MH1810 Math 1 Part 1 Algebra - Vectors

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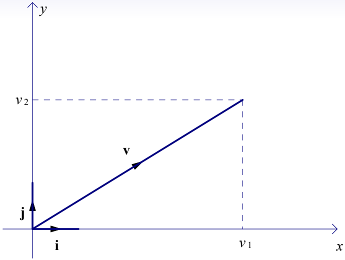
Vectors

Vectors are quantities which have both magnitude and direction. Examples of vectors include acceleration, displacement, force, momentum and velocity.

Vectors in 2-space(the plane)

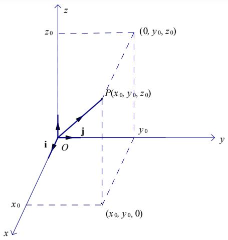


Positioning a vector **v** in the plane $\mathbb{R}^2$ with its initial point at the origin O and the terminal point (v_1, v_2), we may represent or write the vector **v** as

$$\mathbf{v} = \underbrace{(v_1, v_2)}_{\text{row form}}, \quad \text{or } \mathbf{v} = \underbrace{\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{\text{column form}} ..$$

The numbers v_1 and v_2 are called the **components** of **v**.

Vectors in 3-space



Positioning a vector in 3-space with its initial point at the origin O and the terminal point (v_1, v_2, v_3), we may represent or write the vector $\mathbf{v}$ as $\mathbf{v} = (v_1, v_2, v_3)$, or $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

The numbers v_1, v_2 and v_3 are called the **components** of $\mathbf{v}$.
Note: It should be clear from the context whether the coordinate representation (x, y, z) refers to a point or a vector.

Positioning a vector **v** from the point $P_1 = (x_1, y_1, z_1)$ to the point $P_2 = (x_2, y_2, z_2)$, we may express the vector **v** as

$\overrightarrow{P_2 P_1}$

$$\mathbf{v} = \overrightarrow{P_1 P_2} = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \,.$$

In particular, if the starting point is the origin $O = (0, 0, 0)$ and the ending point is $P = (x_0, y_0, z_0)$, then

$$\overrightarrow{OP} = (x_0 - 0, y_0 - 0, z_0 - 0) = (x_0, y_0, z_0) \,.$$

$a = (4, 1, 3)$ $\quad \overrightarrow{Oa}$

Basic Facts

Suppose that $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (u_1, v_2, v_3)$ are two vectors in 3-space. Then we have the following:

1. (Zero vector) $\mathbf{0} = (0, 0, 0)$ is the **Zero vector**
2. (Equality) Two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are equal if and only if $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$.
3. (Addition/subtraction)

   $$\mathbf{u} + \mathbf{v} = (u_1, u_2, u_3) \pm (v_1, v_2, v_3) = (u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3).$$

4. (Scalar Multiplication) For a real number k, we have

   $$k\mathbf{u} = k(u_1, u_2, u_3) = (ku_1, ku_2, ku_3).$$

Length or Norm of a Vector

In 2-space, the **length** of a vector $\mathbf{u} = (u_1, u_2)$ is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2}.$$

In 3-space, the **length** of a vector $\mathbf{u} = (u_1, u_2, u_3)$ is

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

The length of a vector is also known as the **norm** or **magnitude**.
Note: The symbol $\sqrt{a}$ refers to positive square root of the positive real number a. For example, $\sqrt{169} = 13$.
If $a = 0$, then $\sqrt{0} = 0$.
If $a < 0$, then $\sqrt{a}$ is not defined as a real number.

Examples

(a) $\mathbf{u} = (2, -5)$.
$\|\mathbf{u}\| = \sqrt{2^2 + (-5)^2} = \sqrt{29}$.

(b) $\mathbf{v} = (-1, 2, 2)$.
$\|\mathbf{v}\| = \sqrt{(-1)^2 + 2^2 + 2^2} = \sqrt{9} = 3$.

(c) $\mathbf{w} = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$.
$\|\mathbf{w}\| = \sqrt{(\frac{1}{\sqrt{2}})^2 + 0 + (-\frac{1}{\sqrt{2}})^2} = \sqrt{1} = 1$.

