MH1810 Math 1 Part 2 Chapter 6 Integration

Numerical Integration

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Numerical Integration

For some functions, like $\sin\left(x^2\right)$ and e^{-x^2}, their integrals are hard to obtain via antiderivatives.

In order to determine a definite integral such as $\int_a^b \sin\left(x^2\right)\, dx$, we shall use numerical integration, which is a numerical approximation.

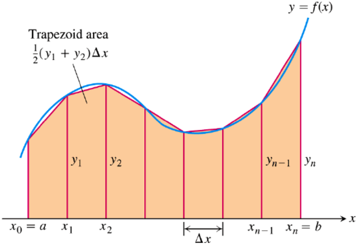
Numerical Integration

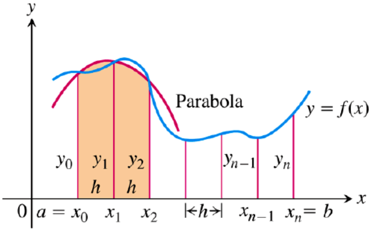
To approximate a definite integral $\int_a^b f(x)\,dx$,

- firstly we partition the interval $[a, b]$ of integration into finitely many small sub-intervals, $[x_i, x_{i+1}]$, of equal width.
- Subsequently, on subintervals, we approximate f with a simpler function, such as a linear function or a polynomial.
- The definite integral of the approximate function on the subintervals shall be used to approximate the integral of f.

This procedure is an example of numerical integration.

Numerical Integration





Approximation via linear functions *via quadratic functions*

In this section, we study two methods of numerical integration, namely the Trapezoidal Rule and Simpson's Rule.

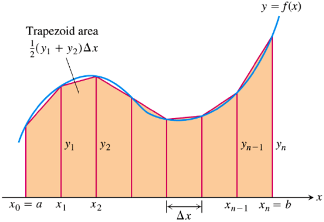
Trapezoidal Rule

As indicated by its name, Trapezoidal Rule uses trapeziums to approximate the definite integral $\int_a^b f(x)\,dx$ the area under the curve.

The graphs of approximate functions are thus straight lines. That is, we use linear functions to approximate f.

Trapezoidal Rule

To understand the idea of Trapezoidal Rule, we may assume $f(x) > 0$ on $[a, b]$. Thus, the definite integral $\int_a^b f(x)\,dx$ is the area under the curve $f(x)$ on $[a, b]$.



To find this area, we first subdivide $[a, b]$ into n subintervals of equal length Δx, where $\Delta x = \frac{b-a}{n}$.

Trapezoidal Rule

The area of the trapezium above the k^{th} interval $[x_{k-1}, x_k]$ is

$$A_k = \frac{\Delta x}{2}\left(y_{k-1} + y_k\right), k = 1, 2, \ldots, n,$$

where $y_{k-1} = f\left(x_{k-1}\right)$ and $y_k = f\left(x_k\right)$.

Trapezoidal Rule

Then the total T_n of areas of n trapezium is

$$T_n = \sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{\Delta x}{2} \left(y_{k-1} + y_k \right)$$

Trapezoidal Rule

$$T_n = \frac{\Delta x}{2}\left(y_0 + y_1\right) + \frac{\Delta x}{2}\left(y_1 + y_2\right)$$

$$+\frac{\Delta x}{2}\left(y_2 + y_3\right) + \cdots + \frac{\Delta x}{2}\left(y_{n-2} + y_{n-1}\right) + \frac{\Delta x}{2}\left(y_{n-1} + y_n\right)$$

$$= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The subscript n in T_n indicates the number of partitions involved.

Trapezoidal Rule

Theorem (The Trapezoidal Rule)

The Trapezoidal Rule approximates the definite integral $\int_a^b f(x)\, dx$ *as follows:*

$$\int_a^b f(x)\, dx \approx T_n = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

where $\Delta x = \dfrac{b-a}{n}$, $x_k = a + k\Delta x$ *and* $y_k = f(x_k)$ *for* $k = 0, 1, 2, \ldots, n$.
Note that $x_0 = a$ *and* $x_n = b$.

Example

Example

Use the Trapezoidal Rule to approximate $\int_1^2 x^2\,dx$ by T_4.

