MH1810 Math 1 Part 2 Chap 5 Differentiation

Maximum and Minimum Problems

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First Derivative and the Growth of a Function

Suppose f is differentiable and f is increasing on (a, b). Then it follows from the definition of derivative that $f'(x) \geq 0$ on (a, b).

How about the converse?
If $f'(x) \geq 0$ on (a, b), does it follow that f is increasing on (a, b)?
The next result says that it is true if $f'(x) > 0$.

Theorem

Theorem

1. *If* $f'(x) > 0$ *on* (a, b)*, then* f *is increasing on* (a, b)*, i.e., for* $x_1, x_2 \in (a, b)$

$$x_1 < x_2 \Rightarrow f(x_1) < f(x_2).$$

2. *If* $f'(x) < 0$ *on* (a, b)*, then* f *is decreasing on* I*, i.e., for* $x_1, x_2 \in (a, b)$

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

Proof.

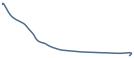
(Use Mean Value Theorem) □

Corollary

Suppose f continuous on $[a, b]$.

1. *If $f'(x) > 0$ on (a, b), then f is increasing on $[a, b]$.*
2. *If $f'(x) < 0$ on (a, b), then f is decreasing on $[a, b]$.*

Example



Example

Find interval(s) where f defined by $f(x) = 2 + 3x - x^3$ is increasing.

Solution

Solution

The function $f(x) = 2 + 3x - x^3$ *is continuous on* $\mathbb{R}$.
Note that

$$f'(x) = 3 - 3x^2 = 3(1-x)(1+x) \text{ on } \mathbb{R}.$$

Thus, $f'(x) > 0$ *for* $x \in (-1, 1)$ *and* $f'(x) < 0$ *for* $x \in (-\infty, -1) \cup (1, \infty)$.
Since f *is continuous* $\mathbb{R}$, *we conclude that* f *is increasing on* $[-1, 1]$.

Using f' for checking one-to-one

If f is increasing or decreasing on (a, b), then f is one-to-one on (a, b).

Example

Show that $f(x) = \sin x$ with domain $[-\pi/2, \pi/2]$ is one-to-one.

Solution

We have

$$f'(x) = \cos x > 0, \quad x \in (-\pi/2, \pi/2).$$

So, f is continuous on $[-\pi/2, \pi/2]$, differentiable on $(-\pi/2, \pi, 2)$ and $f'(x) > 0$ on $(-\pi/2, \pi/2)$.
Thus, f is increasing on $[-\pi/2, \pi/2]$ and hence it is one-to-one.
(And, its inverse is denoted by $\sin^{-1} x$.)

Using f' to Solve Optimization Problems

Theorem (Fermat's Theorem)

Suppose f has a local maximum or minimum at c. If $f'(c)$ exists, then

$$f'(c) = 0.$$

Proof.

(Omitted). $\square$

Example

How do we construct a cylindrical metal can with a given volume V in a way that minimizes the surface area (the amount of metal used)?

Using f' to Solve Optimization Problems

Example

How do we construct a cylindrical metal can with a given volume V in a way that minimizes the surface area (the amount of metal used)?

Solution

Let r be the radius and h the height and A the surface area of the cylindrical can. Then

$$V = \pi r^2 h, \quad A = 2\pi r^2 + 2\pi rh.$$

The first equation gives us $h = \dfrac{V}{\pi r^2}$, which we can substitute into A to get $A(r) = 2\pi r^2 + \dfrac{2V}{r}$.

Solution

$$A(r) = 2\pi r^2 + \frac{2V}{r}.$$

Our objective is to find the minimum of $A(r)$, where the domain of $A(r)$ is $(0, \infty)$.Note that A is continuous on $(0, \infty)$, and we have

$$A'(r) = 4\pi r - \frac{2V}{r^2} = \frac{4\pi}{r^2}\left(r^3 - \frac{V}{2\pi}\right).$$

