Math 1 Midterm Practice

MH1810

AY 21/22

1 Complex Numbers

1. Evaluate $\frac{3+7i}{6-5i}$ in terms of $a + bi$.
2. What is $6 + 7i$ in polar form and exponential form?
3. Express $3e^{4i}$ in standard form.
4. Evaluate $(1 + \sqrt{3}i)^{10}$ in terms of $a + bi$.
5. Find all $x \in \mathbb{C}$ such that $x^4 + 3 = 0$.

2 Vectors

1. Find the angle between $(2, 3, 4)$ and $(1, 1, 1)$.
2. Find the parametric equation of the line through $(5, 2, 0)$ perpendicular to the plane $x + y + z = 11$.
3. Find the vector equation of the plane with normal $(5, 4, 3)$ containing the point $(1, 3, 2)$.
4. Find the distance between the point $(2, 3, 5)$ and the line $(1, 1, 1) + t(0, 1, 0)$, $t \in \mathbb{R}$.
5. Find the distance between the point $(2, 4, 8)$ and the plane $3x - 2y + z = 0$

3 Matrices

1. Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 4 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 2 \\ 8 & 5 & 2 \\ 7 & 4 & 2 \end{bmatrix}$, find the following :

   (a) $A + B$

   (b) AB

   (c) A^3

   (d) $\mathrm{Det}(A)$

   (e) $\mathrm{Tr}(B)$

2. Let $A = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$. Find AB.

3. Find C_{31} of $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}$.

4. Does the following system have a solution? If yes, what is the solution. If no, explain.

$$\begin{aligned} x + y + z &= 5 \\ 2x + 3y + z &= 21 \\ 6x + 7y + 11z &= -21 \end{aligned}$$

5. Find the inverse of $\begin{bmatrix} 5 & 25 \\ 25 & 125 \end{bmatrix}$.

6. (a) Consider the matrix

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

   Compute $P^3 - I$.

   (b) Recall that the $\mathrm{Tr}(A)$ where A is a square matrix is the sum of the entries in its diagonals.

Let a and b be any non-zero real numbers. Calculate the following:

$$\operatorname{Tr}\left(\begin{bmatrix} a & 0 & b \\ b & a & 0 \\ 0 & b & a \end{bmatrix}^9\right)$$

4 Limits

1. Evaluate
$$\lim_{x\to 5} \frac{x^2-2x-15}{x-5}.$$

2. Evaluate
$$\lim_{x\to\infty} \frac{\sin x}{x}$$

3. Let $f(x)$ be defined as
$$f(x) = \begin{cases} \sin(x^2+5) & \text{if } x \leq 0 \\ \frac{x^2+3x+5}{7-x} & \text{if } x > 0 \end{cases}$$
Is f continuous at 0?

4. Recall that the definition of
$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$
if the limit exist. Find $f'(2)$ if $f(x) = x^3 - x^2 + 5$.

5. Evaluate
$$\lim_{x\to\infty} \frac{x^3+5}{6-5x}$$

6. Evaluate
$$\lim_{x\to 2^+} \frac{\sqrt{x^2-4x+8}-2}{2-x}$$

7. Show that $f(x) = e^x + x + 2$ has a root.

8. Show that $f(x) = xe^x$ and $g(x) = x^2 - 1$ intersects.

1. Evaluate $\frac{3+7i}{6-5i}$ in terms of $a+bi$.
2. What is $6+7i$ in polar form and exponential form?
3. Express $3e^{4i}$ in standard form.
4. Evaluate $(1+\sqrt{3}i)^{10}$ in terms of $a+bi$.
5. Find all $x \in \mathbb{C}$ such that $x^4+3=0$.

