NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 8.1 - Relations I

Dr. Guo Jian

Topic Overview

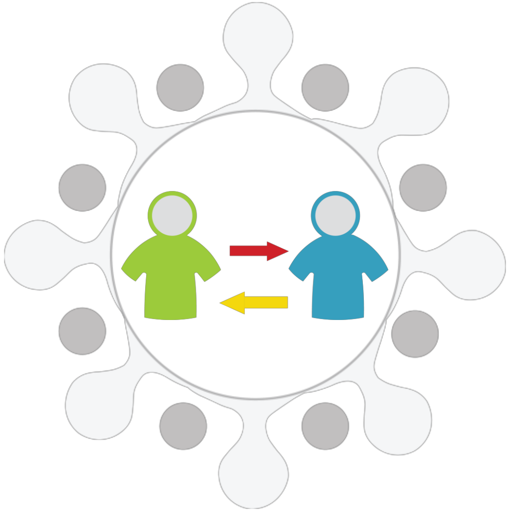
What's in store…

Binary Relations

Reflexivity

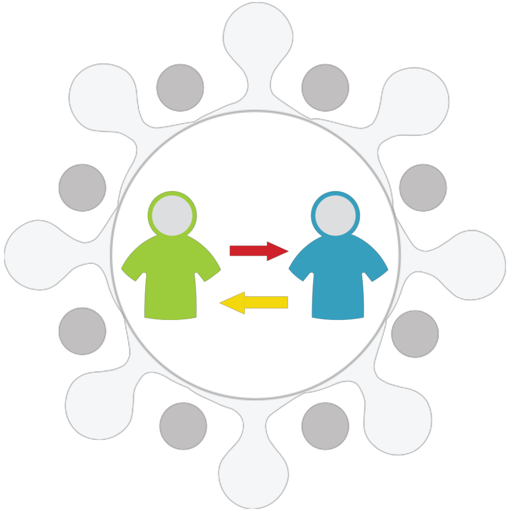
Symmetry

Transitivity



By the end of this lesson, you should be able to…

- Explain the different types of binary relations.
- Explain the concept of reflexivity.
- Explain the concept of symmetry.
- Explain the concept of transitivity.



Binary Relations

Binary Relations: Between Two Sets

Let A and B be sets. A **binary relation R from A to B** is a subset of $A \times B$. Given (x,y) in $A \times B$, **x is related to y by R** $(xRy) \leftrightarrow (x,y) \in R$.

Example

$A = \{1,2\}$, $B = \{1,2,3\}$, $(x,y) \in R \leftrightarrow (x - y)$ is even

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

$(1,1) \in R$, $(1,3) \in R$, $(2,2) \in R$

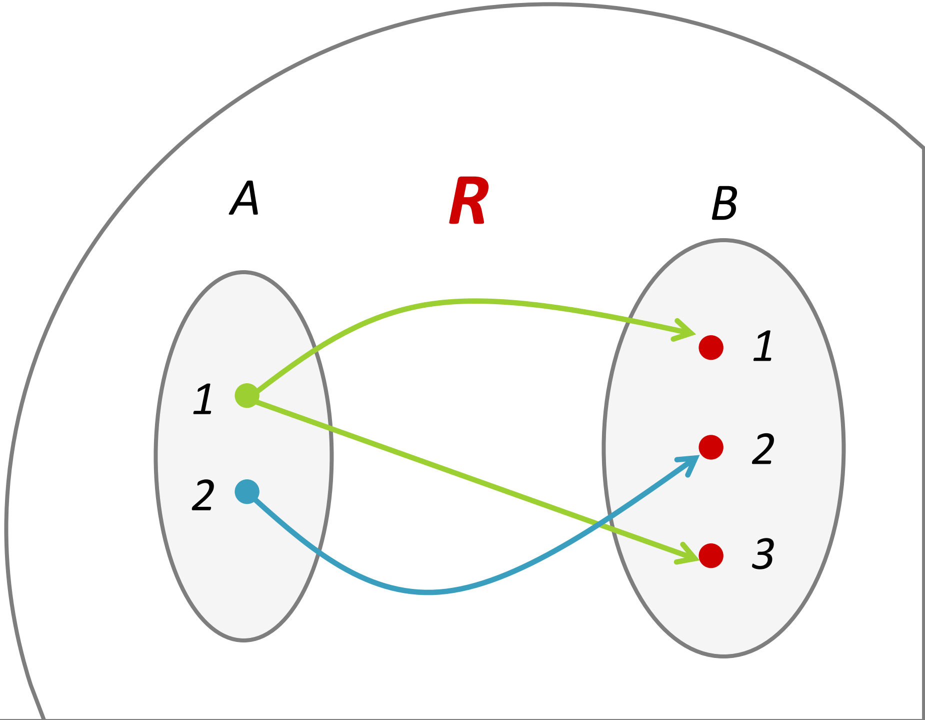
$x > y$, x **owes** y, x **divides** y

Binary Relations: Between Two Sets (Graphically)

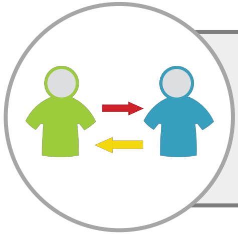
$A = \{1,2\}$, $B = \{1,2,3\}$, $(x,y) \in R \leftrightarrow (x - y)$ **is even**

$A \times B = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3)\}$

$(1,1) \in R$, $(1,3) \in R$, $(2,2) \in R$



Binary Relations: Inverse of a Binary Relation



Let R be a relation from A to B. The inverse relation R^{-1} from B to A is defined as: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.

Binary Relations: Inverse of a Binary Relation (Example)

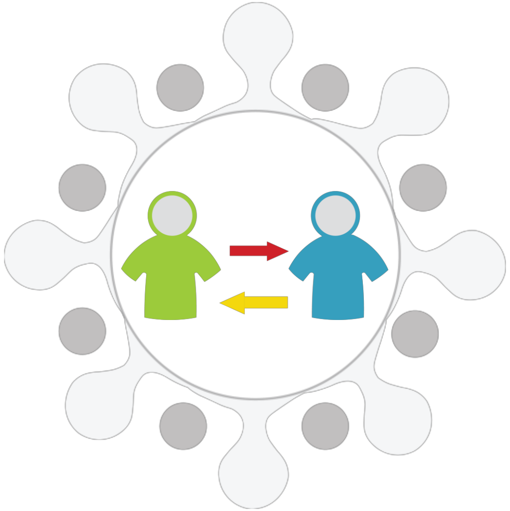
$A = \{2,3,4\}$, $B = \{2,6,8\}$, $(x, y) \in R \leftrightarrow x$ **divides** y

$A \times B = \{(2,2), (2,6), (2,8), (3,2), (3,6), (3,8), (4,2), (4,6), (4,8)\}$

$(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

$(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$

$(y, x) \in R^{-1} \leftrightarrow y$ **is a multiple of** x

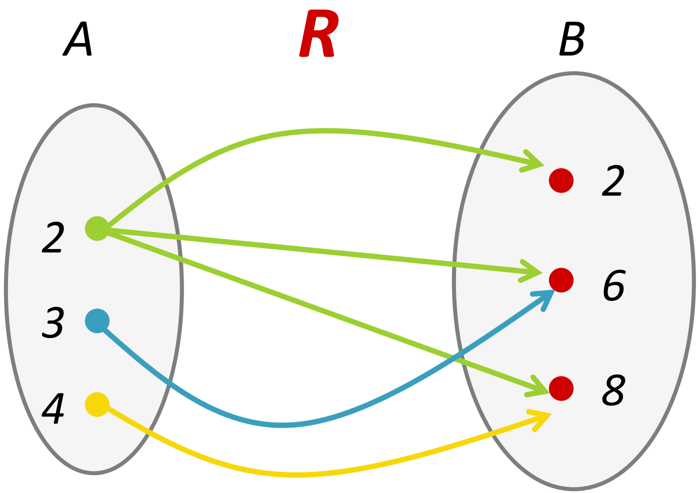


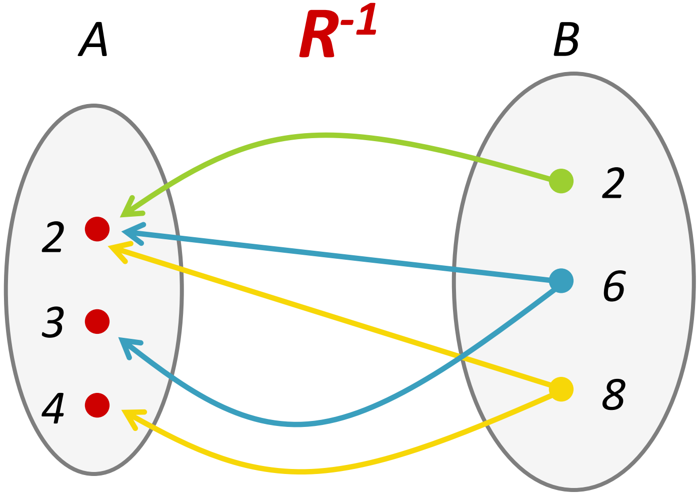
Binary Relations: Inverse of a Binary Relation (Graphically)