Solution

$A'(r) = \frac{4\pi}{r^2}\left(r^3 - \frac{V}{2\pi}\right) = 0 \Leftrightarrow r = \left(\frac{V}{2\pi}\right)^{1/3}$
For $0 < r < \left(\frac{V}{2\pi}\right)^{1/3}$, $A'(r) < 0$. Thus, $A(r)$ is decreasing on $(0, \left(\frac{V}{2\pi}\right)^{1/3})$.
For $r > \left(\frac{V}{2\pi}\right)^{1/3}$, $A'(r) > 0$. Thus, $A(r)$ is increasing on $(\left(\frac{V}{2\pi}\right)^{1/3}, \infty)$. Therefore, $A(r)$ where $r = \left(\frac{V}{2\pi}\right)^{1/3}$ must be a global minimum point. Hence we should choose to make our cans with radius $r = \left(\frac{V}{2\pi}\right)^{1/3}$ and height $h = V/(\pi r^2)$.

Second Derivatives and Shape of Curve

We start with describing the shape of a curve, followed by using the second derivative to classify its shape.

Definition

Suppose f is differentiable.
(a) The graph of a function f concaves upward at a point c if the graph of f lies above its tangent at c, i.e.,

$$f(x) \geq f(c) + f'(c)(x - c)$$

for x in a neighbourhood of c.
The graph of a function f concaves upward on an interval (a, b) if it is concave upward (or convex) at every point in (a, b).

Second Derivatives and Shape of Curve

Definition

Suppose f is differentiable.
(b) The graph of a function f concaves downward at a point c if the graph of f lies below its tangent at c, i.e.,

$$f(x) \leq f(c) + f'(c)(x - c)$$

for x in a neighbourhood of c.
The graph of a function f concaves downward on an interval (a, b) if it is concave downward (or concave) at every point in (a, b).

Inflection Points

Definition

Suppose f is differentiable.

(c) A point P on the curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes concavity, i.e., from concaving upward to concaving downward, or from concaving downward to concaving upward.

Concavity Test

Theorem

(a) *If $f''(x) > 0$ for all x in (a, b), then the graph of f concaves upward on (a, b).*

(b) *If $f''(x) < 0$ for all x in (a, b), then the graph of f concaves downward on (a, b).*

Proof.

(Omitted.) $\square$

This is a consequence of the Mean Value Theorem applied to f'.

Concavity Test : Examples

Example

Let $f\ (x)$ =$2+3x-x^3$. Find the intervals where the graph concave upwards. Find also the intervals where the graph concaves downwards and the points of inflection.

Solution

$f(x)=2+3x-x^3$, $f'(x)=3-3x^2$, $f''(x)=-6x$ *at every* $x\in\mathbb{R}$.

$$f''(x)>0 \Longleftrightarrow x<0,$$

$$f''(x)<0 \Longleftrightarrow x>0.$$

Therefore, the graph of f is concave downward on $(0,\infty)$, and concave upward on $(-\infty,0)$.
There is a change of concavity at $x=0$. So, $x=0$ is a point of inflection.

Second derivatives and the nature of extrema

The next result is useful for solving some optimization problems, especially if the function is twice differentiable.

Theorem

Suppose f is twice differentiable on (a, b) and $f'(c) = 0$ for some $c \in (a, b)$.
(a) If $f''(x) > 0$ on (a, b), then $f(c)$ is a global minimum on (a, b).
(b) If $f''(x) < 0$ on (a, b), then $f(c)$ is a global maximum on (a, b).

[Proof.] Omitted.

Application to an Optimisation Problem

Example

How do we construct a cylindrical metal can with a given volume V in a way that minimizes the surface area (the amount of metal used)?

Solution

Let r be the radius and h the height and A the surface area of the cylindrical can. Then

$$V = \pi r^2 h, \textit{ and } A = 2\pi r^2 + 2\pi rh.$$

The first equation gives us $h = \dfrac{V}{\pi r^2}$, which we can substitute into A to get

$$A(r) = 2\pi r^2 + \frac{2V}{r}.$$

Solution

Note that $A(r)$ is continuous on $(0, \infty)$, and we have

$$A'(r) = \frac{4\pi}{r^2}\left(r^3 - \frac{V}{2\pi}\right) \text{ and } A''(r) = 4\pi + \frac{4V}{r^3} > 0.$$

We also note that

$$A'(r) = 0 \Leftrightarrow r = \left(\frac{V}{2\pi}\right)^{1/3}$$

Since $A''(r) > 0$ for every $r \in (0, \infty)$ and $A'(\left(\frac{V}{2\pi}\right)^{1/3}) = 0$, we conclude that $A(\left(\frac{V}{2\pi}\right)^{1/3})$ is a global minimum.

