NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

Discrete Mathematics

MH1812

Topic 5.1 – Combinatorics

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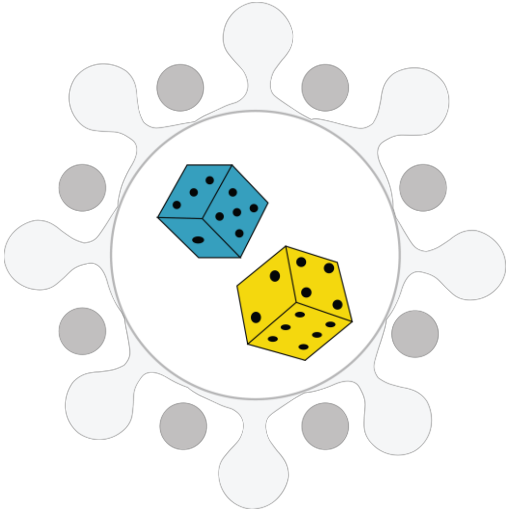
Topic Overview

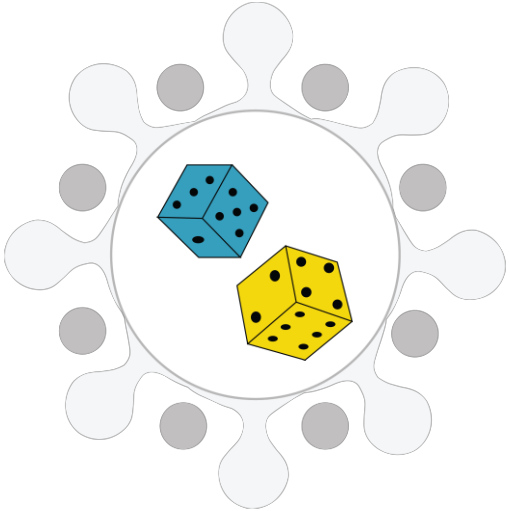
What’s in store…

- **P**rinciple of Counting
- **P**ermutations
- **C**ombinations



By the end of this lesson, you should be able to…

- Explain the concepts of the principle of counting.
- Determine the number of possible permutations using the counting principle, where order does matter.
- Determine the number of possible combinations using the counting principle, where order does not matter.



Principle of Counting

Principle of Counting: Choices and Slots

There are two slots to be filled, there are n_1 choices for slot 1 and n_2 choices for slot 2.

Example

Slot 1: Main Course	Slot 2: Dessert

The total number of unique choices to fill the slots is $n_1 n_2$.

3 Main Course Choices

2 Dessert Choices

Principle of Counting: Choices and Slots

The total number of unique choices for this example is $3 * 2 = 6$.

3 Main Course Choices

2 Dessert Choices

Principle of Counting: Choices and Slots

- **In general:** n_1, n_2, ... n_k **choices for** *k-slots*

- $n_1 * n_2 * \ldots * n_k$ **ways**
  - **Cardinality of the Cartesian product of sets**

Principle of Counting: Choices and Slots

Example

Create a yoghurt dessert with *1* fruit, *1* crunch, and *1* sauce.

- *11* fruits
- *16* crunches
- *15* sauces

Slot *1*: Fruit

Slot *2*: Crunch

Slot *3*: Sauce

Principle of Counting: Cardinality of Power Set

- Consider a set $A = \{a_1,\ldots,a_n\}$ with n elements.
- List all subsets of A. Create a table.

All subsets of A	Binary vectors
$\{a_1\}$	$10\ldots0$

- Each of these n elements are **either in a subset of A or not**: 2 choices.
- Such a choice needs to be made for **each of the n elements**.
- Thus $2*2*\ldots*2 = 2^n$ **choices**.

Principle of Counting: Filling r Slots With n Choices

There are n elements, with which to fill r slots.

Can Be Repeated

When elements can be repeated using the principle of counting:
$n*n*\ldots*n = n^r$ choices.

