MH1810 Math 1 Part 1 Algebra

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Matrices

- Rectangular arrays of real numbers (or complex numbers);
- Wide applications in this modern generation;
  in sciences and engineering as well as social sciences;
- Heavily involved in computer science, computer engineering, statistics;
- Especially useful with large amount of data.
- Provides a neat representation of data;

Matrix Notation and Terminology

- For positive integers m and n, an $m \times n$ matrix A is a rectangular array of mn numbers (real or complex numbers) arranged in m horizontal rows and n vertical columns:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix}.$$

The (i, j)th entry (or simply ij-entry) is the term a_{ij} found in the ith row and jth column.

Matrix Notation and Terminology

The ith row of A, where $1 \leq i \leq m$, is

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix},$$

while the jth column, for $1 \leq j \leq n$, is

$$\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}.$$

Notation and Terminology

Capital letters $A, B, C, \ldots$ are used to denote matrices, and lowercase letters a, b, c, ...to denote numerical quantities. Some examples:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 9 & 6 & 3 \end{bmatrix}, T = \begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix}, Z = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

Both square brackets or round brackets are used to enclose the array of entries.

Terminology

The number m of rows and the number n of columns describe the size of a matrix. We write it as $m \times n$, and read as 'm by n'. The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}.$$

is sometimes written as

$$\left[a_{ij}\right]_{m\times n} \text{ or simply } \left[a_{ij}\right].$$

To refer to the (i,j)th-entry of the matrix A, we may use the notation A_{ij}. So, $A_{ij} = a_{ij}$.

Example

For $A = \begin{bmatrix} 1 & 3 & 2.5 & 7 \\ 9 & 6/11 & 3 & 0 \\ -2 & -4 & -8 & \sqrt{2} \end{bmatrix}$, we have $A_{11} = 1, A_{21} = 9$, $A_{12} = 3$ and $A_{34} = \sqrt{2}$.

NOTE Usually, we match the letter denoting a matrix with the letter denoting its entries.

For a matrix B, its ij-entry is b_{ij}.

Row and Column Matrices

When $m = 1$, the matrix has only one row,i.e.

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n \end{bmatrix}$$

We call this matrix a row matrix .
When $n = 1$, the matrix has only one column,i.e.

$$B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

We call such matrix a column matrix.

Zero Matrix

The $m \times n$ matrix with zeros as its entries is called the zero matrix and we denote it by 0.

Square Matrices

When $m = n$, we call A a square matrix of size n. (The rectangular array now looks like a square.)
For an $n \times n$ square matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

The entries $a_{11}, a_{22}, a_{33}, \ldots, a_{nn}$ are called the diagonal entries. They are on the main diagonal of A.

Identity Matrices

The $n \times n$ square matrix where all the entries along the diagonal from the top left to the bottom right are 1, and 0 elsewhere, is called the identity matrix.
It is often denoted as I_n.
Some examples:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Diagonal Matrices

The $n \times n$ square matrix where all the off-diagonal entries (i.e. entries below and above the main diagonal) are 0 is called an diagonal matrix.

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & g \end{pmatrix} \quad B = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pi \end{pmatrix}.$$

Upper Triangular Matrices

The $n \times n$ square matrix where all the entries below the main diagonal are 0 is called an upper triangular matrix.

$$A = \begin{pmatrix} a & b & c \\ 0 & e & f \\ 0 & 0 & g \end{pmatrix} \quad B = \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 3 & -2 & 9 \\ 0 & 0 & 0 & \pi \\ 0 & 0 & 0 & 0.8 \end{pmatrix}.$$

Lower Triangular Matrices

In a similar way, a lower triangular matrix is a square matrix where all the entries above the main diagonal are 0. Eg.

$$C = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} 53 & 0 & 0 & 0 & 0 \\ -4 & 3 & 0 & 0 & 0 \\ 1/2 & 0 & \pi & 0 & 0 \\ 0 & -1 & 0.4 & \sqrt{2} & 0 \\ 1 & 3 & 5 & 7 & 9 \end{pmatrix}.$$

Scalars

Lastly, the term scalars refer to real numbers or complex numbers in discussing matrices (and vectors).