Remarks on Norm

1. The distance d between two points $P = (u_1, u_2, u_3)$ and $Q = (v_1, v_2, v_3)$ is given by the norm $\|\overrightarrow{PQ}\|$ of the vector $\overrightarrow{PQ}$,

$$d = \sqrt{(v_1 - u_1)^2 + (v_2 - u_2)^2 + (v_3 - u_3)^2}.$$

2. From the definition of $k\mathbf{u}$, it follows that for $k \in \mathbb{R}$,

$$\|k\mathbf{u}\| = |k|\|\mathbf{u}\|.$$

Some examples:

$$\|3(-1, 2, 2)\| = 3\|(-1, 2, 2)\| \;\&\; \|-5(-1, 2, 2)\| = 5\|(-1, 2, 2)\|.$$

Unit vectors

$\hat{z} = \frac{1}{5}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

(a) A vector of length 1 is called a **unit vector.**
For example, the vector $\mathbf{u} = (\frac{-1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$ is a unit vector.

(b) In particular, the following vectors

$$\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0) \text{ and } \mathbf{k} = (0, 0, 1)$$

are unit vectors along the positive direction of x-, y- and z-axes respectively.
Thus, we may represent vectors in terms of unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$:

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Unit vector along a direction

(c) For a nonzero vector $\mathbf{u}$, the vector

$$|u| \quad \hat{\mathbf{u}} = \underbrace{\frac{1}{\|\mathbf{u}\|}}_{\text{scalar product}} \mathbf{u}$$

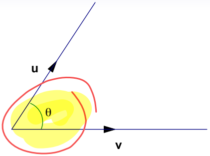
is the unit vector along the direction $\mathbf{u}$.
For example, let $\mathbf{u} = (1, 2, -1)$, where $\|\mathbf{u}\| = \sqrt{6}$. Then

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{6}}(1, 2, -1).$$

We may express

$$\mathbf{u} = \|\mathbf{u}\|\hat{\mathbf{u}}.$$

Dot product



$u \, (\; v \;)$

$= v \cdot v$

(a) The dot product (or scalar product) of two non-zero vectors **u** and **v** is a real number defined as follows

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos \theta,$$

where $\theta \in [0, \pi]$ is the angle (in radians) between the two vectors.

(b) If either **u** or **v** is a zero vector, then we define $\mathbf{u} \cdot \mathbf{v} = 0$.

Dot product

Example

Find the dot product of each pair of vectors.

(a) $\mathbf{u} = (1, -1)$ and $\mathbf{v} = (2, 0)$.

$\sqrt{2}\sqrt{4}\cos\theta$ $\qquad \dfrac{2}{2\sqrt{2}} = \cos\theta$

$2 + 0 = 2$

$\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \theta$

(b) $\mathbf{u} = (1, -1)$ and $\mathbf{w} = (0, 2)$.

(c) $\mathbf{u} = (1, -1)$ and $\mathbf{s} = (2, 2)$.

Basic properties of dot product

1. The dot product of two vectors is a real number.
2. For each vector $\mathbf{u}$,

$$\mathbf{u} \cdot \mathbf{u} = ||\mathbf{u}||^2 .$$

3. Suppose that $\mathbf{u}$ and $\mathbf{v}$ are non-zero vectors. Then

$$\mathbf{u} \cdot \mathbf{v} = 0 \Longleftrightarrow$$

$\mathbf{u}$ and $\mathbf{v}$ are perpendicular, i.e., $\mathbf{u} \perp \mathbf{v}$.

Dot product of unit vectors

We have the following simple facts about unit vectors.

(a) $\mathbf{i} \cdot \mathbf{i} = \|\mathbf{i}\|^2 = 1$,
$\mathbf{j} \cdot \mathbf{j} = \|\mathbf{j}\|^2 = 1$ and
$\mathbf{k} \cdot \mathbf{k} = \|\mathbf{k}\|^2 = 1$.

(b) $\mathbf{i} \cdot \mathbf{j} = 0$, $\mathbf{j} \cdot \mathbf{k} = 0$ and $\mathbf{i} \cdot \mathbf{k} = 0$.

Algebraic properties of dot product

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$. (Commutative)
2. $(\lambda \mathbf{u}) \cdot (\mu \mathbf{v}) = (\lambda \mu)\, \mathbf{u} \cdot \mathbf{v}$.
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
4. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

Computing dot product

The basic properties and algebraic properties of dot product lead to an easy computation of dot product of two vectors which are expressed in coordinate form, without knowing the angle between them.