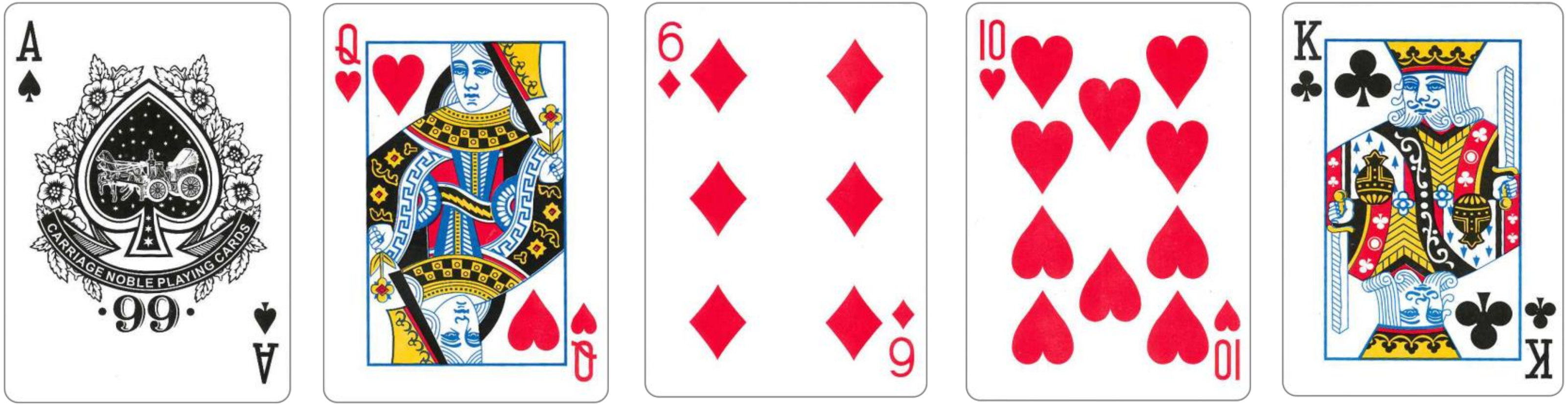
Cannot Be Repeated

When elements cannot be repeated:

- n choices for first slot
- $n - 1$ choices for second slot
- $n - (r - 1)$ choices for last slot
- In total: $n(n - 1)(n - 2)\ldots(n - r + 1)$ choices

