MA1513 TUTORIAL 3

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Link: http://tinyurl.com/MA1513Solutions

KEY CONCEPTS – CHAPTER 2 VECTOR SPACES

Basis and Dimensions

- The **basis** for a space is the smallest possible set of vectors such that every vector in the space is a linear combination of the elements in the set. This is the set of finite vectors (building blocks) which can help to generate all the possible vectors inside that space.

To show $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ is a basis for $\mathbf{R}^n$

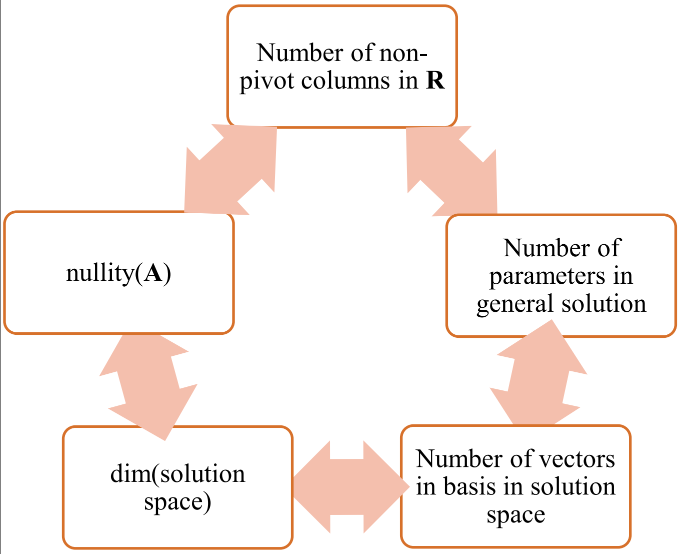
Check: $\text{span}\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\} = \mathbf{R}^n$	Check: $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k\}$ is lin. indep.
$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \boldsymbol{v}$	$c_1\boldsymbol{u}_1 + c_2\boldsymbol{u}_2 + \cdots + c_k\boldsymbol{u}_k = \boldsymbol{0}$
$\boldsymbol{v}$ is any general vector in $\mathbf{R}^n$	$\boldsymbol{0}$ is the zero vector in $\mathbf{R}^n$
check whether the system is always consistent	check whether the system has non-trivial solution
yes: span $= \mathbf{R}^n$; no: span $\neq \mathbf{R}^n$	yes: lin.dep; no: lin.indep

- Every vector in a space can be uniquely defined by a linear combination of basis vectors.
- The number of vectors in the basis for V is called the **dimension** of V, denoted by dim V.

Solution Space of Homogeneous System

$$\mathbf{A}\vec{x} = \vec{0} \to \left(\mathbf{A} \middle| \; \vec{0}\right) \xrightarrow{GE} \left(\underset{REF}{\mathbf{R}} \middle| \; \vec{0}\right) \to \text{non-trivial}$$

(infinitely-many) solutions



Basis for Row and Column Space

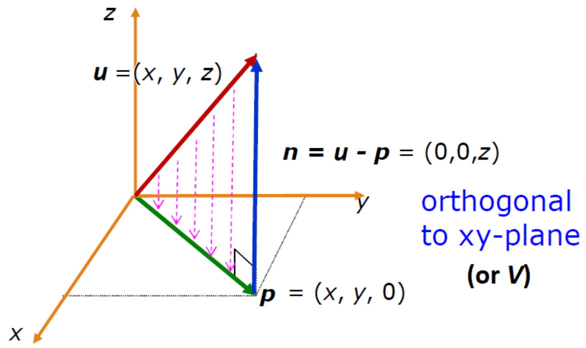
$$\boldsymbol{A} = \begin{array}{c} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 3 & 6 & 6 & 3 \\ 4 & 9 & 9 & 5 \\ -2 & -1 & -1 & 1 \\ 5 & 8 & 9 & 4 \\ 4 & 2 & 7 & 3 \end{pmatrix} \begin{array}{c} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \\ \mathbf{r}_4 \\ \mathbf{r}_5 \\ \mathbf{r}_6 \end{array} \\ \begin{array}{cccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 & \mathbf{c}_4 \end{array} \end{array} \xrightarrow[\text{Elimination}]{\text{Gaussian}} \begin{array}{c} \text{pivot columns} \\ \begin{pmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \\ \\ \\ \end{array} \\ \text{REF} \end{array}$$

