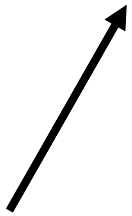
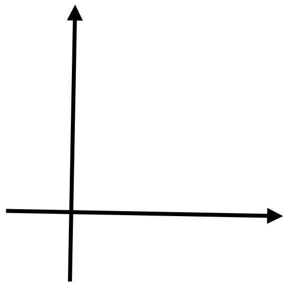
Vector Spaces

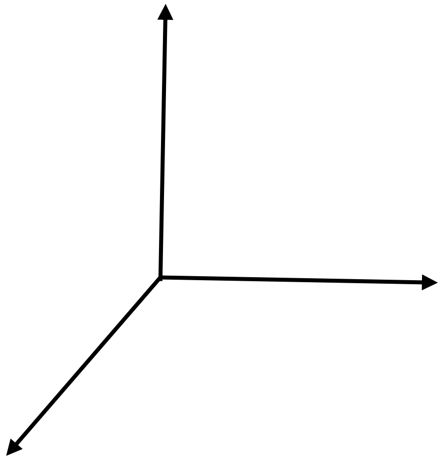
Expanding the definition of vectors and the spaces in which they live

Beyond :



$$\begin{bmatrix} 4 \\ 9 \\ 32.5 \end{bmatrix}$$





Overview and Learning Outcomes

- Vector spaces
  - Determine whether a given set with two operations is a vector space
- Subspaces
  - Determine whether a subset of a vector space is a subspace
  - Determine whether a set S of vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$
- Basis
  - Show that a set of vectors is a basis for a vector space
  - Find the coordinates of a vector relative to a basis

Overview and Learning Outcomes

- Dimension
  - Find a basis for and dimension of null space and column space of a matrix
- Rank
  - Find the rank of a matrix
  - Find the dimension of the row space of a matrix
- Linear Transformation
  - Determine whether a transformation is linear
  - Find the standard matrix for a linear transformation

4.1 Vector spaces

Definition. A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u} = c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$.
10. $1\mathbf{u} = \mathbf{u}$.

Following simple facts can be proved from the axioms: For each $\mathbf{u}$ in V and scalar c,

- $0\mathbf{u} = \mathbf{0}$.
- $c\mathbf{0} = \mathbf{0}$.
- $-\mathbf{u} = (-1)\mathbf{u}$.

Examples of vector spaces

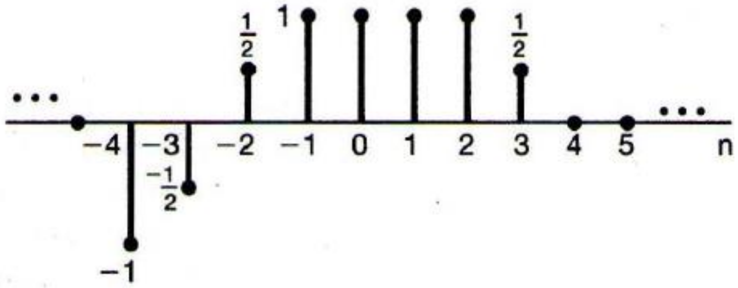
1. $\mathbb{R}^n$

2. $\mathbb{S}$: space of all doubly infinite sequences of numbers

$$\{y_k\} = (\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots)$$

Define addition: If $\{z_k\}$ is another element in $\mathbb{S}$, then sum $\{y_k\} + \{z_k\}$ is the sequence $\{y_k + z_k\}$ formed by adding corresponding terms of $\{y_k\}$ and $\{z_k\}$.

Define scalar multiplication: The scalar multiple $c\{y_k\}$ is the sequence $\{cy_k\}$.



Examples of vector spaces (contd.)

3. $\mathbb{P}_n$: polynomials of degree n, $(n > 0)$

$$\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

Degree : highest power of t in $\mathbf{p}$
If $\mathbf{p}(t) = a_0 \neq 0$, degree of $\mathbf{p}$ is zero.

If all the coefficients are zero, $\mathbf{p}$ is called the *zero polynomial*.

Define addition: If $\mathbf{q}(t) = b_0 + b_1 t + b_2 t^2 + \cdots + b_n t^n$, then the sum $\mathbf{p} + \mathbf{q}$ is defined by

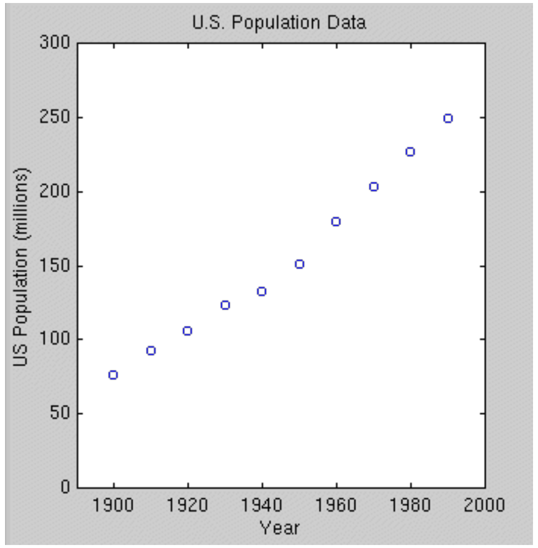
$$\begin{aligned} (\mathbf{p} + \mathbf{q})(t) &= \mathbf{p}(t) + \mathbf{q}(t) \\ &= (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n \end{aligned}$$

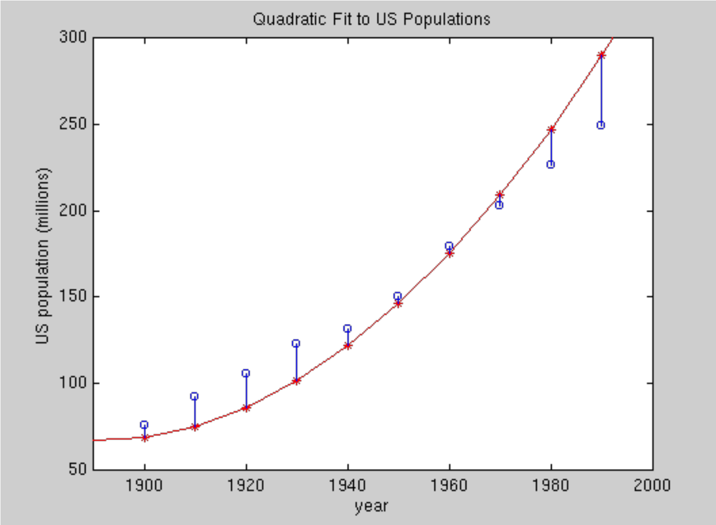
Define scalar multiplication: The scalar multiple $c\mathbf{p}$ is the polynomial

