NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2020–2021

MH1812 – Discrete Mathematics

May 2021 TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

QUESTION 1. **(16 marks)**

(a) Prove or disprove the following logical equivalence.

$$\neg(q \to \neg p) \vee \neg(r \to \neg p) \equiv (\neg q \to r) \wedge p$$

(b) Decide whether or not the following argument is valid:

$$\begin{aligned} &p \vee \neg q; \\ &\neg q \to r; \\ &r \vee p; \\ &r \wedge \neg s; \\ &\therefore p \end{aligned}$$

Justify your answer.

Solution:

(a) We prove it.

1.

p	q	r	$\neg(q \to \neg p)$	$\neg(r \to \neg p)$	$\neg(q \to \neg p) \vee \neg(r \to \neg p)$	$(\neg q \to r)$	$(\neg q \to r) \wedge p$
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	F	T	F
F	T	F	F	F	F	T	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

2.

$$\begin{aligned} \neg(q \to \neg p) \vee \neg(r \to \neg p) &\equiv \neg(\neg q \vee \neg p) \vee \neg(\neg r \vee \neg p) \\ &\equiv (q \wedge p) \vee (r \wedge p) \\ &\equiv (q \vee r) \wedge p \\ &\equiv (\neg q \to r) \wedge p. \end{aligned}$$

(b) The argument is invalid. Counterexample: $p = F$, $q = F$, $r = T$, and $s = F$.

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(b) Decide whether or not the following argument is valid:

$$p \vee \neg q;$$
$$\neg q \to r;$$
$$r \vee p;$$
$$r \wedge \neg s;$$
$$\therefore p$$

Justify your answer.

$\neg(q \Rightarrow \neg p) \vee \neg(r \Rightarrow \neg p) \equiv \neg(\neg q \vee \neg p) \vee \neg(\neg r \vee \neg p)$

$\equiv (q \wedge p) \vee (r \wedge p)$

$\equiv (q \vee r) \wedge p$

$\equiv (\neg q \Rightarrow r) \wedge p$

$r \wedge \neg s$

r

$\neg p \to r$

$\neg p$

p can be F

$r \vee p$ T

$p \vee \neg q$ T

QUESTION 2. **(16 marks)**

(a) Using the characteristic equation, solve the recurrence relation

$$a_0 = 1,\ a_1 = 2, \quad a_n = 5a_{n-2} + 4a_{n-1} \quad \text{for all } n \geqslant 2,$$

that is, write a_n in terms of n. Justify your answer.

(b) Prove by induction that, for all integers $n \geqslant 1$,

$$\sum_{k=1}^{n} \binom{k}{2} = \frac{(n-1)n(n+1)}{6}.$$

Solution:

(a) The characteristic equation is

$$x^2 - 4x - 5 = (x-5)(x+1) = 0.$$

Thus $a_n = u5^n + v(-1)^n$. Since $a_0 = 1$ and $a_1 = 2$, we must have $u + v = 1$ and $5u - v = 2$. Hence $u = v = 1/2$. Thus, $a_n = (5^n + (-1)^n)/2$.

(b) Let $P(n)$ be the hypothesis that

$$\sum_{k=2}^{n} \binom{k}{2} = \frac{(n-1)n(n+1)}{6}.$$

Basis case: $n = 2$ we have both the LHS and RHS are 1. So $P(1)$ is true. Assume that $P(n)$ is true for some $n \in \mathbb{N}$. Now consider $P(n+1)$. Using the

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$x^2 = 4x + 5$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5,\ x = 1$