- Row space of $\mathbf{A}$ = span$\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, \mathbf{r}_5, \mathbf{r}_6\}$
- Column space of $\mathbf{A}$ = span$\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4\}$
- Basis for row space of $\mathbf{A}$ = non-zero rows in REF
  = $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$
- Basis for column space of $\mathbf{A}$
  = corresponding columns of pivot columns in $\mathbf{A}$
  = $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$

dim(column space of **A**) = dim(row space of **A**)
= rank **A**

Dimension Theorem: If **A** is a matrix with n columns, then rank **A** + nullity **A** = n.

Projection and Linear Approximation



Let V be a subspace of $\mathbb{R}^n$ and **u** is a vector in $\mathbb{R}^n$. If **p** is a vector in V such that **u** – **p** is orthogonal to V, then **p** is called the **projection** of **u** onto V.

NB. p is also known as the best approximation of **u** in the subspace V, i.e. $\|\mathbf{u}-\mathbf{p}\| \le \|\mathbf{u}-\mathbf{v}\|$ for all $\mathbf{v} \in V$.

A **least-squares solution** of linear system $\mathbf{A}\boldsymbol{x} = \mathbf{b}$ is a vector $\mathbf{x}_0$ in $\mathbb{R}^n$ that minimises $\|\mathbf{Ax}-\mathbf{b}\|$, i.e.

$$\|\mathbf{Ax}_0 - \mathbf{b}\| \le \|\mathbf{Av}-\mathbf{b}\| \text{ for all } \mathbf{v} \in V.$$

By comparison, (**u** "=" **b** and) **p** = $\mathbf{Ax}_0$, i.e. $\mathbf{Ax}_0$ is the projection of **b** onto column space of **A** (set of all **Av**).

In other words, $\mathbf{x}_0$ is the best approximation to the solution of $\mathbf{A}\boldsymbol{x} = \mathbf{b}$, i.e. $\|\mathbf{Ax}_0 - \mathbf{b}\|$ smallest $\Rightarrow \mathbf{Ax}_0 \approx \mathbf{b}$.

Linear Independence of Vector Functions

- Combine vectors together to form matrix **A**.
- Calculate the Wronskian (determinant of matrix functions) ("Factorise" out common terms in columns, where possible)
- If the Wronskian is non-zero for any t, the vector functions are <u>linearly independent</u>.
- If the Wronskian is zero for all t, use the <u>standard approach</u> to test for linear independence.

Tutorial Problems

Question 1

Determine which of the following set of vectors are bases for $\mathbb{R}^3$. *From Tutorial 2 Question 7*

(a) $S_1 = \{(1, 0, -1), (-1, 2, 3)\}$. (linearly independent)

(b) $S_2 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0)\}$. (linearly independent)

(c) $S_3 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 3)\}$. (linearly dependent)

(d) $S_4 = \{(1, 0, -1), (-1, 2, 3), (0, 3, 0), (1, -1, 1)\}$. (linearly dependent)

Solutions

Recall that for a subset $S = \{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\}$ of a vector space *V*, it is called a **basis** for *V* if

- $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k\} = V$, and
- $\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_k$ are linearly independent.

Moreover, $\dim(\mathbb{R}^3) = 3$, implying that the basis can only contain 3 vectors.

(a) There is insufficient number of vectors inside the set to form a basis.

(b) 3 linearly independent vectors will always form a basis for $\mathbb{R}^3$.