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = ca_0 + (ca_1)t + \cdots + (ca_n)t^n$$

- Axioms 1 and 6: satisfied because $\mathbf{p}+\mathbf{q}$ and $c\mathbf{p}$ are polynomials of degree less than or equal to n
- Axioms 2, 3, and 7-10: satisfied because of properties of real numbers
- Axiom 4: zero polynomial acts as the zero vector
- Axiom 5: $(-1)\mathbf{p}$ acts as the negative of $\mathbf{p}$

Non-linear curve fitting





Examples of vector spaces (contd.)

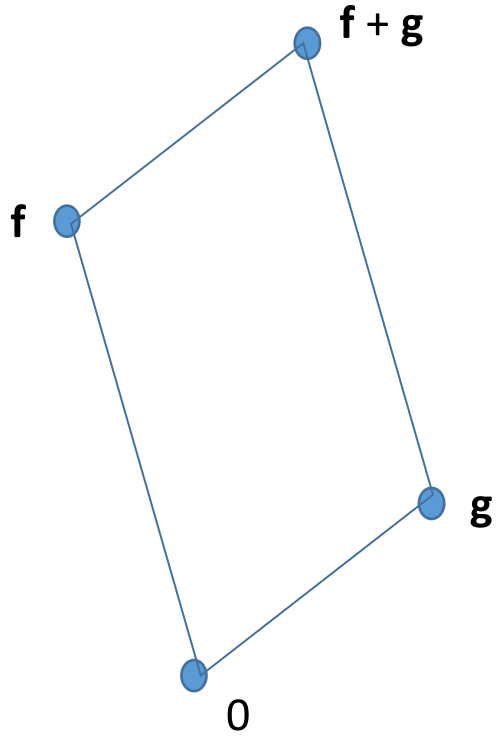
4. Let V be the set of all real-valued functions defined on a set $\mathbb{D}$

Define addition: $\mathbf{f} + \mathbf{g}$ is the function whose value at t in the domain $\mathbb{D}$ is $\mathbf{f}(t) + \mathbf{g}(t)$.

Define scalar multiplication: The scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$.

- Axioms 1 and 6: obvious
- Axiom 4: zero vector is the function that is identically zero, $\mathbf{f}(t) = 0$
- Axiom 5: Negative of $\mathbf{f}$ is $(-1)\mathbf{f}$
- Rest of the axioms: satisfied because of properties of real numbers

Think of an element of a vector space as one “point” or vector in the vector space



4.2 Subspaces

Definition. A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

b. H is closed under vector addition, i.e., for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u} + \mathbf{v}$ is in H.

c. H is closed under scalar multiplication, i.e., for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

In applications, subspaces of $\mathbb{R}^n$ arise

- as the set of all solutions to a system of homogeneous linear equations
- as the set of all linear combinations of certain specified vectors

Subspace H of V is itself a *vector space* under vector space operations already defined in V.

- Axioms 1, 4 and 6: same as (a), (b) and (c)
- Axioms 2, 3, and 7-10: automatically true in H because they apply to all elements of V, including those in H
- Axiom 5: if $\mathbf{u}$ is in H, then $(-1)\mathbf{u}$ is in H [by property (c) and by $(-1)\mathbf{u} = -\mathbf{u}$]

Examples of subspaces

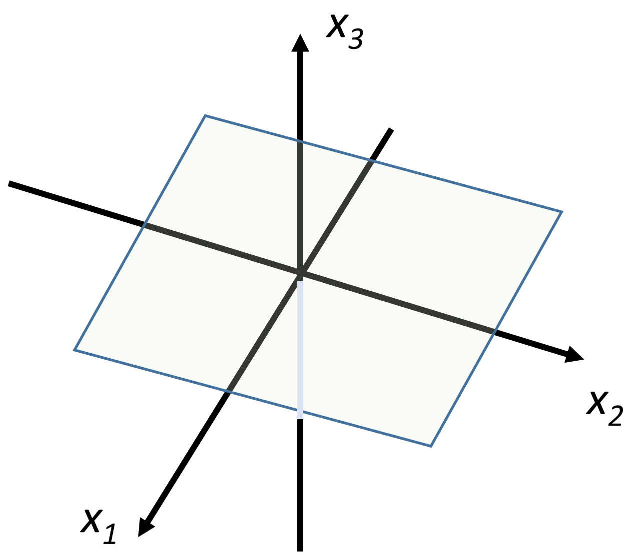
1. The set consisting of only the zero vector in a vector space V : **zero subspace** written as $\{\mathbf{0}\}$.

Examples of subspaces (contd.)

2. $\mathbb{P}$: set of all polynomials with real coefficients, with operations in $\mathbb{P}$ defined as for functions.
For each $n \geq 0$, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$. [properties (a), (b) and (c)]

3. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$?

Is the set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$ a subset of $\mathbb{R}^3$?



4.3 Subspace spanned by a set

Recall: Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

Exercise 4.3.1

Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

Theorem 4.1. *If $\mathbf{v}_1, \cdots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is a subspace of V.*

Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is called **the subspace spanned** by $\mathbf{v}_1, \cdots, \mathbf{v}_p$.
Given any subspace H of V, a **spanning set** for H is a set $\mathbf{v}_1, \cdots, \mathbf{v}_p$ in H such that $H = \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

Exercise 4.3.2

Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$ where a and b are in $\mathbb{R}$. Show that H is a subspace of $\mathbb{R}^4$.

Solution:

$$H = \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \underset{\substack{\uparrow \\ \mathbf{v}_1}}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} + b \underset{\substack{\uparrow \\ \mathbf{v}_2}}{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}.$$

$$H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}.$$

$\therefore H$ is a subspace of $\mathbb{R}^4$.

