-\det\mathbf{A}$
$\mathbf{A}\xrightarrow{R_i+kR_j}\mathbf{B}$	$\det\mathbf{B}=\det\mathbf{A}$

$$|\mathbf{A}|\xrightarrow[REF]{GE}|\mathbf{A}'|\text{ (use property 1 \& 3)}$$

4. $\det(c\mathbf{A})=c^n\det\mathbf{A}$, where n is size of matrix
   $\det(\mathbf{AB})=(\det\mathbf{A})(\det\mathbf{B})$
   $\det\mathbf{A}^T=\det\mathbf{A}$
   $\det\mathbf{A}^{-1}=\dfrac{1}{\det\mathbf{A}}$, where $\mathbf{A}$ is non-singular

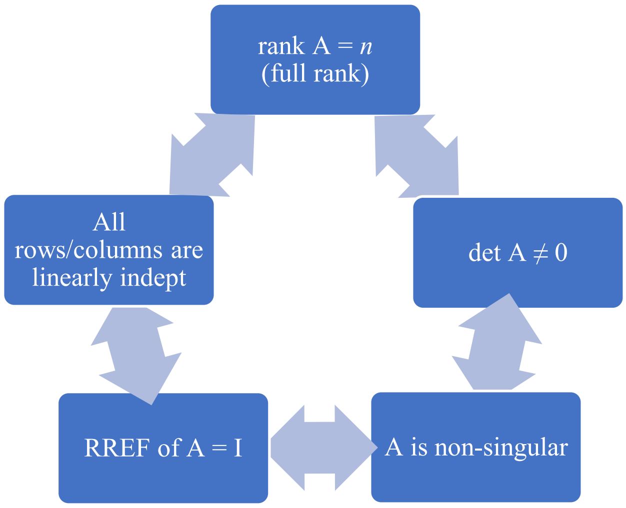
5. $\det\mathbf{A}=0\Leftrightarrow\mathbf{A}$ is singular
   $\det\mathbf{A}\neq0\Leftrightarrow\mathbf{A}$ is non-singular

Rank (for any matrix)

$$\mathbf{A}\xrightarrow{GE}\underset{REF}{\mathbf{R}}$$

The number of non-zero rows after GE is rank **A**.
(or max. # of linearly independent rows/columns)

- For **A** with size $m\times n$, $\operatorname{rank}\mathbf{A}\leq\min\{m,n\}$.
- **A** has <u>full rank</u> when $\operatorname{rank}\mathbf{A}=\min\{m,n\}$.
- For square matrix **A** of size n,



***n*-Space**

The set of all n-vectors is called the n-space, denoted by $\mathbb{R}^n$.

- $\mathbf{u}=(u_1,u_2,\ldots,u_n)\Leftrightarrow\mathbf{u}$ is an n-vector $\Leftrightarrow\mathbf{u}\in\mathbb{R}^n$

Length (norm)	$\lVert\mathbf{u}\rVert=\sqrt{\mathbf{u}\cdot\mathbf{u}}=\sqrt{u_1^2+u_2^2+\ldots+u_n^2}$
Distance between 2 vectors	$\lVert\mathbf{u}-\mathbf{v}\rVert=\sqrt{(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})}=\sqrt{(u_1-v_1)^2+\ldots+(u_n-v_n)^2}$
Angle between 2 vectors	$\theta=\cos^{-1}\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert}\right)$

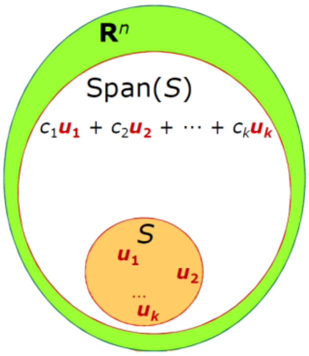
- $\mathbf{u}\cdot\mathbf{v}=0\Leftrightarrow\theta=90°\Leftrightarrow\mathbf{u}$ and $\mathbf{v}$ are <u>orthogonal</u>

Linear Combinations & Span

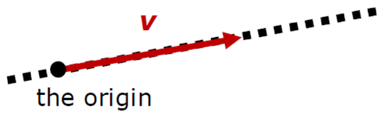
We say $\mathbf{w}$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ if we can find c_1, c_2 and c_3 such that $\mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$.

