Name: ______________________________ Tutorial group: ________________

Matriculation number: | | | | | | | | | |

20 September 2020 **MH1812 Test 1** 60 minutes

QUESTION 1. (30 marks)

Use mathematical induction to show that

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

whenever n is a positive integer.

[Proof:]

Base case: when $n = 1$, the left-hand side is $1^2 = 1$ and the right-hand side is $(-1)^0 \cdot \frac{1 \cdot 2}{2} = 1$. The equality holds.

Inductive step: Suppose that the equality holds for $n = k$, and we shall show that it holds for $n = k + 1$, that is, we shall show that

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{k-1}k^2 + (-1)^k(k+1)^2 = (-1)^k \frac{(k+1)(k+2)}{2}. \tag{1}$$

By induction hypothesis, the LHS of (1) equals

$$(-1)^{k-1} \cdot \frac{k(k+1)}{2} + (-1)^k(k+1)^2 = (-1)^k(k+1) \cdot \left(-\frac{k}{2} + k + 1\right) = (-1)^k(k+1)\frac{k+2}{2},$$

which is exactly the RHS of (1). This completes the proof of the inductive step.

Therefore, by mathematical induction, the original equality holds for all integers $n \geq 1$.

[Grading rules:] The base case is worth 5 points. In the inductive step, stating the equality holds for $n = k$ **and** stating the corresponding equation is worth 10 points. Alternatively, a correct application of the induction hypothesis is worth 10 points. The rest of the derivation is worth 15 points.

For graders only	Question	1	2(a)	2(b)	3(a)	3(b)	Bonus	Total
	Marks							

QUESTION 2. (30 marks)

(a) (10 points) What is $2020^{1812} \bmod 30$?

[Solution:] First, we have $2020 \bmod 30 = 10$. Observe that $10^2 \bmod 30 = 10$, we know that $10^n \bmod 30 = 10$ for all $n \geq 1$. Therefore $2020^{1812} \bmod 30 = 10^{1812} \bmod 30 = 10$.

[Grading rules:] Getting $2020 \bmod 30 = 10$ is worth 4 points and reducing $2020^{1812} \bmod 30$ to $10^{1812} \bmod 30$ is worth another 1 point. The rest is worth 5 points. Note that a direct claim $10^{1812} \bmod 30 = 10$ without any reasoning is not an acceptable argument.

(b) Let $\mathbb{R}$ denote the set of reals. For $x, y \in \mathbb{R}$, let $P(x, y)$ denote the predicate "$x^2 - x + 2020y \geq 0$". What are the truth values of these statements?

(i) (10 points) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x, y)$.

(ii) (10 points) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, P(x, y)$.

[Solution:]

(i) The statement is true. One can just choose $y = -(x^2 - x)/2020$.

(ii) The statement is false. For any $x \in \mathbb{R}$, let $y = -(x^2 - x)/2020 - 1$, then $P(x, y)$ is false.

[Grading rules:] For each subquestion, stating correctly its truth value is worth 3 points and the reason is worth another 7 points.

Use mathematical induction to show that

$$1^2 - 2^2 + 3^2 - \cdots + (-1)^{n-1}n^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$$

whenever n is a positive integer.

Prove : let $f(x) = (-1)^{n-1} \cdot \frac{n(n+1)}{2}$

$\therefore$

Base case, $f(1)$: $(-1)^0 \cdot \frac{1(1+1)}{2} = 1$

$1^2 = 1$ //

Inductive step: let $f(x)$ be true.

$$f(x+1) = (-1)^{(x-1)} \cdot \frac{x(x+1)}{2} + (-1)^x (x+1)^2$$

$$= (-1)^x (-1)^{(-1)} \cdot \frac{x(x+1)}{2} + (-1)^x (x+1)(x+1)$$

$$= (-1)^x (x+1) \left[(-1)^{-1} \cdot \frac{x}{2} + (x+1) \right]$$

$$= (-1)^x (x+1) \left[x+1 - \frac{x}{2} \right]$$

$$= (-1)^x (x+1) \left[\frac{x}{2} + 1 \right]$$

$$= (-1)^x \frac{(x+1)(x+2)}{2}$$

(a) (10 points) What is $2020^{1812} \bmod 30$?

$$2020^{1812} \bmod 30 = (10)^{1812} \bmod 30$$

$$= (10^2)^{906} \bmod 30$$

$$= (10) \bmod 30 = 10$$

(b) Let $\mathbb{R}$ denote the set of reals. For $x, y \in \mathbb{R}$, let $P(x, y)$ denote the predicate "$x^2 - x + 2020y \geq 0$". What are the truth values of these statements?

