NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 3.1 - Predicate Logic I

Dr. Gary Greaves

Limitation of Propositional Logic

- Every SCSE student must study discrete mathematics.
- Jackson is an SCSE student.
  - So, Jackson must study discrete mathematics.

This argument can't be expressed with propositional logic.

What propositional logic allows to express:

- If Jackson is an SCSE student, then he must study discrete mathematics.
- Jackson is an SCSE student.
  - So, Jackson must study discrete mathematics.

Topic Overview

What’s in store…

Predicates

Quantification

By the end of this lesson, you should be able to…

- Identify a statement containing a predicate.

- Use quantifiers to express a property about “all” and “some”.

Predicates

Predicates: Definition

Is the statement "x^2 is greater than x" a proposition?

- Define $P(x)$ = "x^2 is greater than x"
  - Is $P(1)$ a proposition?
  - $P(1)$ = "1^2 is greater than 1"

Predicates: Definition

A **predicate** is a statement that contains variables (**predicate variables**) and that is either true or false depending on the values of these variables.

- $P(x)$ = “x^2 is greater than x”
- $P(1)$ = “1^2 is greater than 1”
- $P(x)$ is a predicate

```
frederique@frederique-desktop:~$ ./bool
Is 10 equal to 3 ? 0
Is 10 different from 3? 1
frederique@frederique-desktop:~$
```

```c
#include <stdio.h>

void main()
{
    int a,b;
    a=10;
    b=3;
    printf("Is %d equal to %d ? %d\n",a,b,a==b);
    printf("Is %d different from %d? %d\n",a,b,a!=b);
```

Predicates: Predicate Instantiated/Domain

A **predicate** is a statement that contains variables (**predicate variables**) and that is either true or false depending on the values of these variables.

A predicate instantiated (where variables are substituted for specific values) is a proposition.

- $P(x)$ = "x^2 is greater than x"
- $P(1)$ = "1^2 is greater than 1"

Predicates: Predicate Instantiated/Domain

The **domain** of a predicate variable is the collection of all possible values that the variable may take.

- E.g., the domain of x in $P(x)$: the integers
- A predicate may have more than one variable
- Different variables may have different domains

Predicates: Example

Let $P(x, y)$ = "$x > y$"

Domain: integers (i.e., both x and y are integers)

$P(4, 3)$

This means "$4 > 3$", so $P(4, 3)$ is **TRUE**.

$P(1, 2)$

This means "$1 > 2$", so $P(1, 2)$ is **FALSE**.

$P(3, 4)$

This means "$3 > 4$", so $P(3, 4)$ is **FALSE**.

In general, $P(x, y)$ and $P(y, x)$ are not equal.

Quantification

Quantification: Statements Like…

Some birds are angry.

The square of any real number is non-negative.

Not all SCSE students study hard.

Quantification: Universal Quantification

A universal quantification is a quantifier (something that tells the amount or quantity) meaning "given any" or "for all".

$\forall$

Symbol

E.g., "$\forall x \in D$, $P(x)$ is true" iff "$P(x)$ is true for every x in D"

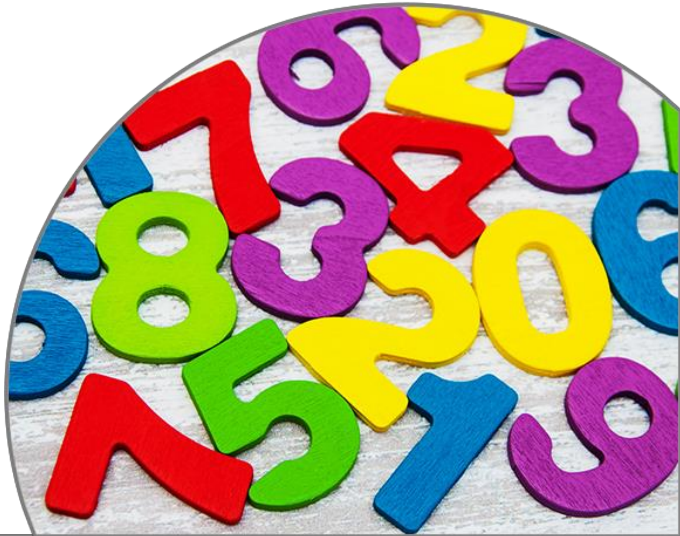
$\forall$	Universal quantifier, "for all", "for every"
$\in$	"Is a member (or) element of", "belonging to"
D	Domain of predicate variable

