

MH1810 Math 1 Part 2 Chapter 6 Integration

Techniques of Integration: Rational Functions

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Integration of Rational Functions

We shall consider integrals like

$$\int \frac{x^3 + 3x^2}{x^2 + 1} dx, \quad \int \frac{2 + 3x + x^2}{x(x^2 + 1)} dx, \dots,$$

or, in general, integrals of the form

$$\int \frac{P(x)}{Q(x)} dx$$

where P and Q are polynomials. How do we handle integrals like

$$\int \frac{x + 2}{x^3 - x} dx?$$

This is done by writing $\frac{P(x)}{Q(x)}$ as “partial fractions”.

General Rational Functions

Consider rational function $\int \frac{P(x)}{Q(x)} dx$ where P and Q are polynomials.

Assume $\deg P < \deg Q$.

Fact from Algebra: Every polynomial

$$Q(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

with real coefficient $a_0, a_1, \cdots, a_{n-1}, a_n$ can be factorized as a product of **linear factors** (i.e., $ax + b$) and **irreducible quadratic factors** (i.e., $ax^2 + bx + c$ with $b^2 - 4ac < 0$.)

Partial Fractions

The rational function $\frac{P(x)}{Q(x)}$ can thus be expressed as a sum of partial fractions given below:

factors of $Q(x)$	corresponding partial fractions
$(ax + b)^k$	$\frac{A_r}{(ax + b)^r}, r = 1, 2, 3, \dots, k$
$(ax^2 + bx + c)^k$	$\frac{A_r x + B_r}{(ax^2 + bx + c)^r}, r = 1, 2, 3, \dots, k$

Case I: Simple Linear Factors

If the denominator $Q(x)$ is a product of distinct linear (degree 1) factors (say $ax + b$), then the corresponding partial fraction representation for each factor $ax + b$ is $\frac{A}{ax + b}$.

Example

Express $\frac{1}{x(x-1)}$ as partial fractions.

Solution

There are two distinct linear factors x and $(x - 1)$ in the denominator.

The corresponding partial fractions are $\frac{A}{x}$ and $\frac{B}{x-1}$.

If $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$, then

$$\frac{1}{x(x-1)} = \frac{A(x-1)+Bx}{x(x-1)} = \frac{(A+B)x-A}{x(x-1)}.$$

Comparing numerators, we have

$1 = (A + B)x - A$ for all $x \neq 0, 1$.

Thus $-A = 1$, and $B = 1$. Hence $\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$.

Example

Example

Evaluate $\int \frac{x+2}{x(x-1)(x+1)} dx$.

Solution

We first express $\frac{P(x)}{Q(x)} = \frac{x+2}{x(x-1)(x+1)}$ in partial fractions.

There are three distinct linear factors in $Q(x)$: x , $(x-1)$ and $(x+1)$. The corresponding partial fractions are

$$\frac{A}{x}, \frac{B}{x-1} \text{ \& } \frac{C}{x+1}$$

Thus, we shall solve constants A , B and C such that

$$\frac{x+2}{x(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1}.$$

Solution

Solution

$$\begin{aligned}\frac{x+2}{x(x-1)(x+1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\ &= \frac{A(x+1)(x-1) + Bx(x+1) + Cx(x-1)}{x(x-1)(x+1)}.\end{aligned}$$

Comparing coefficients on the numerators:

$$\diamond x^2 : 0 = A + B + C$$

$$\diamond x : 1 = B - C,$$

$$\diamond x^0 : 2 = -A.$$

Solution

Solving this we get

$$A = -2, \quad B = 3/2, \quad C = 1/2.$$

Therefore

$$\frac{x+2}{x(x-1)(x+1)} = \frac{-2}{x} + \frac{3/2}{x-1} + \frac{1/2}{x+1}, \text{ and}$$

$$\begin{aligned} \int \frac{x+2}{x(x-1)(x+1)} dx &= \int \left(\frac{-2}{x} + \frac{3/2}{x-1} + \frac{1/2}{x+1} \right) dx \\ &= -2 \int \frac{1}{x} dx + \frac{3}{2} \int \frac{1}{x-1} dx + \frac{1}{2} \int \frac{1}{x+1} dx \\ &= -2 \ln |x| + \frac{3}{2} \ln |x-1| + \frac{1}{2} \ln |x+1| + C. \end{aligned}$$

