MH1810 Math 1 Part 2 Chapter 5 Differentiation

Derivatives

Tang Wee Kee

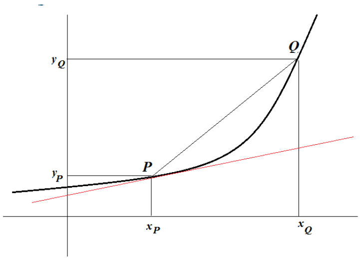
Nanyang Technological University

Tangent

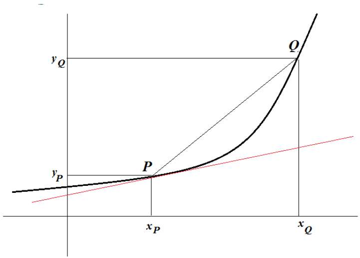
Consider a curve and a fixed point P on a curve. What is the tangent line at P?

Tangent

The tangent to a curve at a point P is a straight line which 'touches' the curve at P. The tangent at P is a line which cuts the curve in one and only one point, namely the point P, in a sufficiently small neighbourhood around P, although it may cut the curve at more than one point.

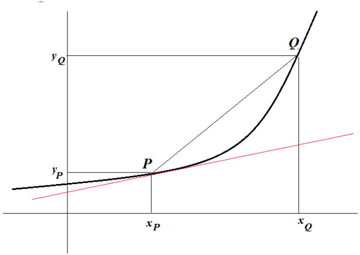


How to define the Tangent?



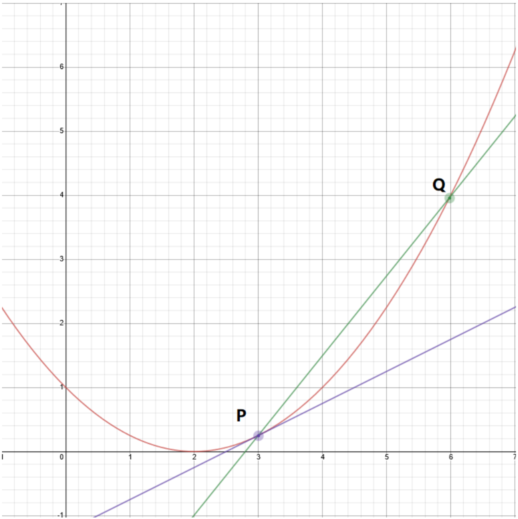
- A straight line passing any two points of a curve is called a chord or segment.

How to define the Tangent?

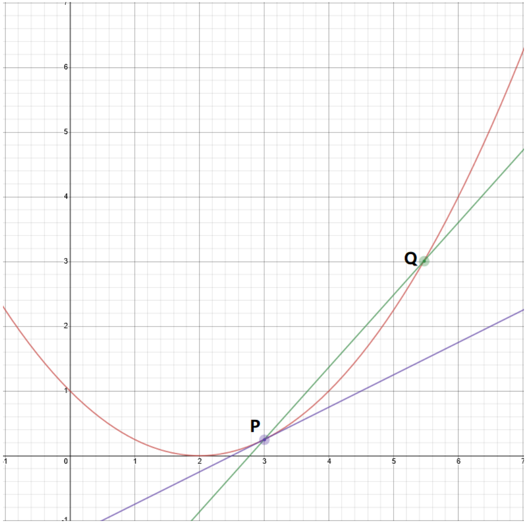


- A straight line passing any two points of a curve is called a chord or segment.
- What happens when Q gets near to P?