Quantification: Universal Quantification

$\forall$	Universal quantifier, "for all", "for every"
$\in$	"Is a member (or) element of", "belonging to"
D	Domain of predicate variable

The square of any real number is non-negative.

$$\forall x \in \mathbb{R}, x^2 \geq 0$$



Quantification: Existential Quantification

An **existential quantification** is a quantifier (something that tells the amount or quantity) meaning "**there exists**", "**there is at least one**" or "**for some**".

E.g., "$\exists\, x \in D$, $P(x)$ is true" iff "$P(x)$ is true for **at least one** x in D"

$\exists$

Symbol

$\exists$	Existential quantifier, "there exists"
$\in$	"Is a member (or) element of", "belonging to"
D	Domain of predicate variable

Quantification: Existential Quantification

∃

∃	Existential quantifier, "there exists"
∈	"Is a member (or) element of", "belonging to"
D	Domain of predicate variable

Some birds are angry.

$\exists\, x \in$ {birds}, x is angry

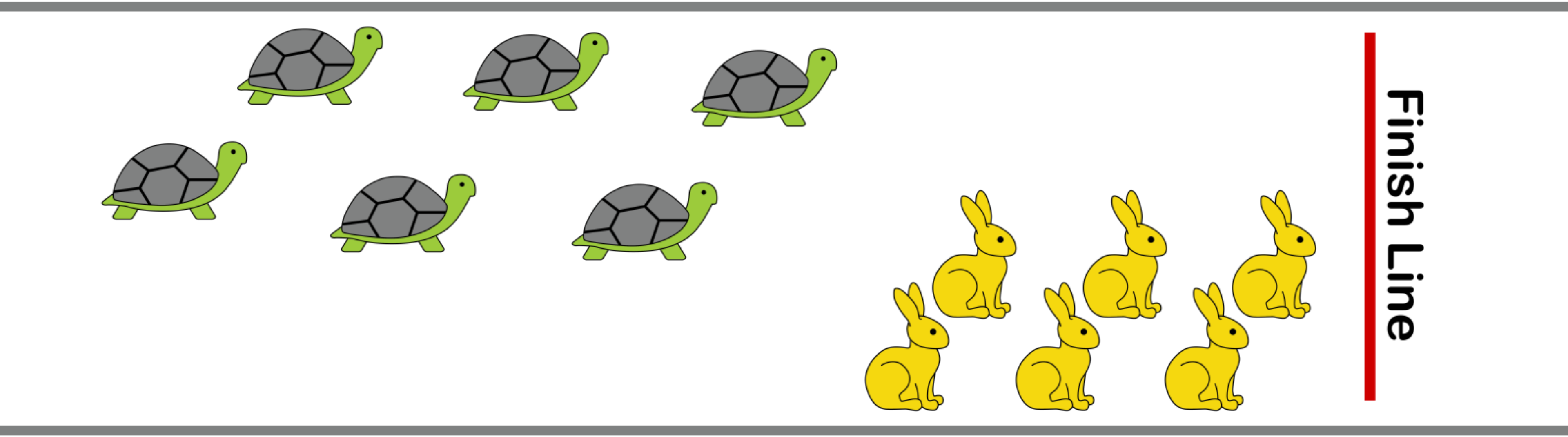
Quantification: Nested Quantification (I)

- A proposition may contain multiple quantifiers:
  - "All rabbits are faster than all tortoises."
  - Domains: R = {rabbits}, T = {tortoises}
  - Predicate $C(x,y)$: Rabbit x is faster than tortoise y

In Symbols	$\forall x \in R, (\forall y \in T, C(x, y))$ or $\forall x \in R, \forall y \in T, C(x, y)$
In Words	For any rabbit x, and for any tortoise y, x is faster than y.

Quantification: Nested Quantification (I)

In Symbols	$\forall x \in R, (\forall y \in T, C(x, y))$ or $\forall x \in R, \forall y \in T, C(x, y)$
In Words	For any rabbit x, and for any tortoise y, x is faster than y.