Note that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$. Thus, we have

$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_1\mathbf{i} + v_2\mathbf{j}) \\
&= (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_1\mathbf{i}) + (u_1\mathbf{i} + u_2\mathbf{j})\cdot(v_2\mathbf{j}) \\
&= (u_1\mathbf{i})\cdot(v_1\mathbf{i}) + (u_2\mathbf{j})\cdot(v_1\mathbf{i}) + (u_1\mathbf{i})\cdot(v_2\mathbf{j}) + (u_2\mathbf{j})\cdot(v_2\mathbf{j}) \\
&= (u_1v_1)(\mathbf{i}\cdot\mathbf{i}) + (u_2v_1)(\mathbf{j}\cdot\mathbf{i}) + (u_1v_2)(\mathbf{i}\cdot\mathbf{j}) + (u_2v_2)(\mathbf{j}\cdot\mathbf{j}) \\
&= u_1v_1 + u_2v_2
\end{aligned}$$

Computing dot product

Therefore we have:

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = u_1 v_1 + u_2 v_2.$$

Similarly

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example

Example

Find the $\mathbf{u} \cdot \mathbf{v}$ if $\mathbf{u} = (2, 3, -1)$ and $\mathbf{v} = (1, 0, 5)$.

Solution

We have

$$\mathbf{u} \cdot \mathbf{v} = (2, 3, -1) \cdot (1, 0, 5) = 2(1) + 3(0) + (-1)(5) = -3.$$

Since $\mathbf{u} \cdot \mathbf{v} < 0$, *the angle* θ *between the two vectors is an obtuse angle.*

Application 1: Find angle between two vectors

The dot product turns out to be very useful, especially in finding the angle between two given non-zero vectors.
Suppose θ is the angle between two vectors $\mathbf{u}$ and $\mathbf{v}$.
Using $\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| \, ||\mathbf{v}|| \cos\theta$, we obtain

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| \, ||\mathbf{v}||}.$$

Example

Example

Find the angle between $\mathbf{u} = (1, 2, -2)$ and $\mathbf{v} = (2, 3, -6)$.

Solution

Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$. Since

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||\;||\mathbf{v}||},$$

we shall compute the values $\mathbf{u}\cdot\mathbf{v}$, $||\mathbf{u}||$ and $||\mathbf{v}||$:

$\mathbf{u}\cdot\mathbf{v} = (1)(2) + (2)(3) + (-2)(-6) = 20.$

$||\mathbf{u}|| = \sqrt{(1)^2 + (2)^2 + (-2)^2} = 3$

$||\mathbf{v}|| = \sqrt{(2)^2 + (3)^2 + (-6)^2} = 7$

Then $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{||\mathbf{u}||||\mathbf{v}||} = \frac{20}{(3)(7)} = \frac{20}{21}$.

Hence, we have $\theta = \cos^{-1}\dfrac{20}{21} \approx 0.310$ *rad*.

Example

Example

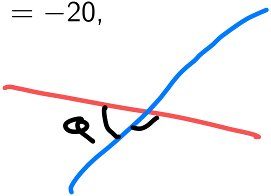
Find the angle between $\mathbf{u} = (-1, -2, 2)$ and $\mathbf{v} = (2, 3, -6)$

Let θ denote the angle between $\mathbf{u}$ and $\mathbf{v}$. We have

$$\mathbf{u} \cdot \mathbf{v} = (-1)(2) + (-2)(3) + (2)(-6) = -20,$$

$\|\mathbf{u}\| = \sqrt{(-1)^2 + (-2)^2 + (2)^2} = 3$ and

$\|\mathbf{v}\| = \sqrt{(2)^2 + (3)^2 + (-6)^2} = 7.$



Thus we have

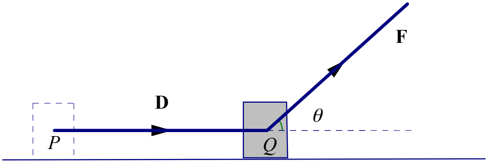
$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \ \|\mathbf{v}\|} = \frac{-20}{(3)(7)} = \frac{-20}{21}.$$

The negative value indicates that θ is an obtuse angle. We have

$$\theta = \pi - \cos^{-1}\frac{20}{21} \approx 2.831748014 \text{ rad}.$$

Application 2: Work Done

One of the many applications of dot product in science and engineering is to compute work done.
Recall that force and displacement are vector quantities, they have magnitude and direction.