4.4 The Null Space of a Matrix

System of homogeneous equations

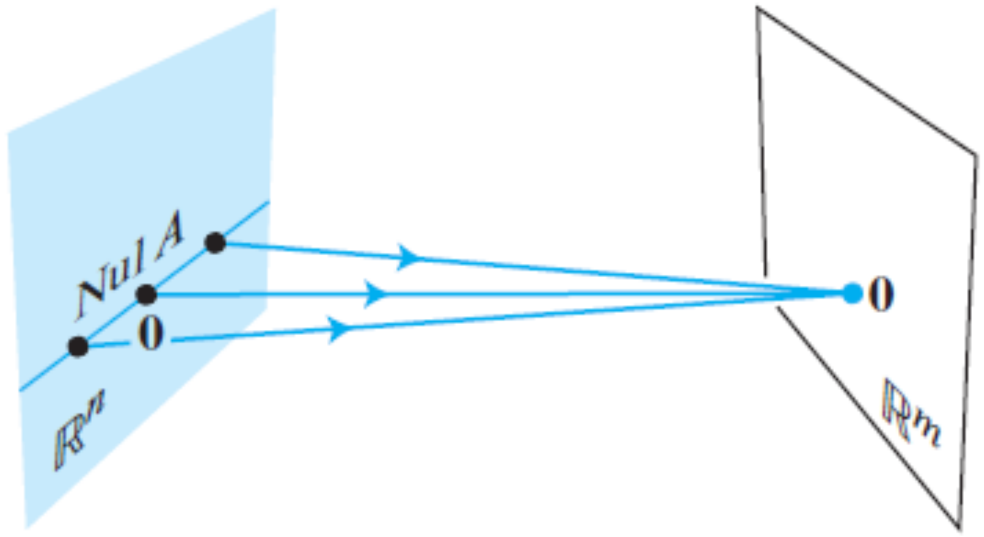
$$\begin{array}{rcrcrcl} x_1 & - & 3x_2 & - & 2x_3 & = & 0 \\ -5x_1 & + & 9x_2 & + & x_3 & = & 0 \end{array}$$

Rewritten as $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

Null space: the set of $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{0}$

Definition. The **null space** of an $m \times n$ matrix A, written as $\mathbf{N}(A)$, is the set of all soutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, i.e.,

$$\mathbf{N}(A) = \{\mathbf{x} : \mathbf{x} \ \text{ is in } \ \mathbb{R}^n \ \text{ and } \ A\mathbf{x} = \mathbf{0}\}$$



All $\mathbf{x}$ in $\mathbb{R}^n$ mapped into the zero vector in $\mathbb{R}^m$ via the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

<u>Exercise 4.4.1</u>

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $\mathbf{u}$ belongs to $\mathbf{N}(A)$.

Exercise 4.4.2

Describe the null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution

Apply elimination on $A\mathbf{x} = \mathbf{0}$.

$$\begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ 3x_1 & + & 6x_2 & = & 0 \end{array} \rightarrow \begin{array}{rcrcl} x_1 & + & 2x_2 & = & 0 \\ & & 0 & = & 0 \end{array}$$

The line $x_1 + 2x_2 = 0$ is $\mathbf{N}(A)$. It contains all solutions (x_1, x_2).

Set free variable x_2 to some value, say, 1. Then $x_1 = -2$.

$\mathbf{N}(A)$ contains all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Theorem 4.2. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Proof. $\square$

Exercise 4.4.3

Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution

Apply elimination on $A\mathbf{x} = \mathbf{0}$ to obtain reduced row echelon form of augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{rcrcrcrcrcl} x_1 & - & 2x_2 & & & - & x_4 & + & 3x_5 & = & 0 \\ & & & & x_3 & + & 2x_4 & - & 2x_5 & = & 0 \\ & & & & & & & & 0 & = & 0 \end{array}$$

General solution : $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + x_5$, with x_2, x_4, and x_5 as free variables.

Decompose the vector giving the general solution into a linear combination of vectors where *the weights are the free variables.*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

Every linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is an element of $\mathbf{N}(A)$ and vice versa. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $\mathbf{N}(A)$.

4.5 The Column Space of a Matrix

Definition. The **column space** of an $m \times n$ matrix A, written as $\mathbf{C}(A)$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then $\mathbf{C}(A) = \text{Span}\{\mathbf{a}_1 \cdots \mathbf{a}_n\}$, i.e.,

$$\mathbf{C}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

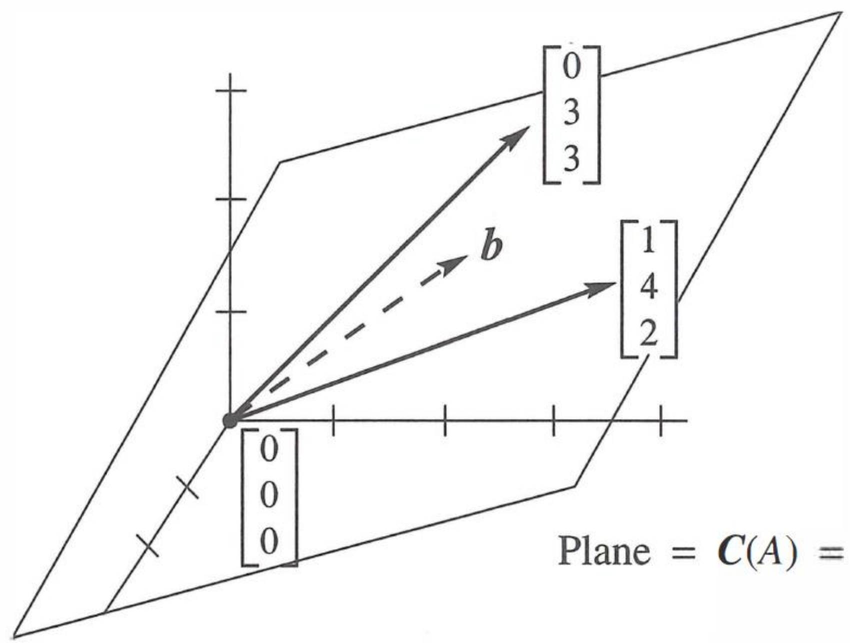
Theorem 4.3. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Proof. $\text{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ is a subspace [Theorem 4.1]

Columns of A are in $\mathbb{R}^m$. $\square$

Recall (from Chapter 1, slide 29)): The columns of A span $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$.