- The set of all linear combinations is called the <u>linear span</u> of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, denoted by

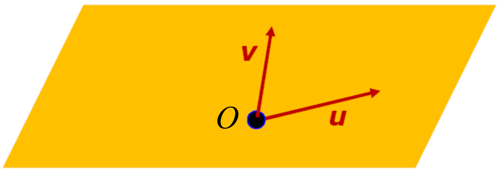
$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \{\mathbf{w} \mid \mathbf{w} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3,\ c_1,\ c_2,\ c_3 \in \mathbb{R}\}$$



- If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are <u>linearly dependent</u>, it means that any vector can be expressed as linear combination of the other two.
- $\text{span}\{\mathbf{u}\}$ is a line parallel to $\mathbf{u}$ passing through O.



- $\text{span}\{\mathbf{u}, \mathbf{v}\}$ is plane containing both vectors $\mathbf{u}$ and $\mathbf{v}$ passing through O.



Linear Independence

To test if $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ are linearly independent:

- Form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n = \mathbf{0}$.

  Check if $c_1 = c_2 = ... = c_n = 0$ is the only solution.
- Use $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ to form square matrix $\mathbf{A}$.

  If det $\mathbf{A} \neq 0$, the vectors are linearly independent.
- If there are only 2 vectors $\mathbf{v}_1, \mathbf{v}_2$,

  Check if there is a k such that $\mathbf{v}_1 = k\mathbf{v}_2$, $k \in \mathbb{R}$.
- If $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n \in \mathbb{R}^m$ and $n > m$,

  then the vectors are linearly dependent.

Subspaces

A subset S of $\mathbb{R}^n$ is a subspace if

- S can be expressed as a linear span
- S satisfies <u>closure properties</u>:
  - $\mathbf{0} \in S$
  - For any $\mathbf{u}, \mathbf{v} \in S$, we have $\mathbf{u} + \mathbf{v} \in S$.
  - For any $\mathbf{v} \in S$, $c \in \mathbb{R}$, we have $c\mathbf{v} \in S$.
- S is a solution set of a homogeneous system (null space of $\mathbf{A}$)
- S represents a line in $\mathbb{R}^2$ or plane in $\mathbb{R}^3$ passing through O.

Tutorial Problems

Question 1

(a) Find the determinant of each of the following matrices.

(i) $\mathbf{A}=\begin{pmatrix}-1 & 3 & -4\\ 2 & 4 & 1\\ -4 & 2 & -9\end{pmatrix}$, (ii) $\mathbf{B}=\begin{pmatrix}2 & 0 & 1\\ 0 & 2 & -1\\ 0 & -1 & 2\end{pmatrix}$, (iii) $\mathbf{C}=\begin{pmatrix}1 & 0 & 0 & 1\\ 1 & 2 & 0 & 0\\ 1 & 2 & 3 & 0\\ 1 & 2 & 3 & 4\end{pmatrix}$.

(b) Which matrix above are non-singular?

(c) Without any further computations, can you tell if **AB** is singular?

(d) Use the answers in (a) to determine whether the following sets are linearly independent.