1) $\frac{3+7i}{6-5i} = \frac{3+7i}{6-5i} \cdot \frac{6+5i}{6+5i} = \frac{(3+7i)(6+5i)}{6^2+5^2} = \frac{18+57i-35}{61} = -\frac{17}{61} + \frac{57}{61}i$

2) $6+7i = \sqrt{85}\, e^{i\tan^{-1}(\frac{7}{6})} = \sqrt{85}\left(\cos(\tan^{-1}(\tfrac{7}{6})) - i\sin(\tan^{-1}(\tfrac{7}{6}))\right)$

$\sqrt{6^2+7^2} = \sqrt{85}$

$\tan^{-1}\frac{7}{6}$

3) $3e^{4i} = 3(\cos(4) + i\sin(4))$
$= -1.9609 - i(2.2704)$

4) $(1+\sqrt{3}i)^{10} = (2e^{i\frac{\pi}{3}})^{10} = 2^{10} e^{i\frac{10\pi}{3}}$

$\sqrt{1^2+3} = 2$

$\tan^{-1}(\frac{\sqrt{3}}{1}) = \frac{\pi}{3}$

$= 1024 e^{i(\frac{4\pi}{3}+2\pi)}$

$= 1024(\cos(\tfrac{4\pi}{3}) + i\sin(\tfrac{4\pi}{3}))$

$= -512 - i\,886.81$

5) $x^4+3=0$ $\quad x \in \mathbb{C}$

$x^4 = -3 = 3e^{i(\pi+2n\pi)}$

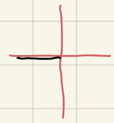
$(x^4)^{\frac{1}{4}} = 3^{\frac{1}{4}} e^{i\frac{\pi}{4}(1+2n)}$

$n=0, \quad \sqrt[4]{3}\, e^{i\frac{\pi}{4}}$

$n=1, \quad \sqrt[4]{3}\, e^{i\frac{3\pi}{4}}$

$n=2, \quad \sqrt[4]{3}\, e^{i\frac{5\pi}{4}}$

$n=3, \quad \sqrt[4]{3}\, e^{i\frac{7\pi}{4}}$



1. Find the angle between $(2,3,4)$ and $(1,1,1)$.
2. Find the parametric equation of the line through $(5,2,0)$ perpendicular to the plane $x+y+z=11$.
3. Find the vector equation of the plane with normal $(5,4,3)$ containing the point $(1,3,2)$.
4. Find the distance between the point $(2,3,5)$ and the line $(1,1,1)+t(0,1,0)$, $t \in \mathbb{R}$.
5. Find the distance between the point $(2,4,8)$ and the plane $3x-2y+z=0$

1) $A \cdot B = \|A\|\|B\| \cos\theta$

$\cos\theta = \frac{A \cdot B}{\|A\|\|B\|}$

$\theta = \cos^{-1}\frac{9}{\sqrt{29}\sqrt{3}}$

$= 15.225^\circ$

2) $r \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 11$

$r = \begin{pmatrix}1\\1\\1\end{pmatrix} t + \begin{pmatrix}5\\2\\0\end{pmatrix}$

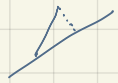
$x = t+5$
$y = t+2$
$z = t$

3) $r \cdot \begin{pmatrix}5\\4\\3\end{pmatrix} = \begin{pmatrix}5\\4\\3\end{pmatrix} \cdot \begin{pmatrix}1\\3\\2\end{pmatrix}$

$r \cdot \begin{pmatrix}5\\4\\3\end{pmatrix} = 23$

$5x+4y+3z = 23$

4) $\begin{pmatrix}2\\3\\5\end{pmatrix}$, $r = \begin{pmatrix}1\\1\\1\end{pmatrix} + t\begin{pmatrix}0\\1\\0\end{pmatrix}$ $t \in \mathbb{R}$



$\begin{pmatrix}3\\1\\1\end{pmatrix}$ exists on the line, (A)

$\vec{AP} = \begin{pmatrix}2\\3\\5\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\4\end{pmatrix}$

$\vec{AP} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\4\end{pmatrix} \times \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}4\\0\\1\end{pmatrix}$

$\therefore$ Distance $= \sqrt{17}$

5) $3x - 2y + z = 0$

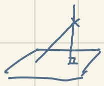
$r \cdot \begin{pmatrix}3\\-2\\1\end{pmatrix} = 0$

$\therefore$ Point A, $\begin{pmatrix}0\\0\\0\end{pmatrix}$ exists

$\vec{PA} = \begin{pmatrix}0\\0\\0\end{pmatrix} - \begin{pmatrix}2\\4\\8\end{pmatrix} = \begin{pmatrix}-2\\-4\\-8\end{pmatrix}$