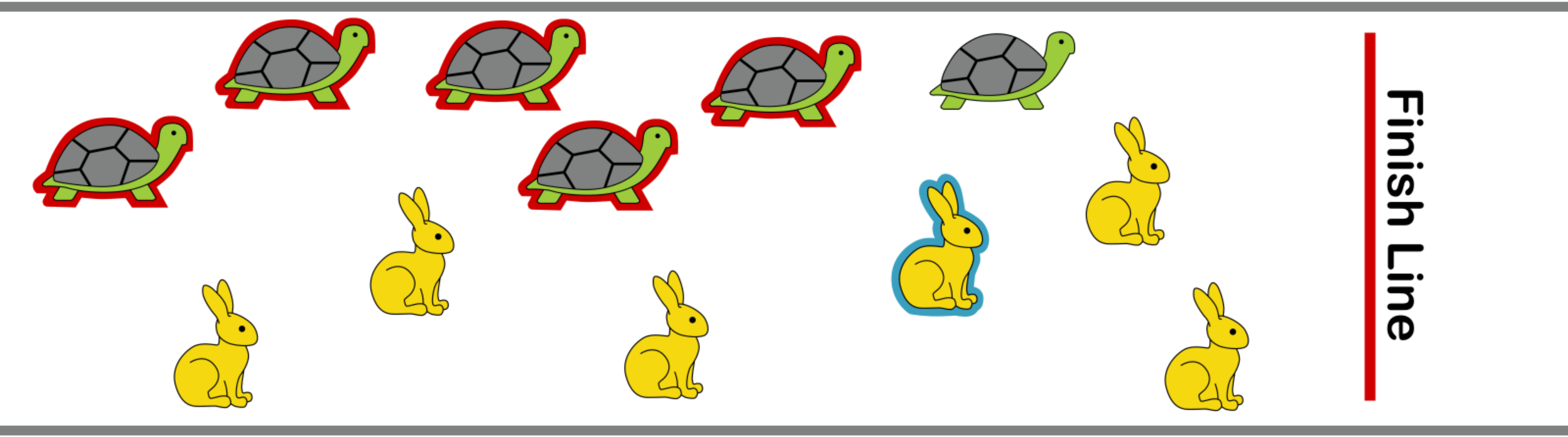
Quantification: Nested Quantification (II)

- Another example:
  - "Every rabbit is faster than some tortoise."
  - Domains: R = {rabbits}, T = {tortoises}
  - Predicate $C(x,y)$: Rabbit x is faster than tortoise y

In Symbols	$\forall x \in R, (\exists y \in T, C(x, y))$ or $\forall x \in R, \exists y \in T, C(x, y)$
In Words	For any rabbit x, there exists a (some) tortoise y, such that x is faster than y.

Quantification: Nested Quantification (II)

In Symbols	$\forall x \in R, (\exists y \in T, C(x, y))$ or $\forall x \in R, \exists y \in T, C(x, y)$
In Words	For any rabbit x, there exists a (some) tortoise y, such that x is faster than y.