$S_1=\{(-1,\ 2,\ -4),\ (3,\ 4,\ 2),\ (-4,\ 1,\ -9)\}$,

$S_2=\{(2,\ 0,\ 1),\ (0,\ 2,\ -1),\ (0,\ -1,\ 2)\}$,

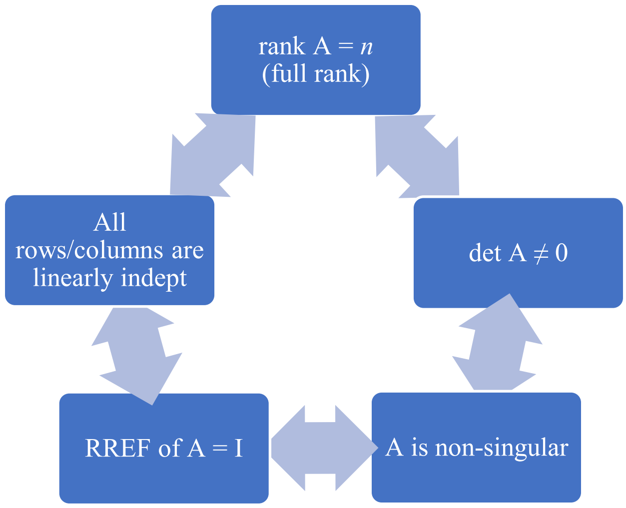
$S_3=\{(1,\ 1,\ 1,\ 1),\ (0,\ 2,\ 2,\ 2),\ (0,\ 0,\ 3,\ 3),\ (1,\ 0,\ 0,\ 4)\}$.

Solutions

(a) (i)
$$\det\mathbf{A}=-1\begin{vmatrix}4 & 1\\ 2 & -9\end{vmatrix}-3\begin{vmatrix}2 & 1\\ -4 & -9\end{vmatrix}-4\begin{vmatrix}2 & 4\\ -4 & 2\end{vmatrix}$$
$$=(-1)(-38)-3(-14)-4(20)=\underline{\underline{0}}$$

(ii)
$$\det\mathbf{B}=2\begin{vmatrix}2 & -1\\ -1 & 2\end{vmatrix}-0+0=2(3)=\underline{\underline{6}}$$

(iii)
$$\det\mathbf{C}=1\begin{vmatrix}2 & 0 & 0\\ 2 & 3 & 0\\ 2 & 3 & 4\end{vmatrix}-1\begin{vmatrix}1 & 2 & 0\\ 1 & 2 & 3\\ 1 & 2 & 3\end{vmatrix}$$
$$=(2)(3)(4)-\left[1\begin{vmatrix}2 & 3\\ 2 & 3\end{vmatrix}-2\begin{vmatrix}1 & 3\\ 1 & 3\end{vmatrix}\right]=\underline{\underline{24}}$$



(b) Since $\det\mathbf{B}\neq 0$ and $\det\mathbf{C}\neq 0$, matrix **B** and **C** are non-singular.

(c) Using $\det(\mathbf{AB})=(\det\mathbf{A})(\det\mathbf{B})=(0)(6)=0$, we can conclude that **AB** is singular.

(d) For S_1, the vectors are columns of matrix **A**. Since $\det\mathbf{A}=0$, it implies S_1 is linearly dependent.

For S_2, the vectors are columns of matrix $\mathbf{B}^T$. Since $\det\mathbf{B}^T=\det\mathbf{B}\neq 0$, it implies S_2 is linearly independent.

For S_3, the vectors are columns of matrix **C**. Since $\det\mathbf{C}\neq 0$, it implies S_3 is linearly independent.

Question 2

Cramer's Rule states that:

If **A** is an $n \times n$ non-singular matrix, then the linear system $\mathbf{A}\vec{x} = \vec{b}$ has exactly one solution for $\vec{x}$ given by $x_i = \dfrac{\det \mathbf{A}_i}{\det \mathbf{A}}$ for every i, where $\mathbf{A}_i$ is the matrix obtained from **A** by replacing i^{th} column of **A** with $\vec{b}$.

