System of Linear Equations

Overview and Learning Outcomes

- Linear Equations
  - Determine whether a given equation is linear
  - Determine whether a linear system is consistent or inconsistent
- Homogeneous Linear Systems
  - Recognize homogeneous linear systems
  - Understand the types of solutions for homogeneous linear systems
- Linear Systems and Matrices
  - Represent linear systems in matrix form

Overview and Learning Outcomes

- Existence of a Solution of Linear Systems
  - Find the augmented matrix of a linear system
  - Perform elementary row operations on an augmented matrix
- Solving Linear Systems
  - Use Gaussian elimination and Gauss-Jordan elimination to find the general solution of a linear system
- Linear Combination of Vectors
  - Write a given vector as a linear combination of other vectors
  - Determine the span of a set of vectors

Overview and Learning Outcomes

- Solution of Linear Systems in the form $A\mathbf{x} = \mathbf{b}$
  - Find parametric vector equation of line and plane
  - Find solution as a sum of a particular solution and of a homogeneous system
- Linear independence
  - Determine if a set of vectors is linearly independent
- Linear Transformations
  - Find the image of a vector under a linear transformation
  - Determine the matrix corresponding to a linear transformation

1.1 Linear Equations

Definition. A **linear equation** in n variables $x_1, x_2, \ldots, x_n$ is one that can be expressed in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where $a_1, a_2, \ldots, a_n$ and b are constants and the $a's$, also called coefficients, are not all zero.

A general linear system of m equations in n unknowns (variables) $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

if one $b \neq 0$, non-homo

In the special case where the $b_1 = b_2 = \ldots = b_m = 0$, the linear system is called a **homogeneous** system; otherwise, it is called **non-homogeneous**.

Example:

Linear system of two equations:

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned} \qquad \text{Solution: } x = 1, y = 2$$

Non-linear equations:

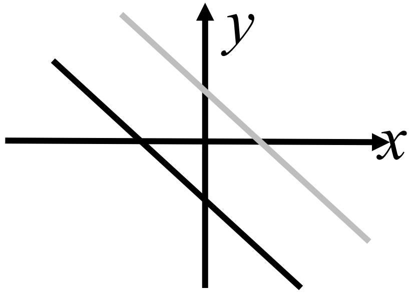
$$\begin{aligned} x + 3y^2 &= 4 \\ \sin x + y &= 0 \end{aligned}$$

A **solution** of a linear system in n unknowns $x_1, x_2, \ldots, x_n$ is a sequence of n numbers $s_1, s_2, \ldots, s_n$ for which the substitution $x_1 = s_1, x_2 = s_2, \ldots, x_n = s_n$ makes each equation a true statement.

A linear system is **consistent** if it has at least one solution and **inconsistent** if it has no solutions.

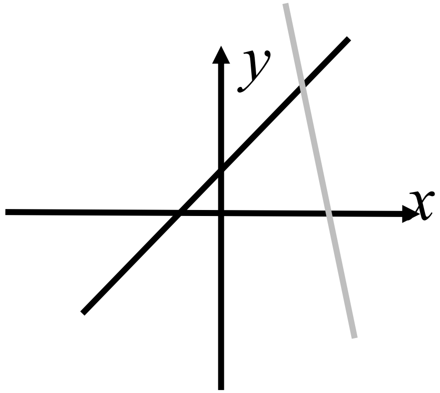
Linear system in two unknowns

Every system of linear equations has *zero, one, or infinitely many* solutions. There is no other possibility (applicable to *any number of unknowns*)



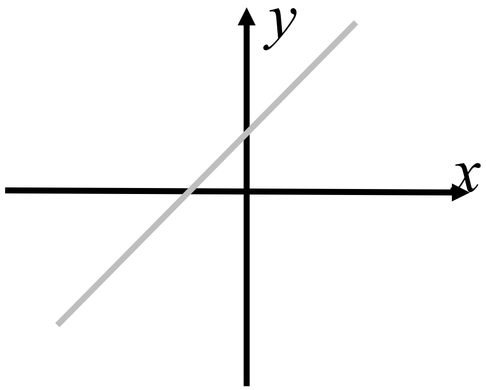
No solution

$$\begin{aligned} x - y &= 4 \\ 3x - 3y &= 6 \end{aligned}$$



One solution

$$\begin{aligned} x - y &= 1 \\ 2x + y &= 6 \end{aligned}$$



Infinitely many solutions (coincident lines)

$$\begin{aligned} 4x - 2y &= 1 \\ 16x - 8y &= 4 \end{aligned}$$

1.2 Homogeneous Linear Systems

A **homogeneous** linear system has the form:

$$\left.\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0 \end{array}\right\}$$

There is a possible solution, where everything = 0.

Every homogeneous system of linear equations is **consistent** because all such systems have $x_1 = 0, x_2 = 0, \ldots, x_n = 0$ as a solution. This solution is called the **trivial solution**; if there are other solutions, they are called **nontrivial solutions**.

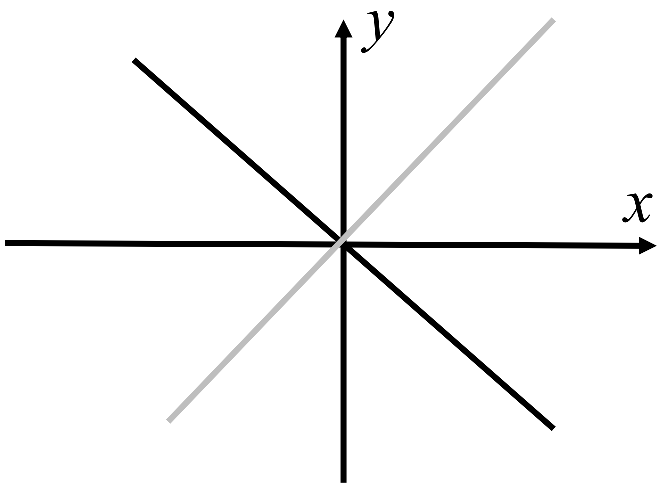
Because a homogeneous linear system always has the trivial solution, there are only two possibilities for its solutions:

- The system has only the trivial solution
- The system has infinitely many solutions in addition to the trivial solution

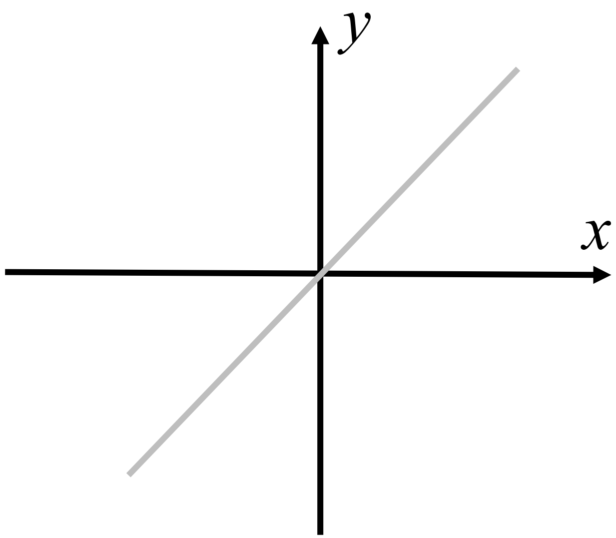
Homogeneous linear system of two equations in two unknowns:

☆ No possible case w/o any solution !

$$\begin{aligned} a_1x + b_1y &= 0 \quad (a_1, b_1 \text{ not both zero}) \\ a_2x + b_2y &= 0 \quad (a_2, b_2 \text{ not both zero}) \end{aligned}$$



Only the trivial solution



Infinitely many solutions (coincident lines)

Matrix notation

The essential information of a linear system is captured in a rectangular array called a **matrix**.

Example:

Linear system of two equations:

$$\begin{aligned} 5x + y &= 3 \\ 2x - y &= 4 \end{aligned}$$

Coefficient matrix : $\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}$

Augmented matrix : $\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

Size of a matrix of m rows and n columns is written as $m \times n$

1.3 Two fundamental questions about a linear system

$\rightarrow$ at least one solution

1. Is the system consistent, i.e., does at least one solution *exist*?
2. If a solution exists, is it the only one, i.e., is the solution *unique*?

To answer these questions: convert augmented matrix into a **row equivalent** matrix

Definition. Two matrices are **row equivalent** if there is a sequence of elementary row operations (EROs) that transforms one matrix into another.

Elementary Row Operations on a matrix/augmented matrix:

1. Multiply a row through by a non-zero constant
2. Interchange two rows
3. Add a constant times one row to another

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \end{bmatrix} \overset{1}{=} \begin{bmatrix} 2 & 4 & 2 \\ 1 & 3 & 4 \end{bmatrix}$$

$$\overset{2}{=} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 2 \end{bmatrix}$$

$$\overset{3}{=} \begin{bmatrix} 1 & 2 & 1 \\ 2\times1+1 & 2\times2+3 & 2\times1+4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$$

Two row equivalent forms:
Row Echelon Form and Reduced Row Echelon Form

A matrix is in Row Echelon Form if it has the following properties:

$$\begin{bmatrix} x & x & x \\ x & x & x \\ 0 & 0 & 0 \end{bmatrix}$$

1. If there are any rows that consist entirely of zeros, then they are grouped together at the bottom of the matrix.
2. In any two consecutive rows that do not consist entirely of zeros, the first nonzero element, called the ***pivot***, in the lower row occurs farther to the right than the pivot in the higher row.

$$\begin{bmatrix} 7 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A matrix is in Reduced Row Echelon Form if it is has the following properties:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is OK

1. The matrix is in row echelon form.
2. In every column containing a pivot, the pivot has value 1 and all other elements in the column are 0.

Start with the 1, look up, make sure all 0

Row Echelon Form and Reduced Row Echelon Form (contd.)

$$\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Row Echelon Form Non-Unique

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced Row Echelon Form Unique

Gaussian elimination: A matrix that is not in row echelon form **can be transformed** into one through a **sequence** of EROs. A different sequence may lead to a different matrix.

Gauss-Jordan elimination: A matrix that is not in reduced row echelon form **can be transformed** into one through a **sequence** of EROs.

Row Echelon Form and Reduced Row Echelon Form (contd.)

Exercise: Transform matrix to row echelon form (Gaussian elimination)

$$\underbrace{\begin{bmatrix}1 & 2 & 3\\ 1 & 1 & -4\\ 7 & 0 & 2\end{bmatrix}}_{r2\leftarrow r2-r1} \rightarrow \underbrace{\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -7\\ 7 & 0 & 2\end{bmatrix}}_{r3\leftarrow r3-7r1} \rightarrow \underbrace{\begin{bmatrix}1 & 2 & 3\\ 0 & -1 & -7\\ 0 & -14 & -19\end{bmatrix}}_{r3\leftarrow r3-14r2} \rightarrow \begin{bmatrix}\boxed{1} & 2 & 3\\ 0 & \boxed{-1} & -7\\ 0 & 0 & \boxed{79}\end{bmatrix}$$

$\square$: pivot

Pivot columns

$$\underbrace{\begin{bmatrix}1 & 2 & 3\\ 1 & 1 & -4\\ 7 & 0 & 2\end{bmatrix}}_{r2\leftrightarrow r1} \rightarrow \underbrace{\begin{bmatrix}1 & 1 & -4\\ 1 & 2 & 3\\ 7 & 0 & 2\end{bmatrix}}_{r2\leftarrow r2-r1} \rightarrow \underbrace{\begin{bmatrix}1 & 1 & -4\\ 0 & 1 & 7\\ 7 & 0 & 2\end{bmatrix}}_{r3\leftarrow r3-7r1} \rightarrow \underbrace{\begin{bmatrix}1 & 1 & -4\\ 0 & 1 & 7\\ 0 & -7 & 30\end{bmatrix}}_{r3\leftarrow r3+7r2} \rightarrow \begin{bmatrix}1 & 1 & -4\\ 0 & 1 & 7\\ 0 & 0 & 79\end{bmatrix}$$

Same here !