The work done by a constant force **F** acting through a displacement **D** is the dot product

$$W = \mathbf{F} \cdot \mathbf{D}.$$

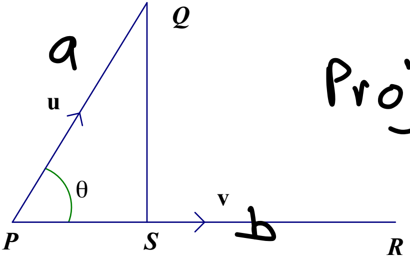
Example

If $\|\mathbf{F}\| = 50N$, $\|\mathbf{D}\| = 30\ m$, and $\theta = 45^\circ$, then the work done by $\mathbf{F}$ acting from P to Q is given by

$$W = \mathbf{F} \cdot \mathbf{D} = \|\mathbf{F}\|\ \|\mathbf{D}\| \cos\theta$$

$$= (50)\,(30)\,\frac{1}{\sqrt{2}} = \frac{1500}{\sqrt{2}}\ \text{J}.$$

Suppose $\mathbf{v}$ is a non-zero vector. The (perpendicular) projection of a vector $\mathbf{u}$ onto $\mathbf{v}$ (or along $\mathbf{v}$) is the vector $\overrightarrow{PS}$ shown in the diagram:



$\text{Proj}_a b = (a \cdot \hat{b})\hat{b}$

$|k|\, b$

If the angle θ is obtuse, then the projection vector is pointing away from $\mathbf{v}$.

Notation: $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Projection of a Vector

Recall the unit vector $\hat{\mathbf{v}}$ along $\mathbf{v}$ is the vector

$$\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}.$$

The vector projection $\text{proj}_{\mathbf{v}}\mathbf{u}$ of $\mathbf{u}$ onto $\mathbf{v}$, is given by

$$\begin{aligned} \text{proj}_{\mathbf{v}}\mathbf{u} &= \left(\|\mathbf{u}\| \cos\theta\right) \hat{\mathbf{v}} \\ &= \left(\mathbf{u} \cdot \hat{\mathbf{v}}\right)\ \hat{\mathbf{v}}. \end{aligned}$$

In terms of $\mathbf{v}$, we have

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\right)\ \mathbf{v} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right)\ \mathbf{v}.$$

Example

Example

Find the vector projection of $\mathbf{u} = (1, -2, 2)$ along $\mathbf{v} = (2, 3, -6)$.

Solution

Note that $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + (-6)^2} = 7.$
Unit vector $\hat{\mathbf{v}}$ *along* $\mathbf{v}$ *is*

$$\hat{\mathbf{v}} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{7}\mathbf{v} = \left(\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}\right).$$

$v \cdot v = (v_1 v_1 + v_2 v_2 + v_3$

Using $proj_{\mathbf{v}}\mathbf{u} = (\mathbf{u} \cdot \hat{\mathbf{v}})\,\hat{\mathbf{v}}$, *we have*

$$proj_{\mathbf{v}}\mathbf{u} = \left((1, -2, 2) \cdot (\frac{2}{7}, \frac{3}{7}, -\frac{6}{7})\right)\hat{\mathbf{v}}$$

$$= \frac{-16}{49}(2, 3, -6).$$

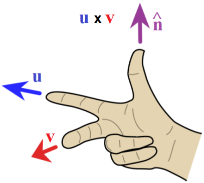
Cross Product

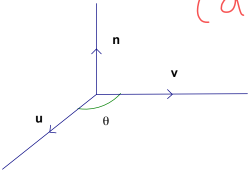
(Important Only defined for vectors in R^3. It is NOT defined for vectors in other dimensions.)

The cross product (or vector product) of two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$ is a vector in $\mathbb{R}^3$. It is denoted by by $\mathbf{u} \times \mathbf{v}$.

Its length (norm) is $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \; \|\mathbf{v}\| \sin\theta$.

Its direction $\hat{\mathbf{n}}$ is the vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, governed by the right hand rule ($\mathbf{u}$-index finger, $\mathbf{v}$-middle finger, $\mathbf{n}$-thumb, of right hand).





$(a \times b) \neq (b \times a)$

Therefore

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\| \; \|\mathbf{v}\| \sin\theta \; \hat{\mathbf{n}}$$

Cross Products of Two Parallel Vectors

If $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\theta = 0$ and thus $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.
In particular, we have

$$\mathbf{u} \times \mathbf{u} = \mathbf{0}.$$

Hence, we have

$$\mathbf{i} \times \ \mathbf{i} = \mathbf{0}, \quad \mathbf{j} \times \ \mathbf{j} = \mathbf{0}, \quad \mathbf{k} \times \ \mathbf{k} = \mathbf{0}.$$

Cross Products of Two Perpendicular Vectors

If $\mathbf{u}$ and $\mathbf{v}$ are perpendicular, then $\theta = \pi/2$ and thus $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \ \|\mathbf{v}\|$.