Restating the above:
The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\boldsymbol{b} = .4 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + .3 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$A\boldsymbol{x} = \boldsymbol{b} \text{ has } \boldsymbol{x} = \begin{bmatrix} .4 \\ .3 \end{bmatrix}$$

Plane $=$ $\boldsymbol{C}(A)$ $=$ all vectors $A\boldsymbol{x}$

Exercise 4.5.1

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$.

a. If the column space of A is a subspace of $\mathbb{R}^k$, what is k?

b. If the null space of A is a subspace of $\mathbb{R}^k$, what is k?

c. Find a nonzero vector in $\mathbf{C}(A)$ and a nonzero vector in $\mathbf{N}(A)$.

d. If $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

i. Determine if $\mathbf{u}$ is in $\mathbf{N}(A)$. Could $\mathbf{u}$ be in $\mathbf{C}(A)$?

ii. Determine if $\mathbf{v}$ is in $\mathbf{C}(A)$. Could $\mathbf{v}$ be in $\mathbf{N}(A)$?

Nul A $\equiv \mathbf{N}(A)$

Col A $\equiv \mathbf{C}(A)$

$\mathbf{N}(A)$	$\mathbf{C}(A)$
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

4.6 Kernel and Range of a Linear Transformation

- Generalize definition of linear transformation to include vector spaces

Definition. A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

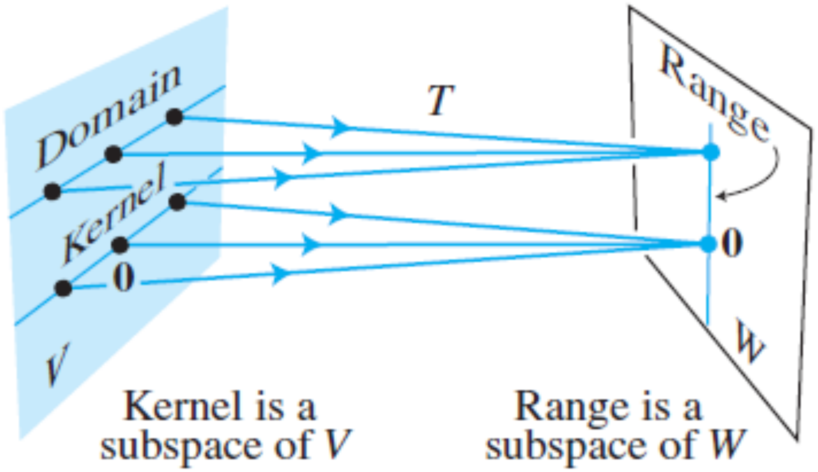
1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

Definition. The **kernel**(or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W).

Definition. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

If $T(\mathbf{x}) = A\mathbf{x}$, then

- Kernel $= \mathbf{N}(A)$
- Range $= \mathbf{C}(A)$



4.7 Bases

- Linear independence (again! introduced in Sec 1.9 for $\mathbb{R}^n$)

Definition. An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

- A set containing a single vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
- A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of other.
- Any set containing the zero vector is linearly dependent.

Theorem 4.4. *An indexed set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *of two or more vectors, with* $\mathbf{v}_1 \neq \mathbf{0}$*, is linearly dependent if and only if some* $\mathbf{v}_j$ *(with* $j > 1$*) is a linear combination of the preceding vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

(Skip proof)

Examples: Linearly independent or not?

- $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$.
- The set $\{\sin t, \cos t\}$ in $C[0,1]$, the space of all continuous functions on $0 \leq t \leq 1$
- The set $\{\sin t \cos t, \sin 2t\}$ in $C[0,1]$, the space of all continuous functions on $0 \leq t \leq 1$

Definition. Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H, i.e.,

$$H = \mathrm{Span}\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$$

- Also true when $H = V$ since every vector space is a subspace of itself.
  $\Rightarrow$ a basis of V is a linearly independent set that spans V.

Examples:

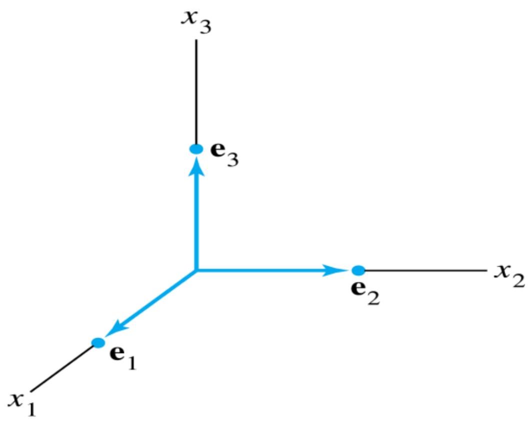
- Invertible matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$

  Columns of A form a basis for $\mathbb{R}^n$

  $\left.\begin{array}{l} \text{– they are linearly independent} \\ \text{– they span } \mathbb{R}^n \end{array}\right\}$ Invertible matrix theorem (Theorem 2.4)

- Columns of I_n

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$



The standard basis for $\mathbb{R}^3$.

The set $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$

Exercise 4.7.1:

Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

Exercise 4.7.2:

Let $S = \{1, t, t^2, \ldots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$.

4.8 The Spanning Set Theorem

Exercise 4.8.1:

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$. Show that $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Then, find a basis for the subspace H.

Theorem 4.5. *Let* $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *be a set in* V, *and let* $H = Span\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

a. If one of the vectors in S *- say,* $\mathbf{v}_k$ *- is a linear combination of the remaining vectors in* S, *then the set formed from* S *by removing* $\mathbf{v}_k$ *still spans* H.

b. If $H \neq \{\mathbf{0}\}$, *some subset of* S *is a basis for* H.