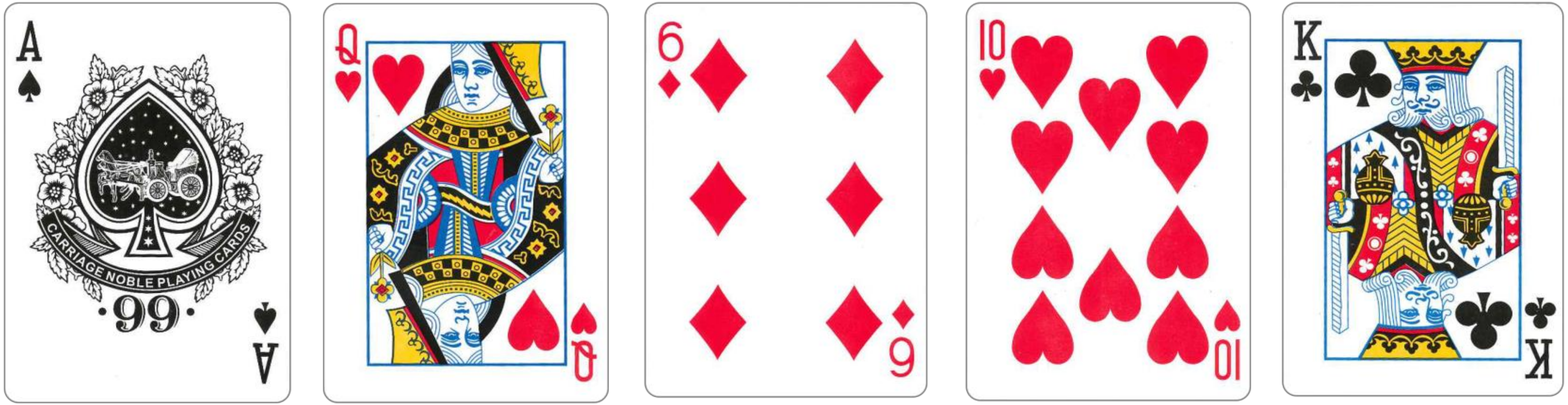
Principle of Counting: Example 1

Sequence of choice of cards from a deck of cards.



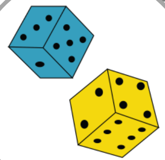
Principle of Counting: Example 2

Sequence of choice of cards from a deck of cards.

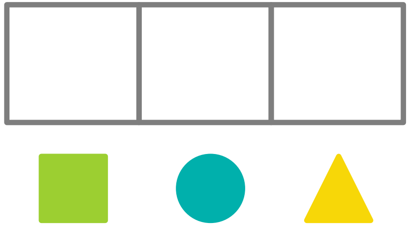


Permutations

Permutations: n!



A **permutation** is an arrangement of all or part of a set of objects, **with** regard to the order of the arrangement.



1^{st} position	*3* choices
2^{nd} position	*2* choices
3^{rd} position	*1* choice

Number of permutations of n objects

$$n*(n-1)*(n-2)\ldots*2*1 = n!$$

where $n!$ is called n factorial.

Permutations: $P(n,r)$

Filling r slots with n choices and no repetition:

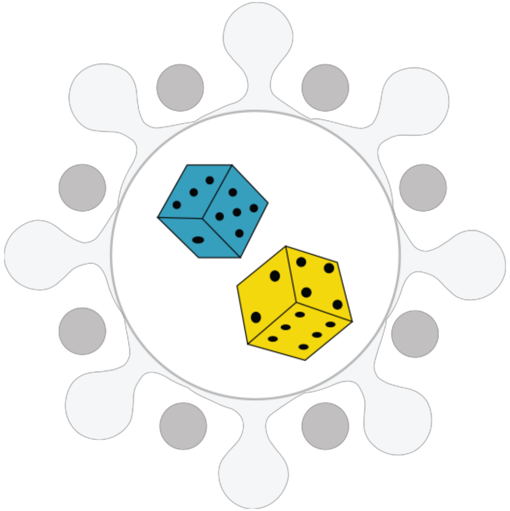
$$n(n-1)(n-2)\ldots(n-r+1)$$

Permutations of n objects: $n!$

Number of permutations of n objects taken r at a time (n objects, the number of ways in which r items can be ordered):

$$P(n,r) = n(n-1)(n-2)\ldots(n-r+1) = n!/(n-r)!$$

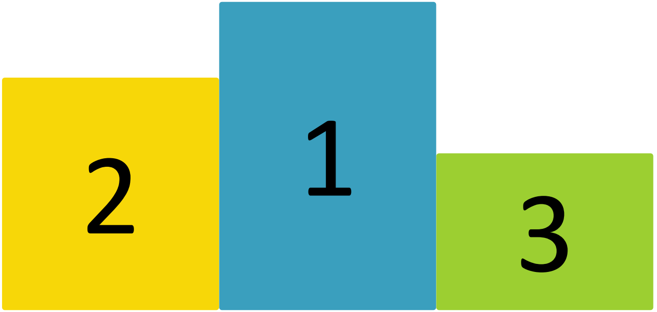
where $n! = n*(n-1)*(n-2)* \ldots *2*1$ (called n factorial).

