Tutorial 1

Systems of Linear Equations

1. Find the values of k for which the equations

$$\begin{array}{rcrcrcl} x & + & 5y & + & 3z & = & 0 \\ 5x & + & y & - & kz & = & 0 \\ x & + & 2y & + & kz & = & 0 \end{array}$$

have a non-trivial solution.

2. Use Gaussian elimination and back substitution to solve the following system of linear equations:

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned}$$

3. Determine the values of a for which the following linear system has (i) no solutions, (ii) infinite solutions, (iii) exactly one solution:

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

4. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\ \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Is $\mathbf{b}$ in W? How many vectors are in W?

5. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppose the vectors $\mathbf{u}$ and $\mathbf{v}$ are in Span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v}$ is also in Span$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

6. Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

   a. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

   b. Do the columns of B span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$?

1. Find the values of k for which the equations

$$\begin{aligned} x &+ 5y &+ 3z &= 0 \\ 5x &+ y &- kz &= 0 \\ x &+ 2y &+ kz &= 0 \end{aligned}$$

have a non-trivial solution.

$$\begin{bmatrix} 1 & 5 & 3 & 0 \\ 5 & 1 & -k & 0 \\ 1 & 2 & k & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & -24 & -(k+15) & 0 \\ 0 & -3 & k-3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 & 3 & 0 \\ 0 & -24 & -(k+13) & 0 \\ 0 & 0 & (k-3)+\frac{k+15}{8} & 0 \end{bmatrix}$$

$$(k-3) + \frac{k+15}{8} = 0$$

$$\frac{9}{8}k - \frac{9}{8} = 0$$

$$k = 1$$

2. Use Gaussian elimination and back substitution to solve the following system of linear equations:

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ x + 2y + 3z &= 1 \\ x + 4y + 5z &= 2 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 1 \\ 1 & 4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 4/3 & 1 & 1/2 \\ 0 & 5/2 & 3 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 4/3 & 1 & 1/2 \\ 0 & 0 & 9/8 & 9/16 \end{bmatrix} = \begin{bmatrix} 1 & 3/2 & 2 & 1/2 \\ 0 & 1 & 3/4 & 3/8 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

$$x = -\frac{1}{2}, \quad y = 0, \quad z = \frac{1}{2}$$

3. Determine the values of a for which the following linear system has (i) no solutions, (ii) infinite solutions, (iii) exactly one solution:

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 14)z &= a + 2 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & (a^2-14) & a+2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & (a^2-2) & a-14 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & 10/7 \\ 0 & 0 & (a^2-16) & a-4 \end{bmatrix}$$

i) no solution, $0x + 0y + 0z = a$

let $(a^2 - 16) = 0$

$a = -4$

ii) ∞ solution, $0 = 0$

$a = 4$

iii) One solution

$a \neq \pm 4$

4. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$. Is $\mathbf{b}$ in W? How many vectors are in W?

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 3 & -2 & 1 \\ -2 & 6 & 3 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & -2/3 & 1/3 \\ 0 & 6 & -5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & -2/3 & 1/3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$\therefore$ $z = -2$

$y =$

$x =$

5. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Suppose the vectors $\mathbf{u}$ and $\mathbf{v}$ are in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v}$ is also in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

$U = a_1 w_1 + b_1 w_2 + c_1 w_3$

$V = a_2 w_1 + b_2 w_2 + c_3 w_3$

$U + V = (a_1 + a_2) w_1 + (b_1 + b_2) w_2 + (c_3 + c_3) w_3$

6. Let $A = \begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix}$

a. How many rows of A contain a pivot position? Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for each $\mathbf{b}$ in $\mathbb{R}^4$?

b. Do the columns of B span $\mathbb{R}^4$? Does the equation $B\mathbf{x} = \mathbf{y}$ have a solution for each $\mathbf{y}$ in $\mathbb{R}^4$?

c. Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

d. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$?

a)
$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ -1 & -1 & -1 & 1 \\ 0 & -4 & 2 & -8 \\ 2 & 0 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -1/2 & 2 \\ 0 & 2 & -1 & 4 \\ 0 & -6 & 3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & -1/2 & 2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

3, No solution

b)
$$\begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 1 & 2 & -3 & 7 \\ -2 & -8 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Not enough, however, there is also not enough to go through R^3

Theorem 1.2. Let A be an $m \times n$ matrix. Then the following statements are logically equivalent, i.e., for a particular A, either they are all true statements or they are all false.

a. For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

b. Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.

c. The columns of A span $\mathbb{R}^m$.

d. A has a pivot position in every row.

7. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.

a) Eq$^{\underline{n}}$ of line : $y = \frac{1}{4}x + 0$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ a & b \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ c \end{bmatrix}$$

as 2x2 matrix $A = l_1$, and l_1 will have ∞ solution, $a, b, c = 0$

$$\begin{bmatrix} 1 & -\frac{1}{4} \\ 1 & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b)
$$\begin{bmatrix} 1 & -\frac{1}{4} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

8. Suppose A is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Why?

If $Ax = y$ has no solution, or $Ax \neq y$, means that there is no linear combination to produce y

9. Find the value of h for which the vectors $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$ are linearly dependent.

$$\begin{bmatrix} 2 & -6 & 8 & 0 \\ -4 & 7 & h & 0 \\ 1 & -3 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 4 & 0 \\ 0 & -5 & 16h & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & (4 - \frac{48h}{5}) & 0 \\ 0 & 1 & -16/5\,h & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3 is a free varible

$x_1 = \left(\frac{48}{5}h - 4\right)x_3$

$x_2 = (16/5\,h)\, x_3$

let $x_3 = 5$, $x_1 = 48h - 20$

$x_2 = 16h$

$(48h - 20)V_1 + 16h V_2 + 5 V_3 = 0$

for any h, it is linearly dependent.

10. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

$T(e_1) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $T(e_2) = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

$$T = T\begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}_{/\!/}$$

$\vec{x} = e_1 x_1 + e_2 x_2$

$T(\vec{x}) = x_1 T(e_1) + x_2 (Te_2)$

$$= x_1 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

c. Can each vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix A? Do the columns of A span $\mathbb{R}^4$?

d. Can every vector in $\mathbb{R}^4$ be written as a linear combination of the columns of the matrix B? Do the columns of B span $\mathbb{R}^3$?

7. Construct a 2×2 matrix A such that the solution set of the equation $A\mathbf{x} = \mathbf{0}$ is the line in $\mathbb{R}^2$ through $(4, 1)$ and the origin. Then, find a vector $\mathbf{b}$ in $\mathbb{R}^2$ such that the solution set of $A\mathbf{x} = \mathbf{b}$ is *not* a line in $\mathbb{R}^2$ parallel to the solution set of $A\mathbf{x} = \mathbf{0}$.

8. Suppose A is a 3×3 matrix and $\mathbf{y}$ is a vector in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{y}$ does *not* have a solution. Does there exist a vector $\mathbf{z}$ in $\mathbb{R}^3$ such that the equation $A\mathbf{x} = \mathbf{z}$ has a unique solution? Why?

9. Find the value of h for which the vectors $\begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$, $\begin{bmatrix} -6 \\ 7 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ h \\ 4 \end{bmatrix}$ are linearly *dependent*.

10. Let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \mathbf{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{e}_1$ into $\mathbf{y}_1$ and maps $\mathbf{e}_2$ into $\mathbf{y}_2$. Find the images of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

11. Find the standard matrix of the linear transformation

a. $T : \mathbb{R}^2 \to \mathbb{R}^2$, which first rotates points through $-3\pi/4$ radian (clockwise) and then reflects points through the horizontal $x-$axis.

b. $T : \mathbb{R}^2 \to \mathbb{R}^2$, which first reflects points through the horizontal $x-$axis and then reflects points through the line $y = x$. Show that the transformation is merely a rotation about the origin. What is the angle of rotation?

Answers

1. $k = 1$
2. $x = -1/2, y = 0, z = 1/2$
3. (i) $a = -4$ (ii) $a = +4$ (iii) $a \neq \pm 4$
4. Yes, Infinite
5.
6. a. 3, No b. No, No c. No, No d.No, No
7. One possibility for $A = \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$. For $\mathbf{b}$, take any vector that is not a linear combination of the columns of A.
8. No
9. All values of h.
10. $\begin{bmatrix} 13 \\ 7, \end{bmatrix}, \begin{bmatrix} 2x_1 - x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$
11. a. $\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, b. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\pi/2$ radians

End

Tutorial 2

Matrix Algebra

1. Consider an $m \times n$ matrix A and an $n \times p$ matrix B. Show that if the columns of B are linearly dependent, then so are the columns of AB.

2. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

3. Suppose A, B and X are $n \times n$ matrices with A, X and $A - AX$ invertible. Also, suppose $(A - AX)^{-1} = X^{-1}B$. Solve for X. If you need to invert a matrix, explain why that matrix is invertible.