In particular, we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \ \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j}.$$

Use the right-hand rule to convince yourself the directions of the above cross products.

Properties of Cross Products

1. The cross product of two vectors **u** and **v** is a vector perpendicular to both **u** and **v** (if it is non-zero).
2. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$ (anti-commutative).
3. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
4. $(\lambda \mathbf{u}) \times (\mu \mathbf{v}) = (\lambda \mu)(\mathbf{u} \times \mathbf{v})$.

Remark For two general vectors in $\mathbb{R}^3$, it is not easy to evaluate their cross vectors via definition.

Determinant Formula for Cross Products

Theorem

If $\mathbf{u} = (u_1, u_2, u_3)$ *and* $\mathbf{v} = (v_1, v_2, v_3)$ *, then*

$$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

Determinant Formula for Cross Products - Proof

$$\mathbf{u} \times \mathbf{v} = (u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}) \times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= u_1\mathbf{i}\times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}) + u_2\mathbf{j}\times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$
$$+u_3\mathbf{k}\times (v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k})$$

$$= (u_1\mathbf{i}\times v_1\mathbf{i} + u_1\mathbf{i}\times v_2\mathbf{j} + u_1\mathbf{i}\times v_3\mathbf{k}) + (u_2\mathbf{j}\times v_1\mathbf{i} + u_2\mathbf{j}\times v_2\mathbf{j} + u_2\mathbf{j}\times v_3\mathbf{k})$$
$$+ (u_3\mathbf{k}\times v_1\mathbf{i} + u_3\mathbf{k}\times v_2\mathbf{j} + u_3\mathbf{k}\times v_3\mathbf{k})$$

Determinant Formula for Cross Products - Proof

$$
\begin{aligned}
\mathbf{u} \times \mathbf{v} &= u_1 v_1 \mathbf{i} \times \mathbf{i} + u_1 v_2 \mathbf{i} \times \mathbf{j} + u_1 v_3 \mathbf{i} \times \mathbf{k} \\
&\quad + u_2 v_1 \mathbf{j} \times \mathbf{i} + u_2 v_2 \mathbf{j} \times \mathbf{j} + u_2 v_3 \mathbf{j} \times \mathbf{k} \\
&\quad + u_3 v_1 \mathbf{k} \times \mathbf{i} + u_3 v_2 \mathbf{k} \times \mathbf{j} + u_3 v_3 \mathbf{k} \times \mathbf{k} \\
&= \mathbf{0} + u_1 v_2 \mathbf{k} + u_1 v_3 \left(-\mathbf{j}\right) + u_2 v_1 \left(-\mathbf{k}\right) + \mathbf{0} + u_2 v_3 \left(\mathbf{i}\right) \\
&\quad + u_3 v_1 \left(\mathbf{j}\right) + u_3 v_2 \left(-\mathbf{i}\right) + \mathbf{0}
\end{aligned}
$$

Determinant Formula for Cross Products - Proof

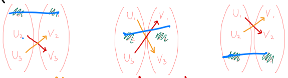
$$\mathbf{u} \times \mathbf{v} = (u_2 v_3 - u_3 v_2)\,\mathbf{i} - (u_1 v_3 - u_3 v_1)\,\mathbf{j} + (u_1 v_2 - u_2 v_1)\,\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k}.$$

② Shortcut method:

$$U \times V = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \times \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix}$$

$$= (U_2V_3 - U_3V_2)\,i + (U_1V_3 - U_3V_1)\,j + (U_1V_2 - U_2V_1)\,k$$



orange first red second

Example

Example

Find $\mathbf{u} \times \mathbf{v}$ if $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 0, 1)$.

Solution

$\mathbf{u} \times \mathbf{v} = (1, 1, 1) \times (1, 0, 1)$

$$= \left(\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} \right)$$

$$= (1, 0, -1)$$

Application 1: Perpendicular Vector

The cross product $\mathbf{u} \times \mathbf{v}$ provides a vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. From this, we may obtain a unit vector in this direction.

Example

Find a unit vector $\hat{\mathbf{n}}$ which is perpendicular to both $\mathbf{u} = (1, 1, 1)$ and $\mathbf{v} = (1, 0, 1)$.