$A = \{2,3,4\}$, $B = \{2,6,8\}$, $(x, y) \in R \leftrightarrow x$ **divides** y

$(2,2) \in R, (2,6) \in R, (2,8) \in R, (3,6) \in R, (4,8) \in R$

$(2,2) \in R^{-1}, (6,2) \in R^{-1}, (8,2) \in R^{-1}, (6,3) \in R^{-1}, (8,4) \in R^{-1}$





Binary Relations: Matrix Representation

$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3, b_{4,}),$

$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$

(i, j)th **entry is** T **if** $a_i R b_j$ **:**

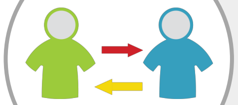
$$\begin{array}{c c} & \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix} \end{array}$$

Example

$A = \{2,3,4\},\ B = \{2,6,8\}, (x, y) \in R \leftrightarrow x$ **divides** y.

$A \backslash B$	2	6	8
2	T	T	T
3	F	T	F
4	F	F	T

Binary Relations: Matrix Representation



R **relation from** *A* **to** *B*: $R^{-1} = \{(y,x) \in B \times A \mid (x,y) \in R\}$.

$$A = (a_1, a_2, a_3),\ B = (b_1, b_2, b_3, b_{4,})$$

$$R = \{(a_1, b_2), (a_2, b_1), (a_3, b_1), (a_3, b_4)\}$$

$$R^{-1} = \{(b_2, a_1), (b_1, a_2), (b_1, a_3), (b_4, a_3)\}$$

The matrix of R^{-1} **is the transpose of the matrix of** R**.**

$$a_i R b_j: \quad \begin{array}{cc} & \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \begin{bmatrix} F & T & F & F \\ T & F & F & F \\ T & F & F & T \end{bmatrix} \end{array}$$

$$b_i R^{-1} a_j: \quad \begin{array}{cc} & \begin{array}{ccc} a_1 & a_2 & a_3 \end{array} \\ \begin{array}{c} b_1 \\ b_2 \\ b_3 \\ b_4 \end{array} & \begin{bmatrix} F & T & T \\ T & F & F \\ F & F & F \\ F & F & T \end{bmatrix} \end{array}$$

Binary Relations: Composition of Relations

Given R in $A \times B$, and S in $B \times C$, the composition of R and S is a relation on $A \times C$ defined by
$R \circ S = \{(a, c) \in A \times C \mid \exists\, b \in B, aRb \text{ and } bSc\}$.

Example

$$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$$

$$R = \{(a_1, b_1), (a_1, b_2)\}$$

$$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$$

What is $R \circ S$?

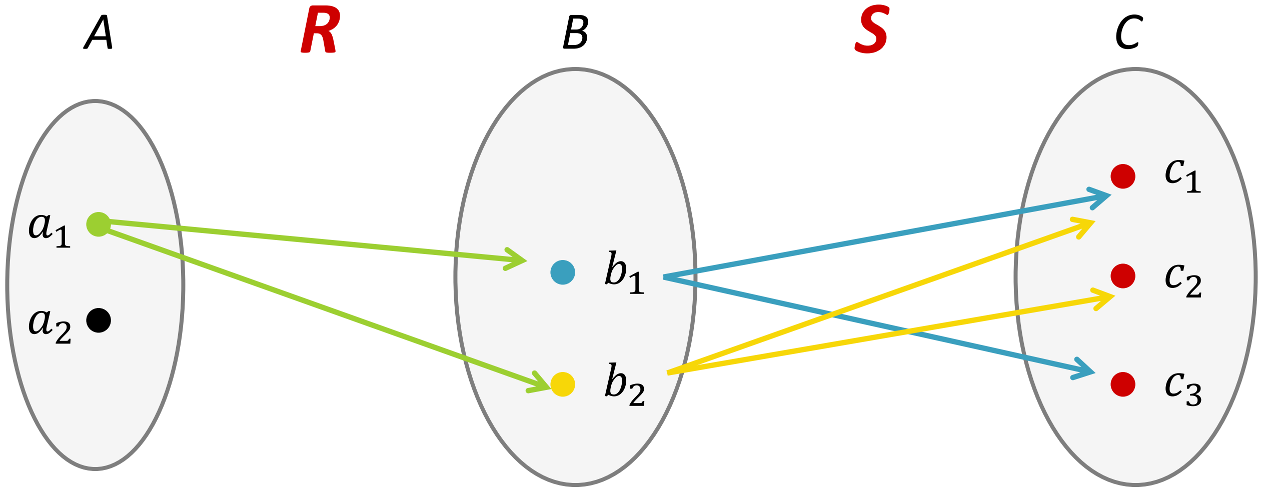
$$R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$$

Binary Relations: Composition of Relations (Graphically)

$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

$R = \{(a_1, b_1), (a_1, b_2)\}$

$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$



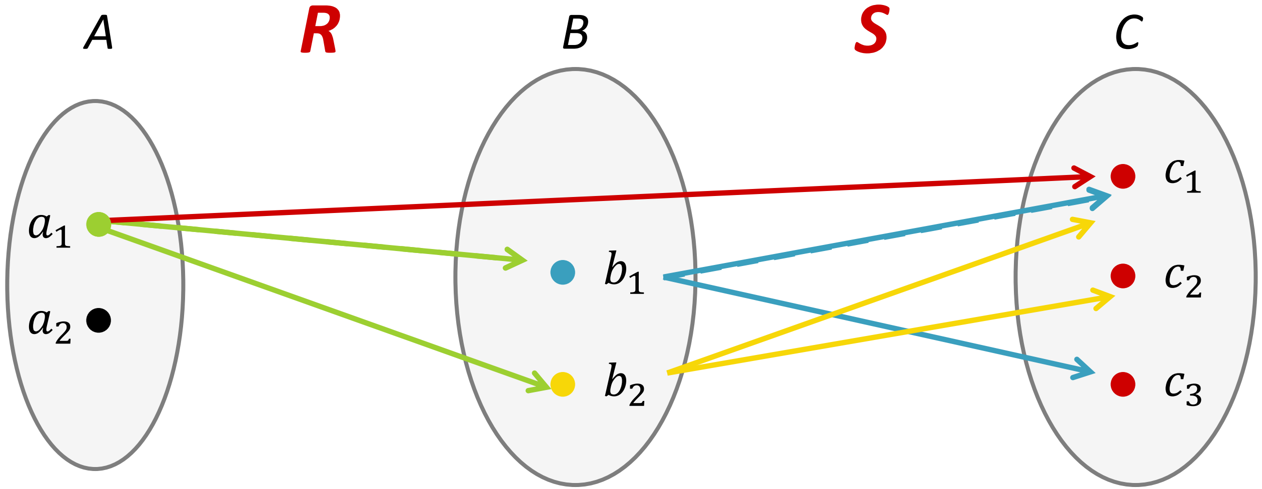
Binary Relations: Composition of Relations (Graphically)