$U(\text{root})^n + V(\text{root})^n$

$U(\text{root})^n + V(n)(\text{root})^n$

$U(5)^n + V \to$ Char eq^n

$a_0 = 1 = U + V$

$a_1 = 2 = 5U + V$

$1 = U + 2 - 5U$

$-1 = -6U$

$U = \frac{1}{6} \quad V = \frac{5}{6}$

b) $\sum_{K=1}^{n} \binom{K}{2} = \frac{(n-1)(n)(n+1)}{6}$

let $P(n) = \sum_{K=1}^{n} \binom{K}{2}$

Base case: $P(2) = \binom{2}{2} = 1 \qquad \frac{(2-1)(2)(2+1)}{6} = \frac{6}{6} = 1$

let $P(n) = \frac{(n-1)(n)(n+1)}{6}$ be T

Inductive case

$$P(n+1) = \sum_{K=1}^{n+1} \binom{K}{2} = \frac{(n-1)(n)(n+1)}{6} + \binom{n+1}{2}$$

$$= \frac{(n-1)(n)(n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{(n-1)(n)(n+1) + 3n(n+1)}{6}$$

$$= \frac{((n-1)+3)(n)(n+1)}{6}$$

$$= \frac{n(n+1)(n+2)}{6}$$

hypothesis $P(n)$ we see that the LHS of $P(n+1)$ is

$$\begin{aligned}
\sum_{k=1}^{n+1} \binom{k}{2} &= \sum_{k=1}^{n} \binom{k}{2} + \binom{n+1}{2} \\
&= \frac{(n-1)n(n+1)}{6} + \binom{n+1}{2} \\
&= \frac{(n-1)n(n+1)}{6} + \frac{n(n+1)}{2} \\
&= \frac{(n-1)n(n+1)}{6} + 3\frac{n(n+1)}{6} \\
&= n(n+1)\frac{(n-1+3)}{6} \\
&= \frac{(n+2)n(n+1)}{6},
\end{aligned}$$

as required.

QUESTION 3. **(17 marks)**

A *bit string* is a sequence of 0s and 1s. How many bit strings of length 11 are there

(i) in total?

(ii) that contain exactly two 0s?

(iii) that contain at most three 0s and every 0 is followed immediately by a 1?

Solution:

(i) $2^{11} = 2048$

(ii) $11!/(9!2!) = 55$

(iii) $1 + 10!/9! + 9!/(2!7!) + 8!/(3!5!) = 1 + 10 + 36 + 56 = 103$

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$\underbrace{01 \quad 01 \quad 01 \quad + 5 \text{ more. ones.}}_{8}$

QUESTION 4. **(12 m**

(a) Is the graph X bipartite? Justify your answer.

(b) Does the graph X have

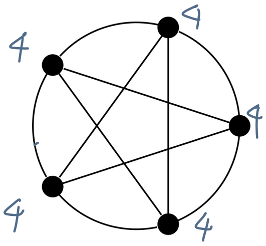
(i) an Euler path?

(ii) a Hamiltonian path?

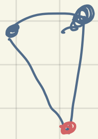
(iii) an Euler circuit?

Justify your answers.

The graph X:



a) Not bipartite. Have triangle



b) i) Yes! all vertices have 4 deg

ii) yes. Trace the star, then the circle

iii) yes, all even.

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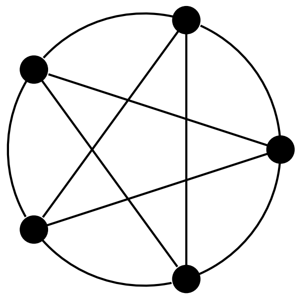

Solution:

(a) (i) no, it has a triangle

(b) (i) no, all degrees are even

(ii) yes, example of a Hamiltonian path

(iii) yes, example of an Euler cycle or state that all degrees are even

QUESTION 4. **(18 marks)**

Let $D = \mathbb{R} - \{0\}$ be the set of real numbers without 0. Let $f : D \to \mathbb{R}$ be given by $f(x) = (2x+1)/x$ and let $g : \mathbb{Z} \to \mathbb{R}$ be given by $g(x) = x/(x^2+1)$.

(a) Show that f is one-to-one.

(b) Is f onto? If yes then prove it, if not then show that there exists an element in the codomain that does not have any preimages.