4. a. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system with n linear equations and n unknowns. If $A\mathbf{x} = 0$ has only the trivial solution, show that for any positive integer k, the system $A^k\mathbf{x} = 0$ also has only the trivial solution.

   b. Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations and let $\mathbf{x}_1$ be a fixed solution. Show that every solution to the system can be written in the from $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$ where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$.

5. a. If the columns of an $n \times n$ matrix A are linearly independent, show that the columns of A^2 span $\mathbb{R}^2$.

   b. Show that if AB is invertible, so is A.

6. Using the notion of pivots and free variables only, answer the following questions:

   a. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why the equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.

   b. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

7. Solve the equation $A\mathbf{x} = \mathbf{b}$ using the LU factorization given for A:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}.$$

8. (*Spectral Factorization*) Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the

1. Consider an $m \times n$ matrix A and an $n \times p$ matrix B. Show that if the columns of B are linearly dependent, then so are the columns of AB.

$m \times n$ - $n \times p$

$\therefore AB = m \times p$

If B is linearly dependent

$$Bx = 0 \quad , \quad x \neq 0$$

$$(AB)x = A(0)$$
$$= 0$$

$(C)x = 0 \Rightarrow C$ has dependent columns,

$\therefore C$ is linearly dependent

2. Suppose A and B are $n \times n$, B is invertible, and AB is invertible. Show that A is invertible.

AB is invertible, let $C = AB$

$CB^{-1} = ABB^{-1} = AI = A$

A is a product of invertible matrix

Or : for (AB) to be invertible

$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1}$
$= AIA^{-1}$
$= AA^{-1}$
$= I$

$\therefore A^{-1}$ exists

3. Suppose A, B and X are $n \times n$ matrices with A, X and $A - AX$ invertible. Also, suppose $(A - AX)^{-1} = X^{-1}B$. Solve for X. If you need to invert a matrix, explain why that matrix is invertible.

We know exists : $A, A^{-1}, x, x^{-1}, A-Ax, (A-Ax)^{-1}, B$

$(A-Ax)^{-1} = x^{-1}B$, $\quad$ B must be invertible

$((A-Ax)^{-1})^{-1} = (x^{-1}B)^{-1}$ $\quad \rightarrow (x^{-1}B)^{-1} = B^{-1}(x^{-1})^{-1}$

$A - Ax = B^{-1}x$ $\quad\quad = B^{-1}x$

$A = B^{-1}x + Ax$

$A = (B^{-1}+A)x$

$A(B^{-1}+A)^{-1} = (B^{-1}+A)^{-1}(B^{-1}+A)x$

$A(B^{-1}+A)^{-1} = x$

non consistent

$1x + 1y = 1$

$1x + 1y = 2$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & -1 \end{array}\right] \leftarrow ?$$

4. a. Let $A\mathbf{x} = \mathbf{0}$ be a homogeneous system with n linear equations and n unknowns. If $A\mathbf{x} = 0$ has only the trivial solution, show that for any positive integer k, the system $A^k\mathbf{x} = 0$ also has only the trivial solution.

$A \in R^{n \times n}$, homogeneous

$Ax = 0$ has only trivol solution,

$\therefore x = 0$

$\therefore A0 = 0$

$A^k x = 0$ ..

By IMT, A^{-1} exists

$$(\underbrace{A^{-1} \ldots A^{-1}A^{-1}}_{K}) A^k x = (\underbrace{A^{-1} \ldots A^{-1}A^{-1}}_{K}) 0$$

$x = 0$ // (shown)

b. Let $A\mathbf{x} = \mathbf{b}$ be any consistent system of linear equations and let $\mathbf{x}_1$ be a fixed solution. Show that every solution to the system can be written in the from $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_0$ where $\mathbf{x}_0$ is a solution to $A\mathbf{x} = \mathbf{0}$.

when $Ax = b$ is consistent

$[A \vdots B]$

$$\left[\begin{array}{cccc|c} \square & \cdots & & & \\ & \square & \cdots & & \\ & & & \square & \\ 0 & 0 & & 0 & 0 \\ 0 & & & & 0 \end{array}\right]$$

" fixed solution "

$Ax^0 = 0$, $Ax' = b$

$Ax^0 + Ax' = b + 0$

$A(x^0 + x') = b$

$\therefore x = x^0 + x'$

$Ax = b$, $Ax_1 = b$

$A(x - x_1) = Ax - Ax_1 = b - b = 0$

$Ax^0 = 0$

$\therefore x^0 = x - x_1$

$x = x_0 + x_1$

5. a. If the columns of an $n \times n$ matrix A are linearly independent, show that the columns of A^2 span $\mathbb{R}^2$.