$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

$R = \{(a_1, b_1), (a_1, b_2)\}$

$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$

$R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$



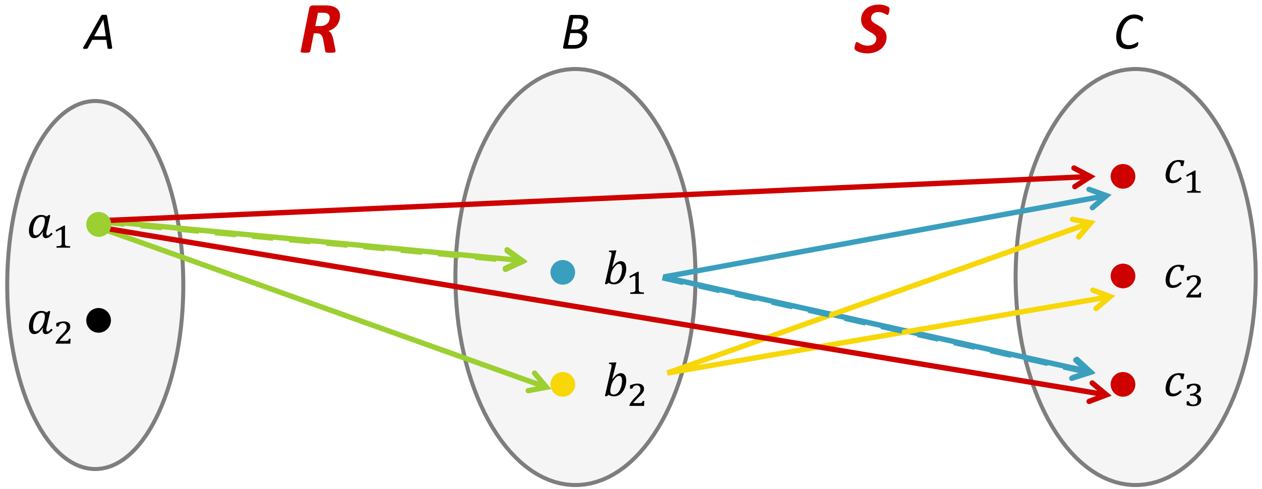
Binary Relations: Composition of Relations (Graphically)

$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

$R = \{(a_1, b_1), (a_1, b_2)\}$

$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$

$R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$



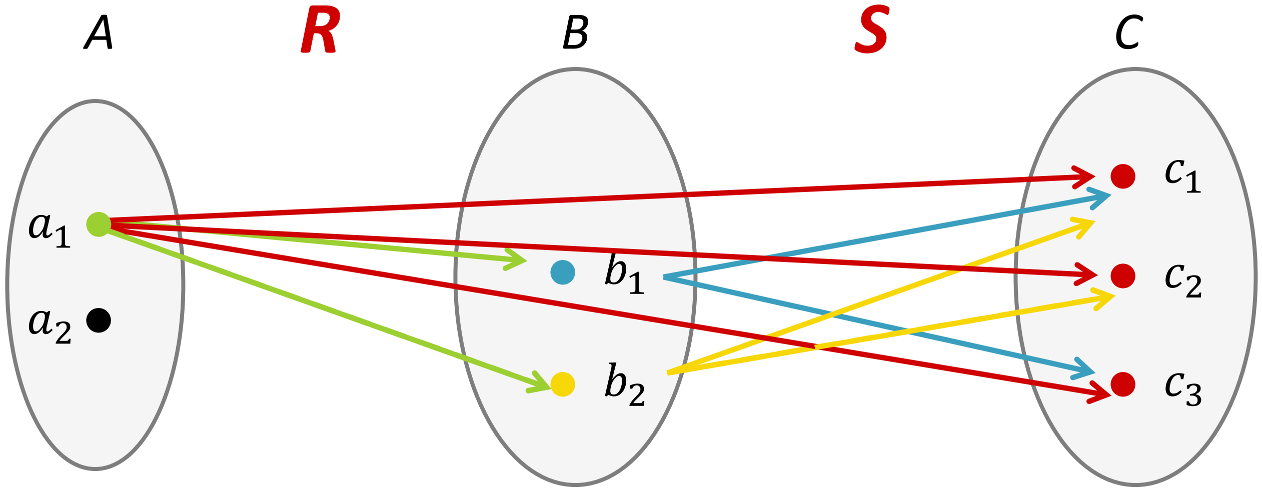
Binary Relations: Composition of Relations (Graphically)

$A = \{a_1, a_2\}, B = \{b_1, b_2\}, C = \{c_1, c_2, c_3\}$

$R = \{(a_1, b_1), (a_1, b_2)\}$

$S = \{(b_1, c_1), (b_2, c_1), (b_1, c_3), (b_2, c_2)\}$

$R \circ S = \{(a_1, c_1), (a_1, c_3), (a_1, c_2)\}$



Reflexivity

Reflexivity: Definition

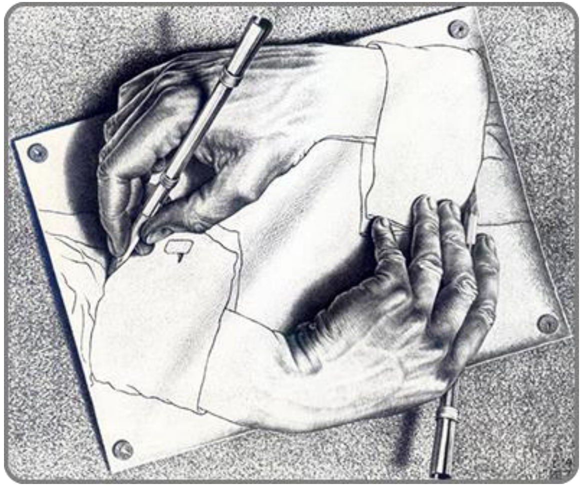
A relation R on a set A is **reflexive** if every element of A is related to itself: $\forall x \in A, xRx$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: reflexive

$A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: not reflexive

What is the reflexivity on the matrix representing R?



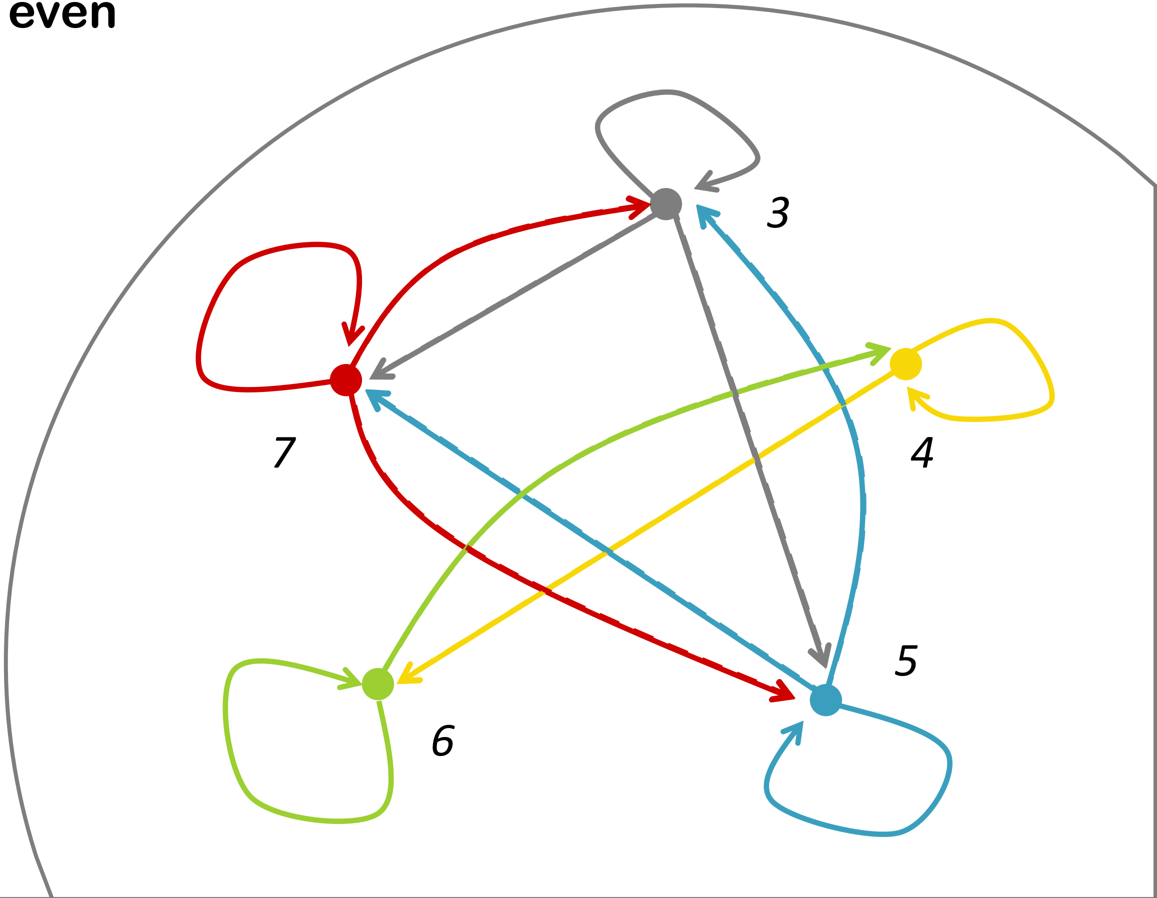
Drawing Hands (M.C. Escher)

