MA1513 TUTORIAL 1

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Link: http://tinyurl.com/MA1513Solutions

KEY CONCEPTS – CHAPTER 1 LINEAR SYSTEMS & MATRIX ALGEBRA

Gaussian & Gauss-Jordan Elimination

Objective: [Gaussian]

To achieve row echelon form:

$$\begin{pmatrix} 5 & 0 & 4 & 0 & \dots \\ 0 & 3 & 0 & 0 & \dots \\ 0 & 0 & 2 & 2 & \dots \\ 0 & 0 & 0 & 1 & \dots \end{pmatrix}$$

"upper-triangular matrix"

[Gauss-Jordan]

To obtain reduced-row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 5 & \dots \\ 0 & 1 & 0 & 0 & 2 & \dots \\ 0 & 0 & 1 & 0 & 3 & \dots \\ 0 & 0 & 0 & 1 & 5 & \dots \end{pmatrix}$$

"identity matrix"

0

Locate the left-most column that does not consist entirely of zeros.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Start the next action from here.

If the top-most entry is zero, switch the top row with another, so that a non-zero entry is now on top.

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

1

To each of the rows below the first row, add a (different) constant multiple of first row so that its leading entry becomes zero. Replace that row with the result.

$$\begin{pmatrix} 2 & 0 & 4 \\ -4 & 1 & 0 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{R_2 + 2R_1 \to R_2} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 8 \\ 1 & 1 & 3 \end{pmatrix} \xrightarrow{R_3 - \frac{1}{2}R_1 \to R_3} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 8 \\ 0 & 1 & 1 \end{pmatrix}$$

If the leading entry (along diagonal) is zero, switch the row with another row below it.

Continue with the 2^{nd} row: To each row below the 2^{nd} row, add a (different) constant multiple of 2^{nd} row so that its leading entry becomes zero. Replace that row with the result.

$$\begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 8 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{pmatrix} 2 & 0 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & -7 \end{pmatrix} \leftarrow \text{row-echelon form}$$

– Gaussian elimination ends here –

2

Multiply a suitable (different) constant to each row so all leading entries becomes 1.

$$\begin{pmatrix} 2 & 0 & 4 & \dots \\ 0 & 1 & 8 & \dots \\ 0 & 0 & -7 & \dots \end{pmatrix} \xrightarrow{-\frac{1}{7}R_3 \to R_3} \begin{pmatrix} 2 & 0 & 4 & \dots \\ 0 & 1 & 8 & \dots \\ 0 & 0 & 1 & \dots \end{pmatrix} \xrightarrow{\frac{1}{2}R_1 \to R_1} \begin{pmatrix} 1 & 0 & 2 & \dots \\ 0 & 1 & 8 & \dots \\ 0 & 0 & 1 & \dots \end{pmatrix}$$

If the leading entry is already a zero, take no action on this row.

3

Starting from the last row with non-zero leading entry, add a (different) constant multiple of the last row to each row above it so that its entries above the leading entry are all zeros.

$$\begin{pmatrix} 1 & 0 & 2 & \dots \\ 0 & 1 & 8 & \dots \\ 0 & 0 & 1 & \dots \end{pmatrix} \xrightarrow{R_2 - 8R_3 \to R_2} \begin{pmatrix} 1 & 0 & 2 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \end{pmatrix} \xrightarrow{R_1 - 2R_3 \to R_1} \begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \end{pmatrix}$$

If the leading entry (along diagonal) is zero, switch the row with another row above it.

Continue the process until the "identity matrix" is obtained.

$$\begin{pmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \end{pmatrix} \leftarrow \text{already obtained}$$

– Gauss-Jordan elimination ends here –

Purpose of Gaussian/Gauss Jordan Elimination

Solve system of linear equations	Find inverse of a square matrix
<u>Question</u>: Solve the following system: $\begin{cases} x-2y+z=0 \\ 2x+y-3z=5 \\ 4x-7y+z=-1 \end{cases}$	<u>Question</u>: Find the inverse of the 3 × 3 matrix: $\begin{pmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{pmatrix}$
Step 1: Form the augmented matrix.	
$(\mathbf{A} \mid \vec{b}) = \left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right)$	$(\mathbf{A} \mid I) = \left(\begin{array}{ccc\|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 1 & 0 \\ 4 & -7 & 1 & 0 & 0 & 1 \end{array}\right)$
Step 2: Perform Gaussian/Gauss-Jordan elimination.	
$\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right) \xrightarrow{\text{GE}} \left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 10 & 10 \end{array}\right)$ $\downarrow \text{GJE}$ $\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$	$\left(\begin{array}{ccc\|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & 1 & -3 & 0 & 1 & 0 \\ 4 & -7 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{GJE}}$ $\left(\begin{array}{ccc\|ccc} 1 & 0 & 0 & 2 & 0.5 & -0.5 \\ 0 & 1 & 0 & 1.4 & 0.3 & -0.5 \\ 0 & 0 & 1 & 1.8 & 0.1 & -0.5 \end{array}\right) = (I \mid \mathbf{A}^{-1})$

