NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2017-2018

MH1812 - DISCRETE MATHEMATICS

December, 2017 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **FIVE (5)** printed pages.

2. Answer **ALL** questions. The marks for each question are indicated at the end of each question.

3. Answer each question beginning on a **FRESH** page of the answer book.

4. This **IS NOT** an **OPEN BOOK** exam.

5. Candidates may use calculators. However, they should write down systematically the steps in the workings.

QUESTION 1.

(a) Show $p \to (q \vee r) \equiv (p \wedge \neg q) \to r$. **(10 marks)**

Solution:$p \to (q \vee r) \equiv \neg p \vee (q \vee r) \equiv (\neg p \vee q) \vee r \equiv \neg(p \wedge \neg q) \vee r \equiv (p \wedge \neg q) \to r$.
An alternative solution is to use the truth table. □

(b) Let $X = \{1, 2, 3\}$ and let $\mathbb{P}(X)$ be the power set of X (the set of all subsets of X). A relation $\mathbf{R}$ is defined on $\mathbb{P}(X)$ as follows: for all $A, B \in \mathbb{P}(X)$, $A\mathbf{R}B$ if and only if the number of elements in A equals the number of elements in B.

(i) Show $\mathbf{R}$ is an equivalence relation on $\mathbb{P}(X)$. **(9 marks)**

Solution:

- $\mathbf{R}$ is reflexive: for $\forall A \in \mathbb{P}(X)$, $|A| = |A|$, therefore $A\mathbf{R}A$.
- $\mathbf{R}$ is symmetric: for $\forall A, B \in \mathbb{P}(X)$, if $A\mathbf{R}B$ $|A| = |B|$, therefore $B\mathbf{R}A$.
- $\mathbf{R}$ is transitive: for $\forall A, B, C \in \mathbb{P}(X)$, if $A\mathbf{R}B$ and $B\mathbf{R}C$, then $|A| = |B|$ and $|B| = |C|$. It follows $|A| = |C|$. Therefore $A\mathbf{R}C$.

□

(ii) List all the equivalence classes of $\mathbf{R}$. **(6 marks)**

Solution:

$[\emptyset] = \{\emptyset\}$
$[\{1\}] = \{\{1\}, \{2\}, \{3\}\}$
$[\{1, 2\}] = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$
$[\{1, 2, 3\}] = \{\{1, 2, 3\}\}$

□

QUESTION 2.

(a) Using the characteristic equation, solve the recurrence relation, $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, with $a_0 = 2, a_1 = 1$. **(10 marks)**

Solution:characteristic equation is $x^2 = 7x - 10$, solving it gives two solutions $x = 2$ and $x = 5$, hence $a_n = u \cdot 2^n + v \cdot 5^n$, substituting $a_0 = 2$ and $a_1 = 1$ in, we get $2 = u \cdot 2^0 + v \cdot 5^0 = u + v$ and $1 = u \cdot 2^1 + v \cdot 5^1 = 2u + 5v$, solving it gives $u = 3$ and $v = -1$, hence $a_n = 3 \cdot 2^n - 5^n$. $\square$

(b) Prove by mathematical induction that

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

for all integers $n \geq 2$. **(15 marks)**

Solution: Let $f(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}}$.

Basis Step: $n = 2$, $f(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$, and RHS $= \sqrt{2} < f(2)$, TRUE.

Inductive Step: assume $n = k$, the conclusion holds, i.e.,

$$f(k) > \sqrt{k}$$

When $n = k + 1$,

$$\begin{aligned} f(k+1) &= \tfrac{1}{\sqrt{1}} + \tfrac{1}{\sqrt{2}} + \cdots + \tfrac{1}{\sqrt{k}} + \tfrac{1}{\sqrt{k+1}} \\ &= f(k) + + \tfrac{1}{\sqrt{k+1}} \\ &> \sqrt{k} + \tfrac{1}{\sqrt{k+1}} \\ &> \sqrt{k} + \tfrac{1}{\sqrt{k+1} + \sqrt{k}} \\ &= \sqrt{k} + (\sqrt{k+1} - \sqrt{k}) \\ &= \sqrt{k+1} \end{aligned}$$

Overall, $f(n) > \sqrt{n}$ for all integer $n \geq 2$. (Take note the trick used by the red step)

$\square$

QUESTION 3.
Let set $A = \{1, 2, 3, 4\}$, functions $f, g : A \to A$ by the rules $f(x) = 3^x \bmod 5$ and $g(x) = 2^x \bmod 5$,

(a) is f one-to-one? **(5 marks)**

Solution: $f : 1 \to 3, 2 \to 4, 3 \to 2, 4 \to 1$, so it is one-to-one. $\square$

(b) is g onto? **(5 marks)**

Solution:$g : 1 \to 2, 2 \to 4, 3 \to 3, 4 \to 1$, so it is onto. $\square$

(c) find the composition $f \circ g$ of f and g. **(5 marks)**

Solution:$f \circ g : 1 \to 4, 2 \to 1, 3 \to 2, 4 \to 3$. $\square$

QUESTION 4.

(a) Let A, B, and C be sets, show $(B - A) \cap (C - A) = (B \cap C) - A$. **(10 marks)**

Solution: There are many ways to prove. One can use subset methods, or membership method or existing identities. Here we us the existing identities:

$$(B - A) \cap (C - A) = (B \cap \bar{A} \cap C \cap \bar{A}) = (B \cap C) \cap \bar{A} = (B \cap C) - A.$$

$\square$

(b) From a group of 7 men and 6 women, 5 persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done? **(15 mark)**

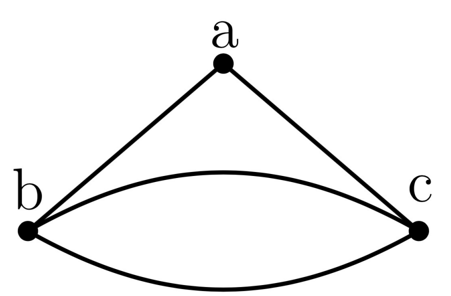
Solution:

$$\binom{7}{3}\binom{6}{2} + \binom{7}{4}\binom{6}{1} + \binom{7}{5}\binom{6}{0}.$$

$\square$

QUESTION 5.

Refer to the graph below, find Euler Circuit and Hamilton Circuit if any, justify your answer if it does not exist. **(10 marks)**



Solution: Since $\deg(a) = 2$, $\deg(b) = 3$, $\deg(c) = 3$, not all degrees are even. Therefore no Euler Circuit.

There is a Hamilton Circuit $bacb$. $\square$

END OF PAPER