Global Extrema

Let f be a function with domain D_f. Recall

Definition

We say that f has a global maximum (respectively global minimum) at c if $f(c) \geq f(x)$ (respectively $f(c) \leq f(x)$) for all $x \in D_f$.

Our aim : find c where $f(c)$ is a global extremum (maximum or minimum).

Local (Relative) Maximum/Minimum

Definition

Let f be a function with domain D_f

(a) f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ for $x \in (u, v) \cap D_f$ where (u, v) is some open interval containing c.

(b) f has a local minimum (or relative minimum) at c if $f(c) \leq f(x)$ for $x \in (u, v) \cap D_f$ where (u, v) is some open interval containing c.

Note that a global maximum (respectively minimum) is a local maximum (respectively minimum).

Local Maximum/Minimum (Diagram)

The First Derivative Test

Theorem

Suppose that f is continuous in a neighbourhood of c where c is a critical point of f and that f' exists in a deleted neighbourhood of c. (Note that $f'(c)$ may not be defined.)
(a) If $f'(x)$ changes from negative to positive as x increases through c, then f has a local minimum at c.

The First Derivative Test

Theorem

Suppose that f is continuous in a neighbourhood of c where c is a critical point of f and that f' exists in a deleted neighbourhood of c. (Note that $f'(c)$ may not be defined.)

(b) If $f'(x)$ changes from positive to negative as x increases through c, then f has a local maximum at c.

(c) If $f'(x)$ does not change sign as x increases through c, then f has no maximum or minimum at c.

Example

Example

Let $f(x) = (x-1)^{2/3}$. Find and classify all critical points of f on $\mathbb{R}$.

Solution

We have

$$f'(x) = \frac{2}{3}(x-1)^{-1/3},$$

which is undefined at $x = 1$. Hence we have a singular point at $x = 1$. Furthermore, since $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, the first derivative test tells us that $f(1) = 0$ is a local minimum for f.

Example

Example

Let $f(x) = (x-1)^{1/3}$. Find and classify all critical points of f on $\mathbb{R}$.

Solution

We have

$$f'(x) = \frac{1}{3}(x-1)^{-2/3},$$

which is undefined at $x = 1$. Hence we have a singular point at $x = 1$.
Furthermore, $f'(x) > 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$, so the first derivative test tells us that $f(1) = 0$ is neither a local maximum nor a local minimum for f.

The Second Derivative Test

Theorem

Suppose $f'(c) = 0$ and f'' is continuous near c.

(a) *If $f''(c) > 0$, then f has a local minimum at c.*

(b) *If $f''(c) < 0$, then f has a local maximum at c.*

(c) *If $f''(c) = 0$, there is no conclusion. We don't know whether f has a local maximum or local minimum at c.*

Graphical explanation:

Example

Example

Let $f(x) = 2 + 3x - x^3$. Classify all critical points of f.

Solution

$f(x) = 2 + 3x - x^3$, $f'(x) = 3 - 3x^2$,$f''(x) = -6x$ *at every* $x \in \mathbb{R}$.

Critical points are $x = 1$ *and* $x = -1$.

At $x = 1$, *note that* $f'(1) = 0$ *and* $f''(1) < 0$. *By the second derivative test,* f *has a local maximum at* $x = 1$.

At $x = -1$, *note that* $f'(-1) = 0$ *and* $f''(1) > 0$. *By the second derivative test,* f *has a local minimum at* $x = -1$.

Example

Example

Classify all critical points of $f(x) = x^4$.

Solution

$f(x) = x^4$, $f'(x) = 4x^3$,$f''(x) = 12x^2$ *at every* $x \in \mathbb{R}$.
It is clear that $x = 0$ *is the only critical point.*
The second derivative test can not be applied here as $f''(0) = 0$.
We shall use first derivative test.
For $x < 0$, $f'(x) < 0$ *whereas* $f'(x) > 0$ *for* $x > 0$. *By the first derivative test, we conclude* f *has a local minimum at* $x = 0$.