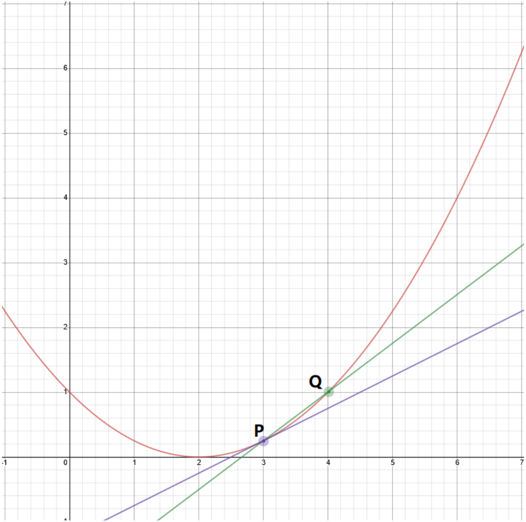
What happens when Q gets near to P?



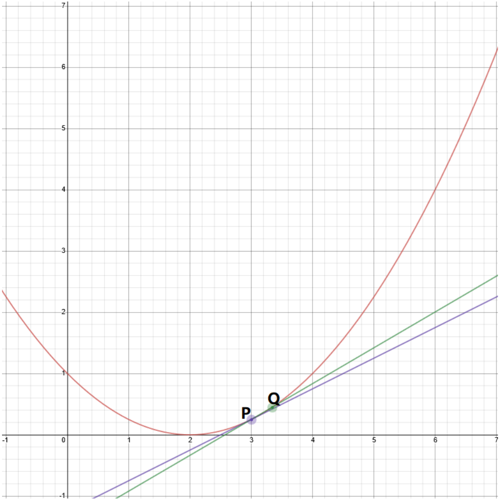
What happens when Q gets near to P?

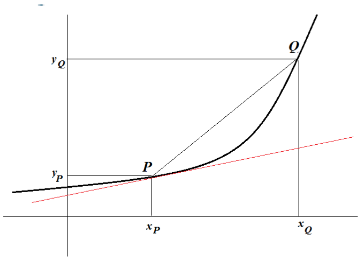


What happens when Q gets near to P?



What happens when Q gets near to P?





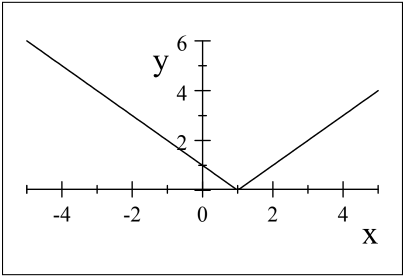
- As the point Q approaches P, the line through PQ approaches the tangent.
- In particular, the slope of PQ approaches the slope of the tangent. Using the notation of limit, we write

$$m = \lim_{Q \to P} \text{ slope of Chord PQ}$$

- The <u>tangent line</u> at P is the line with slope (gradient) m and passes through P.

Example

Consider the following curve:

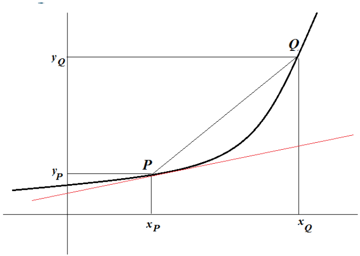


There is no (unique) tangent at P, when there is a 'kink'.

Describe Tangent Mathematically

Question How to describe mathematically the tangent to a curve at a point P?

Describe Tangent Mathematically



To describe the tangent at P, we have to find the equation of the tangent. We have to determine the gradient (or the slope) of the tangent. Since

$$\text{slope of tangent at P } = \lim_{Q \to P} \text{ slope of Chord PQ},$$

we shall determine the slope of the chord PQ:

$$\text{Slope of chord PQ } = \frac{y_Q - y_P}{x_Q - x_P}.$$

Describe Tangent Mathematically

$$\text{Slope of tangent at P } = \lim_{Q \to P} \frac{y_Q - y_P}{x_Q - x_P}.$$

Denoting by Δx the change in x and Δy the corresponding change in y, we have

$$\Delta x = x_Q - x_P \quad \text{and} \quad \Delta y = y_Q - y_P.$$

Thus we have

$$\text{Slope of tangent at P } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

Slope of a Tangent

If the curve is the graph of a function $f(x)$, we have

$$\text{Slope of tangent at P } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{x \to x_P} \frac{f(x) - f(x_P)}{x - x_P}.$$

if limit exists.