b. Show that if AB is invertible, so is A.

a) A is linearly independent, A is square, by IVT $\therefore A^{-1}$ exists

By contradiction

$A^2 x = 0 \;\; \exists \;\; x \neq 0$

when A has dependent columns

$(A^{-1}A^{-1})A^2 x = (A^{-1}A^{-1})(0)$

$x = 0,$

$\therefore$ This implies that A has independent columns

b) $(AB)^{-1}$ exists

$C = (AB)^{-1}$

$(AB)C = I$

$A(BC) = I$

$\therefore$ A is proved to be invertible, as product of invertible matrix is invertible

6. Using the notion of pivots and free variables only, answer the following questions:

a. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$. Explain why the equation $A\mathbf{x} = \mathbf{b}$ has in fact exactly one solution.

b. Suppose A is an $n \times n$ matrix with the property that the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Explain why the equation $A\mathbf{x} = \mathbf{b}$ must have a solution for each $\mathbf{b}$ in $\mathbb{R}^n$.

a) $Ax = b$ for any b, x has one solution

$[A \,\vdots\, b]$

gauss jordan

1) All columns have pivots

2) No free varibles

$[I \,\vdots\, b]$

$\left[{}^{1}{}_{0}{}^{0} \,\vdots\, b\right]$

means each row has only one varible making it easy to solve for a specific b value

as $Ax=b$ has solution for each b, implies A pivots in every row. furthermore, as $A = n \times n$, A pivots in every column.

$\therefore$ no free varibles in $Ax=b \Rightarrow$ Solution is unique

b) $Ax = 0$ only has trival

$\therefore$ A is invertible

$\therefore$ A has unique solutions of $b \in \mathbb{R}^n$,

given by $\boxed{x = A^{-1}b}$

x has no "freedom"

7. Solve the equation $A\mathbf{x} = \mathbf{b}$ using the LU factorization given for A:

$$A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -3 & 1 \\ 3 & 7 & 5 \end{bmatrix}, b = \begin{bmatrix} 0 \\ -5 \\ 7 \end{bmatrix}, A = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 3/2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 4 \\ 0 & -2 & -1 \\ 0 & 0 & -6 \end{bmatrix}.$$

$A = LU$

$LUx = b$ Let $y = Ux$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & -5 \\ 3/2 & -5 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -18 \end{bmatrix}$$

$y = \begin{bmatrix} 0 \\ -5 \\ -18 \end{bmatrix}$

$Ux = y$

$$\begin{bmatrix} 2 & -2 & 4 & 0 \\ 0 & -2 & -1 & -5 \\ 0 & 0 & -6 & -18 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$x = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Find A^2 and A^3 and hence, a simple formula for A^k (where k is a positive integer). This factorization is useful when computing high powers of A.

Answers

7. $\mathbf{x} = \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix}$

8. $A^k = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2^k & 0 \\ 0 & 0 & 1/3^k \end{bmatrix} P^{-1}$

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8. (*Spectral Factorization*) Suppose a 3×3 matrix A admits a factorization as $A = PDP^{-1}$, where P is some invertible 3×3 matrix and D is the diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$. Find A^2 and A^3 and hence, a simple formula for A^k (where k is a positive integer). This factorization is useful when computing high powers of A.

$$A^2 = AA = (PDP^{-1})(PDP^{-1})$$
$$= PD\cancel{P^{-1}P}DP^{-1}$$
$$= PDDP^{-1}$$
$$= PD^2P^{-1}$$