Row Echelon Form and Reduced Row Echelon Form (contd.)

Exercise: Transform matrix to reduced row echelon form (Gauss-Jordan elimination)

$$\underbrace{\begin{bmatrix}1&2&3\\1&1&-4\\7&0&2\end{bmatrix}}_{r2\leftarrow r2-r1}\rightarrow\underbrace{\begin{bmatrix}1&2&3\\0&-1&-7\\7&0&2\end{bmatrix}}_{r3\leftarrow r3-7r1}\rightarrow\underbrace{\begin{bmatrix}1&2&3\\0&-1&-7\\0&-14&-19\end{bmatrix}}_{r3\leftarrow r3-14r2}\rightarrow\underbrace{\begin{bmatrix}1&2&3\\0&-1&-7\\0&0&79\end{bmatrix}}_{r1\leftarrow r1+2r2}\rightarrow$$

← gausian

gauss-Jordan

$$\underbrace{\begin{bmatrix}1&0&-11\\0&-1&-7\\0&0&79\end{bmatrix}}_{r3\leftarrow\frac{1}{79}r3}\rightarrow\underbrace{\begin{bmatrix}1&0&-11\\0&-1&-7\\0&0&1\end{bmatrix}}_{r1\leftarrow r1+11r3}\rightarrow\underbrace{\begin{bmatrix}1&0&0\\0&-1&-7\\0&0&1\end{bmatrix}}_{r2\leftarrow -1r2}\rightarrow\underbrace{\begin{bmatrix}1&0&0\\0&1&7\\0&0&1\end{bmatrix}}_{r2\leftarrow r2-7r3}\rightarrow\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Gauss-Jordan elimination consists of two parts – a **forward phase** in which zeros are introduced below the pivots and a **backward phase** in which zeros are introduced above the pivots and the pivots are transformed to 1. If *only the forward phase* is used, then the procedure produces a row echelon form only and is called Gaussian elimination.

1. Is the system consistent, i.e., does at least one solution *exist*?

Exercise:

2. If a solution exists, is it the only one, i.e., is the solution *unique*?

Determine if the following system is consistent:

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array}$$

Solution:

The augmented matrix is $\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{bmatrix}}_{r2 \leftrightarrow r1} \rightarrow \underbrace{\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{bmatrix}}_{r3 \leftarrow r3 - 2r1} \rightarrow \underbrace{\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{bmatrix}}_{r3 \leftarrow r3 + 2r2} \rightarrow \begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{bmatrix}$$

Corresponding linear system:

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 8 \\ 0x_1 & + & 0x_2 & + & 0x_3 & = & 15 \end{array}$$

Last equation is $0 = 15$. Therefore, Inconsistent, i.e., no solutions.

Here, $0+0+0=15$ cannot exist & no solution :)

1. Is the system consistent, i.e., does at least one solution *exist*?
2. If a solution exists, is it the only one, i.e., is the solution *unique*?

1.4 Solutions of Linear Systems

Row reduction of the augmented matrix directly leads to a description of the solution set of a linear system.

Exercise:

$$
\begin{aligned}
x_1 + 3x_2 - 2x_3 + 2x_5 &= 0 \\
2x_1 + 6x_2 - 5x_3 - 2x_4 + 4x_5 - 3x_6 &= -1 \\
5x_3 + 10x_4 + 15x_6 &= 5 \\
2x_3 + 6x_2 + 8x_4 + 4x_5 + 18x_6 &= 6
\end{aligned}
\quad \Rightarrow M = \left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right]
$$

$$
\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{bmatrix}}_{\substack{r2 \leftarrow r2 - 2r1 \\ r4 \leftarrow r4 - 2r1}} \rightarrow
\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -2 & 0 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}}_{r2 \leftarrow -1r2} \rightarrow
\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 0 & 0 & 4 & 8 & 0 & 18 & 6 \end{bmatrix}}_{\substack{r3 \leftarrow r3 - 5r2 \\ r4 \leftarrow r4 - 4r2}} \rightarrow
$$

$$
\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 2 \end{bmatrix}}_{\substack{r3 \leftrightarrow r4 \\ r3 \leftarrow \frac{1}{6} r3}} \rightarrow
\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{r2 \leftarrow r2 - 3r3} \rightarrow
\underbrace{\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{r1 \leftarrow r1 + 2r2} \rightarrow
$$

Row echelon form

infinite solutions?

Solutions of Linear Systems (contd.)

Because x_1, x_2, x_3 can take any value

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form

From the reduced row echelon form, the corresponding system of equations is

$$\begin{aligned} x_1 + 3x_2 \quad + 4x_4 + 2x_5 &= 0 \\ x_3 + 2x_4 &= 0 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

x_1, x_3, and x_6 are called **leading variables**. The other variables are called **free variables**. Solving for the leading variables

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

The general solution of the system is expressed parametrically by assigning the free variables x_2, x_4 and x_5 arbitrary values r, s, and t, respectively. That is,

$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$

Replace w. r, s, t, u, v

Procedure

1. Is the system consistent, i.e., does at least one solution *exist*?
2. If a solution exists, is it the only one, i.e., is the solution *unique*?

Using row reduction to solve a linear system

1. Write the augmented matrix.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in row echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced row echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Gaussian Elimination and Back-Substitution

For small linear systems that are solved by hand, Gauss-Jordan elimination is good. However, for large linear systems, it is more efficient to use Gaussian elimination with back-substitution.