Retrieved Mar 10, 2018 from https://www.wikiart.org/en/m-c-escher/drawing-hands

Reflexivity: Graphically

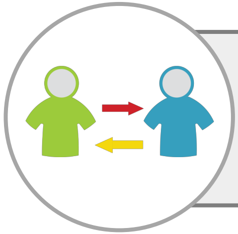
$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ is even

R reflexive

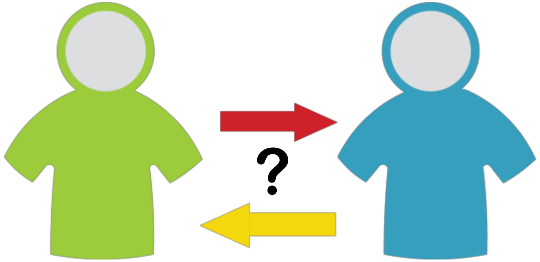


Symmetry

Symmetry: Definition

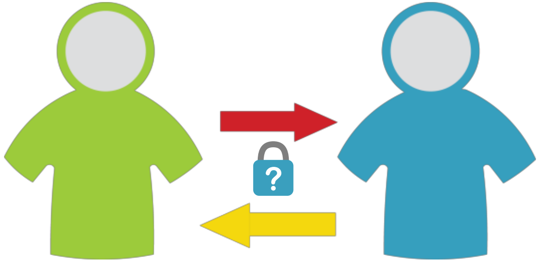


A relation R on a set A is **symmetric** if $(x,y) \in R$ implies $(y,x) \in R$: $\forall x \in A \ \ \forall y \in A,\ xRy \rightarrow yRx$.



Not Symmetric Relationship

E.g., $A = \mathbb{Z}$, $xRy \leftrightarrow x > y$:
not symmetric



Symmetric Relationship

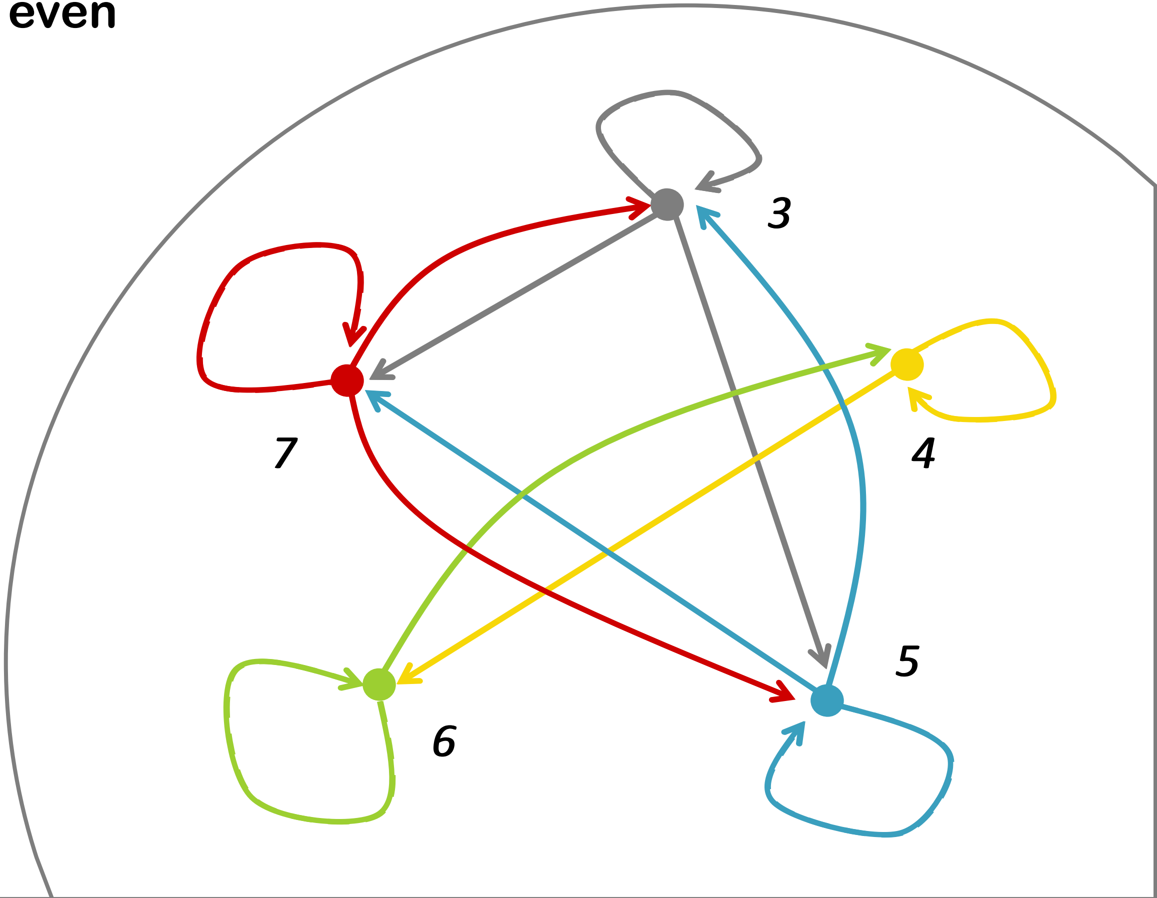
E.g., $A = \mathbb{Z}$, $xRy \leftrightarrow x = y$:
symmetric

Symmetry: Graphically

$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ is even

R reflexive

R symmetric



Transitivity

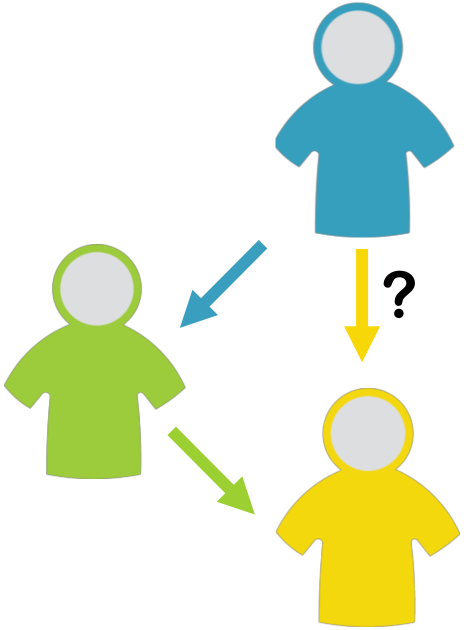
Transitivity: Definition

A relation R on a set A is **transitive** if $(x,y) \in R$ and $(y,z) \in R$ implies $(x,z) \in$ R: $\forall x\ \forall y\ \forall z\ xRy \wedge yRz \rightarrow xRz$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: transitive