$$A^3 = AAA = (PDP^{-1})(PDP^{-1})(PDP^{-1})$$
$$= PDDDP^{-1}$$
$$= PD^3P^{-1}$$

$$\therefore \quad A^K = PD^KP^{-1}$$

CE/CZ1104 Linear Algebra for Computing

Tutorial 3

Determinants

1. If a 3×3 matrix A has $|A| = -1$, find $|\frac{1}{2}A|, |-A|, |A^2|$ and $|A^{-1}|$.

2. Reduce $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ to U to find $|A|$ as the product of pivots.

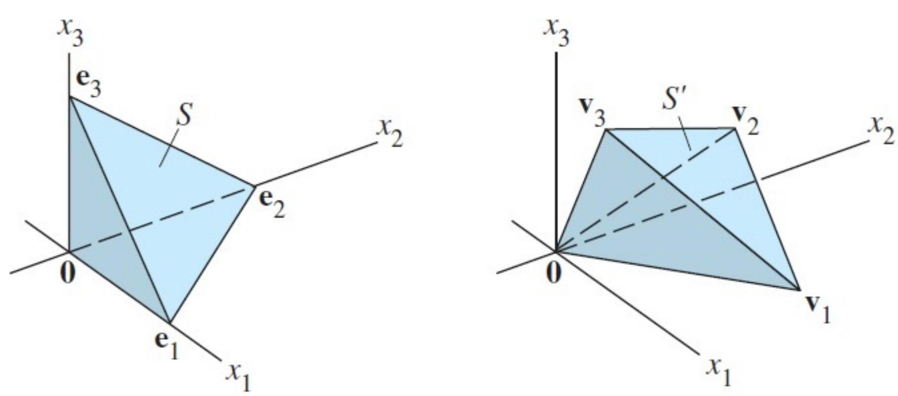
3. Using variables a, b, c, construct a 3×3 skew-symmetric matrix ($A = -A^T$). Show that the determinant of such a matrix is equal to 0.

4. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0), (-2, 0, 2)$ and $(-1, 3, -1)$.

5. In the figure below, let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ ($\mathbf{e}_i$'s are standard unit vectors) and let S' be the tetrahedron with vertices at vectors $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

   a. Find the standard matrix for the linear transformation that maps S to S'.

   b. Find a formula for the volume of the tetrahedron S'. (Volume of a tetrahedron = (1/3) $\times$ (area of base) $\times$ height).



Answers

1. -1/8, 1, 1, -1

2. 1

3.

4. 18

5. a. $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$ b. (1/6) $\times$ abs($|A|$)

End

1. If a 3×3 matrix A has $|A| = -1$, find $|\frac{1}{2}A|, |-A|, |A^2|$ and $|A^{-1}|$.

$|A| = -1$

$|\frac{1}{2}A| = \frac{1}{2^3}|A| = -\frac{1}{2^3}$

2. Reduce $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ to U to find $|A|$ as the product of pivots.

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

$\therefore |A| = 1$

$\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 2 \end{matrix} \rightarrow \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{matrix}$

$\begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{matrix}$

3. Using variables a, b, c, construct a 3×3 skew-symmetric matrix ($A = -A^T$). Show that the determinant of such a matrix is equal to 0.

Skew symmetric matrix, diag. is 0

$A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \Rightarrow |A| = 0\cdot0\cdot0 + a(-b)(-c) + (-a)(b)(c) - (c\cdot0\cdot(-c)) + (-b)(b)(0) + 0(-a)(a) = 0$

OR

$|A| = |-A^T| = (-1)^3|A^T| = -|A^T|$

But $|A| = |A^T|$ always

$\therefore$ must hold $|A| = -|A^T| = -|A|$

$\therefore$ To fufill this criteria, $|A| = 0$

4. Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at $(1, 3, 0), (-2, 0, 2)$ and $(-1, 3, -1)$.

① Put in column form } Vertex is origin
② find det

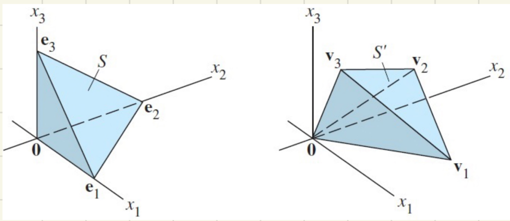
$A = \begin{bmatrix} 1 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{bmatrix}$

$\text{abs}(|A|) = \text{abs}(-18) = 18$ units

5. In the figure below, let S be the tetrahedron in $\mathbb{R}^3$ with vertices at the vectors $\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ ($\mathbf{e}_i$'s are standard unit vectors) and let S' be the tetrahedron with vertices at vectors $\mathbf{0}, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

a. Find the standard matrix for the linear transformation that maps S to S'.

b. Find a formula for the volume of the tetrahedron S'. (Volume of a tetrahedron = (1/3) × (area of base) × height).