Instantaneous Rate of Change

Increments

- Suppose the state of a physical or chemical system is described by a set of variables. E.g.
  - $(x, t) =$ position of a sprinter along a 100m-race at time t track.
  - $(m, T) =$ molarity and temperature of an aqueous chemical solution.
  - $(p, V, T) =$ pressure, volume and temperature of a gas

Increments

- When the system undergoes a change (e.g. when a sprinter runs or the temperature of the gas changes), the values of those state variables (e.g. position x or temperature T) change.
  We denote a change, or increment, in the variable x as

$$\Delta x = \text{new value of } x - \text{ old value of } x.$$

- Note that Δx can be positive, negative or zero, depending on if x increases, decreases or stays the same.

Increments

- Suppose $f(x)$ is a function of x. As x changes from x_1 to x_2, the corresponding $f(x)$ changes from $f(x_1)$ to $f(x_2)$. We have

$$\Delta x = x_2 - x_1, \text{ and } \Delta f = f(x_2) - f(x_1).$$

- Most often, we are interested in the ratio $\dfrac{\Delta f}{\Delta x}$, which is known as the average change in f as x changes from x_1 to x_2. The limit $\displaystyle\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}$ is the instantaneous rate of change in f with respect to x.

Derivatives & Differentiable Functions

Definition

(1) The derivative of a function f at a number c, denoted by $f'(c)$, is defined as follows:

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$

if this limit exists.

We say f is differentiable at c if $f'(c)$ exists.

Derivatives & Differentiable Functions

Definition

(1) A function f is said to be differentiable on an open interval (a, b) if it is differentiable at every number c in (a, b), i.e., $f'(c)$ exists.

(2) The derivative function (or simply derivative) f' is defined by

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

The domain of f' is the set of real numbers x such that $f'(x)$ exists.

Remark

1. Using the symbol $\Delta x = x - c$ for the change in x, we note that $\Delta x \to 0$ whenever $x \to c$. Thus we may express $f'(c)$ as

$$f'(c) = \lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}.$$

2. Sometimes, we use $h = x - c$ instead of Δx to denote the change. Thus, we can also write

$$f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}.$$

Remark

(3) In the expression below,

$$f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}.$$

The variable y on the right is a dummy variable which can be replaced by other symbol except x. For example, we have

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

(4) Leibniz notation for derivative: $\left.\dfrac{df}{dx}\right|_{x=a}$, $\dfrac{df}{dx}(a)$.

(5) To differentiate f (with respect to x) means to determine the derivative f' (i.e., $f'(x)$).

Examples

Example

(a) Is $f(x) = x^2$ differentiable at $x = 1$?
(b) Find $f'(x)$ and the domain of f'.

Solution (a)

To check whether f is differentiable at $x = 1$, we have to check whether the limit $\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1}$ exists. We proceed to compute this limit:

$$\lim_{x\to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1} \frac{x^2 - (1)^2}{x - 1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to 1}(x+1) = 2.$$

Since the limit exists, f is differentiable at $x = 1$ and hence $f'(1) = 2$.

Solution

Solution (b)

We determine $f'(x)$ for $f(x) = x^2$ as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h}$$

$$= \lim_{h \to 0} (2x + h) = 2x.$$

Therefore, we have $f'(x) = 2x$ for $x \in \mathbb{R}$.

Examples

Example

(a) Is the modulus function $f(x) = |x|$ differentiable at 0? Is there a tangent to the graph of $y = |x|$ at $x = 0$?
(b) Find $f'(x)$ and the domain of f'.