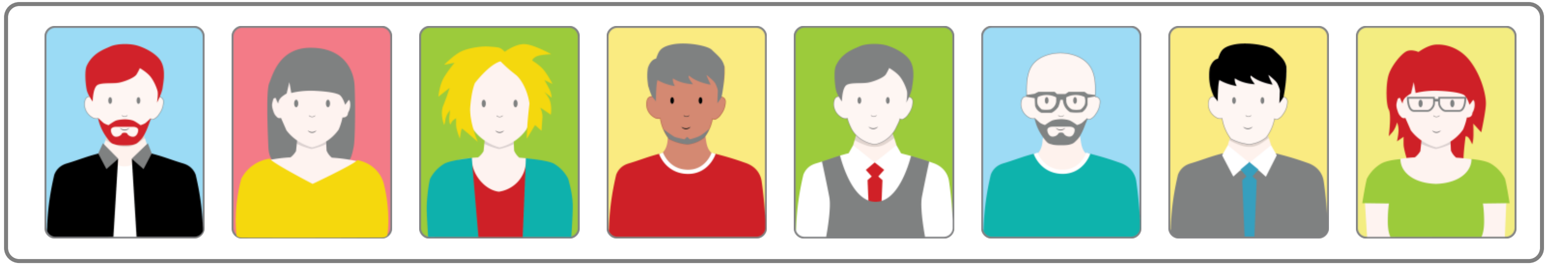


Permutations: Example

In how many ways can we award a 1^{st}, 2^{nd} and 3^{rd} prize to 8 contestants?

- **For the 1^{st} prize, any of the 8.**
- **Then for the 2^{nd} prize, any of the 7 left.**
- **And for the 3^{rd} prize, any of the 6 left.**
- **Hence $8*7*6 = 336$.**

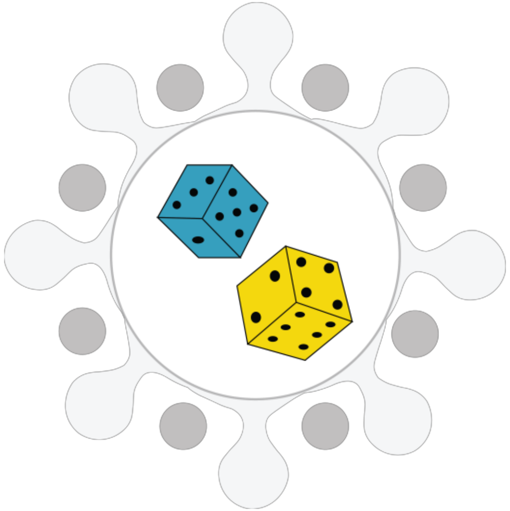




Permutations: Distinguishable Permutations

In general, the number of distinguishable permutations from a collection of objects, where the first object appears (repeats) k_1 times, the second object k_2 times, … for r distinct objects:

$$n!/(k_1!\,k_2!\ldots k_r!)$$



Permutations: Distinguishable Permutations

Example

How many permutations are there of “MISSISSIPPI”?

M	Appears $k_1 = 1$
I	Appears $k_2 = 4$
S	Appears $k_3 = 4$
P	Appears $k_4 = 2$

$$11!/(1!\ 4!\ 4!\ 2!)$$

Combinations

Combinations: $C(n,r)$ or $\binom{n}{r}$

A **combination** is a selection of all or part of a set of objects, **without** regard to the order in which objects are selected.

Example

Team of 4 people from a group of 10.

Number of combinations of n objects taken r at a time:

$$\binom{n}{r} = C(n,r) = n!/r!(n - r)!$$

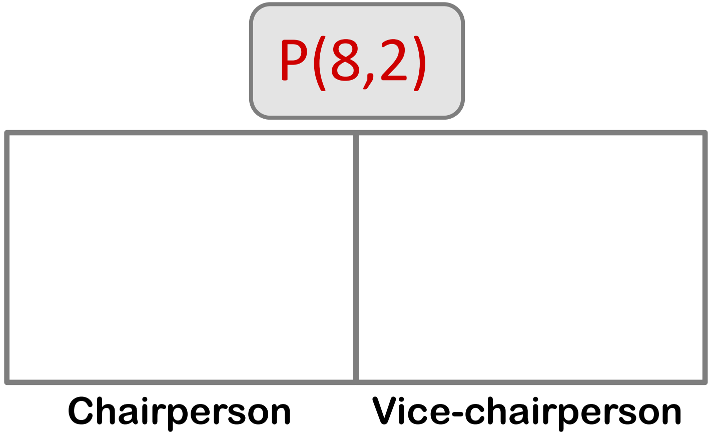
There are $r!$ possible orderings within each combination.

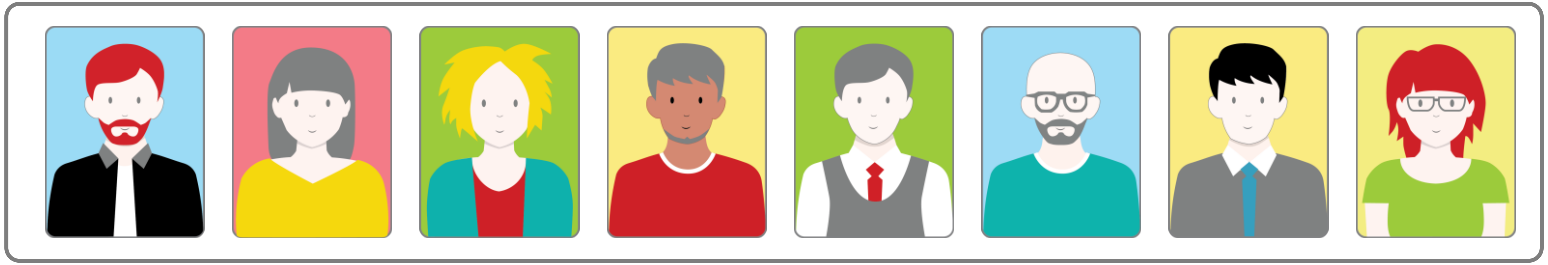
So $r!\,C(n,r) = P(n,r)$ by definition of permutation.

Combinations: Example

From a committee of 8 people, in how many ways can you choose:

- A chairperson and vice-chairperson (one person cannot hold more than one position)?





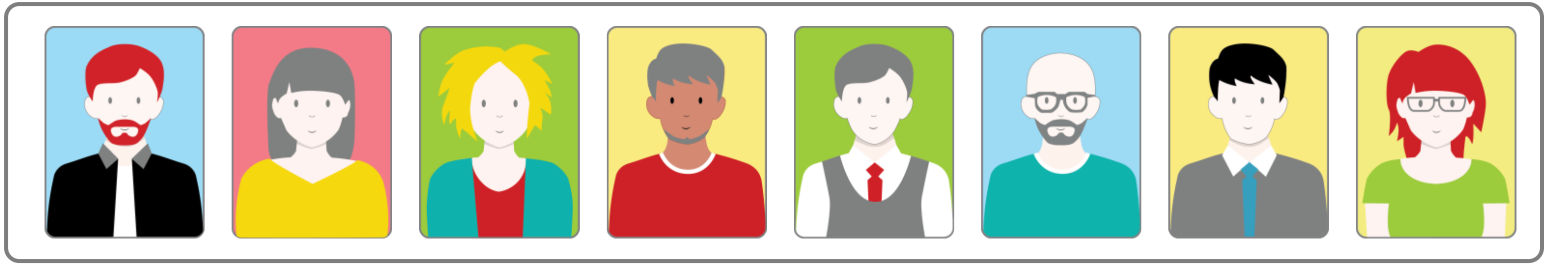
Combinations: Example

From a committee of *8* people, in how many ways can you choose:

- A subcommittee of *2* people?

C(8,2)

Subcommittee



Topic Summary

Let’s recap…

- Principle of counting

- Permutations (with order)

- Combinations (without order)