Standard matrix

$A = [T(e_1) \;\; T(e_2) \;\; T(e_3)] = [v_1 \; v_2 \; v_3]$

for eg. $T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$T(e_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$T(e_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

Volume of $S = (1/3) \times (b) \times (h)$

$S = 1/3 \times 1/2 \times 1 = \frac{1}{6}$

$\text{Vol}(S') = |A| \cdot \text{Vol}(S) = \frac{1}{6}|A|$

CE/CZ1104 Linear Algebra for Computing

Tutorial 4

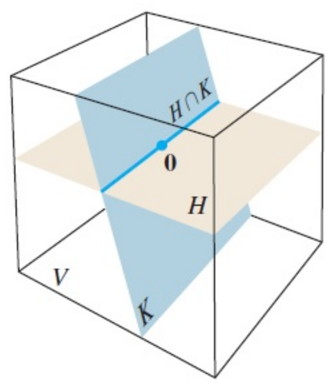
Vector Spaces

1. An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times3}$, the vector space of all 3×3 matrices.

2. (a) Let P be the plane in $\mathbb{R}^3$ with equation $x + y - 2z = 4$. Find two vectors in P and check that their sum is not in P.

   (b) Let P_0 be the plane through $(0, 0, 0)$ and parallel to P. Write the equation for P_0. Find two vectors in P_0 and check that their sum is in P_0.

3. Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V.



4. Determine if the following set is a vector space:

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{lcl} a - 2b & = & 4c \\ 2a & = & c + 3d \end{array} \right\}$$

5. Find the matrix A if the following set is $\mathbf{C}(A)$:

$$\left\{ \begin{bmatrix} 2s + 3t \\ r + s - 2t \\ 4r + s \\ 3r - s - t \end{bmatrix} : r, s, t \ \text{ real} \right\}$$

1. An $n \times n$ matrix A is said to be symmetric if $A^T = A$. Let S be the set of all 3×3 symmetric matrices. Show that S is a subspace of $M_{3\times3}$, the vector space of all 3×3 matrices.

$$S = \{ A = A^T \}$$

1) $\underset{\sim}{0}$ matrix is in S $\therefore$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2) $\underset{\sim}{A}, \underset{\sim}{B}$: $\underset{\sim}{A} = \underset{\sim}{A}^T$, $\underset{\sim}{B} = \underset{\sim}{B}^T$

$(A+B)^T = A^T + B^T = A + B$ $\therefore$ $A+B$ is symmetrical

3) $(cA)^T = c(A^T) = cA$

Symmetrical

2. (a) Let P be the plane in $\mathbb{R}^3$ with equation $x + y - 2z = 4$. Find two vectors in P and check that their sum is not in P.

(b) Let P_0 be the plane through $(0, 0, 0)$ and parallel to P. Write the equation for P_0. Find two vectors in P_0 and check that their sum is in P_0.

a) $\begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$ is in P.

The sum of vector $= \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$

$\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix}$ is not in P $\therefore$ $P \neq$ subspace $\mathbb{R}^3$

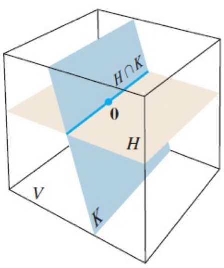
☆ as long as plane passes thru 0,0,0, will be closed under addition & multiplication

b) Eq^n of Plane: $x + y - 2z = 0$

$\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ & $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$ exist.

$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ is within plane P

3. Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set $\mathbf{v}$ in V that belong to both H and K. Show that $H \cap K$ is a subspace of V.



H U K may not be subspace

1) $H \cap K$ has zero vector, as H & K are inside subspace of H & K

2) let $\underset{\sim}{u}$ & $\underset{\sim}{v}$ be vectors in $H \cap K$. $\underset{\sim}{u} + \underset{\sim}{v}$ is found in both H & K as H & K are subspaces. $\therefore$ $H \cap K$ includes $\underset{\sim}{u} + \underset{\sim}{v}$

3) If $\underset{\sim}{v}$ is a vector in H, $\alpha \underset{\sim}{v}$ is in H.
If $\underset{\sim}{u}$ is a vector in K, $\alpha \underset{\sim}{u}$ is in K

4. Determine if the following set is a vector space:

$$\left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} : \begin{array}{lcl} a-2b & = & 4c \\ 2a & = & c+3d \end{array} \right\}$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 4 & 2 \\ 4 & 2 & 7 \end{bmatrix} \to \begin{matrix} 1 & 4 & 2 \\ 3 & 1 & 6 \\ 4 & 7 & 2 \end{matrix}$$

$a - 2b - 4c + 0d = 0$
$a + 0b - c - 3d = 0$

-30

$W = N(A)$
null space of $A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 1 & 0 & -1 & -3 \end{bmatrix}$

W is a subspace of R^4
and therefore is a vector space

$$\begin{bmatrix} 1 & -2 & -4 & 0 \\ 1 & 0 & -1 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

5. Find the matrix A if the following set is $\mathbf{C}(A)$:

$$\left\{ \begin{bmatrix} 2s+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} : r, s, t \text{ real} \right\}$$

$A [C_1 | C_2 | \dots | C_n]$
$C(A) = Col(A) \longrightarrow$ any column can be made through linear combination w. any r, s, t.

$$r\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = A \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$

Column space linearly dependent? can ref.