Use Cramer's rule to solve the system of linear equations:

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3y + 9y = 3. \end{cases}$$

Solutions

Rewriting the linear system in the matrix form:

$$\begin{cases} x + y + 3z = 0 \\ 2x - 2y + 2z = 4 \\ 3y + 9y = 3 \end{cases} \Rightarrow \begin{pmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$$

Using Cramer's Rule,

$$x = \frac{\begin{vmatrix} 0 & 1 & 3 \\ 4 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \underline{\underline{2.2}}, \quad y = \frac{\begin{vmatrix} 1 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \underline{\underline{-0.4}}, \quad z = \frac{\begin{vmatrix} 1 & 1 & 0 \\ 2 & -2 & 4 \\ 3 & 9 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 2 & -2 & 2 \\ 3 & 9 & 0 \end{vmatrix}} = \underline{\underline{-0.6}}.$$

Question 3

Find the rank of each of the following matrices, and determine which ones are full rank.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 3 & 6 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 4 & 5 & 8 \\ -1 & 4 & 3 & 0 \\ 2 & 0 & 2 & 1 \end{pmatrix}.$$

Solutions

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ -1 & 3 & 6 \\ 2 & 1 & 0 \\ 3 & 1 & -1 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the matrix is left with 3 non-zero rows after Gaussian elimination, rank **A** = 3.

Since **A** is a 5 × 3 matrix (maximum rank = 3), **A** is full rank.

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 0 & 0 & -6 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since the matrix is left with 2 non-zero rows after Gaussian elimination, rank **B** = 2.

Since **B** is a 4 × 5 matrix (maximum rank = 4), **B** is not full rank.

$$\mathbf{C} = \begin{pmatrix} 1 & 4 & 5 & 8 \\ -1 & 4 & 3 & 0 \\ 2 & 0 & 2 & 1 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 4 & 5 & 8 \\ 0 & 8 & 8 & 8 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

Since the matrix is left with 3 non-zero rows after Gaussian elimination, rank **C** = 3.

Since **C** is a 3 × 4 matrix (maximum rank = 3), **C** is full rank.

Question 4

Find the distance and angle between each of the following pairs of vectors **u** and **v**. Which pairs are orthogonal to each other?

(a) $\mathbf{u} = (1, -1)$ and $\mathbf{v} = (-1, 3)$.

(b) $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (0, -3, 2)$.

(c) $\mathbf{u} = (1, -1, 1, -1)$ and $\mathbf{v} = (2, 1, 1, 2)$.

Solutions

	Distance $= \|\mathbf{u}-\mathbf{v}\|$	**Angle,** $\theta = \cos^{-1} \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$	**Orthogonal?**
(a)	$2\sqrt{5}$ units	153.4°	No
(b)	$3\sqrt{3}$ units	90°	Yes
(c)	$\sqrt{14}$ units	90°	Yes

Question 5

Let $\mathbf{u}_1 = (2, 1, 0, 3)$, $\mathbf{u}_2 = (3, -1, 5, 2)$, $\mathbf{u}_3 = (-1, 0, 2, 1)$, $\mathbf{v} = (2, 3, -7, 3)$ and $\mathbf{w} = (1, 1, 1, 1)$.

(a) Are **v** and **w** linear combinations of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$?

(b) Do **v** and **w** belong to $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$?

(c) Which of the following subspaces of $\mathbb{R}^4$ are equal to another? Why?

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}, \quad \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}, \quad \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}\}.$$

Solutions

(a) Let us set up the vector equations separately:

$$a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 = \mathbf{v} \text{ and } a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3 = \mathbf{w},$$

giving us two linear systems. We combine their augmented matrices into one (for simplification):

$$\left(\begin{array}{ccc|c|c} 2 & 3 & -1 & 2 & 1 \\ 1 & -1 & 0 & 3 & 1 \\ 0 & 5 & 2 & -7 & 1 \\ 3 & 2 & 1 & 3 & 1 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c|c} 2 & 3 & -1 & 2 & 1 \\ 0 & -5 & 1 & 4 & 1 \\ 0 & 0 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 & -14 \end{array}\right)$$

What is the difference between:
1. Is $\mathbf{w}$ a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$?
2. Are $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ linearly indept?

Looking at the first augmented matrix, we can obtain a unique solution for a, b and c using back-substitution, which means that $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

Looking at the second augmented matrix, the last row is inconsistent. This implies that there are <u>no solutions</u> for a, b and c, which means that $\mathbf{w}$ is <u>not</u> a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$.

