Given the matrix $A=\begin{pmatrix}2&3&5\\4&8&17\\2&9&27\end{pmatrix}$ find it's LU factorization. Using LU factorization find solution for $Ax=b$, $b=\begin{pmatrix}0\\5\\16\end{pmatrix}$.

$$\begin{pmatrix}2&3&5\\4&8&17\\2&9&27\end{pmatrix}\xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-2R_1}\begin{pmatrix}2&3&5\\0&2&7\\0&6&22\end{pmatrix}\xrightarrow{R_3\leftarrow R_3-3R_2}\begin{pmatrix}2&3&5\\0&2&7\\0&0&1\end{pmatrix}$$

$$L_1=\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}\quad L_2=\begin{pmatrix}1&0&0\\0&1&0\\-1&0&1\end{pmatrix}\quad L_3=\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}$$

$$L_1^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\quad L_2^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\quad L_3^{-1}=\begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}$$

Example:

$$\begin{pmatrix}1&0&0\\0&1&0\\-b&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\b&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\-b+b&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$(L_3L_2L_1)^{-1}=L_1^{-1}L_2^{-1}L_3^{-1}=\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}=$$

$$=\begin{pmatrix}1&0&0\\2&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}=\begin{pmatrix}1&0&0\\2&1&0\\1&3&1\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\2&1&0\\1&3&1\end{pmatrix}\begin{pmatrix}2&3&5\\0&2&7\\0&0&1\end{pmatrix}=\begin{pmatrix}2&3&5\\4&8&17\\2&9&27\end{pmatrix}\leftarrow \text{LU factorization}$$

$$\begin{pmatrix}1&0&0\\2&1&0\\1&3&1\end{pmatrix}\boxed{\begin{pmatrix}2&3&5\\0&2&7\\0&0&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}}=\begin{pmatrix}0\\5\\16\end{pmatrix}$$

$$\boxed{\cdots}=\begin{pmatrix}x_1\\y_1\\z_1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1&0&0&0\\2&1&0&5\\1&3&1&16\end{array}\right)\xrightarrow[R_3\leftarrow R_3-R_1]{R_2\leftarrow R_2-2R_1}\left(\begin{array}{ccc|c}1&0&0&0\\0&1&0&5\\0&3&1&16\end{array}\right)\xrightarrow{R_3\leftarrow R_3-3R_2}$$

Given the matrix $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 8 & 17 \\ 2 & 9 & 27 \end{pmatrix}$ find it's LU factorization. Using LU factorization find solution for $Ax = b$, $b = \begin{pmatrix} 0 \\ 5 \\ 16 \end{pmatrix}$.

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 8 & 17 \\ 2 & 9 & 27 \end{bmatrix} \xrightarrow[\substack{L_{13} \\ R_3 = R_3 + R_1(-1)}]{\substack{L_{12} \\ R_2 = R_2 + R_1(-2)}} \begin{bmatrix} 2 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 6 & 22 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{bmatrix} = U$$

L_{23}
$R_3 = R_3 + R_2(-3)$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$LUx = b$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 5 \\ 1 & 3 & 1 & 16 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ 0 & 2 & 7 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right] = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$x = -1, \; y = -1, \; z = 1$$

$$\xrightarrow{R_3 \rightleftarrows R_3 - 3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right) \Longleftrightarrow \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} \Longleftrightarrow \begin{cases} 2x + 3y + 5z = 0 \\ 2y + 7z = 5 \\ z = 1 \end{cases} \Longleftrightarrow \begin{cases} 2x + 3y = -5 \\ 2y = -2 \\ z = 1 \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} 2x = -2 \\ y = -1 \\ z = 1 \end{cases} \Longleftrightarrow \begin{cases} x = -1 \\ y = -1 \\ z = 1 \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Given the matrix A

a) $A = \begin{pmatrix} 3 & 5 \\ 6 & 16 \end{pmatrix}$ b) $A = \begin{pmatrix} 4 & 1 & 2 \\ -4 & 0 & 3 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & 1 & 6 \\ -3 & 0 & -2 \\ 3 & 3 & 17 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$ find it's LU factorization.

Using LU factorization find solution for $Ax = b$

a) $b = \begin{pmatrix} 8 \\ 22 \end{pmatrix}$ b) $b = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$ c) $b = \begin{pmatrix} 9 \\ -5 \\ 20 \end{pmatrix}$ d) $b = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$

Answers:

LU factorization:

a) $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 0 & 6 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 5 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix}$

$Ax = b$ solution:

a) $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

b) $\left\{ \begin{pmatrix} 1/4 + 3/4\, t \\ 7 - 5t \\ t \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$

c) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

a)

$$A = \begin{bmatrix} 3 & 5 \\ 6 & 16 \end{bmatrix} \xrightarrow[R_2 = R_2 + R_1(-2)]{L_{12}} \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix} //$$

$$\begin{bmatrix} 1 & 0 & 8 \\ 2 & 1 & 22 \end{bmatrix} = \begin{matrix} 1 & 0 & 8 \\ 0 & 1 & 6 \end{matrix}$$

$$\begin{bmatrix} 3 & 5 & 8 \\ 0 & 6 & 6 \end{bmatrix} = \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{matrix} \rightsquigarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b)

$$B = \begin{bmatrix} 4 & 1 & 2 \\ -4 & 0 & 3 \end{bmatrix} \xrightarrow[R_2 = R_2 + R_1(1)]{L_{12}} \begin{bmatrix} 4 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & 2 \\ 0 & 1 & 5 \end{bmatrix} //$$

$$\begin{bmatrix} 1 & 0 & 8 \\ -1 & 1 & -1 \end{bmatrix} = \begin{matrix} 1 & 0 & 8 \\ 0 & 1 & 7 \end{matrix}$$

$$\begin{bmatrix} 4 & 1 & 2 & 8 \\ 0 & 1 & 5 & 7 \end{bmatrix} = \left[\begin{array}{ccc|c} 1 & 0 & -\frac{3}{4} & \frac{1}{4} \\ 0 & 1 & 5 & 7 \end{array}\right]$$

$$\begin{aligned} x - \tfrac{3}{4}t &= \tfrac{1}{4} \\ y + 5t &= 7 \\ t &= t \end{aligned} \quad \therefore \left\{ \begin{bmatrix} \frac{1}{4} + \frac{3}{4}t \\ 7 - 5t \\ t \end{bmatrix} \;\middle|\; t \in \mathbb{R} \right\}$$

c)

$$A = \begin{bmatrix} 3 & 1 & 6 \\ -3 & 0 & -2 \\ 3 & 3 & 17 \end{bmatrix} \xrightarrow[L_{13}\; R_3 = R_3 + R_1(-1)]{L_{12}\; R_2 = R_2 + R_1} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 2 & 11 \end{bmatrix} \xrightarrow[R_3 = R_3 + R_2(-2)]{L_{23}} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ -5 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 9 \\ -1 & 1 & 0 & -5 \\ 1 & 2 & 1 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & 6 & 9 \\ 0 & 1 & 4 & 4 \\ 0 & 0 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix} \quad b = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$$

L_{12} $R_2 = R_2 + R_1(-2)$

L_{13} $R_3 = R_3 + R_1(-3)$

$$\to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow[-\frac{2}{3}]{} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 2 & 1 & 0 & 4 \\ 3 & 2 & 1 & 4 \end{bmatrix} \to \qquad \to \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \to$$