$\vec{PA} \cdot \frac{1}{\sqrt{14}}\begin{pmatrix}3\\-2\\1\end{pmatrix} = \frac{1}{\sqrt{14}}(-6)$

$= \frac{-6}{\sqrt{14}}$



1. Given $A = \begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}$ and $B = \begin{bmatrix}4&3&2\\8&5&2\\7&4&2\end{bmatrix}$, find the following :
   (a) $A+B$
   (b) AB
   (c) A^3
   (d) $\text{Det}(A)$
   (e) $\text{Tr}(B)$
2. Let $A = \begin{bmatrix}5\\2\\1\end{bmatrix}$ and $B = \begin{bmatrix}1&2&3\end{bmatrix}$. Find AB.
3. Find C_{31} of $\begin{bmatrix}1&2&3&4\\4&1&2&3\\2&3&4&1\\3&4&1&2\end{bmatrix}$.

1a) $\begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix} + \begin{bmatrix}4&3&2\\8&5&2\\7&4&2\end{bmatrix} = \begin{bmatrix}5&5&5\\10&7&5\\10&8&4\end{bmatrix}$

b) $\begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}\begin{bmatrix}4&3&2\\8&5&2\\7&4&2\end{bmatrix} = \begin{bmatrix}41&25&12\\45&28&14\\59&37&18\end{bmatrix}$

c) $\begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}^3 = \begin{bmatrix}14&18&15\\15&20&18\\17&22&25\end{bmatrix}\begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix} = \begin{bmatrix}95&124&126\\109&142&141\\136&178&167\end{bmatrix}$

d) $\det\begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix} = (1)(1)(-8) - (2)(-5) + (1)(3)(2)$
$= 8$

e) $\text{Tr}\begin{bmatrix}4&3&2\\8&5&2\\7&4&2\end{bmatrix}$

2. $A = \begin{bmatrix}5\\2\\1\end{bmatrix}$ $\quad B = \begin{bmatrix}1&2&3\end{bmatrix}$ $\quad 3\times1 \quad 1\times3$

$AB = \begin{bmatrix}5&10&15\\2&4&6\\1&2&3\end{bmatrix}$

1. Evaluate $\frac{3+7i}{6-5i}$ in terms of $a+bi$.
2. What is $6+7i$ in polar form and exponential form?
3. Express $3e^{4i}$ in standard form.
4. Evaluate $(1+\sqrt{3}i)^{10}$ in terms of $a+bi$.
5. Find all $x \in \mathbb{C}$ such that $x^4+3=0$.

1. $\frac{3+7i}{6-5i} \times \frac{6+5i}{6+5i}$

$= \frac{18+15i+42i+35i^2}{36+30i-30i-25i^2}$

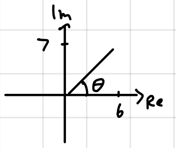
$= \frac{57i-17}{61}$

$= -\frac{17}{61} + \frac{57}{61}i$

2. $6+7i$

$|z| = \sqrt{6^2+7^2}$

$= \sqrt{85}$



$\alpha = \tan^{-1}\frac{7}{6}$

$\theta = \tan^{-1}(\frac{7}{6})$

$\therefore \sqrt{85}\left(\cos\left(\tan^{-1}(\frac{7}{6})\right) + i\sin\left(\tan^{-1}(\frac{7}{6})\right)\right)$