Case II: Repeated Linear Factors

If $Q(x)$ is a product of linear factors, some of which are **repeated**, say $(ax + b)^k$, where $k \geq 2$, then there are k corresponding partial fractions:

$$\frac{A_1}{ax + b}, \frac{A_2}{(ax + b)^2}, \dots, \frac{A_{k-1}}{(ax + b)^{k-1}}, \frac{A_k}{(ax + b)^k}$$

Example

Example

Evaluate $\int \frac{x^2}{(x-3)(x+2)^2} dx$.

Solution

First, we express the integrand $\frac{x^2}{(x-3)(x+2)^2}$ in partial fractions.

For $(x-3)$, the corresponding partial fraction is $\frac{A}{x-3}$.

For $(x+2)^2$, the corresponding partial fractions are $\frac{B}{x+2} + \frac{C}{(x+2)^2}$.

We shall find constants A , B and C such that

$$\frac{x^2}{(x-3)(x+2)^2} = \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

Solution

Solution

$$\begin{aligned}\frac{x^2}{(x-3)(x+2)^2} &= \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \\ &= \frac{A(x+2)^2 + B(x-3)(x+2) + C(x-3)}{(x-3)(x+2)^2}.\end{aligned}$$

Comparing coefficients:

- x^2 : $1 = A + B$. This gives $B = 1 - A$
- x : $0 = 4A - B + C$
- x^0 : $0 = 4A - 6B - 3C$

Solution

Solution

Solving for A , B and C by eliminating B (using $B = 1 - A$), we have

$$0 = 4A - (1 - A) + C \iff 5A + C = 1 \text{ --- (1)}$$

$$0 = 4A - 6(1 - A) - 3C \iff 10A - 3C = 6 \text{ --- (2)}$$

$$(1) \times 2 - (2): 5C = -4 \iff C = -4/5$$

$$\text{From (1): } A = \frac{1}{5}(1 + 4/5) = 9/25$$

$$\text{Therefore, } B = 1 - (9/25) = 16/25.$$

Solution

Solution

So, we have

$$\frac{x^2}{(x-3)(x+2)^2} = \frac{1}{25} \left(\frac{9}{x-3} + \frac{16}{x+2} - \frac{20}{(x+2)^2} \right).$$

Thus,

$$\begin{aligned} \int \frac{x^2}{(x-3)(x+2)^2} dx &= \frac{1}{25} \left(\int \frac{9}{x-3} + \frac{16}{x+2} - \frac{20}{(x+2)^2} dx \right) \\ &= \frac{1}{25} \left(9 \int \frac{1}{x-3} dx + 16 \int \frac{1}{x+2} dx - 20 \int \frac{1}{(x+2)^2} dx \right) \\ &= \frac{1}{25} \left(9 \ln |x-3| + 16 \ln |x+2| + 20 \frac{1}{(x+2)} \right) + C \end{aligned}$$

Irreducible Quadratic Factors

The quadratic expression $ax^2 + bx + c$ is said to be **irreducible** when it cannot be reduced to a product of linear factors.

In this case we have $b^2 - 4ac < 0$.

The quadratic expression $ax^2 + bx + c$ can be expressed in form $(Ax + B)^2 + D^2$, via completing square.