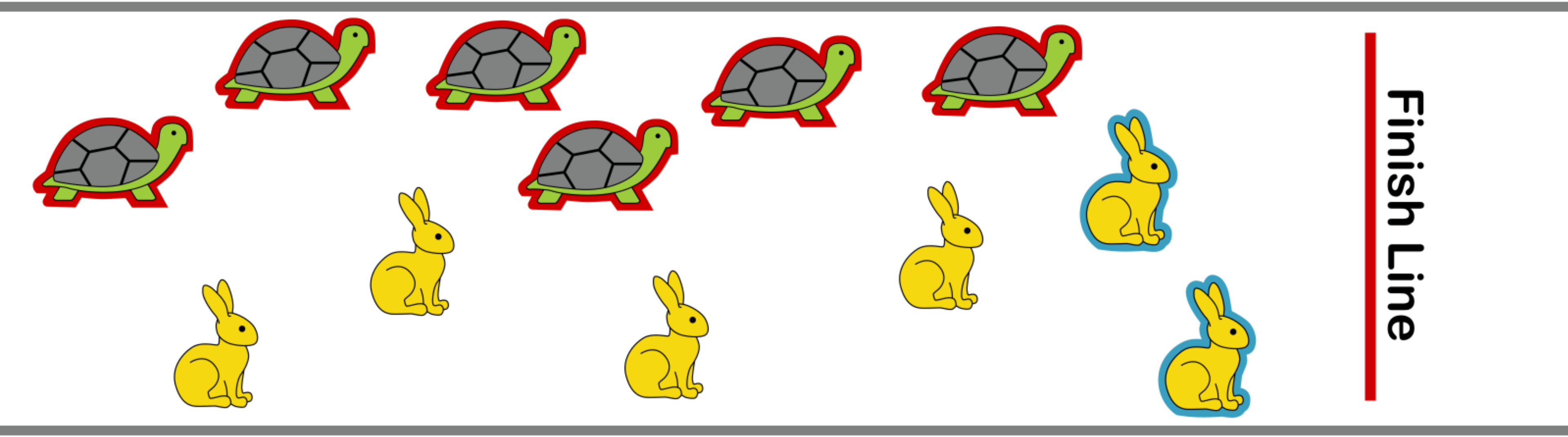
Quantification: Nested Quantification (III)

- Another example:
  - "There is a rabbit that is faster than all tortoises."
  - Domains: R = {rabbits}, T = {tortoises}
  - Predicate $C(x,y)$: Rabbit x is faster than tortoise y

In Symbols	$\exists x \in R, (\forall y \in T, C(x, y))$ or $\exists x \in R, \forall y \in T, C(x, y)$
In Words	There exists a rabbit x, such that for any tortoise y, this rabbit x is faster than y.

Quantification: Nested Quantification (III)

In Symbols	$\exists x \in R, (\forall y \in T, C(x, y))$ or $\exists x \in R, \forall y \in T, C(x, y)$
In Words	**There exists a rabbit *x*, such that for any tortoise *y*, this rabbit *x* is faster than *y*.**



Quantification: Order of Nesting Matters

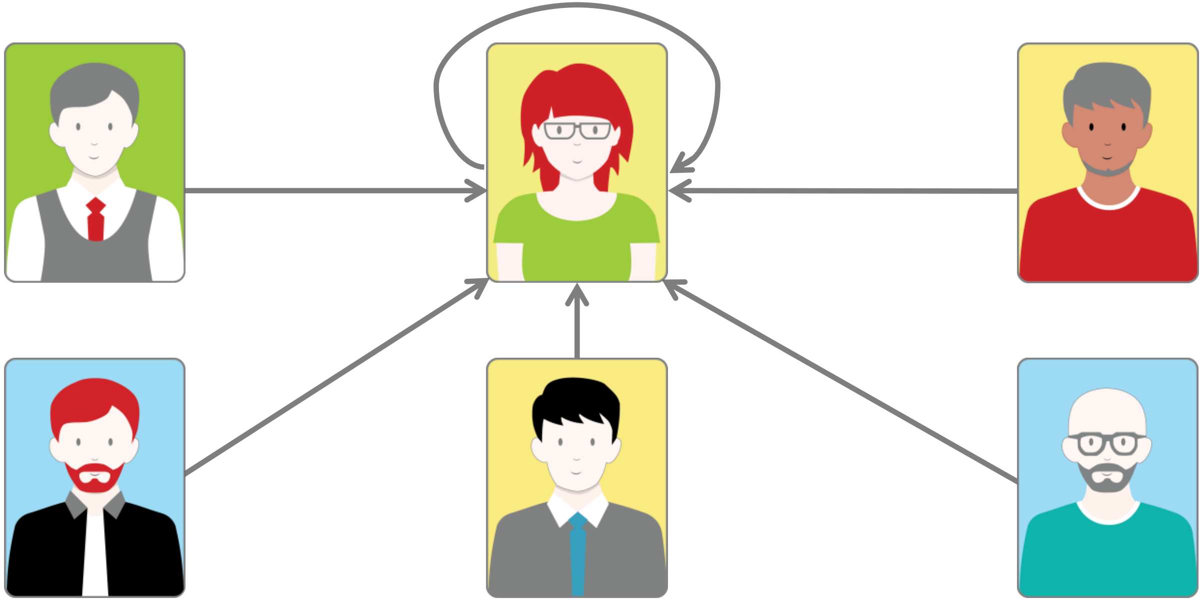
Is $\forall x \in D, \exists y \in D, P(x,y) \equiv \exists y \in D, \forall x \in D, P(x,y)$ in general?

