Nanyang Technological University
SPMS/Division of Mathematical Sciences

2021/22 Semester 1 **MH1810 Math 1** **Take Home Test**

Version L

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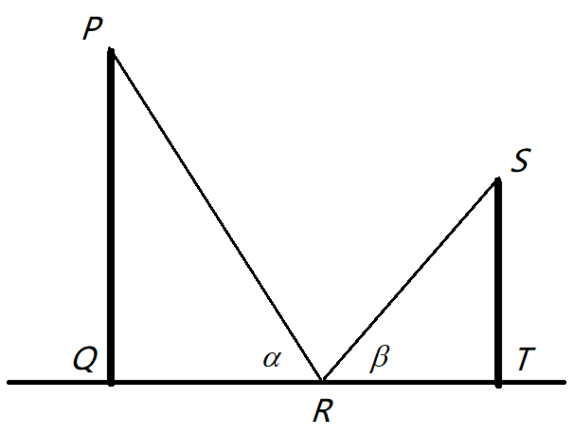
All questions carry the same marks. Answer ALL questions.

1. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the two poles and then to the top of the second pole as shown in the figure. Show that the shortest length of such a rope occurs when $\alpha = \beta$, where $\alpha = \angle PRQ$ and $\beta = \angle SRT$.



2. Let $f(x) = \sqrt{1 + \dfrac{1}{x}}$. Use the *definition of deivatives* to show that

$$f'(x) = -\frac{1}{2x^2\sqrt{\frac{1}{x}+1}}.$$

3. Express the following as a definite integral $\int_0^1 f(x)\, dx$ and find its exact value.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^2}{i^3+n^3}$$

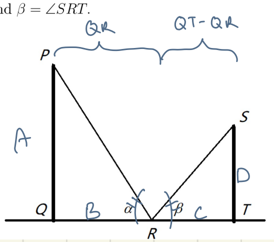
4. Show that

   (a) $\displaystyle\int_0^1 \frac{x+3}{x^2+5x+7} dx = \tfrac{1}{2}\ln a + \tfrac{1}{\sqrt{3}}\left(\tan^{-1} b - \tan^{-1} c\right)$, where the numbers a, b, c are to be determined.

   (b) $\displaystyle\int_0^1 \frac{1}{1+2^x} dx = \frac{\ln A}{\ln B}$, where the numbers A, B are to be determined.

5. Let R be the region bounded by the curve $y = \dfrac{x}{1+3x^2+x^3}$, $x = 1$, $x = 0$ and $y = 0$. Find the volume when R is rotated 2π radians about the the line $x = -2$. Express your answer in terms of π.

1. Two vertical poles PQ and ST are secured by a rope PRS going from the top of the first pole to a point R on the ground between the two poles and then to the top of the second pole as shown in the figure. Show that the shortest length of such a rope occurs when $\alpha = \beta$, where $\alpha = \angle PRQ$ and $\beta = \angle SRT$.



Q1

$PR = \sqrt{QR^2 + PQ^2} \qquad SR = \sqrt{RT^2 + ST^2}$

$\text{Length} = \sqrt{QR^2 + PQ^2} + \sqrt{RT^2 + ST^2}$

$QT = QR + RT$

$\therefore RT = QT - QR$

$L = \sqrt{QR^2 + PQ^2} + \sqrt{(QT - QR)^2 + ST^2}$

let $x = QR$, as QR can change

$L(x) = \left(x^2 + PQ^2\right)^{\frac{1}{2}} + \left((QT - x)^2 + ST^2\right)^{\frac{1}{2}}$

$L'(x) = \frac{1}{2}\left(x^2 + PQ^2\right)^{-\frac{1}{2}}(2x) + \frac{1}{2}\left((QT - x)^2 + ST^2\right)^{-\frac{1}{2}}\left(2(QT - x)(-1)\right)$

$= \dfrac{\cancel{2}x}{\cancel{2}\sqrt{x^2 + PQ^2}} + \dfrac{-\cancel{2}(QT - x)}{\cancel{2}\sqrt{(QT - x)^2 + ST^2}}$

$= \cos(\alpha) - \cos(\beta)$

When $L'(x) = 0$, $\cos(\alpha) - \cos(\beta) = 0$ $\qquad \cos(\alpha)$ & $\cos(\beta) > 0$

(minimum) $\qquad \cos(\alpha) = \cos(\beta)$

$\therefore \alpha = \beta$ //

2. Let $f(x) = \sqrt{1 + \frac{1}{x}}$. Use the *definition of deivatives* to show that

$$f'(x) = -\frac{1}{2x^2\sqrt{\frac{1}{x} + 1}}.$$

Q2

$f'(x) = \lim\limits_{y \to x} \dfrac{f(y) - f(x)}{y - x}$

$= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$f(x) = \sqrt{1 + \frac{1}{x}}$