Equality of Matrices

Definition

Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

Matrices A and B are equal if they have the same size (same number of rows and same number of columns) and $A_{ij} = B_{ij}$ for all i and j. (Here, note that if the size of both matrix is $m \times n$, then $1 \leq i \leq m$ and $1 \leq j \leq n$.)

Example

Example

Find a, b, c and d if

$$\begin{bmatrix} a-b & b+c \\ 3d+c & 2a-4d \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 6 \end{bmatrix}.$$

Example (Solution)

Note that both matrices are 2 by 2. For matrices to be equal, each corresponding entries must be equal. Therefore we have

$$\begin{aligned} a-b &= 8 \\ b+c &= 1 \\ 3d+c &= 7 \\ 2a-4d &= 6 \end{aligned}$$

Solving for a, b, c and d, we obtain
$a = 5, b = -3, c = 4, d = 1$.(Exercise)

Example

Example

Let A be a 3×4 matrix whose (i,j)-th entry is defined by
$A_{ij} = (-1)^{i+j} 2i + j$.
Write down the matrix A.

Solution

$$A_{11} = (-1)^{1+1} 2(1) + 1 = 3, A_{12} = (-1)^{1+2} 2(1) + 2 = 0,$$

$$A_{13} = (-1)^{1+3} 2(1) + 3 = 5, A_{14} = (-1)^{1+4} 2(1) + 4 = 2$$

......

Therefore

$$A = \begin{bmatrix} 3 & 0 & 5 & 2 \\ -3 & 6 & -1 & 8 \\ 7 & -4 & 9 & -2 \end{bmatrix}.$$

Example

Let $B = [b_{ij}]$ be a 3×3 matrix where $b_{ij} = \begin{cases} i+j & \text{if } i > j \\ 0 & \text{if } i = j \\ -j & \text{if } i < j \end{cases}$.

Find B.

Solution

(Exercise)

$$B = \begin{bmatrix} 0 & -2 & -3 \\ 3 & 0 & -3 \\ 4 & 5 & 0 \end{bmatrix}.$$

Addition and Subtraction

Definition

If A and B are two $m \times n$ matrices, then

(a) the addition (or sum) $A + B$ is the matrix obtained by adding entries in the same positions, i.e.,

$$(A+B)_{ij} = A_{ij} + B_{ij}.$$

(b) the difference $A - B$ is the matrix obtained by subtracting entries of B from the corresponding entries of B, i.e.,

$$(A-B)_{ij} = A_{ij} - B_{ij}.$$

Example

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix}$. Then

$$A + B = \begin{pmatrix} 1+7 & 2+8 & 3+9 \\ 4+10 & 5+11 & 6+12 \end{pmatrix} = \begin{pmatrix} 8 & 10 & 12 \\ 14 & 16 & 18 \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1-7 & 2-8 & 3-9 \\ 4-10 & 5-11 & 6-12 \end{pmatrix} = \begin{pmatrix} -6 & -6 & -6 \\ -6 & -6 & -6 \end{pmatrix}.$$

WARNING!!! Matrices of different sizes cannot be added or subtracted.

Scalar Multiple of Matrices

Here, by a scalar, we refer to a real number or a complex number.