Case III: Distinct Irreducible Quadratic Factors

Suppose $Q(x)$ contains the quadratic factor $ax^2 + bx + c$ ($b^2 - 4ac < 0$). Then the partial fraction representation of $\frac{P(x)}{Q(x)}$ will contain the term:

$$\frac{Ax + B}{ax^2 + bx + c}$$

A Useful Formula

Theorem

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C.$$

Example

$$\begin{aligned} \int \frac{1}{9x^2 + 25} dx &= \frac{1}{9} \int \frac{1}{x^2 + 25/9} dx \\ &= \frac{1}{9} \left(\frac{1}{5/3} \tan^{-1} \frac{x}{5/3} \right) + C = \frac{1}{15} \tan^{-1} \frac{3x}{5} + C. \end{aligned}$$

Example

Example

Evaluate $\int \frac{1}{x^2 + 4x + 5} dx$.

Solution

The quadratic expression $x^2 + 4x + 5$ is irreducible as its discriminant, $b^2 - 4ac = 16 - 20 = -4 < 0$. Completing square to obtain

$$x^2 + 4x + 5 = (x + 2)^2 + 1.$$

Thus, we have

$$\int \frac{1}{x^2 + 4x + 5} dx = \int \frac{1}{(x + 2)^2 + 1} dx.$$

Solution

Solution

We use $u(x) = x + 2$, with $u'(x) = 1$.

$$\begin{aligned}\int \frac{1}{x^2 + 4x + 5} dx &= \int \frac{1}{u^2 + 1} du \\ &= \tan^{-1} u + C = \tan^{-1} (x + 2) + C.\end{aligned}$$

Example

Example

Evaluate $\int \frac{1}{4x^2 + 4x + 26} dx$.

Solution

First, completing square to obtain

$$4x^2 + 4x + 26 = (2x + 1)^2 + 25 = (2x + 1)^2 + 5^2$$

Therefore, we have

$$\begin{aligned} \int \frac{1}{4x^2 + 4x + 26} dx &= \int \frac{1}{\underbrace{(2x + 1)}_u^2 + 5^2} dx = \frac{1}{2} \int \frac{1}{u^2 + 5^2} du \\ &= \frac{1}{2} \left(\frac{1}{5} \tan^{-1} \left(\frac{u}{5} \right) \right) + C = \frac{1}{10} \tan^{-1} \frac{2x + 1}{5} + C \end{aligned}$$

Another Useful Integral

Theorem

$$\int \frac{2ax+b}{ax^2+bx+c} dx = \ln |ax^2 + bx + c| + C$$

$$\text{-- from } \int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + C$$

Example

$$\begin{aligned} & \int \frac{2x+1}{4x^2+4x+26} dx \\ &= \frac{1}{4} \int \frac{8x+4}{4x^2+4x+26} dx = \frac{1}{4} \ln |4x^2+4x+26| + C. \end{aligned}$$

Example

Example

Evaluate

$$\int \frac{x}{x^2 + 4x + 13} dx.$$

Solution

Check that $x^2 + 4x + 13$ is irreducible: Its discriminant $b^2 - 4ac = 4^2 - 4(1)(13) < 0$. Next, completing square:

$$x^2 + 4x + 13 = (x + 2)^2 + 9.$$

Solution

Solution

Now, we express

$$\frac{x}{x^2 + 4x + 13} = \underbrace{\frac{A(2x + 4)}{x^2 + 4x + 13}}_{\frac{Af'(x)}{f(x)}} + \frac{B}{x^2 + 4x + 13}$$

which gives $A = \frac{1}{2}$ and $B = -2$.

Solution

Solution

$$\begin{aligned}\int \frac{x}{x^2 + 4x + 13} dx &= \frac{1}{2} \underbrace{\int \frac{(2x + 4)}{x^2 + 4x + 13} dx}_{\ln |x^2 + 4x + 13| + C} - 2 \underbrace{\int \frac{1}{x^2 + 4x + 13} dx}_{\int \frac{1}{(x+2)^2 + 3^2} dx} \\&= \frac{1}{2} \ln |x^2 + 4x + 13| - 2 \left(\frac{1}{3} \tan^{-1} \frac{x+2}{3} \right) + C \\&= \frac{1}{2} \ln |x^2 + 4x + 13| - \frac{2}{3} \tan^{-1} \frac{x+2}{3} + C\end{aligned}$$

Example

Example

Evaluate $\int \frac{x-1}{x^3+x} dx$.