You can also check by setting up a vector equation with a generic 3-vector (x, y, z):

$$a\begin{pmatrix}1\\0\\-1\end{pmatrix}+b\begin{pmatrix}-1\\2\\3\end{pmatrix}+c\begin{pmatrix}0\\3\\3\end{pmatrix}=\begin{pmatrix}x\\y\\z\end{pmatrix}\Rightarrow\left(\begin{array}{ccc|c}1&-1&0&x\\0&2&3&y\\-1&3&0&z\end{array}\right)\xrightarrow{GE}\left(\begin{array}{ccc|c}1&-1&0&x\\0&2&3&y\\0&0&-3&x-y+z\end{array}\right)$$

For any *x*, *y* and *z* values, the system is consistent, hence any vector in $\mathbb{R}^3$ can always be written as a linear combination of the 3 vectors in S_2 (i.e. $\text{span}\{S_2\} = \mathbb{R}^3$). Thus, S_2 is a basis for $\mathbb{R}^3$.

(c) The 3 vectors in S_3 are linearly dependent.

(d) The 4 vectors in S_3 are linearly dependent. Moreover, there are too many vectors in the set to be used as a basis.

Question 2

Find a basis and dimension of the solution space to each of the following homogeneous systems.

(a) $\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ -3x_2 + x_3 = 0 \end{cases}$ (b) $\begin{cases} x_1 + 3x_2 - x_3 + 2x_4 = 0 \\ -3x_2 + x_3 = 0 \\ x_1 - x_4 = 0 \end{cases}$

Why is this typically not done for non-homogeneous systems?

Solutions

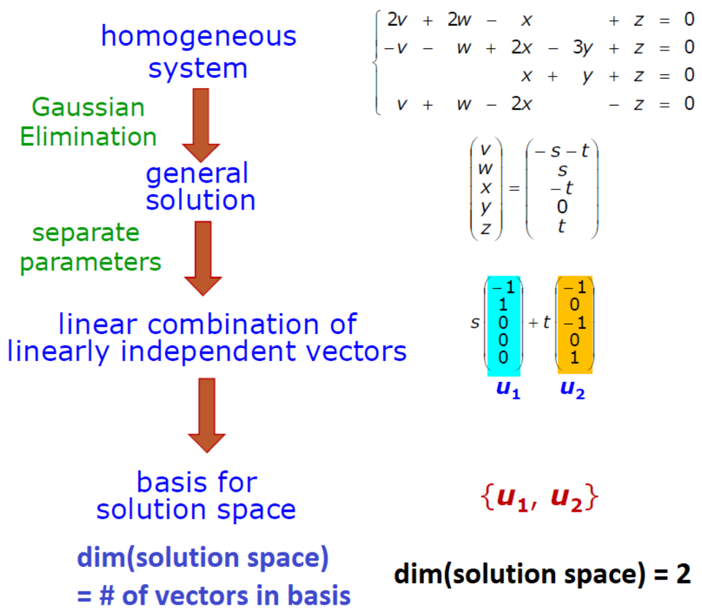
(a) $\left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & 0 & 0 \end{array}\right)$ [REF]

Since the last two columns are non-pivot, we can choose x_3 and x_4 as free variables. Let $x_3 = s$ and $x_4 = t$. Then,

$$x_2 = \frac{1}{3}s \text{ and } x_1 = -2t.$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t \\ s/3 \\ s \\ t \end{pmatrix} = \frac{s}{3}\begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus, $\{(0, 1, 3, 0)^T, (-2, 0, 0, 1)^T\}$ is a basis for the solution space, and dim(solution space) = 2.



(b) $\left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{cccc|c} 1 & 3 & -1 & 2 & 0 \\ 0 & -3 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \end{array}\right)$

Since the 3rd column is a non-pivot column, we can use the 3rd variable as the free variable. Let $x_3 = t$. Then,

$$x_4 = 0,\ x_3 = t,\ x_2 = \frac{1}{3}t,\ x_1 = 0 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{t}{3}\begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \end{pmatrix}$$

As such, $\{(0, 3, 1, 0)^T\}$ is a basis for the solution space, and dim(solution space) = 1.