$A = \mathbb{Z}$, $xRy \leftrightarrow x > y$: transitive



Transitivity: Graphically

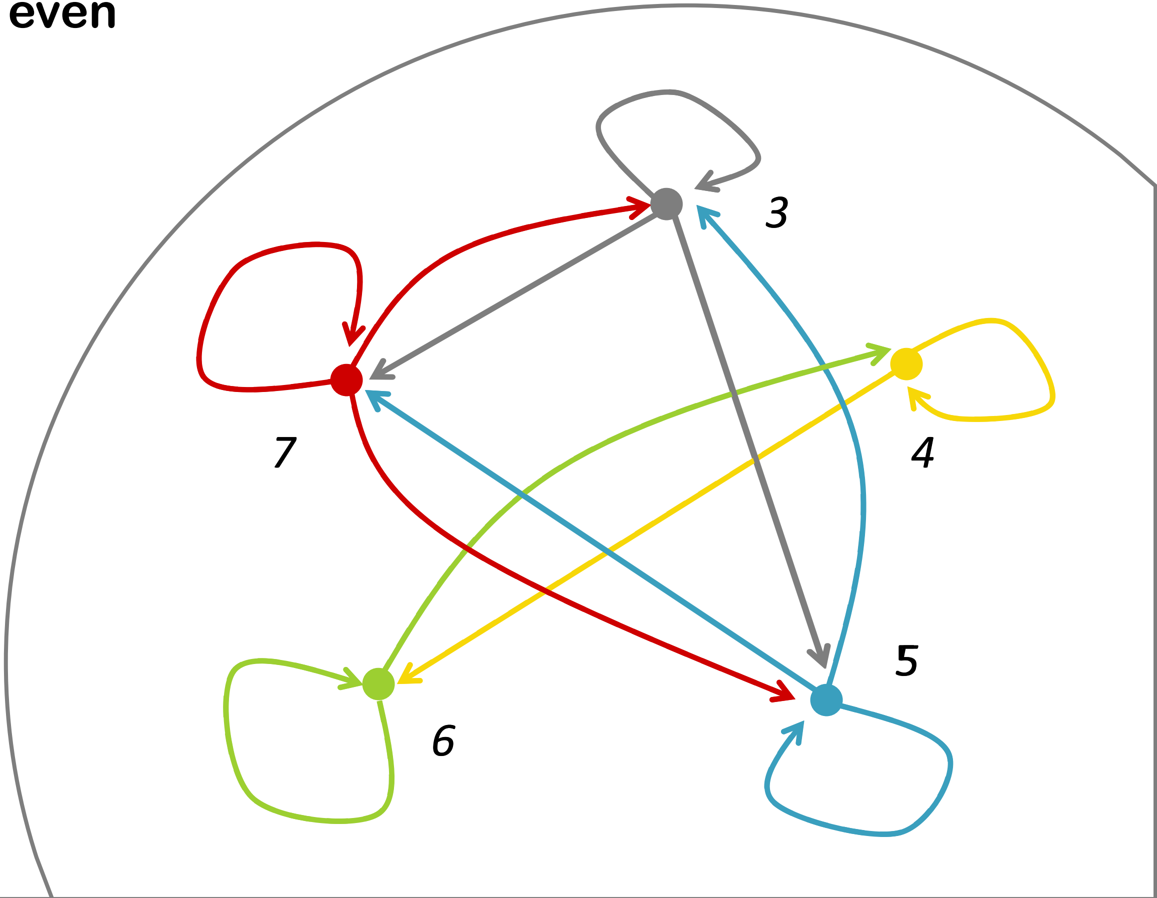
$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ **is even**

$[3] = \{3,5,7\}$, $[4] = \{4,6\}$

R **reflexive**

R **symmetric**

R **transitive**



Topic Summary

Let’s recap…

- Binary relations:
  - Inverse and composition
  - Graphical representation
- Properties:
  - Reflexivity
  - Symmetry
  - Transitivity

Discrete Mathematics

MH1812

Topic 8.2 - Relations II

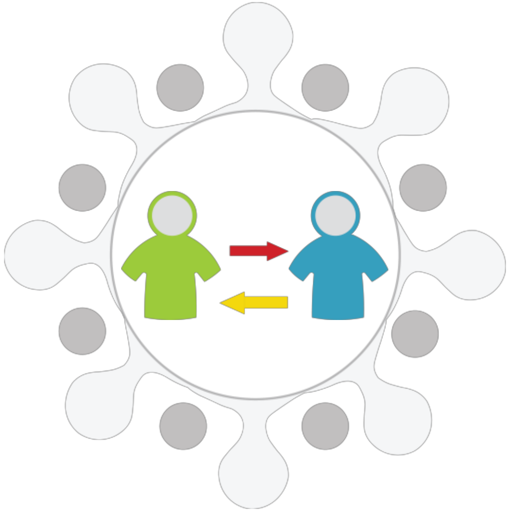
Dr. Guo Jian

Topic Overview

What's in store…

Equivalence Relations

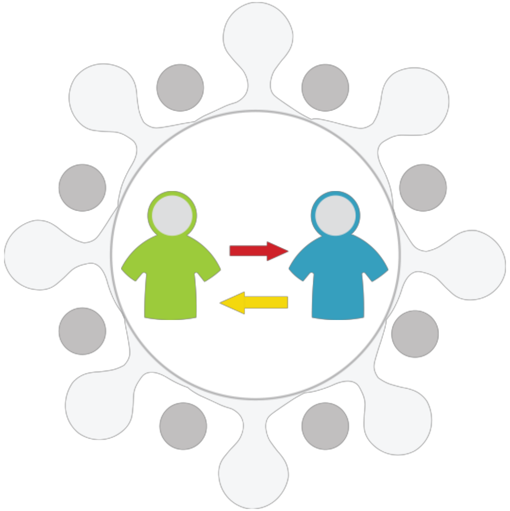
Antisymmetry



By the end of this lesson, you should be able to...

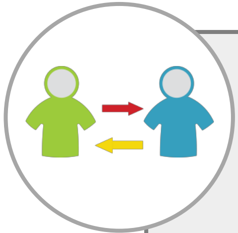
- Explain the conditions for an equivalence relation.

- Explain the concept of antisymmetry.



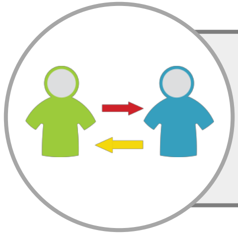
Equivalence Relations

Equivalence Relations: Definition



A relation R on a set A is **an equivalence relation** if:

1. R is reflexive: $\forall x \in A, xRx$
2. R is symmetric: $\forall x\, \forall y\; xRy \rightarrow yRx$
3. R is transitive: $\forall x\, \forall y\, \forall z\; xRy \wedge yRz \rightarrow xRz$



Equivalence class of a in A: $[a] = \{x \in A|\ aRx\}$ for R an equivalence relation.

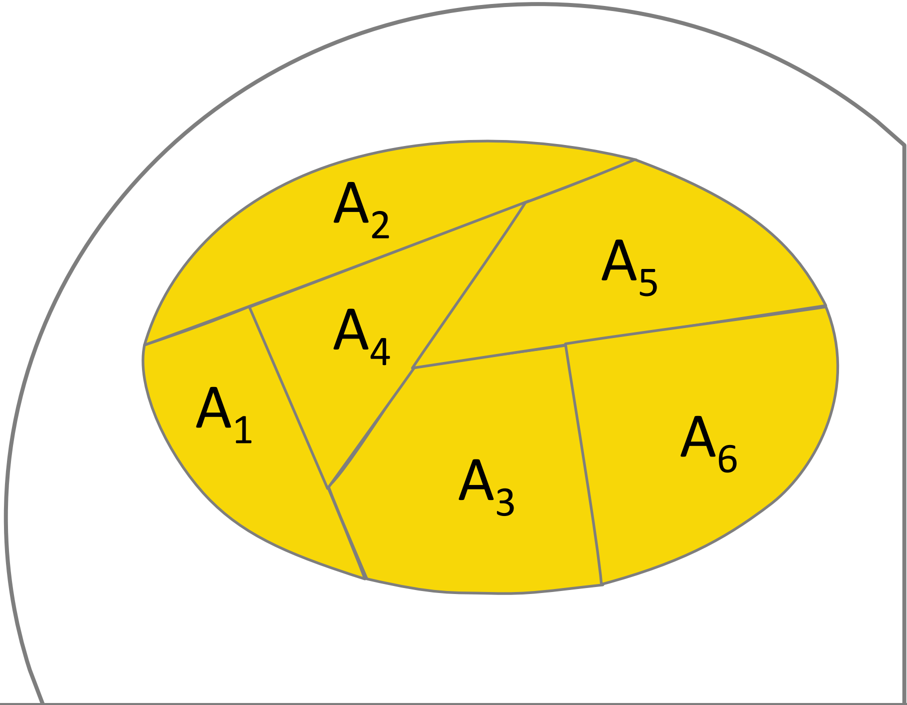
Equivalence Relations: Equivalence Classes