Here, we start with the row echelon form $\begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The corresponding system of equations is

$$\begin{array}{ccccccccccccc} x_1 & + & 3x_2 & - & 2x_3 & & & + & 2x_5 & & & = & 0 \\ & & & & x_3 & + & 2x_4 & & & + & 3x_6 & = & 0 \\ & & & & & & & & & & x_6 & = & \frac{1}{3} \end{array}$$

Solving for the leading variables

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= 1 - 2x_4 - 3x_6 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Beginning with the bottom equation and working upward, successively substitute each equation into all the equations above it.

Gaussian Elimination and Back-Substitution (contd.)

Substituting $x_6 = \frac{1}{3}$ into the second equation

$$\begin{aligned} x_1 &= -3x_2 + 2x_3 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Substituting $x_3 = -2x_4$ into the first equation

$$\begin{aligned} x_1 &= -3x_2 - 4x_4 - 2x_5 \\ x_3 &= -2x_4 \\ x_6 &= \tfrac{1}{3} \end{aligned}$$

Assigning arbitrary values r, s and t to the free variables x_2, x_4 and x_5, we obtain the same solution:

$$x_1 = -3r - 4s - 2t, \quad x_2 = r, \quad x_3 = -2s, \quad x_4 = s, \quad x_5 = t, \quad x_6 = \frac{1}{3}$$

1.5 Linear combination of vectors

Definition. Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and given scalars $c_1, c_2, \ldots, c_p$, the vector $\mathbf{y}$ defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ with **weights** $c_1, c_2, \ldots, c_p$.

Exercise:

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether $\mathbf{b}$ can be written as a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. $\blacksquare$

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ -5 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$. In particular, $\mathbf{b}$ can be generated by a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the above augmented matrix.

1.6 Span

Definition. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ is denoted by Span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ and is called the **subset of $\mathbb{R}^n$ spanned** (or **generated**) **by $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$**.

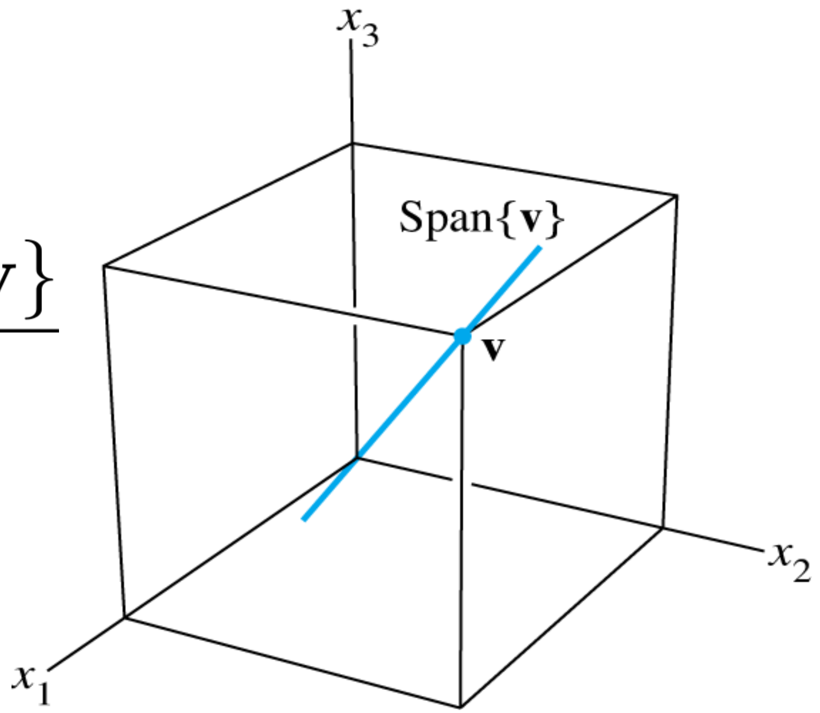
That is, Span $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p$$

with scalars $c_1, c_2, \ldots, c_p$.

Geometric view of Span$\{\mathbf{v}\}$ and Span$\{\mathbf{u}, \mathbf{v}\}$

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^3$. Span $\{\mathbf{v}\}$ is the set of all scalar multiples of $\mathbf{v}$, which is the set of points in the line in $\mathbb{R}^3$ through $\mathbf{v}$ and $\mathbf{0}$.



If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors in $\mathbb{R}^3$, with $\mathbf{v}$ not a multiple of $\mathbf{u}$, then Span$\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{0}$.

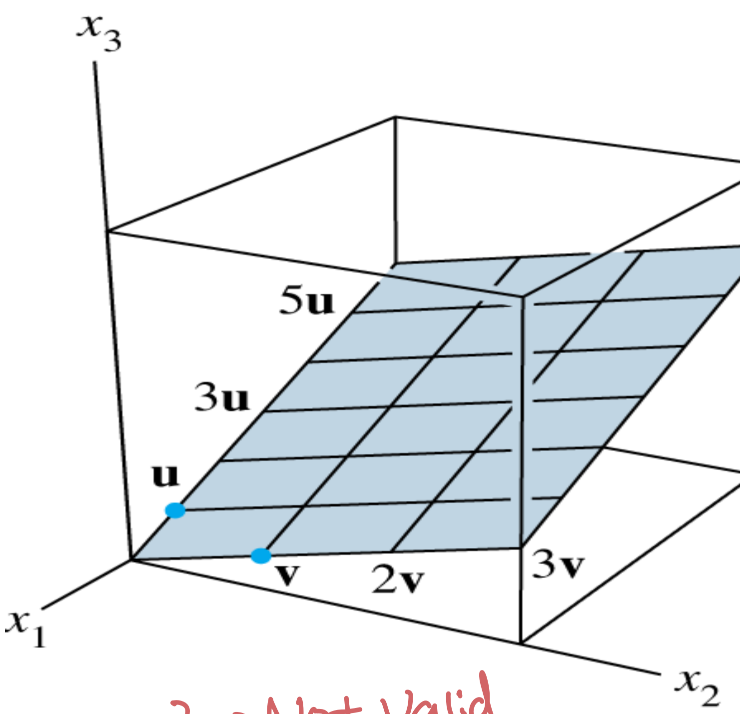
$$\begin{bmatrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 18 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$0x_1 + 0x_2 = 2$? → Not Valid

$\therefore$ No solution



Exercise:

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Determine if $\mathbf{b}$ is in the Span$\{\mathbf{a}_1, \mathbf{a}_2\}$. ■

1.7 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

A general linear system of m equations in n unknowns (variables) $x_1, x_2, \ldots, x_n$ can be written as

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots \quad\quad \vdots \quad\quad\quad \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m, \end{array}$$

which can be written in matrix notation as

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \text{that is,} \quad A\mathbf{x} = \mathbf{b}$$

where A is the matrix of coefficients, $\mathbf{x}$ is the vector of unknowns/variables and $\mathbf{b}$ is the vector of constant terms.