(Skip proof)

4.9 Bases for $\mathbf{N}(A)$ and $\mathbf{C}(A)$

$\mathbf{N}(A)$

Recall from Exercise 4.4.3:

- found vectors that span $\mathbf{N}(A)$
- vectors were linearly independent (when $\mathbf{N}(A)$ contains non-zero vectors)

So, that method produces a *basis* for $\mathbf{N}(A)$.

$\mathbf{C}(A)$

Exercise 4.9.1

Find a basis for $\mathbf{C}(B)$, where

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Solution

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

If $p > n$, the columns are linearly dependent.

Each nonpivot column of B is a linear combination of the pivot columns.

$\mathbf{b}_2 = 4\mathbf{b}_1, \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$

By Theorem 4.5, discard $\mathbf{b}_2$ and $\mathbf{b}_4$; $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span $\mathbf{C}(B)$. Let

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

By Theorem 4.4, since $\mathbf{b}_1 \neq 0$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent.

$\therefore S$ is a basis for $\mathbf{C}(B)$.

- For a matrix A that is *not* in reduced row echelon form

Recall: Any linear dependence relationship among the columns of A can be expressed in the form $A\mathbf{x} = \mathbf{0}$.

If A is row reduced to B, $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have exactly the same set of solutions.

If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$, then

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \text{ and } x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = \mathbf{0}$$

have the same set of solutions.

Columns of A have *exactly the same linear dependence relationships* as the columns of B.

Exercise 4.9.2

It can be shown that

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix B in Exercise 4.9.1. Find a basis for $\mathbf{C}(A)$.

Solution

In Exercise 4.9.1, $\mathbf{b}_2 = 4\mathbf{b}_1, \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.

So, $\mathbf{a}_2 = 4\mathbf{a}_1, \mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$ (Check!)

Discard $\mathbf{a}_2$ and $\mathbf{a}_4$. For $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ to be linearly independent, $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ should also be linearly independent, which is true.

Therefore, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is a basis for $\mathbf{C}(A)$.

Theorem 4.6. *The pivot columns of a matrix A form a basis for $\mathbf{C}(A)$.*

Proof. Let B be the reduced echelon form of A.

The set of pivot columns of B are linearly independent.

Since A is row equivalent to B, the pivot columns of A are linearly independent, AND

Every nonpivot column of A is a linear combination of the pivot columns of A.

So, the nonpivot columns of A can be discarded from the spanning set for $\mathbf{C}(A)$ [Spanning Set theorem].

This leaves the pivot columns of A as the basis for $\mathbf{C}(A)$. □

4.10 Coordinate Systems

Let $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \cdots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Definition. Suppose $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis** $\mathcal{B}$ (or the $\mathcal{B}-$**coordinates of x**) are the weights $c_1, \cdots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Co-ordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$, or the $\mathcal{B}-$**coordinate vector of x**

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Exercise 4.10.1

Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2,\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$. Suppose an $\mathbf{x}$ in $\mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix}-2\\3\end{bmatrix}$. Find $\mathbf{x}$.

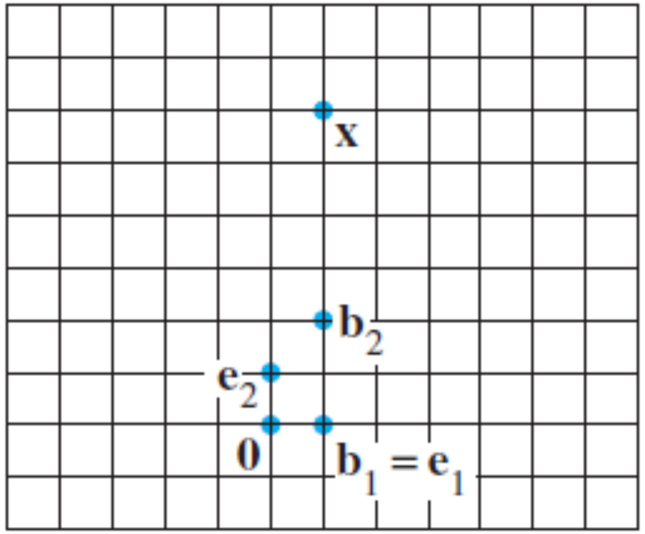
Solution

$$\mathbf{x} = (-2)\mathbf{b}_1 + 3\mathbf{b}_2 = -2\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\6\end{bmatrix}$$

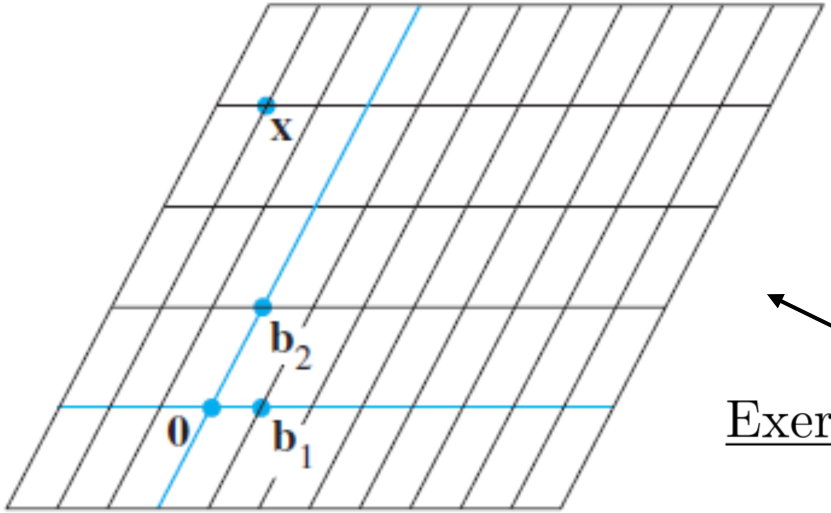
Example

Consider the *standard basis* $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \}$. If $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$,

then $\mathbf{x} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, i.e., $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$.