LHS	RHS
$\forall x \in D, \exists y \in D, P(x,y)$	$\exists y \in D, \forall x \in D, P(x,y)$
y can **vary** with x	y is **fixed**, but x **varies**

Let $P(x,y)$ = "x admires y"

LHS	RHS
"Every person admires someone"	"Some people are admired by everyone"

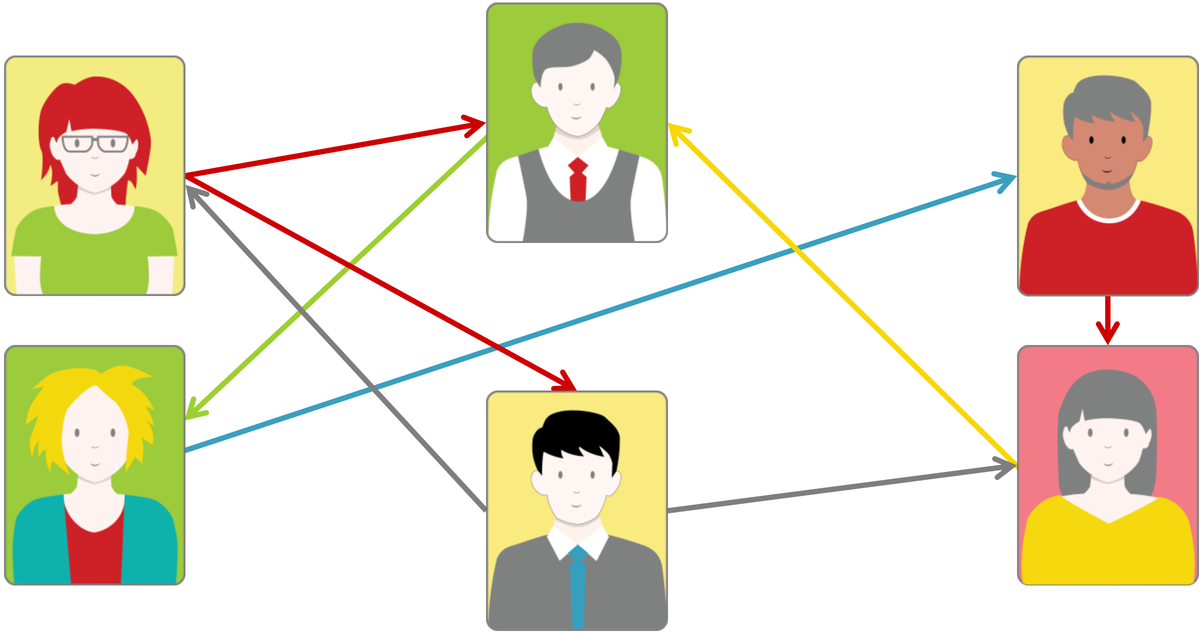
Quantification: Order of Nesting Matters



LHS $\forall x \in D, \exists y \in D, P(x,y)$ "Every person admires someone"

RHS $\exists y \in D, \forall x \in D, P(x,y)$ "Some people are admired by everyone"

Quantification: Order of Nesting Matters



LHS $\forall x \in D, \exists y \in D, P(x,y)$ "Every person admires someone"

RHS $\exists y \in D, \forall x \in D, P(x,y)$ "Some people are admired by everyone"

Topic Summary

Let's recap…

- Predicates:
  - Statements with variables
- Quantifiers:
  - Use to express a property about "all" and "some"
  - Universal
  - Existential

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Topic 3.2 - Predicate Logic II

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Topic Overview

What's in store…

Negation of Quantification

Determining Truth Values

Conditional Quantification

By the end of this lesson, you should be able to…

- Express the negation of a quantified statement.
- Find the truth value of a quantified statement.
- Manipulate quantified conditional statements.