(c) Is g one-to-one? If yes then prove it, if not then find two distinct elements in the domain that have the same image.

Solution:

(a) Assume $f(x) = f(y)$. Then

$$\begin{aligned}(2x+1)/x &= (2y+1)/y\\ 2xy + y &= 2yx + x.\end{aligned}$$

Hence $x = y$.

(b) No. 2 does not have a preimage. Indeed, suppose $f(x) = 2$. Then

$$\begin{aligned}(2x+1)/x &= 2\\ 2x+1 &= 2x,\end{aligned}$$

which is impossible.

(c) Yes. Assume $g(x) = g(y)$. Then

$$\begin{aligned}x/(x^2+1) &= y/(y^2+1)\\ xy^2 + x &= yx^2 + y.\end{aligned}$$

This implies

$$\begin{aligned}xy^2 - yx^2 + x - y &= 0\\ xy(y-x) + x - y &= 0\\ (xy-1)(y-x) &= 0.\end{aligned}$$

Therefore, we must have $x = y$ or $xy = 1$. The only solutions to $xy = 1$ with x and y both integers is when $x = y = \pm 1$.

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a) Show injective

Let $f(x_1) = f(x_2)$

$(2x_1+1)/x_1 = (2x_2+1)/x_2$

$x_2(2x_1+1) = x_1(2x_2+1)$

$\cancel{2x_1x_2} + x_2 = \cancel{2x_1x_2} + x_1$

$x_1 = x_2$ //

b) Show surjective Not onto. $y = 2$,

$2 = \frac{2x+1}{x}$

$2x = 2x+1$

c) $g(x_1) = g(x_2)$

$\frac{x_1}{x_1^2+1} = \frac{x_2}{x_2^2+1}$

$x_1(x_2^2+1) = x_2(x_1^2+1)$

$x_1x_2^2 + x_1 = x_1^2x_2 + x_2$

$(x_1x_2)(x_2 - x_1) + x_1 - x_2 = 0$

$(x_1x_2)(x_2 - x_1) + (-1)(x_2 - x_1) = 0$

$(x_1x_2 - 1)(x_2 - x_1) = 0$

$x_1x_2 = 1$

However, x_1 & x_2 both integer, $\therefore$ $x_1 = 1$, $x_2 = 1$

$\therefore$ yes, one-one

QUESTION 5. **(21 marks)**

(a) Find the transitive closure of the relation $R = \{(1,2),(2,3),(3,1),(3,4)\}$.

(b) Let R be a relation on a set $A = \{1,2,3\}$. Suppose that R is anti-symmetric but not reflexive. Do there exist such relations for which

$$\exists\,(x,y) \in R,\ \big((x,y) \in R\big) \wedge \big((y,x) \in R\big)?$$

If so, give an example of such a relation; if not, explain why.

(c) For an integer $n \geqslant 5$, let $A = \{1,\dots,n\}$. Consider the cartesian product $P = A \times A \times A \times A \times A$. How many elements $(x_1,\dots,x_5) \in P$ satisfy

$$\sum_{i=1}^{5} x_i = n?$$

Justify your answer.

Solution:

(a) $R^t = \{(1,2),(2,3),(3,1),(3,4),(1,1),(2,2),(3,3),(1,3),(2,1),(3,2),(2,4),(1,4)\}$.

(b) Yes. Example: $R = \{(1,1)\}$

(c) $\binom{n-1}{4}$. Write out a sequence of n ones. There are $n-1$ places to place four partitions. For $i \in \{2,\dots,4\}$, the entry x_i is equal to the number of ones between the $i-1$th and the ith partitions. The entries x_1 and x_5 are equal the number of ones before the first partition and after the last partition repectively.

END OF PAPER

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Justify your answer.

a) $RT = \{(1,2), (2,3), (3,1), (3,4), (1,3), (2,1), (2,4)$

b) y. $R = \{(1,1)\}$

c) ?