Standard graph paper



$\mathcal{B}-$graph paper

Exercise 4.10.1

Exercise 4.10.2

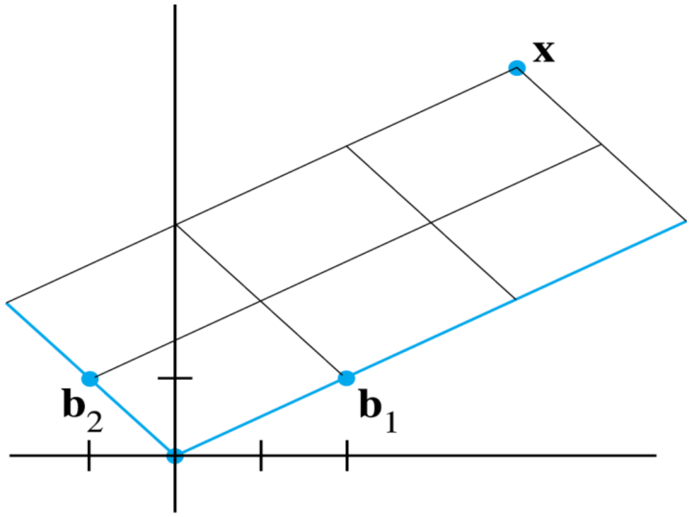
Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.

Solution

Let $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

or $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ or $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$



The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(3, 2)$.

$$\underset{\overset{\uparrow}{P_{\mathcal{B}}}}{\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}} \underset{\overset{\uparrow}{[\mathbf{x}]_{\mathcal{B}}}}{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}} = \underset{\overset{\uparrow}{\mathbf{x}}}{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}$$

The matrix $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ changes the $\mathcal{B}-$coordinates of a vector $\mathbf{x}$ into the standard coordinates of $\mathbf{x}$.

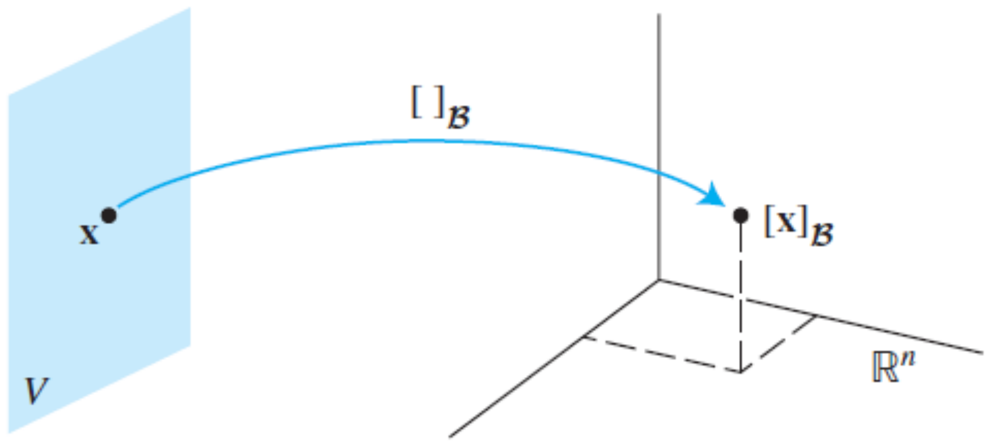
$$\mathbf{x} = P_{\mathcal{B}} \, [\mathbf{x}]_{\mathcal{B}}$$

Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^2$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem). Therefore,

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$ can be seen as a one-to-one linear transformation $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$

Extending to $\mathbb{R}^n$:



If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, c_p$ are scalars, then

$$[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}}$$

The $\mathcal{B}-$coordinate vector of a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ is the *same* linear combinatin of their coordinate vectors.

4.11 Dimension of a Vector Space

Definition. If V is a vector space, the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

Examples

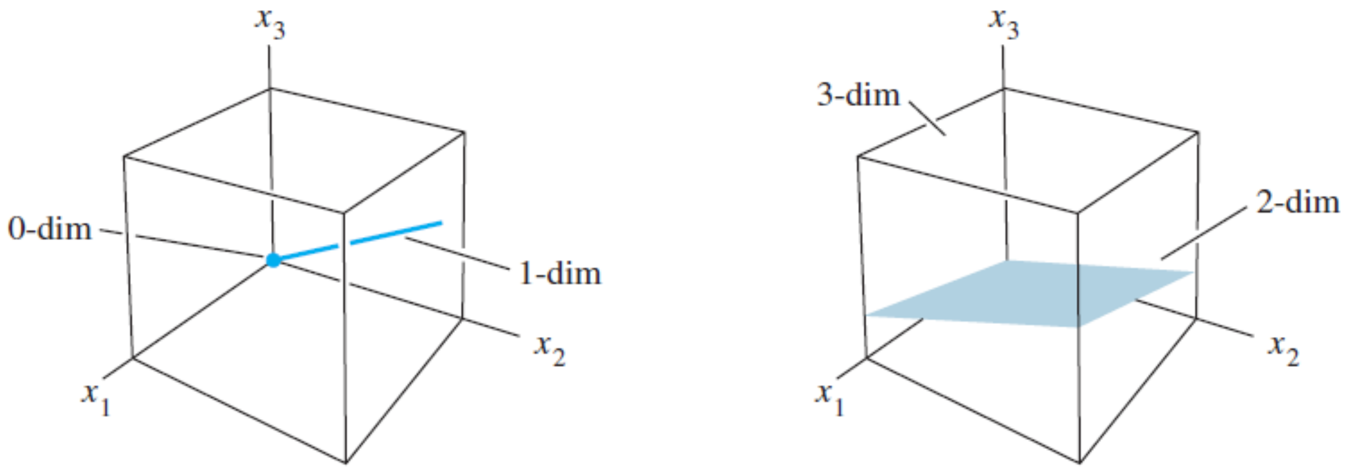
- Standard basis for $\mathbb{R}^n$ contains n vectors $\Rightarrow$ dim $\mathbb{R}^n = n$
- Standard polynomial basis $\{1, t, t^2\} \Rightarrow$ dim $\mathbb{P}_2 = 3$

Exercise 4.11.1 Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

Example

- Subspaces of $\mathbb{R}^3$



(a) (b)

4.12 Dimensions of $\mathbf{C}(A)$ and $\mathbf{N}(A)$

$\mathbf{C}(A)$

Pivot columns of A form a basis for $\mathbf{C}(A)$.

So, $\dim \mathbf{C}(A) =$ number of pivot columns of A.