$f'(x) = \lim\limits_{y \to x} \left(\dfrac{\sqrt{1 + \frac{1}{y}} - \sqrt{1 + \frac{1}{x}}}{y - x}\right) \cdot \left(\dfrac{\sqrt{1 + \frac{1}{y}} + \sqrt{1 + \frac{1}{x}}}{\sqrt{1 + \frac{1}{y}} + \sqrt{1 + \frac{1}{x}}}\right)$

$= \lim\limits_{y \to x} \dfrac{\cancel{1} + \frac{1}{y} - \cancel{1} - \frac{1}{x}}{(y - x)\left(\sqrt{1 + \frac{1}{y}} + \sqrt{1 + \frac{1}{x}}\right)}$

$= \lim\limits_{y \to x} \dfrac{(-1)\left(\frac{\cancel{y - x}}{xy}\right)\left(\frac{1}{\cancel{y - x}}\right)}{\sqrt{1 + \frac{1}{y}} + \sqrt{1 + \frac{1}{x}}}$

$= \lim\limits_{y - x} \dfrac{-1}{\left(\sqrt{1 + \frac{1}{y}} + \sqrt{1 + \frac{1}{x}}\right)xy}$

$= -\dfrac{1}{2x^2\sqrt{1 + \frac{1}{x}}}$ (shown)

//

3. Express the following as a definite integral $\int_0^1 f(x)\,dx$ and find its exact value.

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{i^2}{i^3 + n^3}$$

$\sum_{i=1}^{n} i^2 = \frac{i(i+1)(2i+1)}{6}$

Q3

$\int_0^1 f(x)\,dx$

$\rightsquigarrow x_k^* = 0 + k\frac{1}{n} = \frac{k}{n}$

$\Delta x = \frac{1}{n}$

$\lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{i^2}{i^3 + n^3} = \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{1}{i + \frac{n^3}{i^2}}$

$= \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{1}{\frac{i}{n} + \frac{n^2}{i^2}} \cdot \dfrac{1}{n}$

$= \lim\limits_{n \to \infty} \sum\limits_{i=1}^{n} \dfrac{1}{\frac{i}{n} + \left(\frac{i}{n}\right)^{-2}} \cdot \dfrac{1}{n}$

$= \int_0^1 \dfrac{1}{x + x^{-2}}\,dx$

$= \int_0^1 \dfrac{x^2}{x^3 + 1}\,dx$

$\ln(x^3 + 1) = \dfrac{1}{x^3 + 1}(3x^2)$

$= \left[\frac{1}{3}\ln|x^3 + 1|\right]_0^1$

$= \frac{1}{3}\ln|2| - \frac{1}{3}\ln|1|$

$= \frac{1}{3}\left(\ln|2| - \ln|1|\right)$

$= \dfrac{\ln 2}{3}$ //

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4. Show that

(a) $\displaystyle\int_0^1 \frac{x+3}{x^2+5x+7}dx = \frac{1}{2}\ln a + \frac{1}{\sqrt{3}}(\tan^{-1} b - \tan^{-1} c)$, where the numbers a, b, c are to be determined.

(b) $\displaystyle\int_0^1 \frac{1}{1+2^x}dx = \frac{\ln A}{\ln B}$, where the numbers A, B are to be determined.

Q4

$y = x^2+5x+7$
$y' = 2x+5$

$V' = x+3$
$V = \frac{1}{2}x^2 + 3x$

$U = (x^2+5x+7)^{-1}$
$U' = (-1)(x^2+5x+7)^{-2}(2x+5)$

a)
$$\int_0^1 \frac{x+3}{x^2+5x+7}dx = \int_0^1 \frac{\frac{1}{2}(2x+6)}{x^2+5x+7}dx$$
$$= \frac{1}{2}\int_0^1 \frac{2x+5+1}{x^2+5x+7}dx$$
$$= \frac{1}{2}\left[\int_0^1 \frac{2x+5}{x^2+5x+7}dx + \int_0^1 \frac{1}{x^2+5x+7}dx\right]$$

$$x^2+5x+7 = \left(x+\frac{5}{2}\right)^2 + 7 - \left(\frac{5}{2}\right)^2 = \boxed{\left(x+\frac{5}{2}\right)^2 + \frac{3}{4}}$$
$$\downarrow$$
$$\left(x+\frac{5}{2}\right)^2 + \left(\sqrt{\frac{3}{4}}\right)^2$$

$$= \frac{1}{2}\left[\left[\ln|x^2+5x+7|\right]_0^1 + \int_0^1 \frac{1}{(x+\frac{5}{2})^2 + \left(\sqrt{\frac{3}{4}}\right)^2}dx\right]$$