Solution

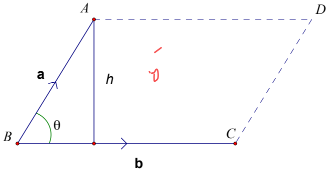
From the previous example,

$$\mathbf{u} \times \mathbf{v} = (1, 0, -1).$$

Thus, we have

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(1, 0, -1).$$

Application 2: Areas



Area of parallelogram with sides **u** and **v** is given by

$$= \text{ base} \times \text{ height}$$

$$= \|\mathbf{u}\| \cdot \|\mathbf{v}\| \sin\theta$$

$$= \|\mathbf{u} \times \mathbf{v}\|.$$

Application 2: Areas

Area of triangle ABC is half of the area of parallelogram with edges AB and AC.

Thus, the area of triangle is

$$\frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|.$$

Scalar Triple Product

For three vectors

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \text{ in } \mathbb{R}^3,$$

the dot product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is called the scalar triple product of vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$.

Scalar Triple Product

The scalar triple product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be computed numerically as follows:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = u_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - u_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + u_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

It can be verified that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

Theorem

(a) For three non-coplanar vectors $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{w}$ *in* $\mathbb{R}^3$*, the absolute value* $|\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})|$ *of the scalar triple product is the volume of the parallelepiped which is a three-dimensional figure formed (by six parallelograms) whose sides are the three given vectors.*

(b) For general three vectors $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$*, if the scalar triple product* $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w})=0$*, we may conclude that either at least one of the vectors* $\mathbf{u}, \mathbf{v}$ *and* $\mathbf{w}$ *is a zero vector or the three vectors are coplanar (they lie on the same plane).*

Example

Example

Find the volume of the parallelepiped formed by vectors $\mathbf{u} = (1, 2, 3)$, $\mathbf{v} = (-1, 1, 0)$ and $\mathbf{w} = (1, 2, 1)$.

Solution

We compute the scalar triple product:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 1 \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 1 - 2(-1) + 3(-3) = -6$$

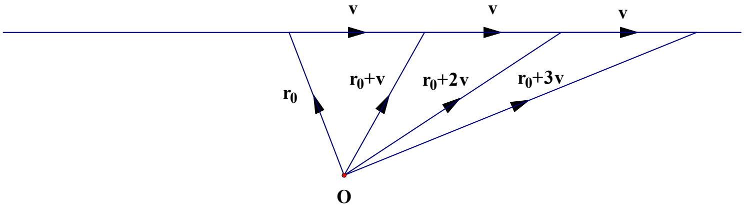
Thus the volume of the parallelepiped is 6 *unit*^{3}.

Applications of Vectors to Lines

Recall that in $\mathbb{R}^2$, a line is determined by a point on the line and the gradient of the line. Knowing the gradient of a line, we are able to find a vector which is parallel to the line.

In $\mathbb{R}^3$, it is not meaningful to define gradient of a line. However, it is useful to know a vector which is parallel to the given line.

Vector Equations of Lines



A line on $\mathbb{R}^2$ or $\mathbb{R}^3$ is uniquely determined by its direction vector and a point lying on the line.

If a line ℓ is parallel to a vector $\mathbf{v}$ and passes through a point with position vector $\mathbf{r}_0$, then a vector equation of ℓ is

$$\ell : \mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v},\ t \in \mathbb{R}.$$

Example

If A, B and C have coordinates $(1,1,0)$, $(0,1,0)$ and $(0,0,1)$respectively. Find a vector equation for the line that is parallel to BC and passes through A.

[SOLUTION] Direction vector
$\overrightarrow{BC} = (0,0,1) - (0,1,0) = (0,-1,1)$.
Thus,a vector equation is given by

$$\mathbf{v} = \underbrace{\overrightarrow{OA}}_{\text{point on line}} + t \underbrace{\overrightarrow{BC}}_{\text{direction}}, t \in \mathbb{R};$$

$$= (1,1,0) + t\,(0,-1,1)\,, t \in \mathbb{R}.$$

Equation of a line: Cartesian form

Suppose that a line ℓ passes through $\mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$ and is parallel to $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. If (x, y, z) is a point on ℓ, then

$$\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}, t \in \mathbb{R}.$$

Equation of a line: Parametric form

Therefore,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 + tv_1 \\ y_0 + tv_2 \\ z_0 + tv_3 \end{pmatrix}, t \in \mathbb{R}.$$

Thus, we obtain the parametric equation of ℓ.

$$x = x_0 + tv_1,\ y = y_0 + tv_2,\ z = z_0 + tv_3, t \in \mathbb{R}.$$

Angle betwee Two Lines

The angle between two lines ℓ_1 and ℓ_2 is the acute angle θ between the two lines.
Suppose ℓ_1 and ℓ_2 have direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.
Then we have

$$\left|\mathbf{v}_1 \cdot \mathbf{v}_2\right| = \left\|\mathbf{v}_1\right\| \cdot \left\|\mathbf{v}_2\right\| \cos\theta.$$