6. For the matrix $D = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$, find a nonzero vector in $\mathbf{N}(D)$ and a nonzero vector in $\mathbf{C}(D)$.

here, columns are linearly dependent.

$\therefore$ Column space $= \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} -6 \\ 3 \\ 12 \\ -9 \end{bmatrix}$, any one?

for null

$$\begin{bmatrix} 2 & -6 & 0 \\ -1 & 3 & 0 \\ -4 & 12 & 0 \\ 3 & -9 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 - 3x_2 = 0$

$\therefore$ nullspace $= \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

because $3 - 3(1) = 0$

6. For the matrix $D = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$, find a nonzero vector in $\mathbf{N}(D)$ and a nonzero vector in $\mathbf{C}(D)$.

7. Find the basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

8. Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$ and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It can be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. Find a basis for H.

9. Consider the polynomials $\mathbf{p}_1(t) = 1 + t, \mathbf{p}_2(t) = 1 - t$ and $\mathbf{p}_3(t) = 2$ (for all t). By inspection, write a linear dependence relation among $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. Then find a basis for $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

10. Use an inverse matrix to find the $\mathcal{B}-$coordinate of the vector $\mathbf{x}$, i.e., $[\mathbf{x}]_{\mathcal{B}}$, for $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

11. Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

12. Determine the dimensions of $\mathbf{N}(A)$ and $\mathbf{C}(A)$ for $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

13. If a 3×8 matrix A has rank 3, find dim $\mathbf{N}(A)$, dim $\mathbf{C}(A^T)$, and rank of A^T.

14. Suppose the solutions of a homogeneous system of 5 linear equations in 6 unknowns are all multiples of 1 nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations?

15. Verify that the rank of $\mathbf{u}\mathbf{v}^T \leq 1$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

16. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for V and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2, \mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.
    (a) Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.
    (b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

Answers

1.

2. (a) e.g., $(4, 0, 0)$ and $(0, 4, 0)$ (b) e.g., $(2, 0, 1)$ and $(0, 2, 1)$

7. Find the basis for the set of vectors in $\mathbb{R}^3$ in the plane $x + 2y + z = 0$.

Solution set for homogeneous solution

= Null space

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$x = -2y - z$

$y = (1)\,y$

$z = (1)\,z$

basis for $N(A) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

8. Let $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$ and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. It can be verified that $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. Find a basis for H.

v_1 v_2 v_3 is not linearly independent

However, any two vectors can become the basis

$\therefore$ answer is (v_1, v_2) or $v_1 v_3$, or $v_2 v_3$ ...

9. Consider the polynomials $\mathbf{p}_1(t) = 1 + t$, $\mathbf{p}_2(t) = 1 - t$ and $\mathbf{p}_3(t) = 2$ (for all t). By inspection, write a linear dependence relation among $\mathbf{p}_1, \mathbf{p}_2$ and $\mathbf{p}_3$. Then find a basis for $\text{Span}\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

10. Use an inverse matrix to find the $\mathcal{B}$-coordinate of the vector $\mathbf{x}$, i.e., $[\mathbf{x}]_{\mathcal{B}}$, for $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 6 \end{bmatrix} \right\}$ and $\mathbf{x} = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$.

B is basis unique

$\therefore\ x = \square\, x_1 + \square\, x_2$ aka, $\begin{pmatrix} 2 \\ -6 \end{pmatrix} = \boxed{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boxed{-6}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Qn 10: $\begin{pmatrix} 2 \\ -6 \end{pmatrix} = \square\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \square\begin{pmatrix} -4 \\ 6 \end{pmatrix}$

$$[x]_B = P_B^{-1}x = \begin{bmatrix} 3 & -4 \\ -5 & 6 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 2 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

11. Find the dimension of the subspace H of $\mathbb{R}^2$ spanned by $\begin{bmatrix} 2 \\ -5 \end{bmatrix}, \begin{bmatrix} -4 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -5 \\ -4 & 10 \\ -3 & 6 \end{bmatrix} \xrightarrow{REF} \begin{bmatrix} 2 & -5 \\ 0 & -3/2 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ only two non zero

$\therefore$ dimention is $\mathbb{R}^2$

Determine the dimensions of $\mathbf{N}(A)$ and $\mathbf{C}(A)$ for $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -4 \end{bmatrix}$.