Definition. If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of A and $\mathbf{x}$**, denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights**, i.e.,

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \ldots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n$$

Theorem 1.1. *If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ and if $\mathbf{b}$ is in $\mathbb{R}^m$, then the matrix equation*

$$A\mathbf{x} = \mathbf{b}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n = \mathbf{b}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.

Exercise:

Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & -2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible values of b_1, b_2, b_3?

For $A\mathbf{x} = \mathbf{b}$ to be consistent, $b_1 - \frac{1}{2}b_2 + b_3 = 0$.

The columns of $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$ span a plane through $\mathbf{0}$ in $\mathbb{R}^3$. ■

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & -2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 - b_2 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{bmatrix}$$

$$0x_1 + 0x_2 + 0x_3 = -2b_1 + b_2 - 2b_3$$

$$0 = -2b_1 + b_2 - 2b_3$$

A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^m$ **spans** (or **generates**) $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_p$, i.e., if Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} = \mathbb{R}^m$.

Theorem 1.2. *Let A be an $m \times n$ matrix. Then the following statements are logically equivalent, i.e., for a particular A, either they are all true statements or they are all false.*

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

Theorem 1.3. *If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^m$, and c is a scalar, then:*

- $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$
- $A(c\mathbf{u}) = c(A\mathbf{u})$

Proof of **Theorem 1.2**

(a), (b) and (c) are logically equivalent.
For an arbitrary matrix A, show that (a) and (d) are either both true or both false.

If U is an echelon form of A and $\mathbf{b}$ is in $\mathbb{R}^m$, then we can row reduce $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ for some $\mathbf{d}$ in $\mathbb{R}^m$.

Assume (d) is true. Then each row of U contains a pivot position $\Rightarrow$ No pivot in augmented column $\Rightarrow$ $A\mathbf{x} = \mathbf{b}$ has a solution for any $\mathbf{b}$ $\Rightarrow$ (a) is true.

Assume (d) is false. Then last row of U is all zeros. Let $\mathbf{d}$ be a vector with a 1 in the last entry. Then $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ represents an *inconsistent* system. Since row operations are reversible, $\begin{bmatrix} U & \mathbf{d} \end{bmatrix}$ can be transformed into $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ $\Rightarrow$ $A\mathbf{x} = \mathbf{b}$ is also inconsistent $\Rightarrow$ (a) is false.

1.8 Solution Sets of Linear Systems

- Solutions of Homogeneous Systems

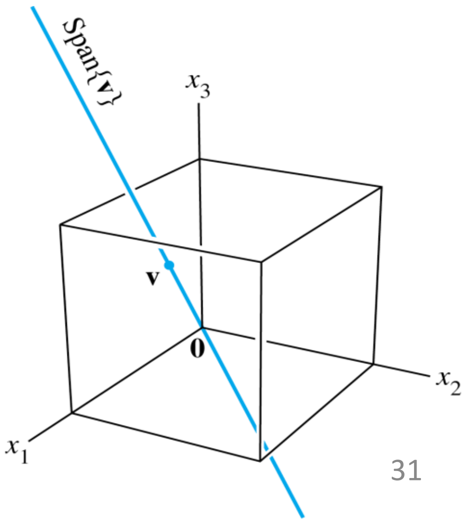
$A\mathbf{x} = \mathbf{b}$: If $\mathbf{b} = \mathbf{0} \Rightarrow$ homogeneous equations and $\mathbf{b} \neq \mathbf{0} \Rightarrow$ non-homogeneous equations

RECALL: For $A\mathbf{x} = \mathbf{0}$, the solution $\mathbf{x}$=$\mathbf{0}$ is called the *trivial solution*. If there is a non zero vector $\mathbf{x}$ that satisfies $A\mathbf{x} = \mathbf{0}$, it is called a *non trivial solution*.

Exercise:

Determine if the following homogeneous system has a nontrivial solution.

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & - & 4x_3 & = & 0 \\ -3x_1 & - & 2x_2 & + & 4x_3 & = & 0 \\ 6x_1 & + & x_2 & - & 8x_3 & = & 0 \end{array}$$



Every solution of $A\mathbf{x} = \mathbf{0}$ in this case is a scalar multiple of $\mathbf{v} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & \boxed{0} & 0 \end{bmatrix}$$

$\rightarrow X_3$ is free varible

$\therefore$ for every choice of X_3 you can have a solution

reduced

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \frac{4}{3}x_3 = 0 \qquad x_2 = 0$$

$$x_1 = \frac{4}{3}x_3$$

$$x = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$= x_3 \cdot \vec{V}$$

Only a single line that gets longer with x_3

- Solutions of Nonhomogeneous Systems

Exercise:

Describe all solutions of $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ -1 \\ -4 \end{bmatrix}$.

