MH1810 Math 1 Part 2 Chap 4 Limits and Continuity

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Intuitive Idea of a Limit

Example (Intuitive Idea)

Let $f(x) = x^2$. What happens to $f(x)$ for values of x near 2?

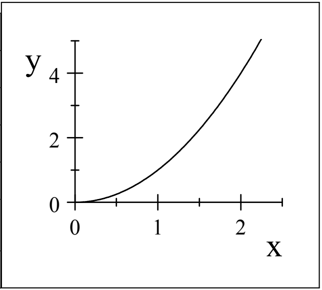
(a) Use a calculator to compute $f(x)$ for some values of x near 2. Does $f(x)$ approach some real number as x approaches 2? [Computational Approach]

(b) Sketch the graph of the function $f(x) = x^2$. Observe the points on the graph of $y = f(x)$ as x approaches 2. [Graphical Approach]

Intuitive Idea of a Limit

For values of x near 2:

x	x^2	x	x^2
2.1	4.41	1.9	3.61
2.01	4.0401	1.99	3.9601
2.001	4.004004	1.999	3.996001
2.0001	4.00040004	1.9999	3.99960001
$\vdots$		$\vdots$	



Numerically and graphically, we observe that $f(x) = x^2$ approaches 4 as x approaches 2.

We write this as

$$\lim_{x \to 2} (x^2) = 4.$$

Limit of a Function at a Point

Suppose that f is defined near $x = a$ but not necessarily at $x = a$. We say that $f(x)$ approaches the limit L as x tends to a, if we can make $f(x)$ become arbitrarily close to L by choosing x sufficiently close to a.

We express this by writing

$$\lim_{x \to a} f(x) = L.$$

Limit of a Function at a Point

(a) When $\lim_{x \to a} f(x)$ exists, which means that there is a real number L such that $\lim_{x \to a} f(x) = L$, and the limit L is unique.

(b) When there is no finite real number L such that $\lim_{x \to a} f(x) = L$, we say that the limit $\lim_{x \to a} f(x)$ does not exist.

Example

Example

Consider the expression $f(x) = \dfrac{1 - x^2}{1 - x}$.

(a) Is $f(1)$ defined?

(b) Guess the value of $\lim_{x \to 1} f(x)$.

Example

$$f(x) = \frac{1 - x^2}{1 - x}.$$

$x > 1$	$f(x)$	$x < 1$	$f(x)$
1.5	2.5	0.5	1.5
1.1	2.1	0.9	1.9
1.01	2.01	0.99	1.99
1.001	2.001	0.999	1.999
1.0001	2.0001	0.9999	1.9999

Note that: $f(1)$ is not defined but $\lim_{x \to 1} f(x) = 2$.

Example

Example

Does $\lim_{x\to 0} \sin(\frac{1}{x})$ exist?

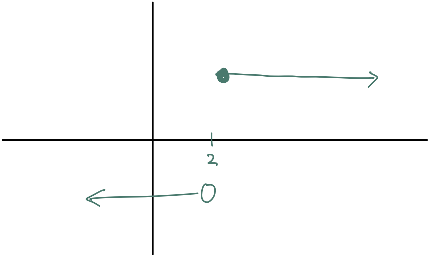
Is there a real number where $\sin(1/x)$ approaches as x approaches 0?

x	$\sin(1/x)$	x	$\sin(1/x)$
$1/\pi$		$2/\pi$	
$1/(2\pi)$	0	$2/(5\pi)$	1
$1/(3\pi)$	0	$2/(9\pi)$	1
$1/(4\pi)$	0	$2/(13\pi)$	1
$1/(5\pi)$	0	$2/(17\pi)$	1

Example

Graph of $y = \sin(\frac{1}{x})$.

Example



Example

Consider the function

$$f(x) = \begin{cases} -1 & x < 2 \\ 1 & x \geq 2 \end{cases}$$

Is there a real number where $f(x)$ approaches as x approaches 2?

Example

Example

Consider the function

$$f(x) = \begin{cases} -1 & x < 2 \\ 1 & x \geq 2 \end{cases}$$

x	$f(x)$	x	$f(x)$
0.5	-1	2.5	1
1.9	-1	2.1	1
1.99	-1	2.01	1
1.999	-1	2.001	1
1.9999	-1	2.0001	1

Example: One sided limit

Left-hand Limit The function $f(x)$, as $x \longrightarrow 2$ from the left, $f(x) \longrightarrow -1$. We shall write

$$\lim_{x \to 2^-} f(x) = -1.$$

Right-hand Limit: As $x \longrightarrow 2$ from the right, $f(x) \longrightarrow 1$. We write

$$\lim_{x \to 2^+} f(x) = 1.$$

These are known as one-sided limits.

One sided limit

There is no single value that $f(x)$ approaches to as $x \longrightarrow 2$. Since $\lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x)$, the limit $\lim_{x \to 2} f(x)$ does not exist.