Negation of Quantification

Negation of Quantification: Truth vs. False

Statement	When True	When False
$\forall x \in D, P(x)$	$P(x)$ is **true** for every x in D.	There is one x for which $P(x)$ is **false**.
$\exists x \in D, P(x)$	There is one x in D for which $P(x)$ is **true**.	$P(x)$ is **false** for every x in D.

Assume that D consists of $x_1, x_2, \ldots, x_n$

$$\forall x \in D, P(x) \equiv P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

$$\exists x \in D, P(x) \equiv P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$$

Negation of Quantification: Example 1

"**Not all** SCSE students study hard." = "**There is at least one** SCSE student who does not study hard."

$\neg(\forall x \in D, P(x))$ $\equiv$ $\exists x \in D, \neg P(x)$

D = {SCSE students}

$P(x)$ = "x studies hard"

Negation of a universal quantification becomes an existential quantification.

Negation of Quantification: Example 2

"It is **not** the case that **some** students in this class are from NUS."

=

"**All** students in this class are **not** from NUS."

$\neg(\exists x \in D, P(x))$ $\equiv$ $\forall x \in D, \neg P(x)$

D = {Students}

$P(x)$ = "x is from NUS"

Negation of an existential quantification becomes an universal quantification.

Negation of Quantification: Example 3

$$\neg\,(\forall x \in D,\ P(x) \wedge Q(x))$$

$$\equiv\ \exists x \in D,\ \neg\,(P(x) \wedge Q(x))$$ Negation of Quantification

$$\equiv\ \exists x \in D,\ (\neg\,P(x) \vee \neg\,Q(x))$$ De Morgan

- Not all students in this class are using Facebook and (also) Google+.
  - There is some (at least one) student in this class who is not using Facebook or not using Google+ (or may be using neither).

Determining Truth Values

Determining Truth Values: Three Methods

Systematic Approaches

Method of:

- Exhaustion
- Case
- Logical derivation

Determining Truth Values: Method of Exhaustion

Let $D = \{5,6,7,8,9\}$

Is $\exists x \in D, x^2 = x$ true or false?

x	x^2	$x^2 = x$
5	$5^2 = 25$	**False**
6	$6^2 = 36$	**False**
7	$7^2 = 49$	**False**
8	$8^2 = 64$	**False**
9	$9^2 = 81$	**False**

Limitation?

- **Domain may be too large to try out all options, e.g., all integers.**

Determining Truth Values: Method of Case

Positive Example to Prove Existential Quantification

Let $\mathbb{Z}$ denote all integers.

Is $\exists x \in \mathbb{Z}, x^2 = x$ true or false?

Take $x = 0$ or 1 and we have it.

True!

Counterexample to Disprove Universal Quantification

Let $\mathbb{R}$ denote all reals.

Is $\forall x \in \mathbb{R}, x^2 > x$ true or false?

Take $x = 0.3$ as a counterexample.

False!

Determining Truth Values: Method of Case

Positive Example

It is not a proof of universal quantification.

Negative Example

It is not disproof of existential quantification.

Note that it may be hard to find suitable "cases" even if such cases do exist!

Determining Truth Values: Method of Logical Derivation

Consider an (arbitrary) domain X with n members.

Is $\exists x \in X, (P(x) \vee Q(x)) \equiv (\exists x \in X, P(x)) \vee (\exists x \in X, Q(x))$?

$\exists x \in X, (P(x) \vee Q(x))$

$\equiv [P(x_1) \vee Q(x_1)] \vee \ldots \vee [P(x_n) \vee Q(x_n)]$

$\equiv [P(x_1) \vee \ldots \vee P(x_n)] \vee [Q(x_1) \vee \ldots \vee Q(x_n)]$

$\equiv (\exists x \in X, P(x)) \vee (\exists x \in X, Q(x))$

Conditional Quantification

Conditional Quantification: Example 1

For any real number x, if $x > 1$ then $x^2 > 1$ (i.e., any real number greater than 1 has a square larger than 1).

- **Let $P(x)$ denote "$x > 1$".**
- **Let $Q(x)$ denote "$x^2 > 1$".**
- **Recall: $\mathbb{R}$ is the collection of all real numbers.**

In Symbolic Form: $\forall x \in \mathbb{R}, (P(x) \rightarrow Q(x))$

Conditional Quantification: Example 2

Many statements can be restated as conditional statements. Consider the statement “lions are fierce animals”.