Solution:

Row operations on $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$:

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \qquad \begin{array}{rcrcl} x_1 & & -\frac{4}{3}x_3 & = & -1 \\ & x_2 & & = & 2 \\ & & 0 & = & 0 \end{array}$$

Thus $x_1 = -1 + \frac{4}{3}x_3, x_2 = 2$, and x_3 is free.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\mathbf{p}} + x_3 \underbrace{\begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}}$$

$$\begin{bmatrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & -9 & 0 & -18 \end{bmatrix} = \begin{bmatrix} 3 & 5 & -4 & 7 \\ 0 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5/3 & -4/3 & 7/3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 4/3\, x_3 = -1$$

$$x_2 = 2$$

x_3 is free

$$x = \begin{bmatrix} 4/3\, x_3 + 1 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3\, x_3 \\ 0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}$$

Parametric vector form of the solution set of $A\mathbf{x} = \mathbf{b}$:

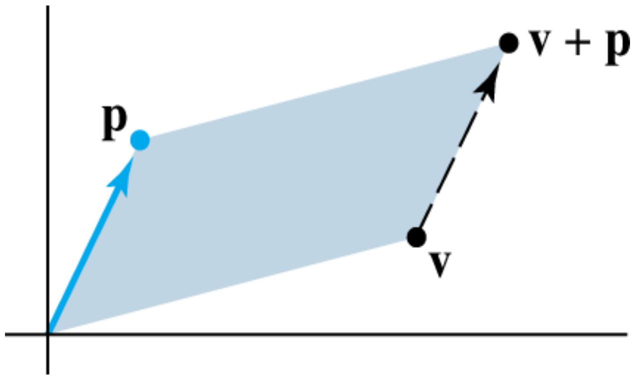
$$\mathbf{x} = \mathbf{p} + x_3\mathbf{v} \quad \text{or} \quad \mathbf{x} = \mathbf{p} + t\mathbf{v}, \quad t \in \mathbb{R}$$

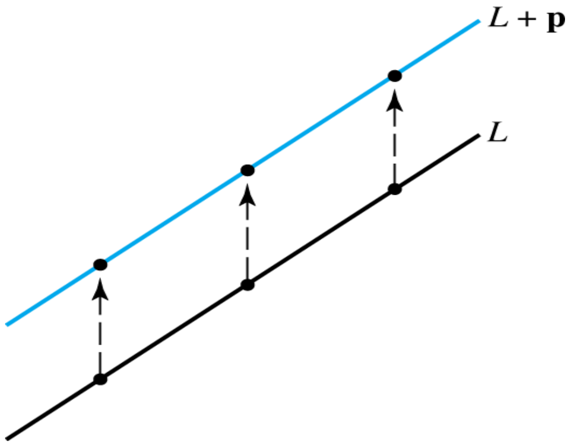
Previous exercise (slide 31): Parametric vector form of the solution set of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t\mathbf{v}$.

Solutions of $A\mathbf{x} = \mathbf{b}$ obtained by adding $\mathbf{p}$ to the solutions of $A\mathbf{x} = \mathbf{0}$.

$\mathbf{p}$ is one particular solution of $A\mathbf{x} = \mathbf{b}$ (corresponding to $t = 0$). ■

Geometrically, vector addition as *translation* $\Longrightarrow$ **v is translated by p to v+p**



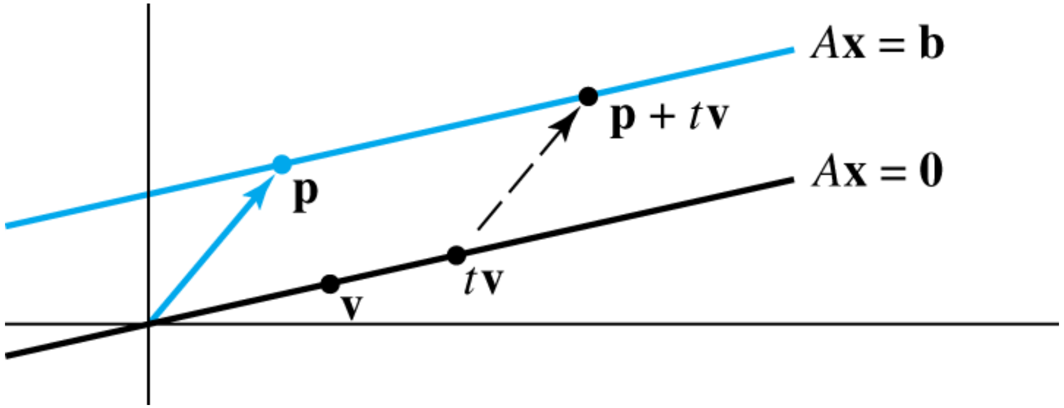


Each point on a line translated by a vector $\mathbf{p}$

Parametric vector form of the solution set of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = t\mathbf{v}$.
This is a line L through $\mathbf{0}$ and $\mathbf{v}$.

Adding $\mathbf{p}$ to each point on L, we get the **the equation of the line through $\mathbf{p}$ parallel to $\mathbf{v}$**.

The solution set of $A\mathbf{x} = \mathbf{b}$ is a line through $\mathbf{p}$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.



Theorem 1.4. *Suppose the equation* $A\mathbf{x} = \mathbf{b}$ *is consistent for some given* $\mathbf{b}$, *and let* $\mathbf{p}$ *be a solution. Then the solution set of* $A\mathbf{x} = \mathbf{b}$ *is the set of all vectors of the form* $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$ *,where* $\mathbf{v}_h$ *is any solution of the homogeneous equation* $A\mathbf{x} = \mathbf{0}$.

1.9 Linear Independence

Definition. An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

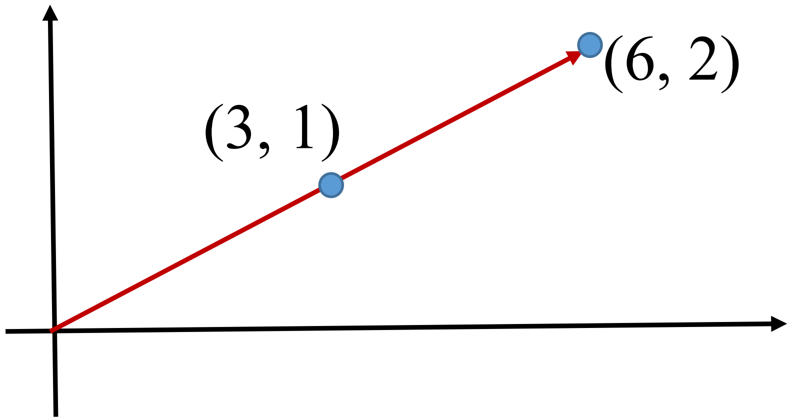
Exercise:

Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$. Determine if the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. If not, what is the linear dependence relation among them? ■

The columns of a matrix A are linearly independent if and only if the equation $A\mathbf{x} = \mathbf{0}$ has *only* the trivial solution.