One sided limit

We also say that the left-hand limit of $f(x)$ as x approaches a is equal to L.
We write

$$\lim_{x \to a^-} f(x) = L.$$

Similarly, the right-hand limit of $f(x)$ is denoted by

$$\lim_{x \to a^+} f(x) = L.$$

Equal One sided limit

The following result provides the relationship between $\lim_{x \to a} f(x)$ and one-sided limits. We use it to determine whether a limit exists.

Theorem (Equal One-sided Limits.)

$\lim_{x \to a} f(x) = L$ *if and only if* $\lim_{x \to a^-} f(x) = L$ *and* $\lim_{x \to a^+} f(x) = L$.

(Proof Omitted.)

Remark This result is useful for the evaluation of limit at a point a if the function takes different mathematical expressions for $x < a$ and $x > a$ when x are near a.

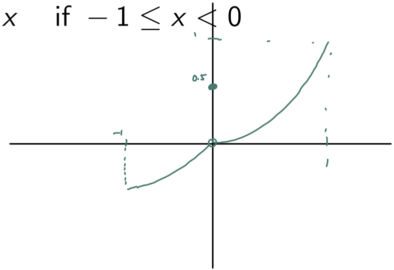
Example

Example

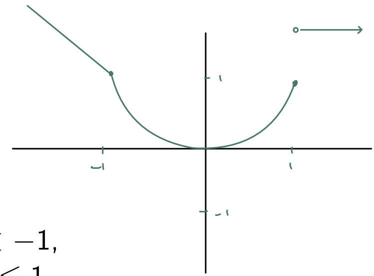
Let g be the function defined by

$$g(x) = \begin{cases} x^2 & \text{if } 0 < x \leq 1, \\ 0.5 & \text{if } x = 0, \\ \sin x & \text{if } -1 \leq x < 0 \end{cases}$$

Does $\lim_{x \to 0} g(x)$ exist?



Example



Example

Sketch the graph of

$$f(x) = \begin{cases} -x & \text{if } x < -1, \\ x^2 & \text{if } |x| \leq 1, \\ 2 & \text{if } x > 1 \end{cases}$$

Use the graph to determine whether each of the following (if exists)

(a) $\lim_{x \to 3} f(x)$ 2

(b) $\lim_{x \to -1^+} f(x)$ 1

(c) $\lim_{x \to -1} f(x)$ 1

(d) $\lim_{x \to 1} f(x)$ NA

Infinite Limit

Let f be a function be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \quad \text{as } x \to a$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we like) by taking x sufficiently close to a but not equal to a.

Similarly for $\lim_{x \to a} f(x) = -\infty$.

Example

Example

What is $\lim\limits_{x \to 0} \dfrac{1}{x^2}$?

We evaluate $f(x) = \dfrac{1}{x^2}$ for some small values of x as shown in the following table.

x	$f(x)$	x	$f(x)$
0.1	100	-0.1	100
0.01	10000	-0.01	10000
0.001	1000000	-0.001	1000000
0.0001	100000000	-0.0001	100000000

As x becomes close to 0, $\frac{1}{x^2}$ becomes very large. We say the limit does not exist. However, to reflect this blow-up behaviour, we write

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Vertical Asymptotes

The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$
$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Vertical Asymptotes

Example

(a) The vertical line with equation $x = 0$ (i.e., the y-axis) is a vertical asymptote of the curve $y = \dfrac{1}{x^2}$.

(b) The lines $x = \pm\dfrac{\pi}{2}$ are vertical asymptotes of the curve $y = \tan x$.

(c) The vertical line $x = 0$ is a vertical asymptote of $y = \ln x$.

Limits at Infinity

Let $f(x)$ be a function be a function defined on some interval (a, ∞) (resp. $(-\infty, a)$). Then

$$\lim_{x \to \infty} f(x) = L(\text{ resp. } \lim_{x \to -\infty} f(x) = L)$$

means that the values of $f(x)$ can be made as close to L as we like by taking x sufficiently large (resp. sufficiently negatively large).

Example

For the function $f(x) = \dfrac{1}{x}$, what happens to the values of $f(x)$ as x increases to large positively large values?

Horizontal Asymptotes

The horizontal line $y = b$ is called a horizontal asymptote of the curve $y = f(x)$ if

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b.$$

Examples

(a) For every positive integer n, note that $\lim_{x\to\infty} \frac{1}{x^n} = 0$, and $\lim_{x\to-\infty} \frac{1}{x^n} = 0$.
The horizontal line $y = 0$ is a horizontal asymptote of the curve $y = \frac{1}{x^n}$.

(b) Note that $\lim_{x\to-\infty} e^x = 0$. The horizontal line $y = 0$ is a horizontal asymptote of the curve $y = e^x$.

Examples

$$\lim_{x\to\infty} \sin x, \lim_{x\to\infty} \cos x, \lim_{x\to\infty} \tan x,$$

$$\lim_{x\to\infty} e^x, \lim_{x\to\infty} \ln x,$$

Examples

$$\lim_{x \to \infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x \to -\infty} \tan^{-1} x = \frac{-\pi}{2}$$