Step 3: Evaluate the answer.	
For GE, do back-substitution. For GJE, the answer is obvious. (One can use inverse matrix to solve a linear system $\mathbf{A}\vec{x}=\vec{b}$ by performing $\vec{x}=\mathbf{A}^{-1}\vec{b}$.)	$\mathbf{A}^{-1}=\begin{pmatrix} 2 & 0.5 & -0.5 \\ 1.4 & 0.3 & -0.5 \\ 1.8 & 0.1 & -0.5 \end{pmatrix}$

What is the formula to find the inverse of 2 × 2 matrix?

System of Linear Equations

Homogeneous System

$$\begin{cases} x-2y+z=0 \\ y-3z=0 \\ 3x+2y-z=0 \end{cases}$$

Non-Homogeneous System

$$\begin{cases} x-2y+z=0 \\ y-3z=3 \\ 3x+2y-z=0 \end{cases}$$

trivial solution

Unique solution	No solutions	Infinitely many solutions
GE gives non-zero leading entries $\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 10 & 10 \end{array}\right)$	GE gives <u>inconsistent matrix</u> $\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 10 \end{array}\right)$	GE gives zero entries for last row $\left(\begin{array}{ccc\|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ First few variables can be made in terms of the last variable.
consistent matrix	inconsistent matrix	consistent matrix
		If there are more variables than equations, the system will always have infinitely many solutions.

Basic Properties of Matrices

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 3 & -5 & 7 \\ -5 & 1 & 0 \\ 7 & 0 & 2 \end{pmatrix}$	$\begin{pmatrix} 3 & -5 & 7 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$
zero matrix, $\mathbf{0}$	diagonal matrix	identity matrix, $\mathbf{I}$	symmetric matrix	upper-triangular

- Matrix multiplication
  - Size matters! **A** (size m × n), **B** (size n × p). Then **AB** is valid but **BA** is invalid.
  - $\mathbf{AB}=\mathbf{0}$ does not imply $\mathbf{A}=\mathbf{0}$ or $\mathbf{B}=\mathbf{0}$. Can you give a counter-example?
  - Not commutative: $\mathbf{AB}\neq\mathbf{BA}$.
  - Associative: $\mathbf{A}(\mathbf{BC})=(\mathbf{AB})\mathbf{C}$.
  - Distributive: $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$, $(\mathbf{A}+\mathbf{B})\mathbf{C}=\mathbf{AC}+\mathbf{BC}$.
  - Scalar multiple: $c(\mathbf{AB})=(c\mathbf{A})\mathbf{B}=\mathbf{A}(c\mathbf{B})$, where c is a scalar.

- Power of matrices: $\mathbf{A}^n = \underbrace{\mathbf{AA}\ldots\mathbf{A}}_{n \text{ times}},\ n \geq 1$ $\qquad \mathbf{A}^0 = \mathbf{I}$
- Transpose of matrix, $\mathbf{A}^T$: interchange rows and columns of **A**
  - If **A** is symmetric, then $\mathbf{A} = \mathbf{A}^T$.
  - $(\mathbf{A}^T)^T = \mathbf{A}$
  - $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$
  - $(a\mathbf{A})^T = a\mathbf{A}^T$
  - $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$
- Inverse of matrix, $\mathbf{A}^{-1}$
  - $(a\mathbf{A})^{-1} = \dfrac{1}{a}\mathbf{A}^{-1}$
  - $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$
  - $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$
  - $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$
- A **singular matrix** is a matrix which does not have an inverse.