Example

Example

Find the acute angle between z-axis and the line ℓ :
$x = 1 - t, y = 3 + \sqrt{2}t,\ z = -5 + t$

Solution

Note that the line ℓ and z-axis have respective direction vectors

$$\mathbf{v}_1 = \left(-1, \sqrt{2}, 1\right) \ \textit{and}\ \mathbf{v}_2 = (0, 0, 1)\,.$$

Let θ be the acute angle between $\mathbf{v}_1$ and $\mathbf{v}_2$. Then

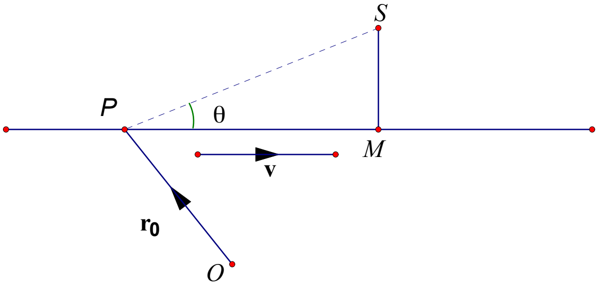
$$|\mathbf{v}_1 \cdot \mathbf{v}_2| = \|\mathbf{v}_1\| \ \ \|\mathbf{v}_2\| \cos\theta,\ \textit{i.e.},\ 1 = 2\cos\theta.$$

Thus, we have

$$\theta = \frac{\pi}{3} \ \ \textit{rad}.$$

Distance from a Point to a Line

The distance d from a point S to a line ℓ is the shortest distance between S and point Q on the line ℓ.



Geometrically, we observe that the distance d is the distance from S to M on ℓ where M is the foot of perpendicular of S to ℓ.

Distance from a Point to a Line

The distance d from a point S to a line ℓ passing though P and with direction vector $\mathbf{v}$ is given by

$$\|\overrightarrow{PS}\| \sin\theta = \|\overrightarrow{PS} \times \hat{\mathbf{v}}\|.$$

Example

Example

Find the distance from a point $S(1,3,2)$ to the line $\ell : \mathbf{r} = (0,1,2) + t\,(1,0,1)\,,\ t \in \mathbb{R}$.

Solution

Let P denote the point $(0,1,2)$ on ℓ, $\mathbf{v} = \mathbf{i} + \mathbf{k}$ (direction vector), with $\|\mathbf{v}\| = \sqrt{2}$. Then $\overrightarrow{PS} = (1,3,2) - (0,1,2) = (1,2,0)$. The distance from P to ℓ is given by

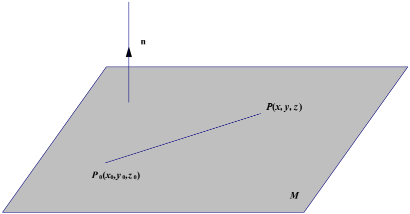
$$\|\overrightarrow{PS}\| \sin\theta = \|\overrightarrow{PS} \times \hat{\mathbf{v}}\| = \left\| \begin{vmatrix} 2 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{vmatrix}, - \begin{vmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ \frac{1}{\sqrt{2}} & 0 \end{vmatrix} \right\|$$

$$= \|(\sqrt{2}, \frac{-1}{\sqrt{2}}, -\sqrt{2})\| = \frac{3}{\sqrt{2}} \textit{units}.$$

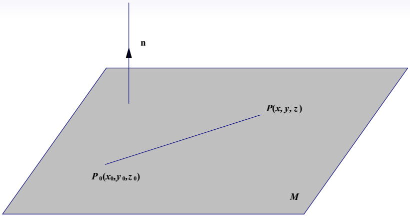
Planes

A plane in a space is uniquely determined by a point and its 'inclination'. This inclination can be specified by a vector **n** that is normal, or perpendicular, to the plane.

Planes: Vector Equation



Suppose that a plane M contains a point $P_0 = (x_0, y_0, z_0)$ and a nonzero vector $\mathbf{n} = (a, b, c)$ is normal to it.
If $P = (x, y, z)$ is any other point lying on this plane, then P_0P and $\mathbf{n}$ are perpendicular.