Question 3

Let $V = \{(a+b+2d,\ a-c,\ -b-d,\ c+d) \mid a, b, c, d \in \mathbb{R}\}$ and $S = \{(1, 1, 0, 0), (1, 0, -1, 0), (0, -1, 0, 1)\}$.

(a) Show that V is a subspace of $\mathbb{R}^4$ and S is a basis for V.
(b) Find the coordinate vector of $\mathbf{u} = (1, 2, 3, 2)$ relative to S.
(c) Find a vector $\mathbf{v}$ such that $(\mathbf{v})_S = (1, 3, -1)$.

Solutions

(a) Note $(a+b+2d,\ a-c,\ -b-d,\ c+d) = a(1, 1, 0, 0) + b(1, 0, -1, 0) + c(0, -1, 0, 1) + d(2, 0, -1, 1)$, which is a linear combination of 4 vectors. Thus,

$$V = \text{span}\{(1, 1, 0, 0), (1, 0, -1, 0), (0, -1, 0, 1), (2, 0, -1, 1)\}.$$

Since V can be expressed in terms of a linear span, V must be a subspace of $\mathbb{R}^4$.

Observe that the 3 vectors in S are defined above in the linear span to give V. This means that one of the vector in the above set used to define the span is redundant. Let us set up the matrix:

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Column Space Method

(to find basis vectors given $V = \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\}$)

- Form matrix $\mathbf{A}$ using $\mathbf{v}_1, ..., \mathbf{v}_n$
- Reduce matrix $\mathbf{A}$ to row echelon form $\mathbf{R}$ (GE)
- Identify pivot columns in $\mathbf{R}$
- The corresponding columns in $\mathbf{A}$ forms the basis in V.

Since the first 3 columns of the REF are pivot columns, this implies that we can use the first 3 vectors as the basis for V, which is simply the set S.

(b) We want to find a, b, c (which are the required coordinates) such that $\mathbf{u}$ is written as a linear combination of the basis vectors. Set up a vector equation which produces the following augmented matrix:

$$a\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & -1 & 2 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Using back substitution, we get $c = 2,\ b = -3,\ a = 4$. Thus, $(\mathbf{u})_S = (4, -3, 2)$.

Note that this is the same method to check if $\mathbf{v}$ is a linear combination of the 3 vectors.

(c) Substitute $a = 1,\ b = 3,\ c = -1$ into the general linear combination

$$\mathbf{v} = a\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 4 \\ 2 \\ -3 \\ -1 \end{pmatrix}}}.$$

Question 4

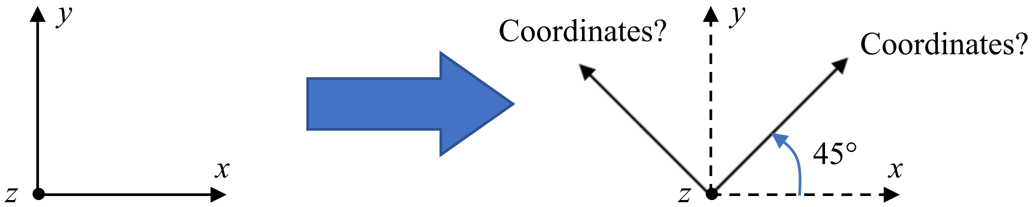
The regular coordinate axes for the xyz-space is rotated about the z-axis through 45° counter-clockwise, viewing from the positive z-axis.

(a) Find a basis S consisting of unit vectors that determine the new coordinate axes.

(b) What are the coordinates of the vector $\mathbf{v} = (1, 1, 1)$ relative to the new axes?

(c) Find a matrix $\mathbf{M}$ such that $\mathbf{Mv} = (\mathbf{v})_S$.

