MH1812 Discrete Mathematics: Quiz (CA) 2

Name: Tutorial Group:

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There are 3 (THREE) questions, please try all of them, and justify all your answers! Best of luck!

Question 1 (40 points)

a) Let A, B, C be sets. Prove the distributivity property

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

without using Venn diagrams.

b) Let p be a prime. Prove or disprove the following:

$$(x + y)^p \equiv x^p + y^p \mod p.$$

Solution.

a) Write the membership table:

A	B	C	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$	$B \cap C$	$A \cup (B \cap C)$
0	0	0	0	0	0	0	0
1	0	0	1	1	1	0	1
0	1	0	1	0	0	0	0
1	1	0	1	1	1	0	1
0	0	1	0	1	0	0	0
1	0	1	1	1	1	0	1
0	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1

Alternatively, one can show reverse inclusions. If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$, then $x \in (A \cup B) \cap (A \cup C)$. Conversely, if $x \in (A \cup B) \cap (A \cup C)$, $x \in A \cup B$ and in $A \cup C$. Then $x \in A$ or $x \in B \cap C$ and $x \in A \cup (B \cap C)$.

b) We have, using the binomial theorem, that

$$(x + y)^p = \sum_{k=0}^{p} \binom{p}{k} x^k y^{n-k}.$$

Now

$$\binom{p}{k} = \frac{p!}{k!(p-k)!} = p\frac{(p-1)!}{k!(p-k)!}$$

which is a multiple of p, except if $k = 0$ and $k = p$. This is because $\binom{p}{k}$ counts a number of choosing k elements out of p, and therefore is a positive integer number, and so will be $\frac{(p-1)!}{k!(p-k)!}$ because $k!(p-k)!$ contains only factors which are smaller than p (whenever $0 < k < p$), therefore they can never be simplified with p at the numerator. Thus $\binom{p}{k}$ is always zero $\mod p$ except when $k = 0, p$ for which $\binom{p}{k} = 1$, which completes the proof.

Question 2 (30 points)

Prove, using mathematical induction, that

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

for $n \geq 1$.

Solution. For $n = 1$, $1^3 = (1 \cdot 2/2)^2 = 1$. Suppose true for k:

$$\sum_{i=1}^{k} i^3 = \left(\frac{k(k+1)}{2}\right)^2.$$

We want to show true for $k+1$. We have

$$\begin{aligned}
\sum_{i=1}^{k+1} i^3 &= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{1}{4}(k^2(k+1)^2 + 4(k+1)(k+1)^2) \\
&= \frac{1}{4}(k+1)^2(k^2 + 4(k+1)) \\
&= \frac{1}{4}(k+1)^2(k+2)^2
\end{aligned}$$

which concludes the proof.

Question 3 (30 points)

Solve the following linear recurrence:

$$a_n = -8a_{n-1} - 16a_{n-2}$$

with initial conditions $a_0 = 2$, $a_1 = -20$.

Solution. Write the characteristic equation:

$$x^n = -8x^{n-1} - 16x^{n-2} \Rightarrow x^n + 8x^{n-1} + 16x^{n-2} = 0 \Rightarrow x^2 + 8x + 16 = 0.$$

We factorize to get

$$(x+4)^2 = 0.$$

A general solution is thus of the form

$$a_n = u(-4)^n + nv(-4)^n.$$

When $n = 0$, we find $u = 2$, and when $n = 1$, we find

$$-20 = 2(-4) + v(-4) \Rightarrow v = -12/-4 = 3.$$

Thus the solution is

$$a_n = 2(-4)^n + 3n(-4)^n.$$