Partition of a set A**:**

$A_i \cap A_j = \varphi$ **whenever** $i \neq j$

$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6 = A$

Equivalence classes of A **form a partition of** A**.**



Equivalence Relations: Integers mod n

$$a \equiv b \pmod{n} \leftrightarrow a = qn + b$$

$\equiv \pmod{n}$ **is an equivalence relation**:

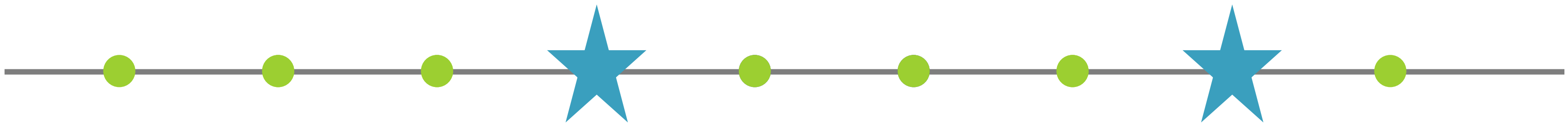
1. $\equiv \pmod{n}$ **is reflexive:** $\forall x \in A, x \equiv x \bmod n$
2. $\equiv \pmod{n}$ **is symmetric:** $\forall x\, \forall y\; x \equiv y \pmod{n} \to y \equiv x \pmod{n}$
3. $\equiv \pmod{n}$ **is transitive:** $\forall x\, \forall y\, \forall z\; x \equiv y \pmod{n} \wedge y \equiv z \pmod{n} \to x \equiv z \pmod{n}$

Equivalence Relations: Integers mod n

Equivalence class of $[0] = \{0, n, 2n, 3n, \ldots, -n, -2n, -3n \ldots\}$

Equivalence class of $[1] = \{1, n+1, 2n+1, 3n+1, \ldots, -n+1, -2n+1 \ldots\}$

Example: Integers mod 4



Integers mod n **can be represented as elements between** 0 **and** $n-1$:
$\{0, 1, 2, \ldots, n-1\}$

Antisymmetry

Antisymmetry: Definition

A relation R on a set A is **antisymmetric** if $(x,y) \in R$ and $(y,x) \in R$ implies $x = y$: $\forall x \, \forall y \; xRy \wedge yRx \rightarrow x = y$.

Example

$A = \mathbb{Z}$, $xRy \leftrightarrow x = y$: antisymmetric

$A = \mathbb{Z}$, $xRy \leftrightarrow x \geq y$: antisymmetric

$BRC \leftrightarrow B \subseteq C$: antisymmetric

Antisymmetry: Graphically

$A = \{3,4,5,6,7\}$, $xRy \leftrightarrow (x - y)$ **is even**

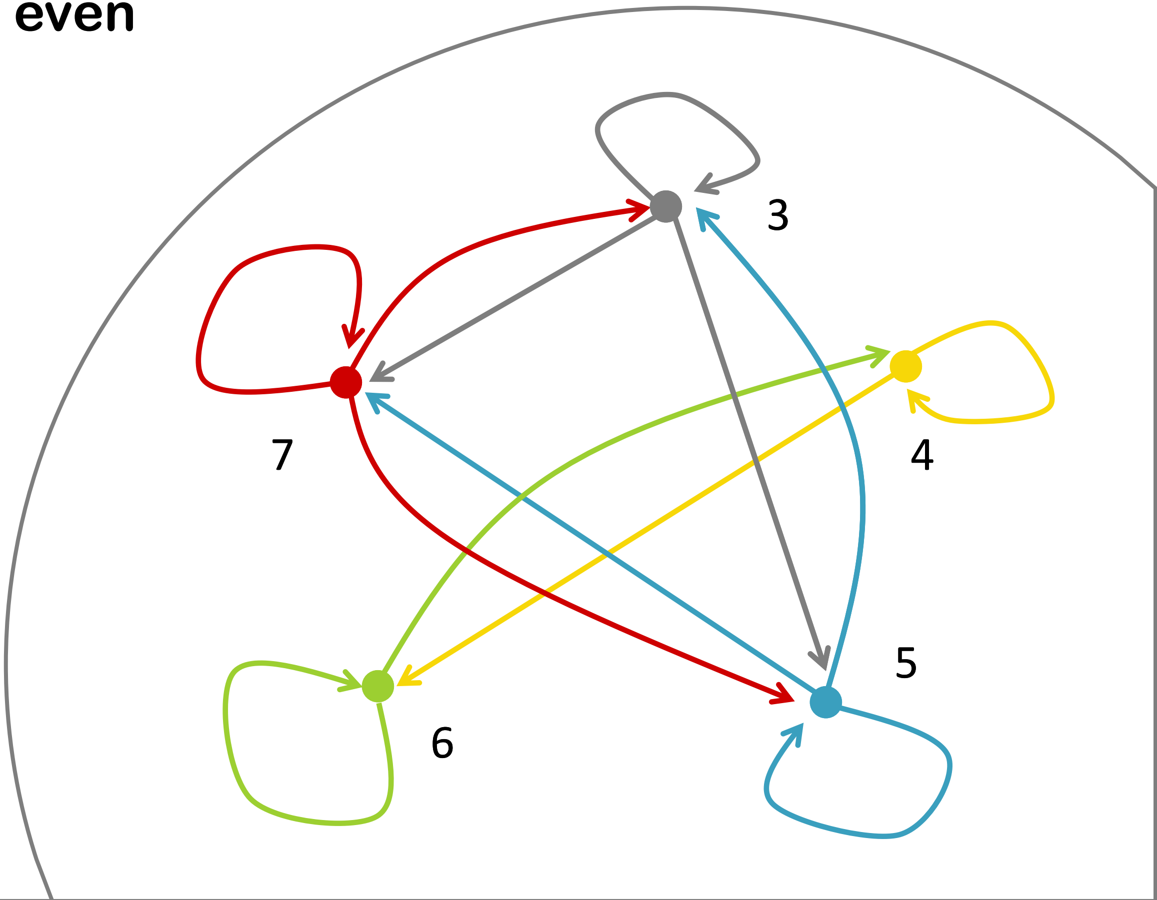
$[3] = \{3,5,7\}$, $[4] = \{4,6\}$

R **reflexive**

R **symmetric**

R **transitive**

R **is not antisymmetric**



Topic Summary

Let's recap…

- Equivalence relations: equivalence class
- Partial order: antisymmetry

NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

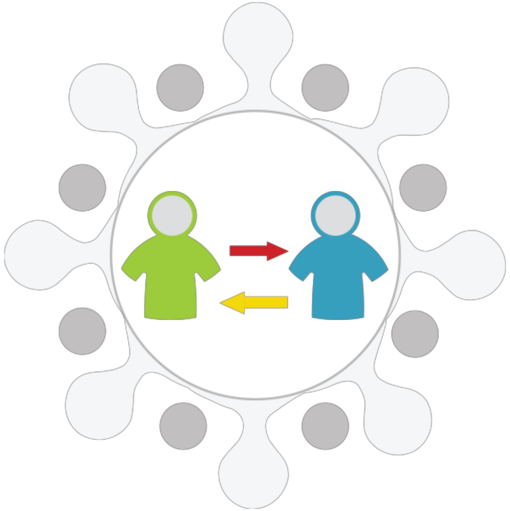
Topic 8.3 - Relations III

Dr. Guo Jian

Topic Overview

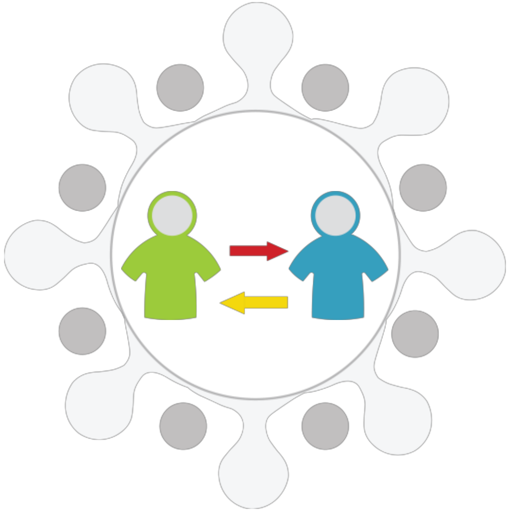
What's in store…

- **P**artial Order
- **T**ransitive Closure
- **B**eyond Binary Relations
- **O**perations on Relations



By the end of this lesson, you should be able to...