Square matrix ***A***

A non-singular matrix	***A*** singular matrix
A has an inverse $\boldsymbol{A}^{-1}$	***A*** does not have an inverse $\boldsymbol{A}^{-1}$
row echelon form of ***A*** does not have zero row	row echelon form of ***A*** has a zero row
reduced row echelon form of ***A*** is the identity matrix	reduced row echelon form of ***A*** is not the identity matrix
homogeneous system ***Ax = 0*** has only trivial solution	homogeneous system ***Ax = 0*** has non-trivial solutions
non-homogeneous system ***Ax = b*** has exactly one solution	non-homogeneous system ***Ax = b*** has either no solution or infinitely many solutions

TUTORIAL PROBLEMS

Question 1

Solve each of the following systems by Gaussian elimination or Gauss-Jordan elimination.

(a) $\begin{cases} x_1+3x_2+3x_3+2x_4=1 \\ 2x_1+6x_2+9x_3+5x_4=5 \\ -x_1-3x_2+3x_3=5 \end{cases}$ (b) $\begin{cases} x+y+2z=4 \\ x-y-z=-1 \\ 2x-4y-5z=1 \end{cases}$ (c) $\begin{cases} u-v+2w=6 \\ 2u+2v-5w=3 \\ 2u+5v+w=9 \end{cases}$

Solutions

(a) Consider the augmented matrix $\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 5 \\ -1 & -3 & 3 & 0 & 5 \end{array}\right)$. Performing Gaussian elimination,

$$\left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 5 \\ -1 & -3 & 3 & 0 & 5 \end{array}\right) \xrightarrow{R_2-2R_1\to R_2} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ -1 & -3 & 3 & 0 & 5 \end{array}\right) \xrightarrow{R_3+R_1\to R_3} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 6 & 2 & 6 \end{array}\right)$$

$$\xrightarrow{R_3-2R_2\to R_3} \left(\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Consequently, we are left with $\begin{cases} x_1+3x_2+3x_3+2x_4=1---(1) \\ 3x_3+x_4=3---(2) \end{cases}$. Since only 2 equations are left with 4 variables, we can define 2 variables freely – let $x_2=s$ and $x_4=t$. Then,

$$x_4=t,\quad x_3=1-\frac{1}{3}t,\quad x_2=s,\quad x_1=-2-3s-t \Rightarrow \underline{\underline{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ -1/3 \\ 1 \end{pmatrix},\ s,t\in\mathbb{R}}}$$

If entries of last row are all zeros, it implies that the rest of the variables can be defined in terms of the last variable. The linear system has infinitely-many solutions.

a) $\begin{cases} x_1+3x_2+3x_3+2x_4=1 \\ 2x_1+6x_2+9x_3+5x_4=5 \\ -x_1-3x_2+3x_3=5 \end{cases}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 1 \\ 2 & 6 & 9 & 5 & 5 \\ -1 & -3 & 3 & 0 & 5 \end{array}\right] = \begin{array}{l} x_1+3x_2+x_4=-2 \\ x_3+\frac{1}{3}x_4=1 \end{array}$$

Solution $\begin{bmatrix} -3x_2-x_4-2 \\ x_2 \\ 1-\frac{1}{3}x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{bmatrix} x_4$

$\begin{cases} x+y+2z=4 \\ x-y-z=-1 \\ 2x-4y-5z=1 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & -1 & -1 & -1 \\ 2 & -4 & -5 & 1 \end{array}\right] =$$

(b) Consider the augmented matrix $\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & -1 & -1 & -1 \\ 2 & -4 & -5 & 1 \end{array}\right)$. Performing Gaussian elimination,

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 1 & -1 & -1 & -1 \\ 2 & -4 & -5 & 1 \end{array}\right) \xrightarrow{R_2 - R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -2 & -3 & -5 \\ 2 & -4 & -5 & 1 \end{array}\right) \xrightarrow{R_3 - 2R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -2 & -3 & -5 \\ 0 & -6 & -9 & -7 \end{array}\right)$$

$$\xrightarrow{R_3 - 3R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 0 & -2 & -3 & -5 \\ 0 & 0 & 0 & 8 \end{array}\right)$$

The last row implies that the system is inconsistent. The linear system has no solutions.