(b) Since $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, then $\mathbf{v} \in \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

Since $\mathbf{w}$ is not a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, then $\mathbf{w} \notin \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$.

(c) Since $\mathbf{v}$ is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, $\mathbf{v}$ is a redundant vector in $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}$. Thus,

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{v}\}.$$

Since $\mathbf{w}$ is not a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, $\mathbf{w}$ is not a redundant vector in $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}\}$.

$$\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} \neq \text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{w}\}.$$

Question 6

Which of the following are subspaces of $\mathbb{R}^3$ or $\mathbb{R}^4$? Justify your answers.

(a) The set of all 3-vectors of the form (a, b, b).

(b) The set of all 3-vectors (a, b, c) such that $abc = 0$.

(c) The set of all 4-vectors (x, y, z, w) which satisfy $w + z = 0$ and $x + y - 4z = 0$.

(d) The set of all 4-vectors of the form $(x, 0, y, 0)$.

Solutions

(a) $\begin{pmatrix} a \\ b \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, where a and b are any real constants.

Thus, $(a, b, b) \in \text{span}\{(1, 0, 0), (0, 1, 1)\}$. As such, it is a subspace of $\mathbb{R}^3$.

Note that $(1, 0, 0), (0, 1, 1)$ are linearly independent.
1. How do you verify that?
2. Is that important?

(b) Note that $(1, 2, 0)$ and $(0, 0, 3)$ belongs to the set because $abc = 0$.

However, $(1, 2, 0) + (0, 0, 3) = (1, 2, 3)$ which does not belong to the set because $abc \neq 0$.

The set does not fulfill the closure properties, hence it is not a subspace of $\mathbb{R}^3$.

(c) We set up the augmented matrix with the two equations:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 \end{array}\right)$$

Note that the set is the solution set to the above homogeneous system of linear equations. Hence, the set is a subspace of $\mathbb{R}^4$.

(d) Similar to (a), note that $(x,\ 0,\ y,\ 0) = x(1,\ 0,\ 0,\ 0) + y(0,\ 0,\ 1,\ 0)$, where $x, y \in \mathbb{R}$.

Thus, $(x,\ 0,\ y,\ 0) \in \text{span}\{(1,\ 0,\ 0,\ 0),\ (0,\ 0,\ 1,\ 0)\}$. As such, the set is a subspace of $\mathbb{R}^4$.

Question 7

Use both the standard approach by setting up the vector equations, and some "shortcut methods" to determine whether the following sets are linearly independent.

(a) $S_1 = \{(1,\ 0,\ -1),\ (-1,\ 2,\ 3)\}$.

(b) $S_2 = \{(1,\ 0,\ -1),\ (-1,\ 2,\ 3),\ (0,\ 3,\ 0)\}$.

(c) $S_3 = \{(1,\ 0,\ -1),\ (-1,\ 2,\ 3),\ (0,\ 3,\ 3)\}$.

(d) $S_4 = \{(1,\ 0,\ -1),\ (-1,\ 2,\ 3),\ (0,\ 3,\ 0),\ (1,\ -1,\ 1)\}$.

Solutions

Standard Approach:

Set up vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n = \mathbf{0}$, and rewrite the system using augmented matrix. Their corresponding row-echelon forms after Gaussian elimination are also given.

(a) $$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ -1 & 3 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Solving using back substitution, we observe that the trivial solution is the only solution. Hence, the two vectors are linearly independent.

(b) $$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 3 & 0 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right)$$

Solving using back substitution, we observe that the trivial solution is the only solution. Hence, the two vectors are linearly independent.

(c) $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 3 & 3 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

> How is the last row of zeros different from the one in (a)?

Since the last row are all zero entries, the last vector can be rewritten as a linear combination of the first two (i.e. the system has non-trivial solutions). Hence, the vectors are linearly dependent.

(d) $\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 \\ -1 & 3 & 0 & 1 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & -1 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right)$

Since there are not enough equations compared with unknowns, the linear system has non-trivial solutions, and thus the vectors are linearly dependent.