(i) (10 points) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x, y)$.

(ii) (10 points) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, P(x, y)$.

i) for any x,

$y = -\frac{(x^2 - x)}{2020}$. valid

ii) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, P(x,y)$

for $x = a$,

$y = -\frac{(a^2 - a)}{2020} - 1$ is F

QUESTION 3. **(40 marks)**

(a) (20 points) Show that $q \wedge \neg(p \to q)$ is a contradiction.

[Proof:]

$$\begin{aligned} q \wedge \neg(p \to q) &\equiv q \wedge \neg(\neg p \vee q) \\ &\equiv q \wedge (p \wedge \neg q) \\ &\equiv (q \wedge \neg q) \wedge p \\ &\equiv F \wedge p \\ &\equiv F \end{aligned}$$

[Grading rules:] Each equality above is worth 4 points. For a truth table of 2 variables and 4 rows, each row is worth 5 points and no partial credits are given for an incorrect row.

(b) (20 points) Determine whether the following argument is valid[1].

$$\begin{aligned} &(p \wedge q) \to (r \vee s); \\ &\neg r; \\ &p \to q; \\ &p; \\ \therefore\ &s. \end{aligned}$$

[Solution:] The argument is valid, as shown by the following inference table.

Step	Formula	Reason
(1)	$p \to q$	Premise
(2)	p	Premise
(3)	q	(1)+(2), modus ponens
(4)	$p \wedge q$	(2)+(3), conjunctive addition
(5)	$(p \wedge q) \to (r \vee s)$	Premise
(6)	$r \vee s$	(4)+(5), modus ponens
(7)	$\neg r$	Premise
(8)	s	(6)+(7), disjunctive syllogism

[Grading rules:] One needs to apply the inference rules four times. Each application is worth 5 points. A correct argument based on the truth table is also acceptable (for which one may assume that p is true and r is false and have only two variables q and s in the table).

[1]The inference rules you may use are: modus ponens, modus tollens, conjunctive simplification, conjunctive addition, disjunctive addition, disjunctive syllogism, rule of contradiction and disjunction elimination.

BONUS QUESTION. **(10 marks)**

[Points will be given to *fully* correct solutions only. The total mark of this test is capped at 100 marks.]

Prove that there are no integers x and y such that $x^2 + y^2 = 444444443$.

[Proof:] Note that $n^2 \equiv 0, 1 \pmod 4$ for any integer n (this can be readily verified by calculating $n^2 \bmod 4$ for $n = 0, 1, 2, 3$). Observe that the right-hand side is congruent to 3 modulo 4, therefore it cannot be written as a sum of two squares.

[Grading rules:] This is an all-or-nothing question. The marks given should be either 0 or 10 marks.

(a) (20 points) Show that $q \wedge \neg(p \to q)$ is a contradiction.

$$q \wedge \neg(\neg p \vee q)$$
$$\equiv q \wedge (p \wedge \neg q)$$
$$\equiv (q \wedge \neg q) \wedge p$$
$$\equiv F \wedge p$$
$$\equiv F$$

(b) (20 points) Determine whether the following argument is valid[1].

$$\begin{aligned} &(p \wedge q) \to (r \vee s); \\ &\neg r; \\ &p \to q; \\ &p; \\ \therefore\ &s. \end{aligned}$$

$p \Rightarrow q$ is T

p is T

q is T

$(p \wedge q)$ is T

$(r \vee s)$ is T

$\neg r$ is T

r is F

s is T