(c) Consider the augmented matrix $\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 2 & 2 & -5 & 3 \\ 2 & 5 & 1 & 9 \end{array}\right)$. Performing Gaussian elimination,

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 2 & 2 & -5 & 3 \\ 2 & 5 & 1 & 9 \end{array}\right) \xrightarrow{R_2 - 2R_1 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 2 & 5 & 1 & 9 \end{array}\right) \xrightarrow{R_3 - 2R_1 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 7 & -3 & -3 \end{array}\right)$$

$$\xrightarrow{R_3 - \frac{7}{4}R_2 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & 51/4 & 51/4 \end{array}\right)$$

<u>Option 1</u>: Back-substitution

$$\frac{51}{4}w = \frac{51}{4} \Rightarrow \underline{\underline{w = 1}}$$

$$4v - 9w = -9 \Rightarrow \underline{\underline{v = 0}}$$

$$u - v + 2w = 6 \Rightarrow \underline{\underline{u = 4}}$$

<u>Option 2</u>: Gauss-Jordan elimination

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 4 & -9 & -9 \\ 0 & 0 & 51/4 & 51/4 \end{array}\right) \xrightarrow[\frac{1}{4}R_2 \to R_2]{\frac{4}{15}R_3 \to R_3} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 1 & -9/4 & -9/4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 + \frac{9}{4}R_3 \to R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 - 2R_3 \to R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

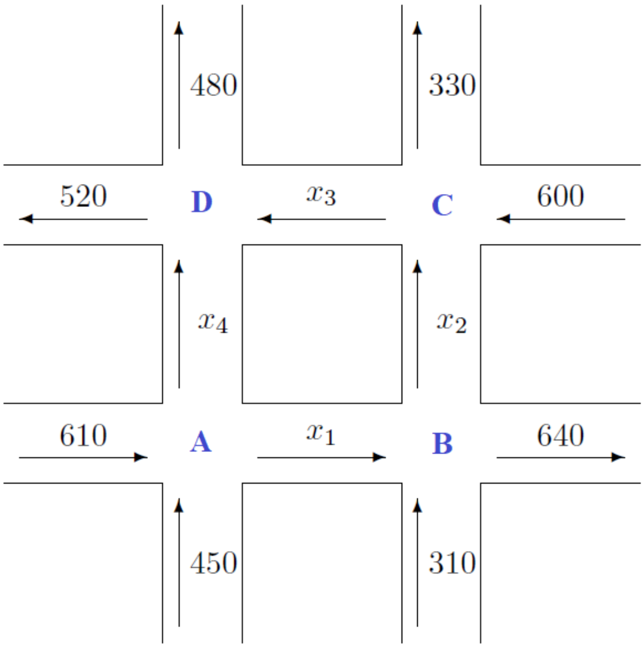
$$\xrightarrow{R_1 + R_2 \to R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Thus, $\underline{\underline{u = 4,\ v = 0,\ w = 1.}}$

The linear system has a unique solution.

Question 2

In the downtown section of a certain city, two sets of one-way streets intersect as shown below. The average hourly volume of traffic entering and leaving this section during rush hour is also given in the diagram.



Note that the average hourly volume of traffic entering an intersection must be equal to the volume of traffic leaving.

(a) Set up a linear system with four equations to find the traffic volume x_1, x_2, x_3 and x_4.

(b) Do we have enough information to find the traffic volumes x_1, x_2, x_3 and x_4?

(c) Given that $x_4 = 500$, find x_1, x_2, x_3.

(d) What is the range for the possible values of x_4?

Solutions

(a) Perform a balance for all the intersection points A, B, C and D.

A: $610+450 = x_1 + x_4 \quad \Rightarrow x_1 + x_4 = 1060 \text{ --- } (1)$

B: $x_1 + 310 = x_2 + 640 \quad \Rightarrow x_1 - x_2 = 330 \text{ --- } (2)$

C: $x_2 + 600 = x_3 + 330 \quad \Rightarrow x_2 - x_3 = -270 \text{ --- } (3)$

D: $x_3 + x_4 = 520 + 480 \quad \Rightarrow x_3 + x_4 = 1000 \text{ --- } (4)$

We can set up the augmented matrix $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 1 & -1 & 0 & 0 & 330 \\ 0 & 1 & -1 & 0 & -270 \\ 0 & 0 & 1 & 1 & 1000 \end{array}\right)$.