- Explain the concept of partial order.
- Explain the three properties of transitive closure.
- Explain the concept of non-binary relations.
- Explain the different operations on relations.



Partial Order

Partial Order: Definition

R is **a partial order** on A if R is reflexive, antisymmetric and transitive.

Example

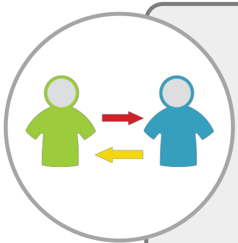
$A = \mathbb{Z}$, $xRy \leftrightarrow x \leq y$

Notion of partial order is useful for scheduling problems across possibly different domains.

Transitive Closure

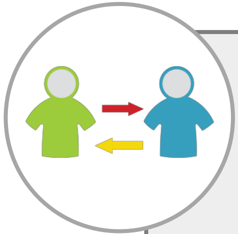
Transitive Closure: What is Closure?

Let A be a set and R a binary relation on A.



The closure of a relation $R \subseteq A \times A$ with respect to a property P (P being reflexive, symmetric, or transitive) is the relation obtained by adding the minimum number of ordered pairs to R to obtain property P.

Transitive Closure: Definition



Let A be a set and R a binary relation on A. The **transitive closure** of R is the binary relation R^t on A that satisfies the following three properties:

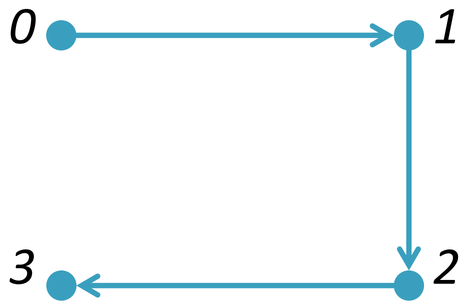
1. R^t is transitive
2. $R \subseteq R^t$
3. If S is any other transitive relation that contains R then $R^t \subseteq S$

Transitive Closure: Example

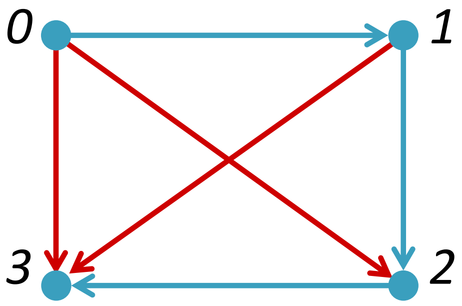
Let $A = \{0,1,2,3\}$

Consider a relation $R = \{(0,1),(1,2),(2,3)\}$ on A

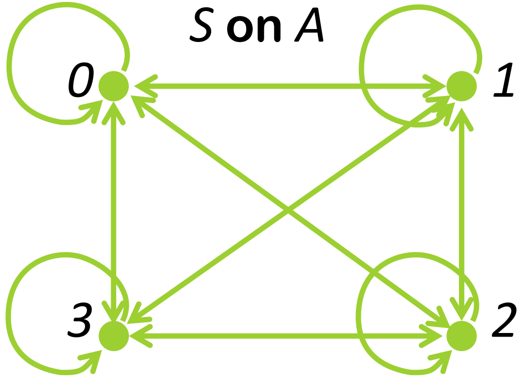
R on A



R^t on A



S on A



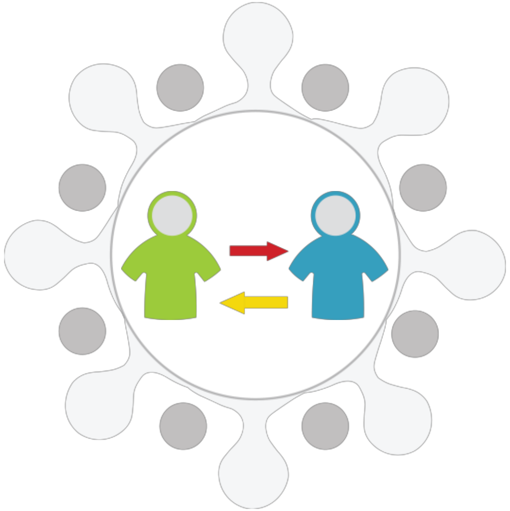
S is transitive and $R \subseteq S$

Thus $R^t \subseteq S$

$R^t = \{(0,1),(1,2),(2,3),(0,2),(0,3),(1,3)\}$

Transitive Closure: Construction

- Let A be a set and R a binary relation on A.
- Start with R, and do the following: $\forall x, y, z \in A$, if $(xRy \wedge yRz \wedge x\not R z)$ then add (x,z).
- Repeat until the obtained relation is transitive (will stop if $|A|$ is finite).
- The ordering in which the edges are added does not matter.

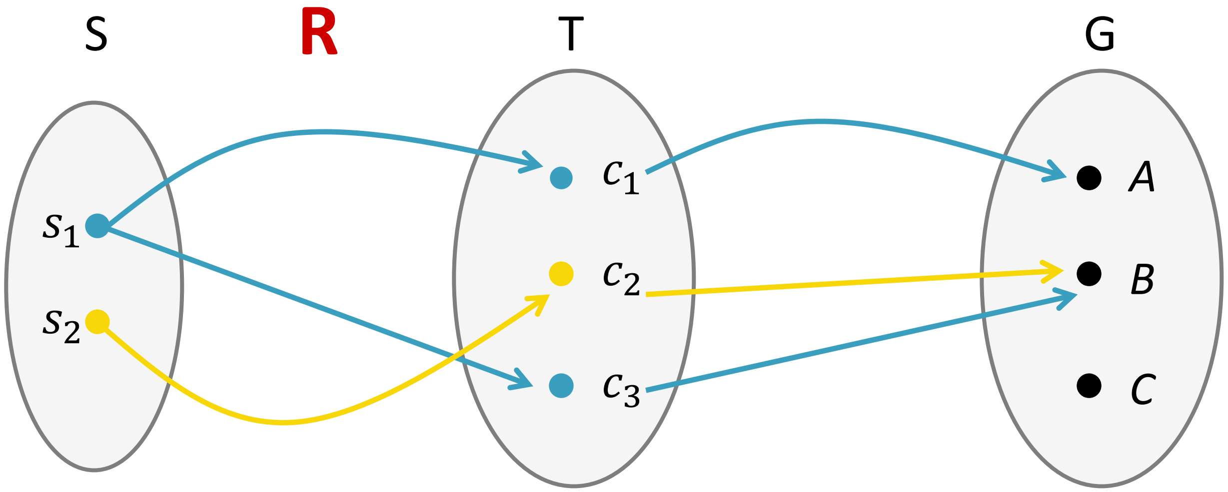


Beyond Binary Relations

Beyond Binary Relations: Non-binary Relations (Example)

$S = \{s_1, s_2\}$ **students**, $T = \{c_1, c_2, c_3\}$ **courses**

$G = \{A, B, C\}$ **grades**, (s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)



Beyond Binary Relations: *n-ary* Relations

Let $A_1, \dots, A_n$ be sets. A *n-ary* relation R is a subset of $A_1 \times \cdots \times A_n$. $a_1, \dots, a_n$ are related if $(a_1, \dots, a_n) \in R$.

Example

$S = \{s_1, s_2\}$ **students,** $T = \{c_1, c_2, c_3\}$ **courses**

$G = \{A, B, C\}$ **grades,** $(s_1, c_1, A), (s_1, c_3, B), (s_2, c_2, B)$

Operations of Relations: Complement of a Relation

Let $R \subseteq A_1 \times \cdots \times A_n$ **be a relation.**
$\overline{R} = (A_1 \times \cdots \times A_n - R)$ **is the relational complement of** R**,**
i.e., $(a_1, a_2, a_3,\ldots, a_n) \in \overline{R} \Leftrightarrow (a_1, a_2, a_3,\ldots, a_n) \notin R$.

Example

$A = \{1,2\}$, $B = \{3,5\}$ **and** $R = \{(1,3), (2,5)\}$

Then $\overline{R} = A \times B - R = \{(1,5), (2,3)\}$

Operations on Relations

Operations of Relations: Union of Relations

Let $R, S \subseteq A_1 \times \cdots \times A_n$ **be two relations.** $R \cup S$ **is the relation such that** $(a_1, a_2, a_3, \ldots, a_n) \in R \cup S \Leftrightarrow (a_1, a_2, a_3, \ldots, a_n) \in R \vee (a_1, a_2, a_3, \ldots, a_n) \in S$.