Solution

Partition $[1,2]$ into 4 subintervals of equal length $\Delta x = \frac{(2-1)}{4} = \frac{1}{4}$.
We tabulate the values involved.

k	x_k	$y_k = f(x_k) = x_k^2$
0	1	1
1	$5/4$	$25/16$
2	$6/4 = 3/2$	$36/16$
3	$7/4$	$49/16$
4	$8/4 = 2$	4

Solution

Solution

We have

$$\int_1^2 x^2\,dx \approx T_4 = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right)$$

$$= \frac{\frac{1}{4}}{2}\left(1 + \frac{25}{8} + \frac{36}{8} + \frac{49}{8} + 4\right) = \frac{75}{32} = 2.34375$$

Note The exact value of the definite integral $\int_1^2 x^2\,dx$ is $\frac{7}{3}$.
The relative error is thus

$$\left|\frac{T_4 - I}{I}\right| = \frac{\frac{75}{32} - \frac{7}{3}}{\frac{7}{3}} \approx 0.0044643 \text{ or } 0.446\%.$$

Example

Example

Estimate $\int_1^2 \sin\left(\pi x^2\right)\, dx$ by Trapezoidal Rule with $n = 4$.

Solution

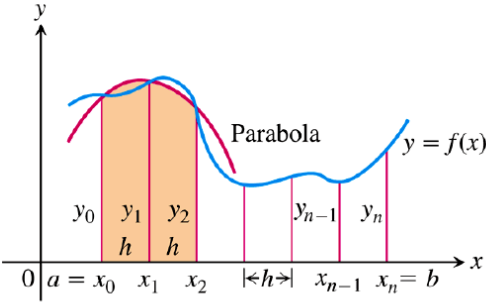
With $n = 4$, the width of each subinterval is $\Delta x = \frac{(2-1)}{4} = \frac{1}{4}$.

i	x_i	$y_i = f(x_i) = \sin\left(\pi x_i^2\right)$
0	1	0
1	$5/4$	$\sin(25\pi/16) \approx -0.980785280$
2	$6/4 = 3/2$	$\sin(36\pi/16) \approx 0.707106781$
3	$7/4$	$\sin(49\pi/16) \approx -0.1950903220$
4	$8/4 = 2$	$\sin(4\pi) = 0$

Thus we have $\int_1^2 \sin\left(\pi x^2\right)\, dx \approx T_4 \approx -0.117192205254$.

Simpson's Rule

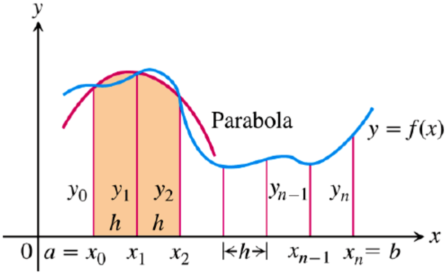
Another numerical method for approximating a definite integral is the Simpson's Rule. It uses parabolas instead of straight line segments.



As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = \dfrac{b-a}{n}$, but n must be an even number.

Simpson's Rule

On each consecutive pair of subintervals, we approximate the curve $y = f(x)$ by a parabola $y = Q(x)$ which passes through three consecutive points (x_{k-1}, y_{k-1}), (x_k, y_k) and (x_{k+1}, y_{k+1}). Here, we note that $x_{k-1} = x_k - h$ and $x_{k+1} = x_k + h$.



The first consecutive pair of subintervals is $[x_0, x_1]$ and $[x_1, x_2]$, where $x_0 = x_1 - h$ and $x_2 = x_1 + h$.

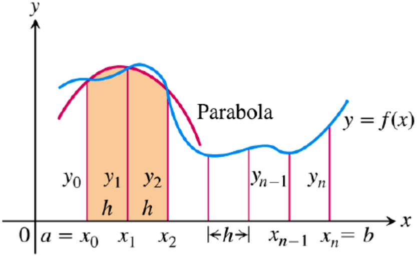
Simpson's Rule

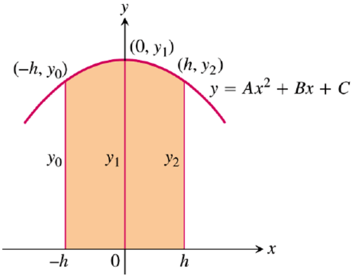
Our aim is to approximate $\int_{x_0}^{x_2} f(x)\,dx$ by $\int_{x_0}^{x_2} Q(x)\,dx$. If $f(x) > 0$ on $[x_0, x_2]$, then we are approximating $\int_{x_0}^{x_2} f(x)\,dx$ by the area under the parabola.