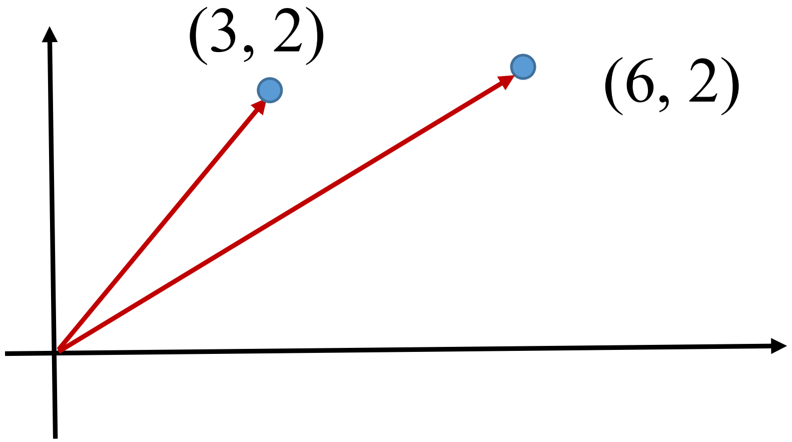
By inspection for simple cases

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$



Linearly Dependent

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

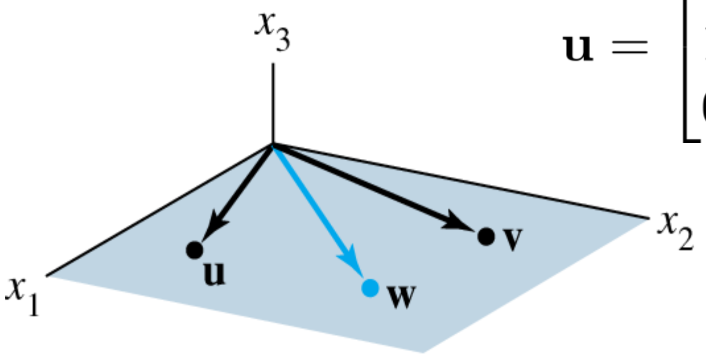


Linearly Independent

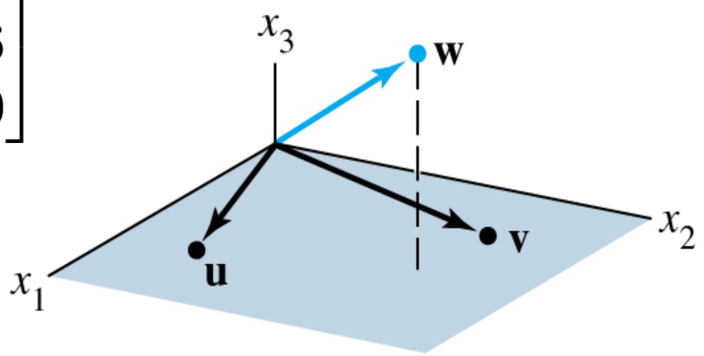
Characterization of linearly dependent sets

An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. In fact, if S is linearly dependent and $\mathbf{v}_1 \neq \mathbf{0}$, then some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$$



Linearly dependent,
w in Span{**u**, **v**}



Linearly independent,
w *not* in Span{**u**, **v**}

$\mathbf{w}$ is in Span$\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly dependent.

Theorem 1.5. *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent, i.e., any set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *in* $\mathbb{R}^n$ *is linearly dependent if* $p > n$.

Proof

Let $A = \begin{bmatrix} \mathbf{v}_1 & \ldots & \mathbf{v}_p \end{bmatrix}$. A is $n \times p$.
$A\mathbf{x} = \mathbf{0}$ corresponds to n equations in p unknowns.
$p > n \implies$ more variables than equations $\implies$ there must be a free variable $\implies$
$A\mathbf{x} = \mathbf{0}$ has non trivial solution $\implies$ columns of A are linearly dependent. ■

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If $p > n$, the columns are linearly dependent.

Example

The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent. Note: none of the vectors is a multiple of one of the other vectors.

Theorem 1.6. *If a set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.*

Proof

By renumbering the vectors, we may suppose $\mathbf{v}_1 = \mathbf{0}$. Then the equation $1\mathbf{v}_1 + 0\mathbf{v}_2 + \ldots + 0\mathbf{v}_p = \mathbf{0}$ shows that S is linearly dependent. ■

1.10 Introduction to Linear Transformations

Think of $A\mathbf{x} = \mathbf{b}$ as :

matrix A as an object that "acts" on a vector $\mathbf{x}$ by multiplication to produce a new vector called $A\mathbf{x}$.