Solution (a)

Note that $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x}$ *(*).*
To determine the limit (*) we note that $f(x)$ *takes different expression according to* $x > 0$ *or* $x < 0$ *because*

$$f(x) = \begin{cases} x & \textit{if } x \geq 0 \\ -x & \textit{if } x < 0 \end{cases}$$

For x *close to* 0, x *can be negative or positive.*

Solution (a)

Solution (a (con'td))

Thus, we have to consider both $\lim_{x\to 0^+} \frac{f(x)}{x}$ *and* $\lim_{x\to 0^-} \frac{f(x)}{x}$, *and check whether these limits are equal.*

- $\lim_{x\to 0^+} \frac{f(x)}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1$
- $\lim_{x\to 0^-} \frac{f(x)}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1$

Since $\lim_{x\to 0^+} \frac{f(x)}{x} \neq \lim_{x\to 0^-} \frac{f(x)}{x}$, *we conclude that* $\lim_{x\to 0} \frac{f(x)}{x}$ *does not exist. Therefore, the function* $f(x) = |x|$ *is not differentiable at* 0.

Solution (b)

Solution (b)

For $x > 0$, note that

$$\lim_{y\to x}\frac{f(y)-f(x)}{y-x}=\lim_{y\to x}\frac{y-x}{y-x}=1.$$

We have used $f(y)=y$ because we are interested in y very close to x, we may thus consider $0<x/2<y<3x/2$ so that $f(y)=y$. Similarly, for $x<0$, note that

$$\lim_{y\to x}\frac{f(y)-f(x)}{y-x}=\lim_{y\to x}\frac{-y-(-x)}{y-x}=-1.$$

We have used $f(y)=-y$ because we are interested in y very close to x, we may thus consider $3x/2<y<x/2<0$ so that $f(y)=-y$.

Solution (b)

Solution (b (cont'd))

As discussed in part (a), f is not differentiable as $f'(0)$ does not exist. In conclusion, we have

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The domain of f' is $\mathbb{R}\backslash\{0\}$

Remark

1. If the derivative $f'(c)$ exists, then the graph of f has a tangent at $x = c$. And, the finite number $f'(c)$ is the slope of this tangent.
   The equation of the tangent is given by
   $$\frac{y - f(c)}{x - c} = f'(c),$$
2. It is however possible for the graph of f to have a tangent even when the derivative does not exist.

Example

Example

(a) Is $f(x) = x^{1/3}$ differentiable at $x = 0$? Is there a tangent to the graph at $x = 0$?
(b) Find the derivative of $f(x) = x^{1/3}$.

Solution (a)

Solution (a)

At $x = 0$, we look at the following limit of the fraction:

$$\frac{f(0+h)-f(0)}{h} = \frac{h^{1/3}-0}{h} = \frac{1}{h^{2/3}} = \frac{1}{(h^{1/3})^2}.$$

For h close to 0, the denominator $(h^{1/3})^2$ is close to 0 and positive (because of the square). Hence $\frac{1}{(h^{1/3})^2}$ is large and positive and we have

$$\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \infty.$$

Since this is not a finite number, $f'(0)$ does not exist.
However, the graph of f has a (vertical) tangent at $x = 0$.

Solution (b)

Solution (b)

For $x \neq 0$, we evaluate the following limit

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = \lim_{y \to x} \frac{y^{1/3} - x^{1/3}}{y - x}$$

$$= \lim_{y \to x} \frac{y^{1/3} - x^{1/3}}{y - x} \cdot \frac{(y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2}{(y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2}$$

$$= \lim_{y \to x} \frac{y - x}{(y - x)} \left((y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2 \right)^{-1}$$

$$= \lim_{y \to x} \frac{1}{(y^{1/3})^2 + y^{1/3}x^{1/3} + (x^{1/3})^2} = \frac{1}{3x^{2/3}}.$$

Thus, $f'(x) = \frac{1}{3x^{2/3}}$, *i.e.,* $\frac{d}{dx}\left(x^{1/3}\right) = \frac{1}{3x^{2/3}}$, *for* $x \in \mathbb{R} \setminus \{0\}$.