<u>Shortcut Methods:</u>

(a) Since we are unable to find a real value of k such that $\mathbf{v}_1 = k\mathbf{v}_2$, this implies that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(b) We can compute the determinant of the matrix $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ -1 & 3 & 0 \end{vmatrix} = -6 \neq 0$. As such, the linear system has only the trivial solution $\Rightarrow$ the vectors are linearly independent.

> Why does the determinant method work in (b) and (c) only?

(c) We can compute the determinant of the matrix $\begin{vmatrix} 1 & -1 & 0 \\ 0 & 2 & 3 \\ -1 & 3 & 3 \end{vmatrix} = 0$. As such, the linear system has non-trivial solution $\Rightarrow$ the vectors are linearly dependent.

(d) There are 4 vectors, but they are from $\mathbb{R}^3$ (insufficient number of equations compared to variables). Hence, these vectors are linearly dependent.

Question 8

(a) For the matrix $\mathbf{C} = \begin{pmatrix} 1 & 4 & 5 & 8 \\ -1 & 4 & 3 & 0 \\ 2 & 0 & 2 & 1 \end{pmatrix}$, write down its row space and column space.

(b) Let $\mathbf{v} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ be a general vector in $\mathbb{R}^4$. Write down the product $\mathbf{Cv}$ in terms of a, b, c and d.

(c) Explain why the vector $\mathbf{Cv}$ belongs to the column space of $\mathbf{C}$ for any $\mathbf{v} \in \mathbb{R}^4$.

Solutions

(a) Row space $= \text{span}\{(1, 4, 5, 8), (-1, 4, 3, 0), (2, 0, 2, 1)\}$

Column space $= \text{span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix} \right\}$

(b) $\mathbf{Cv} = \underline{\underline{\begin{pmatrix} a+4b+5c+8d \\ -a+4b+3c \\ 2a+2c+d \end{pmatrix}}}$

(c) Observe that **Cv** can be rewritten as a linear combination of the columns of **C**:

$$\mathbf{Cv} = a\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} + c\begin{pmatrix} 5 \\ 3 \\ 0 \end{pmatrix} + d\begin{pmatrix} 8 \\ 0 \\ 1 \end{pmatrix}.$$

This implies that **Cv** belongs to the linear span of columns of **C**, which is the column space of **C**.

Question 9

The two linear equations $x+y+z=0$ and $x-y+2z=0$ are represented by two planes in the *xyz*-space. These two planes intersect at a line.

(a) Express this line in terms of a linear span.

(b) Find a matrix **B** such that the nullspace of **B** is given by the linear span in (a).

Solutions

(a) Rewrite the linear equations as augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right)$$

Since the last row "disappeared", the last variable can be chosen to be the free variable. As such, the first two variables can be expressed in terms of the last. Let $z=t$. Then,

$$y = \frac{1}{2}t, \quad x = -\frac{3}{2}t \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{t}{2}\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$$

The solution space is the span of $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$, which is essentially the line of intersection.

(b) Recall that the nullspace of **B** is the solution space of $\mathbf{B}\boldsymbol{x} = \mathbf{0}$. This is the homogeneous system in (a), hence we can take $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix}$.

Can we choose $\mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$?

GLOSSARY

The **column space** of a matrix **A** is the linear span of all columns of **A**, i.e. the set of all linear combinations of the columns of **A**.
The ***n*-space** contains all n-vectors, which are vectors with n components.
The **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is given by $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_n\mathbf{v}_n$, where $c_1, c_2, \ldots, c_n$ are real numbers.
A set of vectors is said to be **linearly dependent** if one of the vectors in a set can be defined as a linear combination of the others; if no vector in the set can be written this way, then the vectors are said to be **linearly independent**.
The **linear span** of set S is the set of all linear combinations of the vectors in set S.
The **null-space** of matrix **A** is the solution set of the homogeneous linear system $\mathbf{A}\boldsymbol{x} = \mathbf{0}$.
The **row space** of a matrix **A** is the linear span of all rows of **A**, i.e. the set of all linear combinations of the rows of **A**.