Solutions



(a) Rotating the x- and y-axis 45° counter-clockwise about the z-axis will cause them to point in the direction (1, 1, 0) and (–1, 1, 0) respectively. These 2 vectors will form basis S that defines the new coordinate axes. The last vector is still the vector in the z-direction. The additional requirement that these vectors must be unit vectors gives

$$\underline{\underline{S = \left\{ \frac{1}{\sqrt{2}}(1, 1, 0), \frac{1}{\sqrt{2}}(-1, 1, 0), (0, 0, 1) \right\}}}.$$

(b) Set up the $\mathbf{v}$ as a linear combination of the basis vectors. The unknown scalars a, b and c will give the coordinates required.

$$\frac{a}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{b}{\sqrt{2}}\begin{pmatrix}-1\\1\\0\end{pmatrix} + c\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Leftrightarrow \begin{pmatrix}1 & -1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}a/\sqrt{2}\\b/\sqrt{2}\\c\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\left(\begin{array}{ccc|c}1 & -1 & 0 & 1\\1 & 1 & 0 & 1\\0 & 0 & 1 & 1\end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c}1 & -1 & 0 & 1\\0 & 2 & 0 & 0\\0 & 0 & 1 & 1\end{array}\right)$$

You can also obtain a, b and c by "observation" (inspection method).

By back-substitution, $c = 1, b = 0,\ a/\sqrt{2} = 1 \Rightarrow a = \sqrt{2}$. Thus, $(\mathbf{v})_S = \underline{\underline{(\sqrt{2}, 0, 1)}}$.

(c) From the matrix equation in (b),

$$\begin{pmatrix}1 & -1 & 0\\1 & 1 & 0\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}a/\sqrt{2}\\b/\sqrt{2}\\c\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \Leftrightarrow \underbrace{\begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2} & 0\\1/\sqrt{2} & 1/\sqrt{2} & 0\\0 & 0 & 1\end{pmatrix}}_{\mathbf{A}}\underbrace{\begin{pmatrix}a\\b\\c\end{pmatrix}}_{(\mathbf{v})_S} = \underbrace{\begin{pmatrix}1\\1\\1\end{pmatrix}}_{\mathbf{v}}$$

To obtain coordinates (a, b, c), we pre-multiply the inverse of the matrix on both sides to get:

$$\begin{aligned}\mathbf{A}(\mathbf{v})_S &= \mathbf{v}\\ \mathbf{A}^{-1}\mathbf{A}(\mathbf{v})_s &= \mathbf{A}^{-1}\mathbf{v}\\ (\mathbf{v})_S &= \underbrace{\mathbf{A}^{-1}}_{\mathbf{M}}\mathbf{v}\end{aligned} \qquad \Rightarrow \mathbf{M} = \mathbf{A}^{-1} = \underline{\underline{\begin{pmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\-1/\sqrt{2} & 1/\sqrt{2} & 0\\0 & 0 & 1\end{pmatrix}}}$$

Question 5

Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Show that the linear system $\mathbf{Ax} = \mathbf{b}$ has no solution.
(b) Find the least squares solution to $\mathbf{Ax} = \mathbf{b}$.
(c) Find the projection of $\mathbf{b}$ onto $\text{span}\{(1, 0, -1, 1), (-1, 1, 0, 1), (0, -1, 1, 1)\}$.

<u>Solutions</u>

(a) Set up the augmented matrix:

$$\left(\mathbf{A} \middle| \ \mathbf{b}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right) \xrightarrow{GE} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 3 & -2 \end{array}\right)$$

From the third row, the matrix is inconsistent. Hence the linear system $\mathbf{Ax} = \mathbf{b}$ has no solutions.

(b)

Approach: Find the solution to $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$.

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus, we obtain the solutions $\underline{\underline{x = \frac{1}{3},\ y = \frac{1}{3},\ z = \frac{1}{3}}}$.