3. $3e^{4i}$

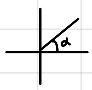
$= 3(\cos 4 + i\sin 4)$

$= -1.961 - 2.270i$

4. $(1+\sqrt{3}i)^{10}$

$|z| = \sqrt{1^2+(\sqrt{3})^2}$

$= 2$



$\alpha = \tan^{-1}\frac{\sqrt{3}}{1}$

$= \frac{\pi}{3}$

$\theta = \frac{\pi}{3}$

$\therefore (2e^{i\frac{\pi}{3}})^{10}$

$2^{10}e^{i\frac{10\pi}{3}}$

$= 1024e^{i\frac{10\pi}{3}}$

$= 1024(\cos\frac{10\pi}{3} + \sin\frac{10\pi}{3}i)$

$= -512 - 886.810i$

5. $x^4+3=0$

$x^4 = -3$

$x = \sqrt[4]{-3}$

$\hookrightarrow -3+0i$

$= 3(\cos\pi + i\sin\pi)$

$= 3e^{i\pi}$

$x_k = \sqrt[4]{3}e^{i(\frac{\pi+2k\pi}{4})}, \ k=0,1,2,3$

$x = \underset{k=0}{\sqrt[4]{3}e^{i\frac{\pi}{4}}}, \underset{k=1}{\sqrt[4]{3}e^{i\frac{3\pi}{4}}}, \underset{k=2}{\sqrt[4]{3}e^{i\frac{5\pi}{4}}}, \underset{k=3}{\sqrt[4]{3}e^{i\frac{7\pi}{4}}}$

1. Find the angle between $(2,3,4)$ and $(1,1,1)$.
2. Find the parametric equation of the line through $(5,2,0)$ perpendicular to the plane $x+y+z=11$.
3. Find the vector equation of the plane with normal $(5,4,3)$ containing the point $(1,3,2)$.
4. Find the distance between the point $(2,3,5)$ and the line $(1,1,1)+t(0,1,0)$, $t \in \mathbb{R}$.
5. Find the distance between the point $(2,4,8)$ and the plane $3x-2y+z=0$

1. $\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\4\end{pmatrix} = 2+3+4$

$= 9$

$\|(1,1,1)\| \cdot \|(2,3,4)\| \cos\theta = 9$

$(\sqrt{1^2+1^2+1^2})(\sqrt{2^2+3^2+4^2})\cos\theta = 9$

$(\sqrt{3})(\sqrt{29})\cos\theta = 9$

$\cos\theta = \frac{9}{\sqrt{87}}$

$\theta = 0.266$

2. $x+y+z=11$,

$\underset{\sim}{n}: (1,1,1)$

$l: (5,2,0)+t\vec{v}$

$\therefore l: (5,2,0)+t(1,1,1)$

Since $\underset{\sim}{n}$ is perpendicular to plane

line parallel to $\underset{\sim}{n}$, same direction

$l: \begin{cases} x = 5+t \\ y = 2+t \\ z = t \end{cases}, t \in \mathbb{R}$

3. $\underset{\sim}{n} = (5,4,3)$

$P_0 = (x,y,z) \quad P = (1,3,2)$

$PP_0 = (x-1, y-3, z-2)$

$(x-1, y-2, z-2)(5,4,3) = 0$

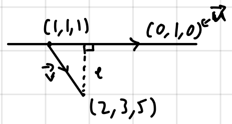
$5(x-1)+4(y-3)+3(z-2) = 0$

$5x-5+4y-12+3z-6 = 0$

$5x+4y+3z = 23$

$(5,4,3)\cdot r = 23$

4.



$l = \|\vec{v} \times \hat{u}\| = \|(1,2,4)\times(0,1,0)\|$

$= \|(-4,0,1)\|$

$= \sqrt{(-4)^2+0^2+1^2}$

$= \sqrt{17}$

5. $3x-2y+z=0$

$P(0,0,0)$, $S(2,4,8)$

distance $= |\vec{PS}\cdot\hat{n}|$

$= \left|(-2,-4,-8)\cdot\frac{1}{\sqrt{3^2+(-2)^2+(1)^2}}(3,-2,1)\right|$

$= \left|(-2,-4,-8)\cdot\frac{(3,-2,1)}{\sqrt{14}}\right|$

$= \frac{1}{\sqrt{14}}|-6+8-8|$

$= \frac{6}{\sqrt{14}}$

Matrices

1. Given $A = \begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}$ and $B = \begin{bmatrix}4&3&2\\8&5&2\\7&4&2\end{bmatrix}$, find the following :

(a) $A+B$
(b) AB
(c) A^3
(d) Det(A)
(e) Tr(B)

a) $A+B = \begin{bmatrix}5&5&5\\10&7&5\\10&8&4\end{bmatrix}$

b) $AB = \begin{bmatrix}41&25&12\\45&28&14\\58&37&18\end{bmatrix}$

c) $A^3 = \begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}\begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}\begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}$