Definition

If α is a scalar and A is an $m \times n$ matrix, then the scalar multiple αA is the $m \times n$ matrix obtained by multiplying each entry of A by α, i.e.,

$$(\alpha A)_{ij} = \alpha(A_{ij}).$$

Example

Example

Let $A = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}$. Then

$$2A = \begin{pmatrix} 2 & 6 & 10 \\ 14 & 18 & 22 \end{pmatrix}, (-3)A = \begin{pmatrix} -3 & -9 & -15 \\ -21 & -27 & -33 \end{pmatrix}, \text{ and}$$

$$\frac{1}{3}A = \begin{pmatrix} \frac{1}{3} & 1 & \frac{5}{3} \\ \frac{7}{3} & 3 & \frac{11}{3} \end{pmatrix},$$

Some Algebraic Laws of Matrix Operations

Theorem

Assuming the sizes of matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

1. $A + B = B + A$ *(Commutative Law for addition)*
2. $(A + B) + C = A + (B + C)$ *(Associative Law for addition)*
3. $A + 0 = 0 + A = A$ *(Additive Identity)*
4. $A + (-A) = 0$ *(Additive Inverse)*
5. $0 - A = -A$

Some Algebraic Laws of Matrix Operations

Theorem

Assuming the sizes of matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

1. $\alpha(A \pm B) = \alpha A \pm \alpha B$
2. $(\alpha \pm \beta)A = \alpha A \pm \beta A$
3. $\alpha(\beta A) = (\alpha\beta)A$

Matrix Multiplication

Definition

If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then the matrix product (or simply the product) AB is the $m \times n$ matrix whose (i,j)-th entry is determined by

$$(AB)_{ij} = A_{i1}B_{1j} + A_{i2}B_{2j} + \cdots + A_{ir}B_{rj}$$

$$= \sum_{k=1}^{r} A_{ik}B_{kj}.$$

Matrix Multiplication and Dot Product

Note that the (i, j)th entry of AB is the value obtained by taking the dot product of the vector formed by the ith row of A and that formed by the jth column of B.

$$AB = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ A_{21} & A_{22} & \cdots & A_{2r} \\ \vdots & \vdots & & \vdots \\ A_{i1} & A_{i2} & \cdots & A_{ir} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mr} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1j} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & B_{2j} & \cdots & B_{2n} \\ \vdots & \vdots & & \vdots & & \\ B_{r1} & B_{r2} & \cdots & B_{rj} & \cdots & B_{rn} \end{pmatrix}$$

Example

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 \\ 4 \\ 2 \end{pmatrix}$. Note that A is 2×3 matrix and B is 3×1, the product AB will be a 2×1 matrix.

To find the entries of AB:

$(1, 1)$-th entry:

$(2, 1)$-th entry:

$(3, 1)$-th entry:

Thus, we have $AB = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

Example

Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} -3 \\ 0 \\ 2 \\ \sqrt{3} \end{pmatrix}$.

Note that A is 2×3 matrix and C is 4×1, the product AC is NOT defined.

Remark

1. Matrix multiplication is not commutative.
2. It is not true that if $AB = 0$, then $A = 0$ or $B = 0$. (Exercise: Find two non-zero 2×2 matrices A and B with $AB = 0$.)

Identity Matrices

Recall that the identity matrix I_n is the square matrix I_n of size n with (i, j)th-entry

$$(I_n)_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

Thus,

$$I_1 = \begin{pmatrix} 1 \end{pmatrix}, I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ldots$$

The role of identity matrices in matrix multiplication is like the number 1 in usual multiplication.

Algebraic Properties

Theorem

Assuming the sizes of matrices are such that the indicated operations can be performed, the following rules of matrix arithmetic are valid.