Number of free varibles
$= N - \text{rank}$

$A_{m\times n}$ ( ) $\longrightarrow$ $m\times 1$ ( )

$R^n \longrightarrow R^m$

$\text{rank}(A) + \boxed{\dim \text{Null}(A)} = 4 - 2 = 2$

$\dim(\mathcal{C}(A)) = \dim \text{Row}(A) = 2$!

13. If a 3×8 matrix A has rank 3, find dim $\mathbf{N}(A)$, dim $\mathbf{C}(A^T)$, and rank of A^T.

$\dim N(A) = 8 - \text{rank}(A) = 8-3 = 5$

$\dim \underbrace{C(A^T)}_{\text{row space}} = \text{rank}(A) = 3$

Since $\text{rank}(A^T) = \dim C(A^T) = 3$

14. Suppose the solutions of a homogeneous system of 5 linear equations in 6 unknowns are all multiples of 1 nonzero solution. Will the system necessarily have a solution for every possible choice of constants on the right sides of the equations?

5×6

$\underset{5\times 6}{A} : \mathbb{R}^6 \longrightarrow \mathbb{R}^5$

$Ax = 0 \Rightarrow \dim[\text{Null}(A)] = 1$ $\quad t\begin{pmatrix} a \\ b \end{pmatrix}, t \in \mathbb{R}$

if $\underbrace{Ax = b}$ for any b,

will this always have solution x?

$\text{RANK}(A) = \text{num varible} - \dim(\text{Null}(A))$

$\text{RANK}(A) = 6 - 1 = 5$

$\text{Rank}(A) = \dim(\text{col}(A)) = 5$

Column Space of A is $\mathbb{R}^5$ already.

$\therefore$ Since image of A is also $\mathbb{R}^5$,

implies $\text{col}(A)$ is whole thing

$\therefore$ any vector is a linear combination

15. Verify that the rank of $\mathbf{u}\mathbf{v}^T \leq 1$ if $\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$uv^T = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix} = \begin{bmatrix} 2a & 2b & 2c \\ -3a & -3b & -3c \\ 5a & 5b & 5c \end{bmatrix}$$

$\therefore$ uv^T spans $\left\{ \begin{matrix} 2 \\ -3 \\ 5 \end{matrix} \right\}$

$$\dim(\text{span}\{v_1\}) \leq 1$$

16. Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be bases for V and suppose $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$.

(a) Find the change-of-coordinates matrix from $\mathcal{A}$ to $\mathcal{B}$.

(b) Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$.

a)

V - VS

2 bases of V

$A = \{a_1, a_2, a_3\}$ $\quad$ $B = \{b_1, b_2, b_3\}$

Relationship?

T ?

$$A \longrightarrow B$$

$a_1 = 4b_1 - b_2$

$a_2 = -b_1 + b_2 + b_3$

$a_3 = b_2 - 2b_3$

$$[a_1]_B = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}, \quad [a_2]_B = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \quad [a_3]_B = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$$

$$P_{B \leftarrow A} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

b)

$$[x]_B = \boxed{} = \boxed{8}\, b_1 + \boxed{2}\, b_2 + \boxed{2}\, b_3$$

$x = 3a_1 + 4a_2 + a_3$

$= 3(4b_1 - b_2) + 4(-b_1 + b_2 + b_3) + 1(b_2 - 2b_3)$

$= 8b_1 + 2b_1 + 2b_3$

$$\therefore [x]_B = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$$

3.

4. Yes

5. $D = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$

6. e.g., $\mathbf{N}(A) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\mathbf{C}(D)$ is either column of D.

7. $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

8. e.g., $\{\mathbf{v}_1, \mathbf{v}_2\}, \{\mathbf{v}_1, \mathbf{v}_3\}, \{\mathbf{v}_2, \mathbf{v}_3\}$.

9. $\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 = \mathbf{0}$, $\{\mathbf{p}_1, \mathbf{p}_2\}$

10. $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$

11. 2

12. $\dim \mathbf{N}(A) = 2$, $\dim \mathbf{C}(A) = 2$.

13. $\dim \mathbf{N}(A) = 5$, $\dim \mathbf{C}(A^T) = 3$, rank of $A = 3$.

14. Yes

15.

16. (a) $\underset{\mathcal{C} \leftarrow \mathcal{B}}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 4 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$

End