Performing Gaussian elimination,

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 1 & -1 & 0 & 0 & 330 \\ 0 & 1 & -1 & 0 & -270 \\ 0 & 0 & 1 & 1 & 1000 \end{array}\right) \xrightarrow{R_2-R_1\to R_2} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 0 & -1 & 0 & -1 & -730 \\ 0 & 1 & -1 & 0 & -270 \\ 0 & 0 & 1 & 1 & 1000 \end{array}\right) \xrightarrow{R_3+R_2\to R_3}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 0 & -1 & 0 & -1 & -730 \\ 0 & 0 & -1 & -1 & -1000 \\ 0 & 0 & 1 & 1 & 1000 \end{array}\right) \xrightarrow{R_4+R_3\to R_4} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1060 \\ 0 & -1 & 0 & -1 & -730 \\ 0 & 0 & -1 & -1 & -1000 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Since the last equation "disappeared", it implies that the first three variables can be defined in terms of the last variable x_4. Let $x_4 = t$. Then,

$$\begin{cases} x_4 = t \\ x_3 = 1000 - t \\ x_2 = 730 - t \\ x_1 = 1060 - t \end{cases}$$

Since t can take any real values, the linear system has infinitely many solutions.

(b) From (a), we deduce that there are infinitely many solutions. As such, we are unable to determine a unique solution for x_1, x_2, x_3 and x_4, unless one of the variables x_1, x_2, x_3 and x_4 is specified.

(c) When $x_4 = t = 500$, we have

$$x_1 = \underline{\underline{560}},\ x_2 = \underline{\underline{230}},\ x_3 = \underline{\underline{500}}.$$

(d) Since traffic volumes cannot be negative, we need to impose the condition $x_1, x_2, x_3, x_4 \geq 0$. Thus,

$$\begin{array}{ll} x_1 \geq 0 \Rightarrow t \leq 1060 & \quad x_3 \geq 0 \Rightarrow t \leq 1000 \\ x_2 \geq 0 \Rightarrow t \leq 730 & \quad x_4 \geq 0 \Rightarrow t \geq 0 \end{array}$$

Taking the intersection of all 4 inequalities, we have the range $\underline{\underline{0 \leq t \leq 730}}$.

Question 3

When propane gas burns, the propane combines with oxygen to form carbon dioxide and water.

$$wC_3H_8 + xO_2 \rightarrow yCO_2 + zH_2O,$$

where w and x are the number of propane and oxygen molecules required for combustion respectively; and y and z are the number of carbon dioxide and water molecules produced respectively.

(a) By equating the number of carbon, hydrogen and oxygen atoms on both sides of the chemical equation respectively, write down a homogeneous system of three equations in terms of w, x, y and z.

(b) Find a general solution for the homogeneous system obtained in (a).

(c) Find the (non-trivial) solution of w, x, y and z with the smallest values.

Solutions

(a) Equating the number of each elements,

$$\begin{array}{lll} \text{C:} & 3w = y & \Rightarrow 3w - y = 0 \\ \text{H:} & 8w = 2z & \Rightarrow 4w - z = 0 \\ \text{O:} & 2x = 2y + z & \Rightarrow 2x - 2y - z = 0 \end{array}$$

Is the linear system homogeneous? How can you tell?

(b) The augmented matrix $\left(\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right)$ is obtained. Performing Gaussian elimination,

$$\left(\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 4 & 0 & 0 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right) \xrightarrow{R_2 - \frac{4}{3}R_1 \to R_2} \left(\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \\ 0 & 2 & -2 & -1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 & 0 \end{array}\right)$$

Since the last equation "disappeared", it implies that the first three variables can be defined in terms of the last variable z. Let $z = t$. Then,

$$\begin{cases} z = t \\ y = \dfrac{3}{4}t \\ x = \dfrac{5}{4}t \\ w = \dfrac{1}{4}t \end{cases}$$

Make sure you know how to formulate this answer!

There are infinitely many solutions.

(c) We note that w, x, y and z must be <u>positive integers</u>. While t can take any real values, we choose $t = 4$ in order to fulfil our condition and obtain the smallest possible solution. Thus,

$$\underline{\underline{w = 1,\ x = 5,\ y = 3,\ z = 4.}}$$

Question 4

Given a quadric surface in the *xyz*-space with equation $ax^2+by^2+cz^2=d$, where a, b, c and d are real constants, that passes through the points $(1, 1, -1)$, $(1, 3, 3)$ and $(-2, 0, 2)$, find a formula for the surface.