We have the vector equation of the plane as follows:

$$\mathbf{n} \cdot \overrightarrow{P_0P} = \mathbf{0},$$

i.e.,

$$\mathbf{n} \cdot \overrightarrow{OP} = \underbrace{\mathbf{n} \cdot \overrightarrow{OP_0}}_{\text{constant}}$$

or simply:

$$\mathbf{r} \cdot \mathbf{n} = d.$$

Planes: Scalar Equation

The vector equation $\mathbf{n} \cdot \overrightarrow{P_0 P} = \mathbf{0}$ is simply

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = 0.$$

Expanding the above, we obtain the scalar equation of the plane through $P_0 = (x_0, y_0, z_0)$ with normal vector $\mathbf{n} = (a, b, c)$:

$$a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0$$

which can be simplified to

$$ax + by + cz = d,$$

where $d = ax_0 + by_0 + cz_0$.

Planes

Vector Equation:

$$\mathbf{n} \cdot \overrightarrow{P_0P} = \mathbf{0} \text{ or } \mathbf{r} \cdot \mathbf{n} = d.$$

Scalar Equation:

$$ax + by + cz = d,$$

The coefficients $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a normal vector to the plane.

Example

Example

Find the scalar equation for the plane through $P_0\,(1,2,3)$ and perpendicular $\mathbf{n}=(4,5,6)$.

Solution

The required equation is

$$4\,(x-1)+5\,(y-2)+6\,(z-3)=0.$$

Simplifying, we have

$$4x+5y+6z=32.$$

Example

Example

Find the scalar equation of the plane passing through $A\,(1,0,0)$, $B\,(0,1,0)$ and $C\,(0,0,1)$.

Solution

The vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel to the plane so that their cross product $\mathbf{n}$ *is a vector normal to the plane, where*

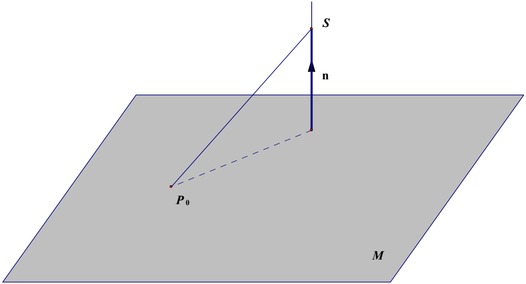
$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (1,1,1)$$

The scalar equation of the plane is

$$1\cdot(x-1) + 1\cdot(y-0) + 1\cdot(z-0) = 0, \textit{i.e.,} x+y+z=1.$$

Distance from a Point to a Plane

The distance from a point S to a plane Π is the shortest distance between point S and points on the plane Π.



If P is a point lying on the plane Π with normal $\mathbf{n}$, then the distance from a point S to the plane Π is the length of the vector projection of $\overrightarrow{PS}$ to $\mathbf{n}$, which is equal to $\left|\overrightarrow{PS} \cdot \hat{\mathbf{n}}\right|$.

Example

Example

Find the distance from the point $S\,(1, 0, -1)$ to the plane $x - 2y + z = 1$.

A normal vector of the plane $x - 2y + z = 1$ is $\mathbf{n} = (1, -2, 1)$, with $\|\mathbf{n}\| = \sqrt{6}$.
We find a point P on the given plane by finding values x, y and z which satisfy the equation $x - 2y + z = 1$.
Let $x = 0$ and $y = 0$, we have $z = 1$.
Thus, we may take $P = (0, 0, 1)$.
The required distance is given by $\left|\overrightarrow{PS} \cdot \hat{\mathbf{n}}\right|$ which is

$$\left|((1, 0, -1) - (0, 0, 1)) \cdot \frac{1}{\sqrt{6}}(1, -2, 1)\right| = \frac{1}{\sqrt{6}}.$$

Angle between two planes

The angle between two planes is defined to be the (acute) angle between their respective normal vectors.

Example

Find the angle between the planes $x - 2y + 2z = 1$ and $6x - 4y + 3z = 7$

The respective normal vectors are $\mathbf{n}_1 = (1, -2, 2)$ and $\mathbf{n}_2 = (6, -4, 3)$.
Note that $\mathbf{n}_1 \cdot \mathbf{n}_2 = 1(6) + (-2)(-4) + 2(3) = 20$,
$\|\mathbf{n}_1\| = \sqrt{9} = 3$ and $\|\mathbf{n}_2\| = \sqrt{61}$.
Thus, the angle between the given planes is

$$\theta = \cos^{-1}\left(\frac{20}{3\sqrt{61}}\right) \approx 0.547978869 \text{ rad}.$$