- Let A denote the collection of all animals.
- Let $P(x)$ denote “x is a lion”.
- Let $Q(x)$ denote “x is fierce”.
- The statement can be rephrased as: “If an animal x is a lion then x is fierce”.

In Symbolic Form: $\forall x \in A, (P(x) \rightarrow Q(x))$

Conditional Quantification: Definitions

Given a conditional quantification such as…

$$\forall x \in A\ (P(x) \to Q(x))$$

Then, we define…

Contrapositive	$\forall x \in A,\ \neg Q(x) \to \neg P(x)$
Converse	$\forall x \in A,\ Q(x) \to P(x)$
Inverse	$\forall x \in A,\ \neg P(x) \to \neg Q(x)$

Note: a conditional proposition is logically equivalent to its contrapositive.

Conditional Quantification: Negation

What is $\neg (\forall x \in X,\ P(x) \rightarrow Q(x))$**?**

$\neg (\forall x \in X,\ P(x) \rightarrow Q(x))$

$\equiv \exists x \in X, \neg (P(x) \rightarrow Q(x))$ **Negation of Quantified Statements**

$\equiv \exists x \in X, \neg (\neg P(x) \vee Q(x))$ **Conversion of Conditionals**

$\equiv \exists x \in X, P(x) \wedge \neg Q(x)$ **De Morgan**

Topic Summary

Let’s recap…

- Negation of quantification

- Determining truth value of a quantification:
  - Methods for proving quantified statements

- Conditional quantification

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Topic 3.3 - Predicate Logic III

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Topic Overview

What's in store…

B asic Inference Rules

By the end of this lesson, you should be able to…

Apply inference rules to verify an argument.

Basic Inference Rules

Basic Inference Rules

Existential Instantiation

Universal Generalisation

Existential Generalisation

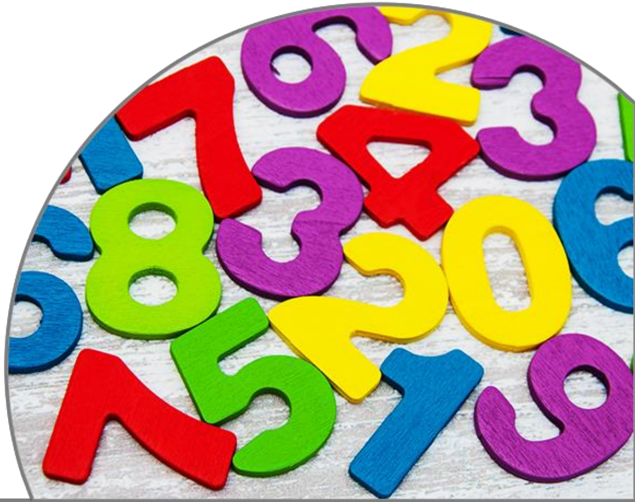
Universal Instantiation

Basic Inference Rules: Universal Generalisation

$P(c)$ for **any arbitrary** c from the domain D.
$\therefore\ \forall x \in D,\ P(x)$

x^2 is non-negative

- $P(x)$ = "x^2 is non-negative"
- $P(c)$ for an arbitrary real c
- Therefore $P(x)$ for all x in $\mathbb{R}$

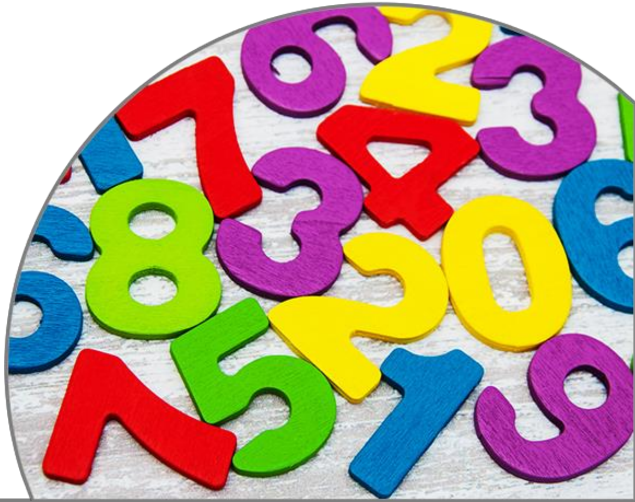


Basic Inference Rules: Universal Generalisation

Domain = $\mathbb{R}$

$P(x)$ **= "**x^2 **is non-negative"**

1	$P(c)$ **for an arbitrary real** c	**Hypothesis**
2	$\forall x \in \mathbb{R}, P(x)$	**Universal Generalisation on 1**