$= \begin{bmatrix}95&124&126\\109&142&141\\136&178&167\end{bmatrix}$

d) $Det(A) = \begin{bmatrix}1&2&3\\2&2&3\\3&4&2\end{bmatrix}\begin{matrix}1&2\\2&2\\3&4\end{matrix}$

$= 46-38$

$= 8$

e) $Tr(B) = 4+5+2$

$= 11$

2. Let $A = \begin{bmatrix}5\\2\\1\end{bmatrix}$ and $B = \begin{bmatrix}1&2&3\end{bmatrix}$. Find AB.

$3\times1 \qquad 1\times3$

$AB = \begin{bmatrix}5\\2\\1\end{bmatrix}\begin{bmatrix}1&2&3\end{bmatrix}$

$= \begin{bmatrix}5&10&15\\2&4&6\\1&2&3\end{bmatrix}$

3. Find C_{31} of $\begin{bmatrix}1&2&3&4\\4&1&2&3\\2&3&4&1\\3&4&1&2\end{bmatrix}$.

$\begin{bmatrix}\cancel{1}&2&3&4\\\cancel{4}&1&2&3\\\cancel{2}&\cancel{3}&\cancel{4}&\cancel{1}\\\cancel{3}&4&1&2\end{bmatrix}$

$C_{31} = + \begin{vmatrix}2&3&4\\1&2&3\\4&1&2\end{vmatrix}$

$= (8+4+36)-(32+6+6)$

$= 4$

4. Does the following system have a solution? If yes, what is the solution. If no, explain.

$$x+y+z=5$$
$$2x+3y+z=21$$
$$6x+7y+11z=-21$$

$\begin{bmatrix}1&1&1\\2&3&1\\6&7&11\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5\\21\\-21\end{bmatrix}$

$\det\left(\begin{bmatrix}1&1&1\\2&3&1\\6&7&11\end{bmatrix}\right) = (33+14+6)-(18+7+22)$

$= 6$

Since $6 \neq 0$, Yes

$x = \frac{\begin{vmatrix}5&1&1\\21&3&1\\-21&7&11\end{vmatrix}}{6}$

$= 14\frac{2}{3} = \frac{44}{3}$

$y = \frac{\begin{vmatrix}1&5&1\\2&21&1\\6&-21&11\end{vmatrix}}{6}$

$= \frac{2}{3}$

$z = \frac{\begin{vmatrix}1&1&5\\2&3&21\\6&7&-21\end{vmatrix}}{6}$

$= -10\frac{1}{3} = -\frac{31}{3}$

5. Find the inverse of $\begin{bmatrix}5&25\\25&125\end{bmatrix}$.

$\begin{bmatrix}5&25\\25&125\end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix}125&-25\\-25&5\end{bmatrix}$

Since $\det A = 0$

$\therefore$ no inverse

6. (a) Consider the matrix

$$P = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

Compute $P^3 - I$.

(b) Recall that the Tr(A) where A is a square matrix is the sum of the entries in its diagonals.

Let a and b be any non-zero real numbers. Calculate the following:

$$\text{Tr}\left(\begin{bmatrix}a&0&b\\b&a&0\\0&b&a\end{bmatrix}^9\right)$$

a) $P^3 = \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

$= \begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$

$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$P^3 - I = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

$= 0$

b) $\begin{bmatrix}a&0&b\\b&a&0\\0&b&a\end{bmatrix}^9 = (bP+aI)^9 \qquad Tr(A+B) = TrA + TrB$

$= \binom{9}{0}(bP)^9(aI)^0 + \binom{9}{1}(bP)^8(aI)^1 + \binom{9}{2}(bP)^7(aI)^2$

$+ \binom{9}{3}(bP)^6(aI)^3 + \binom{9}{4}(bP)^5(aI)^4 + \binom{9}{5}(bP)^4(aI)^5$

$+ \binom{9}{6}(bP)^3(aI)^6 + \binom{9}{7}(bP)^2(aI)^7 + \binom{9}{8}(bP)^1(aI)^8$