Solutions

Substituting the following points into the equation, we have

$$(1, 1, -1): \qquad a+b+c-d=0$$

$$(1, 3, 3): \qquad a+9b+9c-d=0$$

$$(-2, 0, 2): \qquad 4a+4c-d=0$$

We thus obtain the augmented matrix $\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right)$. Performing Gaussian elimination,

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right) \xrightarrow{R_2-R_1\to R_2} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right) \xrightarrow{R_3-4R_1\to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 0 & -4 & 0 & 3 & 0 \end{array}\right)$$

$$\xrightarrow{R_3+\frac{1}{2}R_2\to R_3} \left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 0 & 8 & 8 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right)$$

Since the last equation "disappeared", it implies that the first three variables can be defined in terms of the last variable z. Let $d=t$. Then,

$$\begin{cases} d=t \\ c=-\dfrac{3}{4}t \\ b=\dfrac{3}{4}t \\ a=t \end{cases}$$

Since t can take any real values, the linear system has infinitely many solutions. An example could be to take $t=4$, giving the formula

$$\underline{\underline{4x^2+3y^2-3z^2=4.}}$$

Question 5

Let $\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{E}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, $\mathbf{E}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$.

(a) Pre-multiply each of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ to $\mathbf{A}$.

(b) Each of the multiplications in (a) has the same effect on $\mathbf{A}$ as performing an elementary row operation. Identify the elementary row operation that corresponds to each of $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$.

Solutions

(a) $\mathbf{E}_1\mathbf{A} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, $\mathbf{E}_2\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 3 & 3 & 3 & 3 \end{pmatrix}$, $\mathbf{E}_3\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 \end{pmatrix}$.

(b) $\mathbf{E}_1$ corresponds to interchanging rows 1 and 2.

$\mathbf{E}_2$ corresponds to multiplying row 3 by constant 3.

$\mathbf{E}_3$ corresponds to adding –2 times of row 1 to row 3.

Question 6

Consider the population of certain endangered species of wild animals: On average, each adult gives birth to one baby each year; 50% of the new-born babies will survive the first year; 60% of one-year old cubs will survive the second year and become adults; and 70% of the adults will survive each year.

Let x_0, y_0 and z_0 be the number of babies, one-year old cubs and adults at the end of 2017 respectively, and let x_1, y_1 and z_1 be the respective numbers at the end of 2018.

(a) Based on the information given above, find a 3×3 matrix $\mathbf{A}$ such that $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$.

(b) Let $\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \mathbf{A}^n \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, where n is a positive integer. What do the numbers x_n, y_n and z_n represent?

(c) Suppose initially, $x_0 = 0$, $y_0 = 0$ and $z_0 = 100$. What is the total population three years later?

Solutions

(a) Based on the information above, we can formulate the linear system:

$$\begin{cases} x_1 = z_0 \\ y_1 = 0.5x_0 \\ z_1 = 0.6y_0 + 0.7z_0 \end{cases} \Rightarrow \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0.7 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \Rightarrow \mathbf{A} = \underline{\underline{\begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0.7 \end{pmatrix}}}.$$

(b) x_n, y_n and z_n represent the number of babies, one-year old cubs and adults at the end of year (2017 + n) respectively.

(c) To find the population 3 years later, we use $n = 3$:

$$\begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \mathbf{A}^3 \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0.7 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0.7 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0.5 & 0 & 0 \\ 0 & 0.6 & 0.7 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} = \begin{pmatrix} 49 \\ 35 \\ 64.3 \end{pmatrix}$$

Thus, the total population is about $49 + 35 + 64.3 \approx \underline{\underline{148}}$.

Question 7

A manufacturer makes three types of chairs A, B and C. The company has 260 units of wood, 60 units of upholstery and 240 units of labour available. The manufacturer wants a production schedule that uses all of these resources. The various products require the following amount of resources.

	A	**B**	**C**
Wood	4	4	3
Upholstery	0	1	2
Labour	2	4	5

(a) Find the inverse of the data matrix above and hence determine how many pieces of each product should be manufactured.

(b) If the amount of wood is increased by 10 units, how will this change the number of type C chairs produced?