Basic Inference Rules: Universal Instantiation

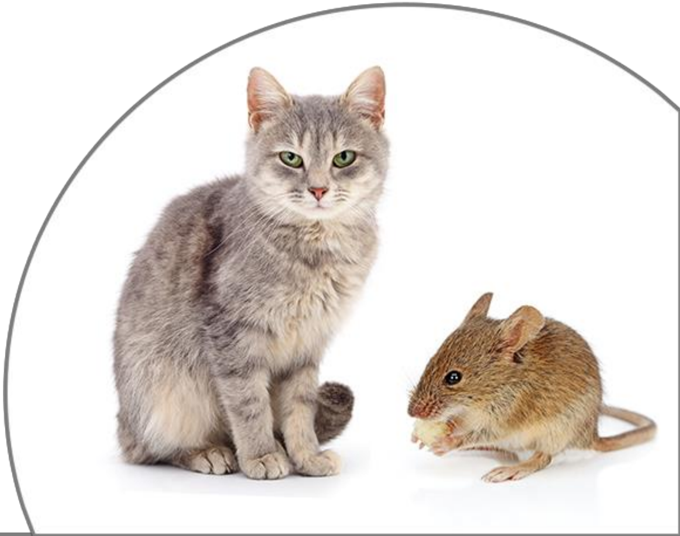
$\forall x \in D, P(x)$

$\therefore P(c)$

where c is **any** element of the domain D.

Tom and Jerry

- No cat can catch Jerry.
- Tom is a cat.
- Therefore, Tom cannot catch Jerry.



Basic Inference Rules: Universal Instantiation

D = {all animals}

Cat(x) = x is a Cat

Catch(x) = x can catch Jerry

1	$\forall x \in D$, [Cat(x) → ¬ Catch(x)]	Hypothesis
2	Cat(Tom)	Hypothesis
3	Cat(Tom) → ¬ Catch(Tom)	Universal Instantiation on 1
4	¬ Catch(Tom)	Modus Ponens on 2 and 3

Basic Inference Rules: Existential Generalisation

$P(c)$

$\therefore \exists x \in D, P(x)$

for c **some** specific element of the domain D.

Selling Stocks

If everyone is selling stocks, then someone is selling stocks.

Basic Inference Rules: Existential Generalisation

D = **{all people}**

Sell(x) = "x **is selling stocks**"

$\forall x \in D$, Sell(x) $\rightarrow$ $\exists x \in D$, Sell(x)

1	$\forall x \in D$, Sell(x)	**Hypothesis**
2	Sell(c)	**Universal Instantiation on 1**
3	$\exists x \in D$, Sell(x)	**Existential Generalisation on 2**

Basic Inference Rules: Existential Instantiation

$\exists x \in D,\ P(x)$

$\therefore\ P(c)$ **for some** c **in the domain** D.

Final Exam

- If any student scores > 80 in the final exam, then s/he receives an A.
- There are students who score > 80 in the final exam.
- Therefore, there are students who receive an A.

Basic Inference Rules: Existential Instantiation

D = {all students}

$A(x)$ = "x receives an A"

$M(x)$ = "x scores > 80 in the final exam"

1	$\forall x \in D, [M(x) \rightarrow A(x)]$	Hypothesis
2	$\exists x \in D, M(x)$	Hypothesis
3	$M(c)$	Existential Instantiation on 2
4	$M(c) \rightarrow A(c)$	Universal Instantiation on 1
5	$A(c)$	Modus Ponens on 4 and 3
6	$\exists x \in D, A(x)$	Existential Generalisation on 5

Topic Summary

Let’s recap…

- More inference rules to verify arguments