(c) Note that the linear span given is the column space of $\mathbf{A}$. The projection of $\mathbf{b}$ onto the linear span is simply given by

$$\mathbf{Ax} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}}.$$

To find projection of $\mathbf{b}$ onto subspace V:

- Find any basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for V.
- Form matrix $\mathbf{A}$ using $\mathbf{u}_1, \ldots, \mathbf{u}_k$.
- Solve $\mathbf{A}^T\mathbf{Ax}_0 = \mathbf{A}^T\mathbf{b}$ for $\mathbf{x}_0$.
- Projection required = $\mathbf{Ax}_0$

Question 6

There are two costs involved if we want to publish a book. C is a fixed cost due to typesetting and editing and D is the printing and binding cost for each additional book we want to produce.

Suppose we expect b, the total cost of producing t books to be a linear function of t. We shall apply the least squares method to find a straight-line $b = C + Dt$ that "best fits" the following set of data:

$$b_1 = 3 \text{ when } t_1 = 1, \qquad b_2 = 5 \text{ when } t_2 = 2, \qquad b_3 = 6 \text{ when } t_3 = 3.$$

(a) Write down a linear system with three equations and two variables using the data set.

(b) Find the equation of the best fit straight-line for the given data set.

Solutions

(a) $$\begin{cases} C+D=3 \\ C+2D=5 \\ C+3D=6 \end{cases} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$$

How can we tell that this linear system is obviously inconsistent?

(b) Using the same approach as Question 5(b), we solve

$$\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 14 \\ 31 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 14 \\ 31 \end{pmatrix}$$

$$\begin{pmatrix} C \\ D \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 10 \\ 9 \end{pmatrix} = \begin{pmatrix} 5/3 \\ 3/2 \end{pmatrix}$$

Thus, the least squares line is given by $\underline{\underline{b = \frac{5}{3} + \frac{3}{2}t}}$.

Question 7

$$\mathbf{A}=\begin{pmatrix}1&2&0\\0&1&1\\-1&3&6\\2&1&0\\3&1&-1\end{pmatrix},\quad \mathbf{B}=\begin{pmatrix}2&1&4&1&2\\4&2&2&3&2\\2&1&-2&2&0\\6&3&6&4&4\end{pmatrix},\quad \mathbf{C}=\begin{pmatrix}1&4&5&8\\-1&4&3&0\\2&0&2&1\end{pmatrix}.$$

For each matrix from Tutorial 2 Question 3, find

(i) a basis for the row space and column space,

(ii) a basis for the nullspace,

(iii) the nullity of the matrix.

<u>Solutions</u>

	$\mathbf{A}=\begin{pmatrix}1&2&0\\0&1&1\\-1&3&6\\2&1&0\\3&1&-1\end{pmatrix}\xrightarrow{GE}\begin{pmatrix}1&2&0\\0&1&1\\0&0&1\\0&0&0\\0&0&0\end{pmatrix}$, rank $\mathbf{A}=3$	$\mathbf{C}=\begin{pmatrix}1&4&5&8\\-1&4&3&0\\2&0&2&1\end{pmatrix}\xrightarrow{GE}\begin{pmatrix}1&4&5&8\\0&8&8&8\\0&0&0&-7\end{pmatrix}$ rank($\mathbf{C}$) = 3
(i)	The basis of row space is simply the non-zero rows of REF form of matrix **A**. Row space basis $=\{(1,\ 2,\ 0),\ (0,\ 1,\ 1),\ (0,\ 0,\ 1)\}$. Column space basis $=\left\{\begin{pmatrix}1\\0\\-1\\2\\3\end{pmatrix},\begin{pmatrix}2\\1\\3\\1\\1\end{pmatrix},\begin{pmatrix}0\\1\\6\\0\\-1\end{pmatrix}\right\}$.	Row space basis = $\{(1,\ 4,\ 5,\ 8),\ (0,\ 8,\ 8,\ 8),\ (0,\ 0,\ 0,\ -7)\}$. Since 1st, 2nd and 4th columns of **R** are pivot columns, its corresponding columns in **C** will form the column space basis. Column space basis $=\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix},\begin{pmatrix}4\\4\\0\end{pmatrix},\begin{pmatrix}8\\0\\1\end{pmatrix}\right\}$.
(ii)	The solution to $\mathbf{Ax}=\mathbf{0}$ is just the trivial solution (because **A** is non-singular). The basis for the nullspace is the empty set.	The augmented matrix in REF form is $\left(\begin{array}{cccc\|c}1&4&5&8&0\\0&8&8&8&0\\0&0&0&-7&0\end{array}\right)$ Since the 3rd column is non-pivot, we choose the 3rd variable as free variable. Let $x_3=t$. Then, $x_4=0,\ x_3=t,\ x_2=-t,\ x_1=-t\Rightarrow\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=t\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}$ Thus, $\{(-1,-1,\ 1,\ 0)^T\}$ is a basis for the nullspace.
(iii)	nullity($\mathbf{A}$) = 0.	nullity($\mathbf{C}$) = 1.