Solutions

(a) Let x, y and z be the number of chairs A, B and C respectively. We can set up the linear system by balancing each of the resources:

$$\begin{cases} \text{Wood}: 4x + 4y + 3z = 260 \\ \text{Upholstery}: \quad y + 2z = 60 \\ \text{Labour}: 2x + 4y + 5z = 240 \end{cases} \Rightarrow \begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}$$

Performing Gauss-Jordan Elimination on the augmented matrix:

$$\left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 4 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - \frac{1}{2}R_1 \to R_3} \left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 3.5 & -0.5 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - 2R_2 \to R_3} \left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -0.5 & -0.5 & -2 & 1 \end{array}\right) \xrightarrow{\frac{1}{-0.5}R_3 \to R_3}$$

$$\left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right) \xrightarrow{R_2 - 2R_3 \to R_2} \left(\begin{array}{ccc|ccc} 4 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -7 & 4 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right) \xrightarrow{R_1 - 3R_3 \to R_1} \left(\begin{array}{ccc|ccc} 4 & 4 & 0 & -2 & -12 & 6 \\ 0 & 1 & 0 & -2 & -7 & 4 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right) \xrightarrow{R_1 - 4R_2 \to R_1}$$

$$\left(\begin{array}{ccc|ccc} 4 & 0 & 0 & 6 & 16 & -10 \\ 0 & 1 & 0 & -2 & -7 & 4 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right) \xrightarrow{\frac{1}{4}R_1 \to R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1.5 & 4 & -2.5 \\ 0 & 1 & 0 & -2 & -7 & 4 \\ 0 & 0 & 1 & 1 & 4 & -2 \end{array}\right)$$

Thus, the inverse of the data matrix is $\begin{pmatrix} 1.5 & 4 & -2.5 \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}$. Thus, pre-multiplying the inverse matrix on both sides of the linear system, we get

$$\mathbf{A}^{-1}\mathbf{A}x = \mathbf{A}^{-1}b$$

$$\begin{pmatrix} 1.5 & 4 & -2.5 \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 2 & 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1.5 & 4 & -2.5 \\ -2 & -7 & 4 \\ 1 & 4 & -2 \end{pmatrix}\begin{pmatrix} 260 \\ 60 \\ 240 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 30 \\ 20 \\ 20 \end{pmatrix}$$

The manufacturer should make <u>30</u> chairs of type A, <u>20</u> chairs of type B and <u>20</u> chairs of type C.

(b) If the total amount of wood becomes 270 instead of 260, the new z will be (1)(270) + (4)(60) – 2(240) = 30. Thus, <u>10 more</u> chairs of type C will be made.

Note that the inverse of the matrix remains the same.

Question 8

Consider the following matrices:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -1 & 3 & -4 \\ 2 & 4 & 1 \\ -4 & 2 & -9 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

(a) Determine whether each matrix above is non-singular. For those that are non-singular, find their inverses.

(b) Find the inverses of $\mathbf{A}^T$, $\mathbf{B}^T$ and $\mathbf{C}^T$ if they exist.

(c) Which of the homogeneous systems $\mathbf{A}\vec{x} = \vec{0}$, $\mathbf{B}\vec{x} = \vec{0}$ and $\mathbf{C}\vec{x} = \vec{0}$ have non-trivial solutions? Find all non-trivial solutions for these systems.

(d) Without performing Gaussian elimination, can you tell whether the homogeneous systems $\mathbf{AB}\vec{x} = \vec{0}$ and $\mathbf{AC}\vec{x} = \vec{0}$ have non-trivial solutions? Why?

Solutions

(a) Performing Gauss-Jordan (or Gaussian) elimination on augmented matrix to find inverse, we have

$$\mathbf{A}: \left(\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{GJ Elimination}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.5 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0.5 & -0.5 & 0.5 \\ 0 & 0 & 1 & 0.5 & 0.5 & -0.5 \end{array}\right)$$

Thus, $\mathbf{A}$ is <u>non-singular</u> and $\mathbf{A}^{-1} = \underline{\underline{\begin{pmatrix} -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{pmatrix}}}$.

$$\mathbf{B}:\left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Gaussian Elimination}} \left(\begin{array}{ccc|ccc} -1 & 3 & -4 & 1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{array}\right)$$

Based on the last line, the inverse cannot exist. Hence **B** is <u>singular</u>.

$$\mathbf{C}:\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{GJ Elimination}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/6 & -1/3 \\ 0 & 1 & 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 & 2/3 \end{array}\right)$$

Thus, **C** is <u>non-singular</u> and $\mathbf{C}^{-1} = \underline{\underline{\begin{pmatrix} 1/2 & -1/6 & -1/3 \\ 0 & 2/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{pmatrix}}}$.

(b) We will make use of the following result:

For non-singular matrix **A**, its transpose $\mathbf{A}^T$ is also non-singular, and $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T$.
For singular matrix **A**, its transpose $\mathbf{A}^T$ is also singular.

For **A**, $(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T = \underline{\underline{\begin{pmatrix} -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \end{pmatrix}}}$.