1. $AI_n = A$ and $I_m A = A$ if A is $m \times n$. *(Identity)*
2. $(AB)C = A(BC)$ *(Associative Law for multiplication)*;
3. $A(B + C) = AB + AC$ *(Left distributive law)*
4. $(A + B)C = AC + BC$; *(Right distributive law)*
5. $A0 = 0, 0A = 0$

Transpose

Definition

For an $m \times n$ matrix A, the matrix A^T obtained by interchanging the rows and columns of A is called the transpose of A. Thus, the (i, j)th-entry of A^T is

$$(A^T)_{ij} = A_{ji}$$

Example

Example

Let $A = \begin{pmatrix} 4 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -3 & -2 \end{pmatrix}$. Then

$$A^T = \begin{pmatrix} 4 & 1 \\ 3 & 3 \\ 2 & 1 \end{pmatrix}, \ B^T = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{pmatrix}.$$

Properties of Transpose

Theorem

If the sizes of the matrices are such that the stated operations can be performed, then

(a) $\left(A^T\right)^T = A$

(b) $(A \pm B)^T = A^T \pm B^T$

(c) $(\alpha A)^T = \alpha A^T$*, where* α *is a scalar*

(d) $(AB)^T = B^T A^T$ *(Note the change in order.)*

Invertible Matrices

Definition

Let A be a $n \times n$ square matrix. If there is another square matrix B such that

$$AB = I_n \text{ and } BA = I_n,$$

then A is said to be invertible (or non-singular), and B is called an inverse of A.

If no such matrix B can be found, then A is said to be not invertible (or singular).

Invertible Matrices

Example

Let $A = \begin{pmatrix} -1 & -2 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 2 \\ -3 & -1 \end{pmatrix}$.

Note that $AB = I_2$ and $BA = I_2$ (Verify these as an exercise).

Since $AB = I_2$ and $BA = I_2$, we conclude that A is invertible, by definition.

Example

Example

Let $C = \begin{pmatrix} 0 & 0 & 0 \\ a & b & c \\ d & e & f \end{pmatrix}$. Is there a matrix D such that $CD = I$?

Example

Example

Let $E = \begin{pmatrix} a & b & c \\ 2a & 2b & 2c \\ d & e & f \end{pmatrix}$.

Is there a matrix F such that $EF = I$?

Singular Matrices

Note that matrix invertibility is defined for square matrices only. It is clear that zero square matrices are singular.

(a) A square matrix with a row (or a column) of zeros is singular (or not invertible).

(b) A square matrix with a row (or a column) which is a multiple of another row (or column) is singular. In particular, a square matrix with identical rows (or columns) is singular.

Example

A row R_i is a linear combination of other rows R_j, where $j \neq i$, means that there are scalars α_j's such that

$$R_i = \sum_{j \neq i} \alpha_j R_j = \alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_{i-1} R_{i-1} + \alpha_{i+1} R_{i+1} + \cdots \alpha_n R_2 n.$$

A square matrix in which a row (or a column) is a linear combination of other rows (or columns) is singular.

Example

Example

Consider

$$G = \begin{pmatrix} a & b & c \\ d & e & f \\ 2a-5d & 2b-5e & 2c-5f \end{pmatrix}.$$

Note that $R_3 = 2R_1 + (-5)R_2$.
The matrix G singular.

The Inverse Matrix

Now we prove that an invertible matrix cannot have more than one inverses. In other words, if an inverse exists, it is unique.
If B and $\widehat{B}$ are both inverses of A, then $B = \widehat{B}$.
[Proof.] Note that

$$AB = I \;\&\; BA = I, \text{ and } A\widehat{B} = I \;\&\; \widehat{B}A = I.$$

(Our aim is to show that $B = \widehat{B}$.)

Since $B = BI$, we have

$$B = BI = B(A\widehat{B}) = (BA)\widehat{B} = I\widehat{B} = \widehat{B}.$$

The Inverse Matrix

The above proposition says that the inverse of an invertible matrix A is unique.

In view of this, we shall denote the inverse of A by A^{-1}. Thus, we have

$$AA^{-1} = I \text{ and } A^{-1}A = I.$$

The Inverse Matrix

It follows immediately from the above equation and definition of matrix invertibility that the inverse A^{-1} of a matrix A is invertible. If A is an invertible matrix, then the inverse A^{-1} is invertible and

$$\left(A^{-1}\right)^{-1} = A.$$

2 x 2 Inverse Matrix

For a 2×2 matrix, there is a good characterization of invertibility. Moreover, there is a nice formula for its inverse when it is invertible.