Let $u = x + \frac{5}{2}$ for
$\frac{du}{dx} = 1$

$$= \frac{1}{2}\left[\left[\ln|13| - \ln|7|\right] + \int_{0+\frac{5}{2}}^{1+\frac{5}{2}} \frac{1}{u^2 + \left(\sqrt{\frac{3}{4}}\right)^2}du\right]$$
$$= \frac{1}{2}\left[\ln\left(\frac{13}{7}\right) + \left[\frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\left(\frac{u}{\sqrt{\frac{3}{4}}}\right)\right]_{\frac{5}{2}}^{\frac{7}{2}}\right]$$
$$= \frac{1}{2}\ln\left(\frac{13}{7}\right) + \frac{1}{2}\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\times\frac{5}{2}}{\sqrt{3}}\right) - \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\times\frac{7}{2}}{\sqrt{3}}\right)\right]$$
$$= \frac{1}{2}\ln\left(\frac{13}{7}\right) + \frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{5}{\sqrt{3}}\right) - \tan^{-1}\left(\frac{7}{\sqrt{3}}\right)\right)$$

$a = \frac{13}{7}$ // $b = \frac{5}{\sqrt{3}}$ // $c = \frac{7}{\sqrt{3}}$ //

$2^0 + 1 = 2$

b)
$$\int_0^1 \frac{1}{1+2^x}dx = \int_2^3 \frac{1}{u}\cdot\frac{1}{(u-1)\ln(2)}du$$

Sub $u = 2^x + 1$ $\quad 2^1 + 1 = 3$

$\frac{du}{dx} = 2^x\ln(2)$

$dx = \left(\frac{1}{2^x\ln(2)}\right)du$

$dx = \frac{1}{(u-1)\ln(2)}du$

~~Sub $u = e^x$~~
~~$\frac{du}{dx} = e^x$~~
~~$dx = e^{-x}du$~~
~~$2^x = u^{\ln(2)}$~~

$$= \int_2^3 \frac{1}{u(u-1)\ln(2)}du$$
$$= \frac{1}{\ln(2)}\int_2^3 \frac{1}{u(u-1)}du$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$
$$1 = A(u-1) + B(u)$$
let $u = 0$, $A = -1$
$u = 1$, $B = 1$

$$= \frac{1}{\ln(2)}\left[\int_2^3 \frac{1}{u-1}du - \int_2^3 \frac{1}{u}du\right]$$
$$= \frac{1}{\ln(2)}\left[\left[\ln(u-1)\right]_2^3 + \left[-\ln u\right]_2^3\right]$$
$$= \frac{1}{\ln(2)}\left[\ln\left(\frac{u-1}{u}\right)\right]_2^3$$
$$= \frac{1}{\ln(2)}\left[\ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{2}\right)\right]$$
$$= \frac{\ln\left(\frac{4}{3}\right)}{\ln(2)}$$

$A = \frac{4}{3}$, $b = 2$

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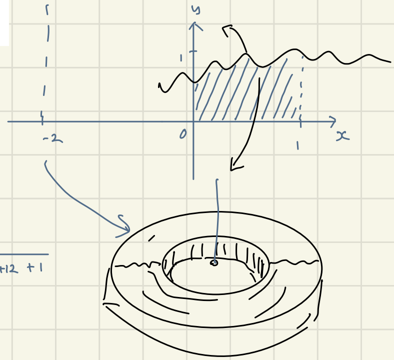
5. Let R be the region bounded by the curve $y = \dfrac{x}{1+3x^2+x^3}$, $x = 1$, $x = 0$ and $y = 0$. Find the volume when R is rotated 2π radians about the the line $x = -2$. Express your answer in terms of π.

Q5



$$f(x) = \frac{x}{1+3x^2+x^3}$$

To rotate around y axis, shift Right

Sub $x = x-2$

$$g(x) = \frac{x-2}{(x-2)^3+3(x-2)^2+1} = \frac{x-2}{x^3-6x^2+12x-8+3x^2-12x+12+1}$$

$$= \frac{x-2}{x^3-3x^2+5}$$

new bound, $x=3$
$x=2$

$$\text{Volume} = 2\pi\int_2^3 \frac{x^2-2x}{x^3-3x^2+5}\,dx$$

Let $U = x^3-3x^2+5$

$$\frac{du}{dx} = 3x^2-6x$$

$$dx = \frac{1}{3x^2-6x}\,du$$

when $x=3$, $U=5$
$x=2$, $U=1$

$$= 2\pi\int_1^5 \frac{\frac{\cancel{x^2-2x}}{3(\cancel{x^2-2x})}}{U}\,du$$

$$= \frac{2}{3}\pi\int_1^5 \frac{1}{U}\,du$$

$$= \frac{2}{3}\pi\left[\ln|U|\right]_1^5$$

$$= \frac{2}{3}\pi(\ln 5)$$

$$= \frac{2\ln 5}{3}\pi$$

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