You can confirm this using the Dimension Theorem.

$$\mathbf{B} = \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 4 & 2 & 2 & 3 & 2 \\ 2 & 1 & -2 & 2 & 0 \\ 6 & 3 & 6 & 4 & 4 \end{pmatrix} \xrightarrow{GE} \begin{pmatrix} 2 & 1 & 4 & 1 & 2 \\ 0 & 0 & -6 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ rank } \mathbf{B} = 2$$

(i) Row space basis $= \{(2, 1, 4, 1, 2), (0, 0, -6, 1, -2)\}$.

Column space basis $= \text{span}\left\{\begin{pmatrix} 2 \\ 4 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \\ -2 \\ 6 \end{pmatrix}\right\}$.

(ii) From the REF, since the 2nd, 4th and 5th columns are non-pivot, we can choose them as free variables. Let $x_2 = s,\ x_4 = t,\ x_5 = u$. Then,

$$x_5 = u,\ x_4 = t,\ x_3 = \frac{1}{6}t - \frac{1}{3}u,\ x_2 = s,\ x_1 = -\frac{1}{2}s - \frac{5}{6}t - \frac{1}{3}u \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \frac{s}{2}\begin{pmatrix} -1 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{t}{6}\begin{pmatrix} -5 \\ 0 \\ 1 \\ 6 \\ 0 \end{pmatrix} + \frac{u}{3}\begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 3 \end{pmatrix}$$

The basis for the nullspace is $\{(-1, 2, 0, 0, 0)^T, (-5, 0, 1, 6, 0)^T, (-1, 0, -1, 0, 3)^T\}$.

(iii) The nullity is given by the number of free variables used. Thus, nullity($\mathbf{B}$) = 3.

Question 8

An engineer has found two solutions to a homogeneous system $\mathbf{Ax} = \mathbf{0}$ of 40 equations and 42 variables. These two solutions are not scalar multiple of each other, and every solution of the system can be constructed by a linear combination of these two solutions.

Suppose now the engineer needs to replace the homogeneous system with an associated non-homogeneous system $\mathbf{Ax} = \mathbf{b}$ for some non-zero vector $\mathbf{b}$. Can the engineer be certain that he will be able to find a solution?

Solutions

Information	**Deduction**
"solutions are not scalar multiple of each other"	Solutions/vectors are linearly independent.
"every solution of the system can be constructed by a linear combination of these two solutions"	The two solutions/vectors will span the solution space, i.e. the solution space can be written as a span of the two solutions/vectors. ⇒ two solutions form a basis for solution space. ⇒ they form basis for nullspace of $\mathbf{A}$. ⇒ nullity($\mathbf{A}$) = 2

- Since matrix has 42 columns, by Dimension Theorem, rank($\mathbf{A}$) = 42 – nullity($\mathbf{A}$) = 40.
- The row-echelon form of $\mathbf{A}$ has 40 non-zero rows.
- Any non-homogeneous system $\mathbf{Ax} = \mathbf{b}$ will have a solution regardless of $\mathbf{b}$. (Specifically, infinitely many solutions)

Question 9

Find the Wronskian for each of following sets of vector functions and determine whether they are linearly independent.