Notice that $(\mathbf{A}^T)^{-1} = \mathbf{A}^{-1}$. This is because **A** is symmetric.

Since **B** is singular, $\mathbf{B}^T$ will also be singular. Hence, its inverse does not exist.

For **C**, $(\mathbf{C}^T)^{-1} = (\mathbf{C}^{-1})^T = \begin{pmatrix} 1/2 & -1/6 & -1/3 \\ 0 & 2/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{pmatrix}^T = \underline{\underline{\begin{pmatrix} 1/2 & 0 & 0 \\ -1/6 & 2/3 & 1/3 \\ -1/3 & 1/3 & 2/3 \end{pmatrix}}}$.

(c) Since only **B** is singular, only the homogeneous system $\mathbf{B}\vec{x} = \vec{0}$ has non-trivial solutions.

The row echelon form of **B** can be obtained from (a): $\begin{pmatrix} -1 & 3 & -4 \\ 0 & 10 & -7 \\ 0 & 0 & 0 \end{pmatrix}$. Since the last row "disappeared", we can make the first two variables as a function of the last variable.

$$z = t, \quad y = \frac{7}{10}t, \quad x = -\frac{19}{10}t, \text{ where } t \text{ can be any real parameter.}$$

Hence, the general solution for $\mathbf{B}\vec{x} = \vec{0}$ is $\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{\underline{t\begin{pmatrix} -19 \\ 7 \\ 10 \end{pmatrix}}}$, $t \in \mathbb{R}$.

(d) We note that $\mathbf{B}\vec{x} = \vec{0}$ has non-trivial solutions. Pre-multiplying the equation by matrix **A**, we have $\mathbf{AB}\vec{x} = \vec{0}$, and it should be expected that non-trivial solutions are also obtained.

Since **A** and **C** are non-singular matrices, **AC** is also non-singular. Thus, $\mathbf{AC}\vec{x} = \vec{0}$ will only have trivial solutions.

GLOSSARY

An **augmented matrix** is a coefficient matrix adjoined with column(s) separated within the matrix brackets. For example,

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & -1 & 0 \\ 1 & 9 & 9 & -1 & 0 \\ 4 & 0 & 4 & -1 & 0 \end{array}\right) \text{ is an augmented matrix.}$$

Elementary row operations are either (1) adding one row to another; (2) multiplying a row by a constant; or (3) interchanging rows; with the goal of achieving (reduced) row-echelon form.

A system of linear equations is said to be **homogeneous** if constant terms in all equations are zero. For example,

$$\begin{cases} x - 2y + z = 0 \\ y - 3z = 0 \\ 3x + 2y - z = 0 \end{cases} \quad \text{is a homogeneous system.}$$

If a linear system has some non-zero constant terms, we say that the system is **non-homogeneous**.

An **inconsistent system/matrix** is a system/matrix that makes no mathematical sense. For example,

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 10 \end{array}\right) \Rightarrow \begin{cases} x - 2y + z = 0 \\ y - 3z = -1 \\ 0 = 10 \ (?!) \end{cases}$$

A **leading entry** in a row of a matrix is the entry that is on the main diagonal.

A **linear equation** in n variables is defined as $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$, where $x_1, x_2, \ldots, x_n$ are variables and $a_1, a_2, \ldots, a_n$ are coefficients.

The **main diagonal** entries are from the upper left corner diagonally to the lower right of a matrix. For example, the main diagonals are underlined in the example below:

$$\begin{pmatrix} \underline{\underline{1}} & 2 & 8 & 9 \\ 5 & \underline{\underline{3}} & 4 & 8 \\ 6 & 7 & \underline{\underline{0}} & 5 \end{pmatrix}$$

Pivot columns are columns that containing leading entries.

The **reduced row-echelon form** is the matrix form which contains either zero or one entries along the main diagonal and zero entries above and below the main diagonal after performing elementary row operations. For example, $\left(\begin{array}{ccc\|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$ is in its reduced row-echelon form.
The **row-echelon form** is the matrix form which contains zero entries below the main diagonal entries after performing elementary row operations. For example, $\begin{pmatrix} 4 & 4 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{pmatrix}$ is in its row-echelon form.
A **singular matrix** is a matrix which does not have an inverse.
The **trivial solution** is the zero solution (i.e. $x_1 = x_2 = ... = x_n = 0$). Any other solutions are known as the **non-trivial solutions**.