The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $ad - bc \neq 0$. In this case the inverse A^{-1} is given by the formula

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{1}$$

2 x 2 Inverse Matrix

Suppose $ad - bc \neq 0$. We verify that the matrix equations are satisfied

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = I_2$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\right] = I_2.$$

By the definition of matrix invertibility, A is invertible and its inverse A^{-1} is

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

2 x 2 Inverse Matrix (Independent Reading)

It remains to prove that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $ad - bc \neq 0$.

We prove it by contradiction.

Suppose on the contrary that $ad - bc = 0$.
We shall consider two cases : (a) $a = 0$ (b) $a \neq 0$.

Suppose on the contrary that $ad - bc = 0$.
Case (a) where $a = 0$, note that either $b = 0$ or $c = 0$.
If $b = 0$, then A has a row of zero and this it is not invertible.
If $c = 0$, then the first row and second row of A are multiple of each other. Hence A is not invertible.
Both contradict A being invertible.

Case (b) where $a \neq 0$, we have $d = \frac{bc}{a}$. Then the second row of A is a scalar multiple of the fist row of A. ($\frac{c}{a}$ of the first row) Thus, A is not invertible, contradicting A being invertible. .

We conclude that if A is invertible, then $ad - bc \neq 0$.

Remark

The above result provides a useful characterization of invertible 2×2 matrices. For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the value $ad - bc$ is called the determinant of A. The above result also provides a formula for the inverse A^{-1} if the matrix A is invertible.

Example

Example

Determine whether each of the following 2×2 matrices is invertible. If it is, find its inverse.

(a) $A = \begin{pmatrix} 5 & 3 \\ 7 & 9 \end{pmatrix}$

(b) $B = \begin{pmatrix} -8 & 4 \\ 6 & -3 \end{pmatrix}$.

(c) $C = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$.

Remark

(a) For matrices of other sizes, there is similar result on invertibility via determinant which will be discussed in latter section. There is also a formula for the inverse of an invertible matrix. However, the formula is more complicated than the case for 2×2 matrices.

(b) We may use Gaussian method to determine whether a square matrix is invertible and also to find its inverse. This will be dealt with in another mathematics course.

Inverse for Diagonal Matrices

Example

Find the inverse of the following diagonal matrix, if $abcd \neq 0$.

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

Inverse for Diagonal Matrices

More generally, we have the following result for diagonal matrices. Suppose D is a diagonal matrix. Then D is invertible if and only of its diagonal entries are non-zero, i.e., $D_{ii} = d_i \neq 0$ for every i. When D is invertible, D^{-1} is a diagonal matrix whose ith diagonal entry is $\frac{1}{d_i}$ for each i.

Inverse for Diagonal Matrices

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}, D^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \vdots & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{d_n} \end{pmatrix}$$

Power Matrices

Definition

Let A be a square matrix. We define the nonnegative integer powers of A to be

$$A^0 = I, \quad A^n = \underbrace{AA\cdots A}_{\text{n times}}(n > 0).$$

Moreover, if A is invertible, then we define the negative integer powers to be

$$A^{-n} = \left(A^{-1}\right)^n = \underbrace{A^{-1}A^{-1}\cdots A^{-1}}_{\text{n times}}(n > 0)$$

Laws of Exponents for Matrices

As in the case of real numbers, we have

$$A^r A^s = A^{r+s}, (A^r)^s = A^{rs}.$$

Example

Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Find $A^2, A^3, A^4, A^{-2}, A^{-3}$.

(Exercise.)

Laws of Exponents for Matrices

We also have the following laws of exponents.
Let n be an integer. If A be an invertible matrix, then A^n is invertible and

$$\left(A^n\right)^{-1} = \left(A^{-1}\right)^n.$$

Proof. (Optional.)