# MH1810 Math 1 Part 2 Chap 5 Differentiation

## Closed Interval Method

Tang Wee Kee

Nanyang Technological University

# Optimization Problem

– problems aiming to find global extreme values
– very practical and important in many different areas

- Finding the shortest time or shortest path or least cost in a transportation problem;
- Fermat's Principle in optics: Light follows path that takes the least time;
- Finding the least material required to construct something subject to some constraints;
- Obtaining the maximum profit to produce a commodity;
- Constructing cylindrical metal can with a given volume $V$ in a way that minimizes the surface area (the amount of metal used).

# Closed Interval Method

In this section, we discuss the closed interval method to solve an optimization problem where the function involved is continuous and the domain is a closed and bounded interval.

# Critical Points

Critical points are points $c$ at which $f'(c) = 0$ or $f'(c)$ fails to exist.

1. A point $c$ where $f'(c) = 0$ is called a stationary point.
2. A point where $f'(c)$ fails to exist is called a singular point.

# Closed Interval Method

Recall that the Extreme Value Theorem states that a continuous function $f$ on a closed and bounded interval $[a, b]$ attains its global maximum and global minimum.

The following three-step procedure can be used to find global maximum and absolute minimum of a continuous function $f$ on a closed and bounded interval $[a, b]$.

# Closed Interval Method

1: Determine all critical points of $f$ in $(a, b)$ and find the corresponding $f$-values.

2: Compute $f(a)$ and $f(b)$.

3: The largest (respectively smallest) value of $f$ from Steps 1 and 2 is the global maximum (respectively global minimum) of $f$ on $[a, b]$.

# Example

## Example

Find the global maximum and global minimum of $f(t) = \sqrt[3]{t}\,(8-t)$ on $[-1, 8]$.

## Solution

*Note that $f$ is continuous on $[-1, 8]$, since it is the product function of continuous functions $\sqrt[3]{t}$ and $8-t$. By the Extreme Value Theorem, $f$ has a global maximum and a global minimum on $[-1, 8]$.*

# Solution

### Solution

*To find critical points of $f$, we have to find $c$ at which $f'(c)$ does not exist or $f'(c) = 0$.*
*For $-1 < t < 0$ or $0 < t < 8$, we have*
$f'(t) = \frac{1}{3}t^{-2/3}(8-t) - \sqrt[3]{t} = \frac{8-4t}{3(\sqrt[3]{t})^2}$.
*Singular point:* $t = 0$ *is a singular point of $f$, since $f$ is not differentiable at $t = 0$.*
*Stationary Point:* $f'(t) = 0 \iff 8 - 4t = 0 \iff t = 2$.
*End Points:* $t = -1$ *and* $t = 8$.

# Solution

Comparing values of $f$:

$$f(2) = \sqrt[3]{2}(6), \quad f(0) = 0, \quad f(-1) = -9, \quad f(8) = 0.$$

Conclusion: Global maximum of $f$ on $[-1, 8]$ is $f(2) = \sqrt[3]{2}(6)$
Global minimum of $f$ on $[-1, 8]$ is $f(-1) = -9$.

# Example

## Example

Let $f(x) = \left(x^2 - 1\right)^{2/3}$. Find the global maximum and global minimum values of $f$ on the interval $[-3, 3]$

## Solution

*The function $f$ is continuous on $[-3, 3]$.*

By the Extreme Value Theorem, it has a global maximum and a global minimum. We have

$$f'(x) = \frac{2}{3}(x^2 - 1)^{-1/3} \cdot 2x = \frac{4x}{3(x^2 - 1)^{1/3}}.$$

# Solution

## Solution

*We have* $f'(x) = \dfrac{4x}{3(x^2-1)^{1/3}}$.

Critical points:

– Stationary point: $f'(x) = 0 \quad \Leftrightarrow \quad x = 0$ and $f(0) = 1$.
– Singular points: $f'(x)$ fails to exist when $x = \pm 1$ and $f(-1) = 0, \quad f(1) = 0$.

Endpoints: $f(-3) = 4, \quad f(3) = 4$

Since these are all candidates for extreme values, we see that the largest value of $f$ on $[-3, 3]$ is $f(-3) = f(3) = 4$ and the smallest value is $f(-1) = f(1) = 0$.