Example

$A = \{1,2,3\}$, $B = \{1,2,3,4\}$ **and** $R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4)\}$

Then $R \cup S = \{(1,1), (2,2), (3,3), (1,2), (1,3), (1,4)\}$

Operations of Relations: Intersection of Relations

Let $R, S \subseteq A_1 \times \cdots \times A_n$ **be two relations.** $R \cap S$ **is the relation such that** $(a_1, a_2, a_3,\ldots, a_n) \in R \cap S \Leftrightarrow (a_1, a_2, a_3,\ldots, a_n) \in R \wedge (a_1, a_2, a_3,\ldots, a_n) \in S$.

Example

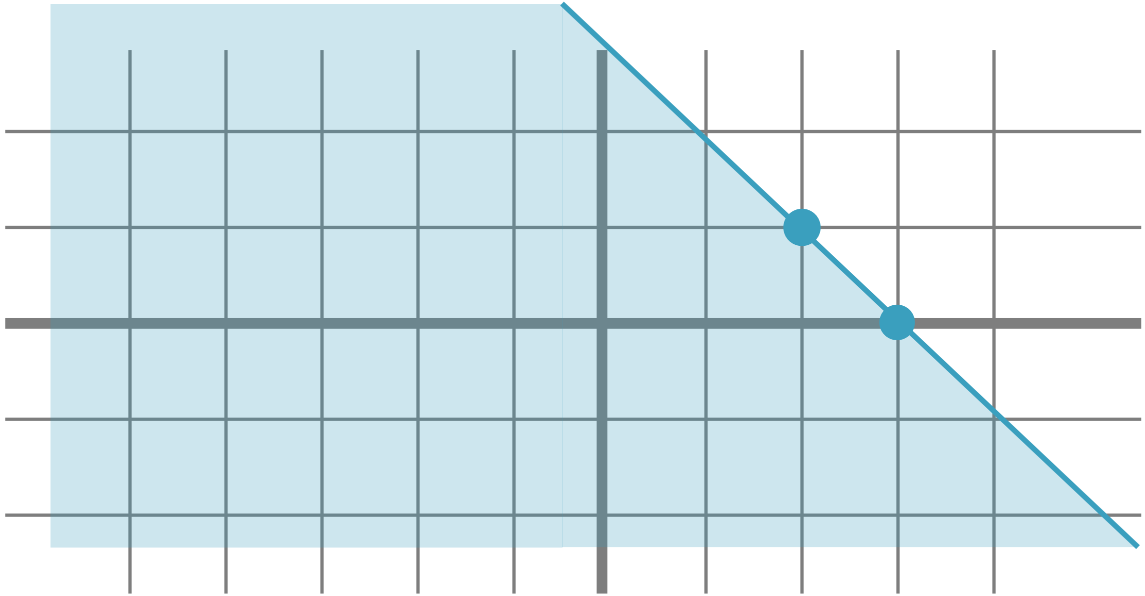
$A = \{1,2,3\}$, $B = \{1,2,3,4\}$ **and** $R = \{(1,1), (2,2), (3,3)\}$

$S = \{(1,1), (1,2), (1,3), (1,4)\}$

Then $R \cap S = \{(1,1)\}$

Operations of Relations: Example

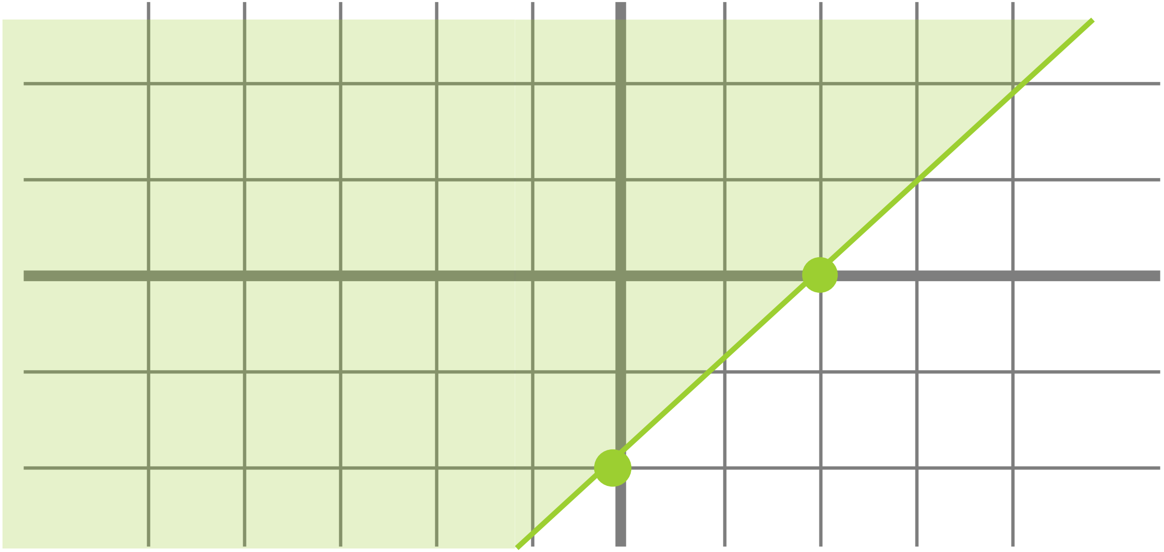
$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$



Operations of Relations: Example

$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$

$S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x - y \leq 2 \}$

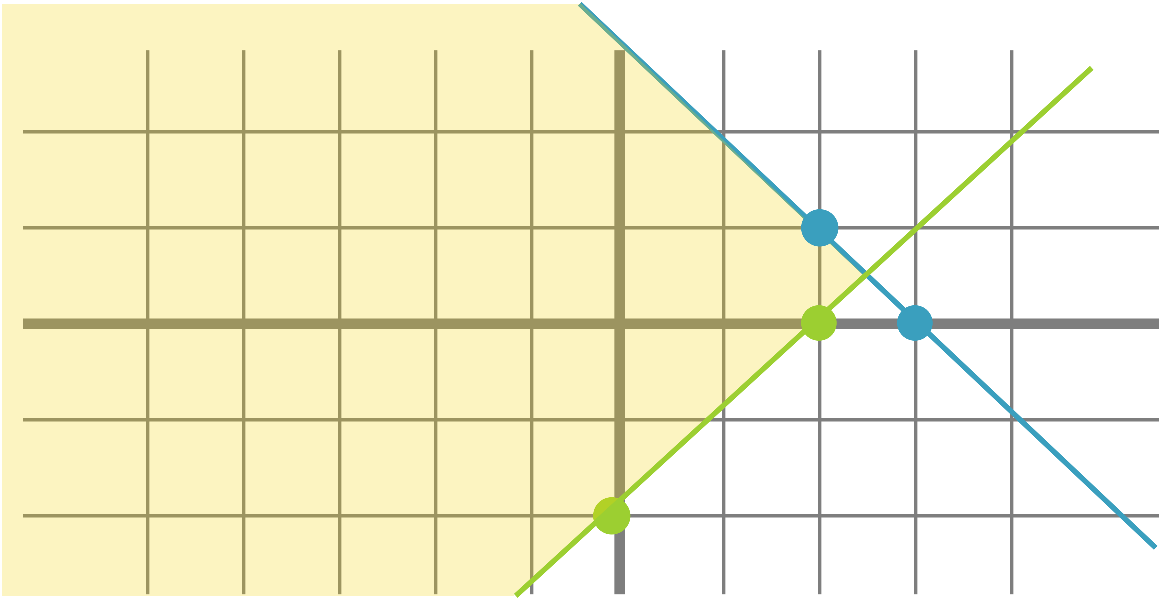


Operations of Relations: Example

$T = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x + y \leq 3 \}$

$S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid x - y \leq 2 \}$

$T \cap S = \{ (x,y) \in \mathbb{R} \times \mathbb{R} \mid (x + y \leq 3) \wedge (x - y \leq 2)\}$



Topic Summary

Let’s recap…

- Partial Order
- Transitive Closure
- Beyond binary relations
- Operations on relations
  - Complement
  - Union
  - Intersection