Solution

The denominator $x^3 + x = x(x^2 + 1)$, where $x^2 + 1$ is irreducible.

Partial fraction representation:

$$\frac{x-1}{x^3+x} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1) + (Bx+C)(x)}{x(x^2+1)}$$

Comparing coefficients:

$$x^2: 0 = A + B$$

$$x: 1 = C$$

$$x^0: -1 = A$$

Solving for A , B and C , we have $A = -1$, $B = 1$ and $C = 1$.

Solution

$$\int \frac{x-1}{x^3+x} dx = \int -\frac{1}{x} + \frac{x+1}{x^2+1} dx = -\int \frac{1}{x} dx + \int \frac{x+1}{x^2+1} dx$$

- $\int \frac{1}{x} dx = \ln|x| + C_1$
- $\int \frac{x+1}{x^2+1} dx = \int \frac{1}{2} \underbrace{\frac{2x}{x^2+1}}_{\ln|x^2+1|} + \underbrace{\frac{1}{x^2+1}}_{\tan^{-1} x} dx$

So, we have

$$\int \frac{x-1}{x^3+x} dx = -\ln|x| + \frac{1}{2} \ln|x^2+1| + \tan^{-1} x + C$$

Case IV: Repeating Irreducible Factors (Optional)

Suppose $Q(x)$ contains the repeating irreducible quadratic factor $(ax^2 + bx + c)^k$. Then the partial fraction representation of $\frac{P(x)}{Q(x)}$ will contain the term:

$$\frac{A_i x + B_i}{(ax^2 + bx + c)^i}, i = 1, 2, 3, \dots, k$$

Example

Example

Evaluate $\int \frac{x^2}{x(x^2 + 4)^3} dx$.

Solution

Partial fractions:

$$\frac{x^2}{x(x^2 + 4)^3} = \frac{A}{x} + \frac{A_1x + B_1}{x^2 + 4} + \frac{A_2x + B_2}{(x^2 + 4)^2} + \frac{A_3x + B_3}{(x^2 + 4)^3}$$

Proceed like the above examples to solve for A_i , B_i and A , and take care of each partial fraction.

Deg of Numerator is at least Deg of Denominator

Example

$$\int \frac{x^3 + 3x^2}{x^2 + 1} dx.$$

Solution

Applying long division to $\frac{x^3 + 3x^2}{x^2 + 1}$, we obtain that

$$\frac{x^3 + 3x^2}{x^2 + 1} = x + 3 + \frac{-x - 3}{x^2 + 1} = x + 3 - \frac{x}{x^2 + 1} - \frac{3}{x^2 + 1}.$$

Solution

This we can integrate

$$\int \frac{x^3 + 3x^2}{x^2 + 1} dx = \underbrace{\int (x + 3) dx}_{\frac{x^2}{2} + 3x + C_1} - \underbrace{\int \frac{x}{x^2 + 1} dx}_{\frac{1}{2} \ln(x^2 + 1) + C_2} - \underbrace{\int \frac{3}{x^2 + 1} dx}_{3 \tan^{-1} x + C_3}.$$

So

$$\int \frac{x^3 + 3x^2}{x^2 + 1} dx = \frac{x^2}{2} + 3x - \frac{1}{2} \ln(x^2 + 1) - 3 \tan^{-1} x + C.$$

Solution

$$\begin{aligned}\int \frac{x^3 + 2}{x^3 - x} dx &= \int \left(1 + \frac{x + 2}{x^3 - x} \right) dx \\ &= x + \int \frac{x + 2}{x(x - 1)(x + 1)}.\end{aligned}$$

The last integral can be handled with the method of partial fractions ...