(a) $\begin{pmatrix} e^t \\ 2e^t \\ e^t \end{pmatrix}, \begin{pmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{pmatrix}, \begin{pmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{pmatrix}$ (b) $\begin{pmatrix} e^{-t} \\ -4e^{-t} \\ e^{-t} \end{pmatrix}, \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}, \begin{pmatrix} 2e^{-t} \\ -4e^{-t} \\ 0 \end{pmatrix}$

Solutions

To determine linear independence of vector functions:

- Combine vectors to form **A**.
- Calculate Wronskian.
- If the Wronskian is non-zero for any *t*, the vector functions are linearly independent.
- If the Wronskian is zero for all *t*, use the standard approach to test for linear independence.

(a) Calculating the Wronskian:

$$W\left[\begin{pmatrix} e^t \\ 2e^t \\ e^t \end{pmatrix}, \begin{pmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{pmatrix}, \begin{pmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{pmatrix}\right] = \begin{vmatrix} e^t & e^{-t} & e^{2t} \\ 2e^t & 2e^{-t} & 3e^{2t} \\ e^t & e^{-t} & e^{2t} \end{vmatrix} = (e^t)(e^{-t})(e^{2t})\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Since the first two columns are identical (or first and third row), the Wronskian (determinant) will be zero. As such, we cannot use the Wronskian to determine if the vectors are linearly independent.

We need to use the standard approach. Set up the following vector equation:

$$a\begin{pmatrix} e^t \\ 2e^t \\ e^t \end{pmatrix} + b\begin{pmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{pmatrix} + c\begin{pmatrix} e^{2t} \\ 3e^{2t} \\ e^{2t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Sub $t = 0$ into top equation: $a + b + c = 0$ $\quad --(1)$

Sub $t = 0$ into middle equation: $2a + 2b + 3c = 0$ $\quad --(2)$

Sub $t = \ln 2$ into top equation: $2a + \frac{1}{2}b + 4c = 0$ $\quad --(3)$

The coefficient matrix of the new homogeneous system $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 2 & 0.5 & 4 \end{pmatrix}$ has non-zero determinant, and is hence non-singular. The system has only the trivial solution, implying that the vector functions are linearly independent.

(b) We concatenate (combine) the vectors into a matrix, which is used to find the Wronskian:

$$W\left[\begin{pmatrix} e^{-t} \\ -4e^{-t} \\ e^{-t} \end{pmatrix}, \begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix}, \begin{pmatrix} 2e^{-t} \\ -4e^{-t} \\ 0 \end{pmatrix}\right] = \begin{vmatrix} e^{-t} & e^{-t} & 2e^{-t} \\ -4e^{-t} & 0 & -4e^{-t} \\ e^{-t} & -e^{-t} & 0 \end{vmatrix} = (e^{-t})^3 \begin{vmatrix} 1 & 1 & 2 \\ -4 & 0 & -4 \\ 1 & -1 & 0 \end{vmatrix} = 0$$

Similarly, we are unable to use the Wronskian to determine linear independence of the vectors. We then set up the vector equation

$$a\begin{pmatrix} e^{-t} \\ -4e^{-t} \\ e^{-t} \end{pmatrix} + b\begin{pmatrix} e^{-t} \\ 0 \\ -e^{-t} \end{pmatrix} + c\begin{pmatrix} 2e^{-t} \\ -4e^{-t} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Sub $t=0$ into top equation: $a+b+2c=0$ $--(1)$

Sub $t=0$ into middle equation: $-4a-4c=0$ $--(2)$

Sub $t=0$ into last equation: $a-b=0$ $--(3)$

The coefficient matrix of the new homogeneous system $\begin{pmatrix} 1 & 1 & 2 \\ -4 & 0 & -4 \\ 1 & -1 & 0 \end{pmatrix}$ has zero determinant, and is hence singular. The system has non-trivial solution, implying that the vector functions are linearly dependent.