$+ \binom{9}{9}(bP)^0(aI)^9$

$= ???$

1. Evaluate
$$\lim_{x\to 5} \frac{x^2-2x-15}{x-5}.$$
2. Evaluate
$$\lim_{x\to\infty} \frac{\sin x}{x}$$
3. Let $f(x)$ be defined as
$$f(x) = \begin{cases} \sin(x^2+5) & \text{if } x \le 0 \\ \frac{x^2+3x+5}{7-x} & \text{if } x > 0 \end{cases}$$
Is f continuous at 0?
4. Recall that the definition of
$$f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$
if the limit exist. Find $f'(2)$ if $f(x) = x^3 - x^2 + 5$.
5. Evaluate
$$\lim_{x\to\infty} \frac{x^3+5}{6-5x}$$
6. Evaluate
$$\lim_{x\to 2^+} \frac{\sqrt{x^2-4x+8}-2}{2-x}$$
7. Show that $f(x) = e^x + x + 2$ has a root.
8. Show that $f(x) = xe^x$ and $g(x) = x^2 - 1$ intersects.

1. $\lim_{x\to 5} \frac{(x-5)(x+3)}{(x-5)}$

$= \lim_{x\to 5} x+3$

$= 8$

2. $-1 \le \sin x \le 1$

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$

$\lim_{x\to\infty} -\frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{1}{x} = 0$

$\therefore$ By squeeze theorem,
$\lim_{x\to\infty} \frac{\sin x}{x} = 0$

3. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \sin(x^2+5)$

$= \sin 5$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x^2+3x+5}{7-x}$

$= \frac{5}{7}$

$\lim_{x\to 0} f(x)$ does not exist

$\therefore f(x)$ is not continuous at $x=0$

4. $f'(2) = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2}$

$= \lim_{x\to 2} \frac{x^3-x^2+5-9}{x-2}$

$= \lim_{x\to 2} \frac{x^3-x^2-4}{x-2}$

$= \lim_{x\to 2} \frac{(x-2)(x^2+x+2)}{x-2}$

$= \lim_{x\to 2} x^2+x+2$

$= 8$

5. $\lim_{x\to\infty} \frac{x^3+5}{6-5x} = \lim_{x\to\infty} \frac{1+\frac{5}{x^3}}{\frac{6}{x^3}-\frac{5}{x^2}} = \frac{1}{0}$

$= -\infty$ $\quad \left(\frac{x^3}{-5x} \Rightarrow -ve\right)$

6. $\lim_{x\to 2^+} \frac{\sqrt{x^2-4x+8}-2}{2-x} \cdot \frac{\sqrt{x^2-4x+8}+2}{\sqrt{x^2-4x+8}+2}$

$= \lim_{x\to 2^+} \frac{x^2-4x+4}{(2-x)(\sqrt{x^2-4x+8}+2)}$

$= \lim_{x\to 2^+} \frac{(x-2)(x-2)}{(2-x)(\sqrt{x^2-4x+8}+2)}$

$= \lim_{x\to 2^+} \frac{-(2-x)(x-2)}{(2-x)(\sqrt{x^2-4x+8}+2)}$

$= \lim_{x\to 2^+} \frac{-(x-2)}{\sqrt{x^2-4x+8}+2}$

$= 0$

7. $f(0) = 3$

$f(-3) = -0.950$

$[-3, 0]$

$f(x)$ is continuous on $[-3,0]$

$f(0) \ne f(-3)$ & $f(-3) < 0 < f(0)$

$\therefore$ By IVT, $\exists c \in (-3,0)$

$f(c) = 0$

$e^c + c + 2 = 0$

$\therefore c$ is a root of $f(x)$

8. $h(x) = xe^x - x^2 + 1$

$h(0) = 1$

$h(-1) = -0.368$

$h(x)$ is continuous on $[-1,0]$

$h(0) \ne h(-1)$ & $h(-1) < 0 < h(0)$

$\therefore$ By IVT, $\exists c \in (-1,0)$

$h(c) = 0$

$ce^c - c^2 + 1 = 0$

$ce^c = c^2 - 1$

$f(c) = g(c)$

$\therefore \exists c$ such that $f(c) = g(c)$,

$\therefore f(x)$ and $g(x)$ intersects.