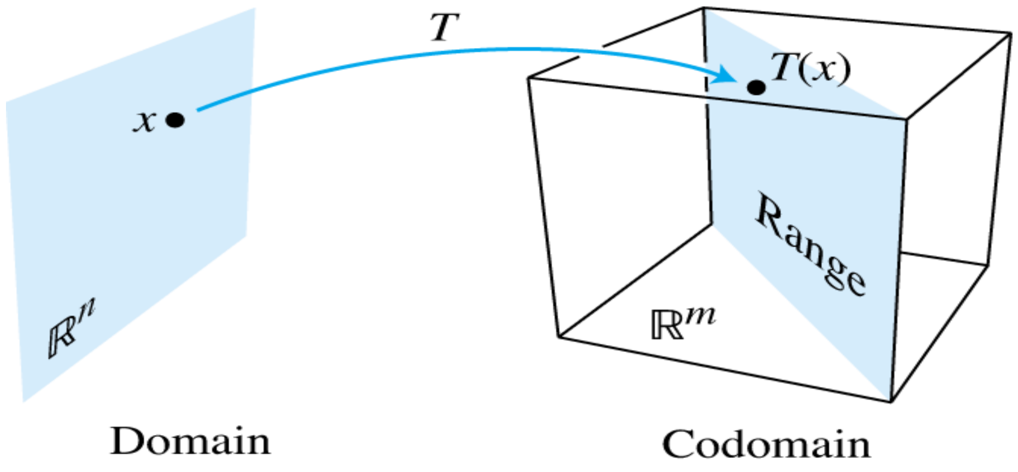
A **transformation** T from $\mathbb{R}^n$ to $\mathbb{R}^m$ (denoted by $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$) is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ a vector $T(\mathbf{x})$ in $\mathbb{R}^m$.

$\mathbb{R}^n$ - **domain** of T

$\mathbb{R}^m$ - **codomain** of T

$T(\mathbf{x})$ - **image** of $\mathbf{x}$

Set of all images $T(\mathbf{x})$ - **range** of T



Domain, codomain, and range of $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

Exercise:

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

a. Find $T(\mathbf{u})$.

b. Find an $\mathbf{x}$ in $\mathbb{R}^2$ whose image under T is $\mathbf{b}$.

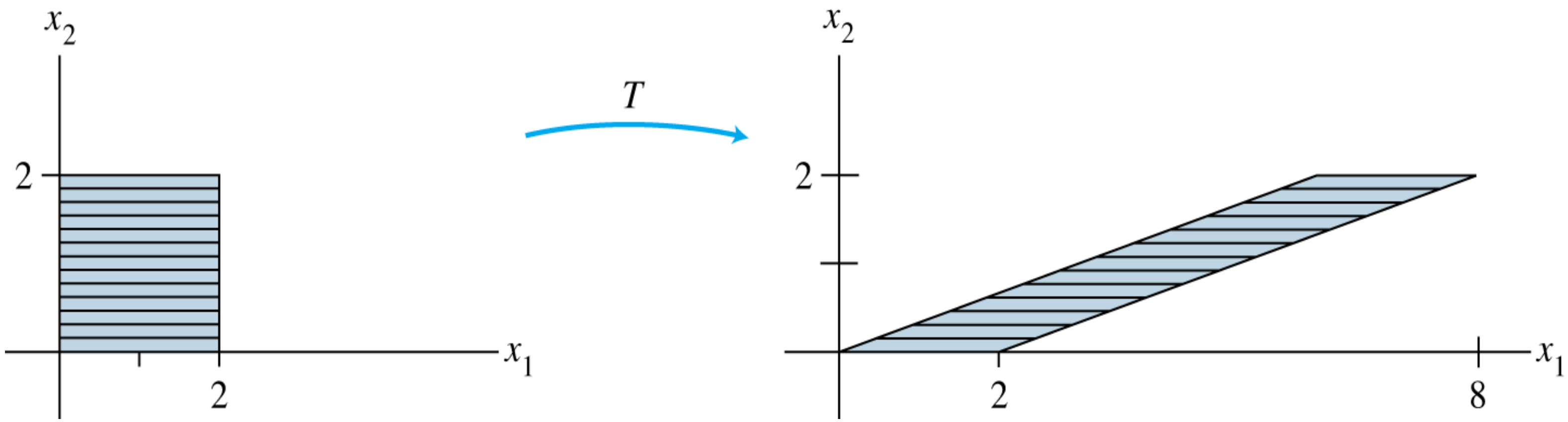
c. Is there more than one $\mathbf{x}$ whose image under T is $\mathbf{b}$?

d. Determine if $\mathbf{c}$ is in the range of T.

Examples:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. What is this transformation?

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. What is this transformation?



1.11 Linear Transformations

Definition. A tranformation T is **linear** if:

i. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in the domain of T;

ii. $T(c\mathbf{u}) = cT(\mathbf{u})$

Generalization:

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p)$$

Examples:

- $T(\mathbf{x}) = r\mathbf{x}, \quad T : \mathbb{R}^2 \to: \mathbb{R}^2$ is called a **contraction** when $0 \leq r \leq 1$ and a **dilation** when $r > 1$.
- Show that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a linear transformation by finding the images of $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

1.12 The Matrix of a Linear Transformation

Given the description of a linear transformation 'in words', how to obtain the corresponding matrix?

Key observation: T is completely determined by what it does to the columns of the $n \times n$ identity matrix I_n.

Exercise:

Let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ such that $T(\mathbf{e}_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(\mathbf{e}_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. Find the formula for the image of an arbitrary $\mathbf{x}$ in $\mathbb{R}^2$.

Theorem 1.7. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all} \quad \mathbf{x} \textit{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose j^{th} column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the identity matrix in $\mathbb{R}^n$:

$$A = \left[T(\mathbf{e}_1) \quad \ldots \quad T(\mathbf{e}_n) \right].$$

■

A is called the **standard matrix for the linear transformation** T.

Exercise:

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle Φ, with counterclockwise rotation for a positive angle. Find the standard matrix A for this transformation.

***** END OF CHAPTER *****

Theorem 1.7. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists a unique matrix A such that*

$$T(\mathbf{x}) = A\mathbf{x} \quad \textit{for all} \quad \mathbf{x} \textit{ in } \mathbb{R}^n.$$

In fact, A is the $m \times n$ matrix whose j^{th} column is the vector $T(\mathbf{e}_j)$, where $\mathbf{e}_j$ is the j^{th} column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & \dots & T(\mathbf{e}_n) \end{bmatrix}.$$

■

A is called the **standard matrix for the linear transformation** T.

Exercise:

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle Φ, with counterclockwise rotation for a positive angle. Find the standard matrix A for this transformation.

***** END OF CHAPTER *****