$\mathbf{N}(A)$

For $m \times n$ matrix A, suppose $A\mathbf{x} = \mathbf{0}$ has k free variables.

Spanning set for $\mathbf{N}(A)$ will have k linearly independent vectors - one for each free variable.

So, $\dim \mathbf{N}(A) =$ number of free variables, k.

Exercise 4.12.1

Find the dimension of the null space and column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solution

Echelon form of augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- 3 free variables $(x_2, x_4, x_5) \Rightarrow \dim \mathbf{N}(A) = 3$
- A has two pivot columns $\Rightarrow \dim \mathbf{C}(A) = 2$

4.13 The Row Space

The set of all linear combinations of the rows (row vectors) of a matrix A is called the **row space** of A.

Since rows of A are columns of A^T, row space is denoted by $\mathbf{C}(A^T)$.

For an $m \times n$ matrix, each row has n entries $\Rightarrow$ $\mathbf{C}(A^T)$ is a subspae of $\mathbb{R}^n$.

Theorem 4.7. *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.*

(Skip proof)

Exercise 4.13.1

Find bases for the row space, column space and null space of

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

Solution

Reduce A to echelon form:

$$A \sim B = \begin{bmatrix} 1 & 3 & -5 & 1 & 5 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -4 & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Row space $\mathbf{C}(A^T)$

From Theorem 4.7,

$$\text{Basis for } \mathbf{C}(A^T) : \{(1,3,-5,1,5),(0,1,-2,2,-7),(0,0,0,-4,20)\}$$

Column space $\mathbf{C}(A)$

Pivots are in columns 1, 2 and 4. Hence, columns 1, 2, and 4 of A (not B!) form the basis:

$$\text{Basis for } \mathbf{C}(A) : \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

Null space $\mathbf{N}(A)$

Need reduced row echelon form of A:

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{0}$ is equivalent to $C\mathbf{x} = \mathbf{0}$

$$\begin{array}{rcrcrcrcrcl} x_1 & + & & & x_3 & & & + & x_5 & = & 0 \\ & & x_2 & - & 2x_3 & & & + & 3x_5 & = & 0 \\ & & & & & & x_4 & - & 5x_5 & = & 0 \end{array}$$

Similar to Exercise 4.4.3 (slide 21),

$$\text{Basis for } \mathbf{N}(A) : \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

4.14 Rank

Definition. The **rank** of A is the dimension of the column space of A.

Recall: $A\mathbf{x} = \mathbf{b}$ is consistent when $\mathbf{b}$ is in $\mathbf{C}(A)$.

Rank of A is the dimension of the set of $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.

Suppose $A \in \mathbb{R}^{n \times n}$ has rank p. This means that if we take all vectors $\mathbf{x} \in \mathbb{R}^{n \times 1}$, then $A\mathbf{x}$ spans p dimensional space $\Rightarrow$ no unique solution.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$$

Rank of $A = 1 \Rightarrow A\mathbf{x}$ spans a 1 dimensional space, i.e., a line $y_2 = 2y_1$ passing through the origin.

Points on $x_1 - x_2$ plane are mapped onto a line $y_2 = 2y_1$.

(Also see Exercise 4.13.1)

- The dimensions of the column space and the row space of an $m \times n$ matrix A are equal.

Example

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \text{Echelon form of } A\text{: } A \sim B = \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & \boxed{-4} & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem 4.6 (slide 42), rank of A is the number of pivot columns in A, i.e., the number of pivot positions in B.

Rows with pivots in B form a basis for the row space of A.

Therefore, rank of A is also the dimension of the row space of A.

- Rank of $A = \dim \mathbf{C}(A) =$ the number of columns with pivots
- $\dim \mathbf{N}(A) =$ the number of free variables (in $A\mathbf{x} = \mathbf{0}$)
  $=$ the number of columns without pivots
- Obviously,

$$\#(\text{columns with pivots}) + \#(\text{columns without pivots}) = \#(\text{columns})$$

Theorem 4.8. *The Rank Theorem*

If A is a matrix with n columns then,

$$\text{rank of } A + \dim \mathbf{N}(A) = n.$$

- Gives a relationship between the solution set of $A\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{N}(A)$ and the set of vectors $\mathbf{b}$ that make $A\mathbf{x} = \mathbf{b}$ consistent, i.e., $\mathbf{C}(A)$

Exercise 4.14.1

A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The solutions are linearly independent and span $\mathbf{N}(A)$.

Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution?

Solution

A: 40×42 coefficient matrix.

The two solutions are a basis for $\mathbf{N}(A)$.

Therefore, the dim $\mathbf{N}(A) = 2 \Rightarrow$ dim $\mathbf{C}(A) = 42 - 2 = 40$.
Since $\mathbb{R}^{40}$ is the only subspace of $\mathbb{R}^{40}$ whose dimension is 40, $\mathbf{C}(A)$ must be all of $\mathbb{R}^{40}$

$\Rightarrow$ Every homogeneous equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Theorem 4.9. *The Invertible Matrix Theorem [Theorem 2.4](continued)*

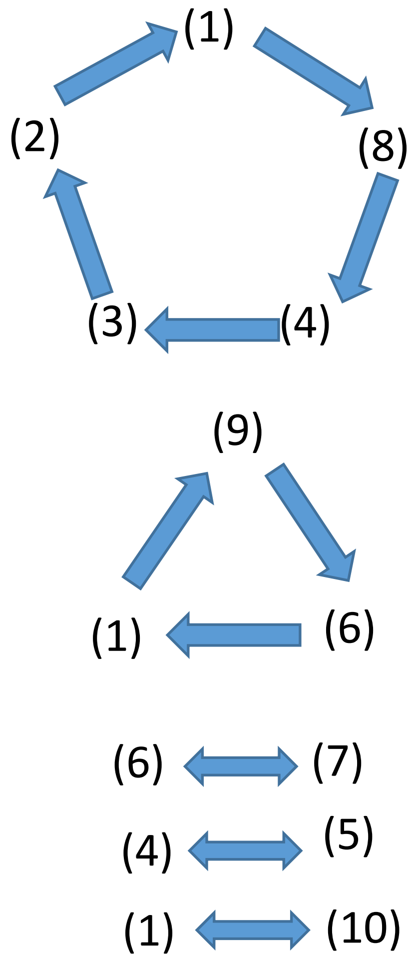
Let A by an $n \times n$ matrix. Then the statement 'A is invertible' is equivalent to the following statements:

11. *The columns of A form a basis for $\mathbb{R}^n$*
12. $\mathbf{C}(A) = \mathbb{R}^n$
13. *dim* $\mathbf{C}(A) = n$
14. *Rank of* $A = n$
15. $\mathbf{N}(A) = \{\mathbf{0}\}$
16. *dim* $\mathbf{N}(A) = 0$

$$(6) \Rightarrow (12) \Rightarrow (13) \Rightarrow (14) \Rightarrow (16) \Rightarrow (15) \Rightarrow (4)$$

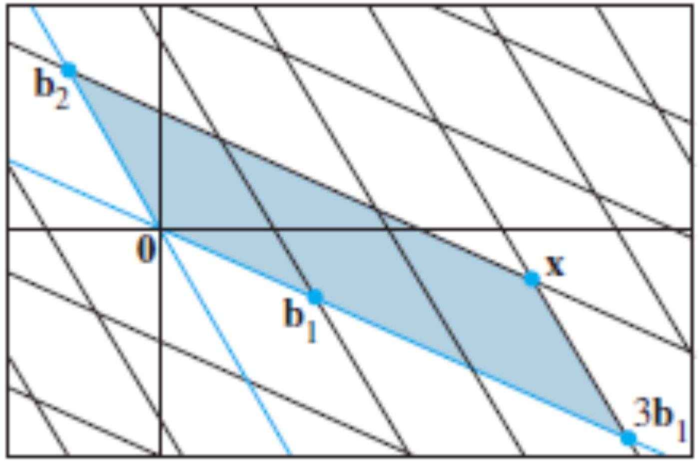
Theorem 2.4. *The Invertible Matrix Theorem*
Let A be a square $n \times n$ matrix. Then the following statements are equivalent, i.e., for a given A, the statements are either all true or all false.

1. *A is an invertible matrix.*
2. *A is row equivalent to I_n.*
3. *A has n pivot positions.*
4. *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *$A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The columns of A span $\mathbb{R}^n$.*
8. *There is an $n \times n$ matrix C such that $CA=I$.*
9. *There is an $n \times n$ matrix D such that $AD=I$.*
10. *A^T is an invertible matrix.*

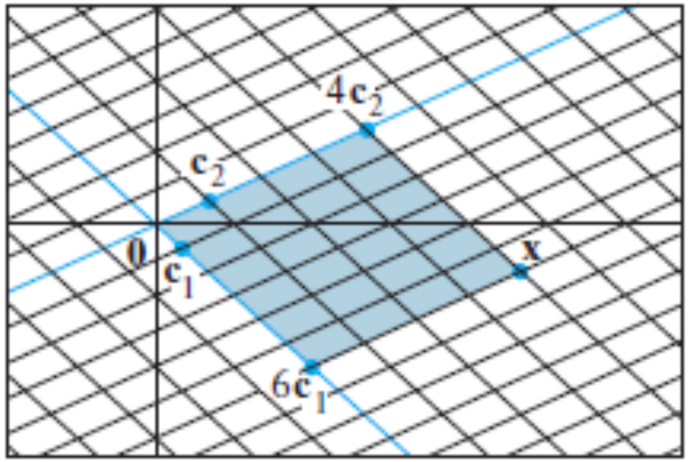


4.15 Change of basis

- A problem could be more easily solved by changing the basis from $\mathcal{B}$ to $\mathcal{C}$
- How $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$ are related for each $\mathbf{x}$ in V.



(a)



(b)

Two co-ordinate systems for the same vector space

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xleftrightarrow{\text{Relation ?}} [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Exercise 4.15.1

Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \text{ and } \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

If $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, i.e., $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{C}}$.

Solution

Coordinate mapping is a linear transformation (slide 48)

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} \\ &= 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \end{aligned}$$

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

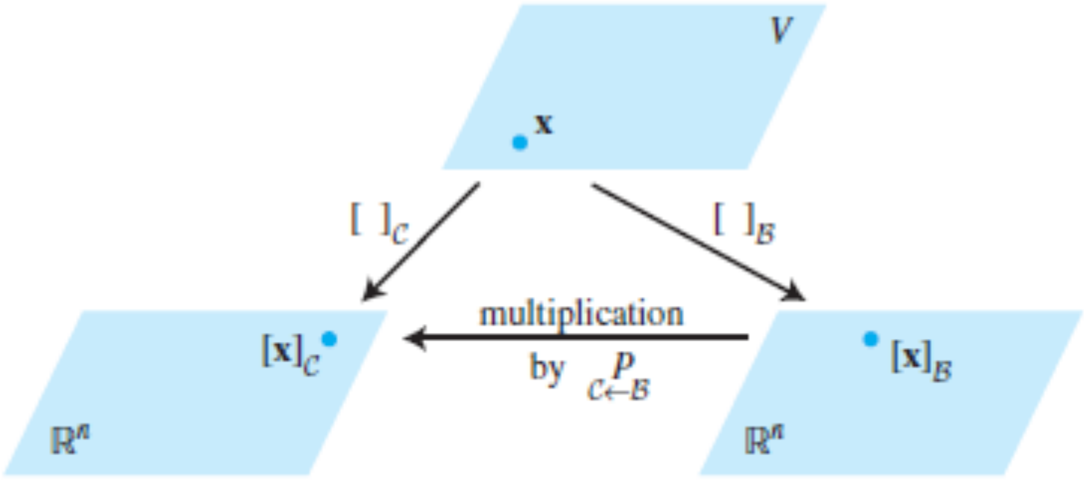
$$[\mathbf{x}]_{\mathcal{C}} = \underbrace{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}}_{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[\mathbf{x}]_{\mathcal{B}}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Theorem 4.9. *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that*

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}-$coordinate vectors of the vectors in the basis B, i.e.,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$



Two coordinate systems for *V*

***** END OF CHAPTER *****