Example

Example

Use the definition of derivative to prove $\frac{dC}{dx} = 0$ for any constant C.

Derivatives of powers of x

Theorem

Let $f(x) = x^n$, where n is a positive integer. Then,

$$f'(x) = nx^{n-1}, \text{ i.e., } \frac{dx^n}{dx} = nx^{n-1}.$$

Derivatives of powers of x

For $f(x) = x^n$, we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\overbrace{\left(x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right)}^{\text{Binomial Expansion}} - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

Derivatives of powers of x

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+nxh^{n-2}+h^{n-1}\right)h}{h}$$

$$=\lim_{h\to 0}\left(nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+nxh^{n-2}+h^{n-1}\right)$$

$$=nx^{n-1}.$$

Thus, f is differentiable at every real number x and $f'(x)=nx^{n-1}$.

Derivatives of a multiple

Theorem

Let α be a real constant. Consider the function αf, defined by $(\alpha f)(x) = \alpha \cdot f(x)$. If f is differentiable at $x = c$, i.e., $f'(c)$ exists, then the function αf is differentiable at $x = c$ and its derivative at c is

$$(\alpha f)'(c) = \alpha \cdot f'(c).$$

Derivatives of a multiple

Proof.

We shall prove that the limit $\lim_{x\to c}\dfrac{\alpha f(x)-\alpha f(c)}{x-c}$ exists as follows:

$$\lim_{x\to c}\frac{(\alpha f)(x)-(\alpha f)(c)}{x-c}=\lim_{x\to c}\frac{\alpha\cdot f(x)-\alpha\cdot f(c)}{x-c}$$

$$=\lim_{x\to c}\frac{\alpha\cdot(f(x)-f(c))}{x-c}=\lim_{x\to c}\alpha\left(\frac{f(x)-f(c)}{x-c}\right)$$

$$=\alpha\lim_{x\to c}\ \frac{(f(x)-f(c))}{x-c}=\alpha f'(c).$$

Therefore, (αf) is differentiable at $x=c$ and

$$(\alpha f)'(x)=\alpha\cdot f'(x).$$

$\square$

Example

We have proved that $\frac{d}{dx}(x^{1/3}) = \frac{1}{3x^{2/3}}$, $x \neq 0$. By the above proposition, we have

$$\frac{d}{dx}(179x^{1/3}) = 179\frac{d}{dx}(x^{1/3}) = \frac{179}{3x^{2/3}}.$$

Higher Derivatives

Once we obtain the derivative f' of f, we may proceed to discuss the derivative of f' and obtain the second derivative of f', and so on. These are known as higher Derivatives of f. Higher derivatives are used in Taylor Series.

Example

Example

Consider $f(x) = x^5$.Then $f'(x) = \frac{dy}{dx} = 5x^4$.

The derivative of f' is the second derivative f'' (also denoted by $f^{(2)}(x), \frac{d^2y}{dx^2}$). We have

$$f^{(2)}(x) = (f')'(x) = \frac{d}{dx}(5x^4) = 5\frac{d}{dx}(x^4) = 5(4x^3) = 20x^3.$$

Similarly, we may also have other higher derivatives:

$$f'(x), f^{(2)}(x), f^{(3)}(x), f^{(4)}(x), \ldots$$

By letting $y = f(x)$, the higher derivatives are denoted by

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots$$

Remark

(a) In the study of kinematics, suppose $x(t)$ is the distance travelled by an object at time t. Then the speed of the object is $x'(t)$ and the acceleration at time t is $x''(t)$.

(b) The shape of the graph of a function $f(x)$ (which is twice differentiable) can be determined by the first and second derivatives. This will be discussed in the next chapter.

(c) Higher derivatives are very useful in approximation. The Taylor series of a function involves higher derivatives.