Key Ideas of Simpson's Rule

(i) By shifting horizontally the curve $y = Q(x)$ to the interval $[-h, h]$, we note that the area under the parabola remains the same.

The area under the parabola $y = Q(x)$ over $[x_1 - h, x_1 + h]$, where $Q(x_1 - h) = y_0$, $Q(x_1) = y_1$ and $Q(x_1 + h) = y_2$, is the same as the area under the parabola $y = Q^*(x) = Q(x - x_1)$ on $[-h, h]$, where $Q^*(-h) = y_0$, $Q^*(0) = y_1$ and $Q^*(h) = y_2$.





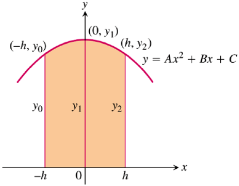
Key Ideas of Simpson's Rule

(ii) Suppose $Q^*(x) = Ax^2 + Bx + C$ on $[-h, h]$ where $Q^*(-h) = y_0$, $Q^*(0) = y_1$ and $Q^*(h) = y_2$. Then . The area under the parabola $y = Q^*(x)$ is

$$\text{Area } = \int_{-h}^{h} \left(Ax^2 + Bx + C\right)\, dx = \frac{h}{3}\left(2Ah^2 + 6C\right).$$

We shall express the above Area in terms of y_0, y_1 and y_2:

$$\begin{aligned} y_0 &= A(-h)^2 + B(-h) + C = Ah^2 - Bh + C, \\ y_1 &= A(0)^2 + B(0) + C = C, \\ y_2 &= A(h)^2 + B(h) + C = Ah^2 + Bh + C. \end{aligned}$$



Key Ideas of Simpson's Rule

Thus, we have $2Ah^2 = y_0 + y_2 - 2y_1$ and $6C = 6y_1$ so that

$$\text{Area } = \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

Therefore, we approximate $\displaystyle\int_{x_0}^{x_2} f(x)\,dx$ by

$$\int_{x_0}^{x_2} f(x)\,dx \approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$

Key Ideas of Simpson's Rule

(iii) Computing the areas under all such parabolas and adding them gives an approximation of $\int_a^b f(x)\,dx$.

$$\int_a^b f(x)\,dx \approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots$$

$$\frac{h}{3}(y_{n-4} + 4y_{n-3} + y_{n-2}) + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

$$= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n)$$

The above gives rise to Simpson's Rule.

Simpson's Rule

Partitioning $[a, b]$ into even number of subintervals, say n subintervals:

$$\int_a^b f(x)\,dx \approx S_n$$

$$= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)$$

Note: To use Simpson's Rule, after we select an even number n, we have $x_k = a + k\Delta x$, where $k = 0, 1, 2, \ldots, n$, $\Delta x = \frac{b-a}{n}$, and we evaluate each $y_k = f(x_k)$.

Simpson's Rule

- Pattern of the coefficients: $1, 4, 2, 4, 2, 4, 2, \ldots, 4, 2, 4, 1$.
- Coefficients of y_0 and y_n are 1. (the first and last y's)
- Coefficients y_k when k is odd is 4.
- Coefficients y_k when k is even where $k \neq 0$ and $k \neq n$, is 2.

Example

Example

Use Simpson's Rule, with $n = 4$, to approximate $\int_1^2 \sin\left(\pi x^2\right)\, dx$.

Solution

With $n = 4$, we have $h = \frac{1}{4}$:

k	x_k	$y_k = f(x_k) = \sin(\pi x_k^2)$
0	1	$y_0 = 0$
1	$5/4$	$y_1 = \sin(25\pi/16) \approx -0.980785280$
2	$6/4 = 3/2$	$y_2 = \sin(36\pi/16) \approx 0.707106781$
3	$7/4$	$y_3 = \sin(49\pi/16) \approx -0.1950903220$
4	$8/4 = 2$	$y_4 = \sin(4\pi) = 0$

Solution

Solution

By Simpson's Rule, we have

$$\int_1^2 \sin\left(\pi x^2\right)\, dx \approx S_4 = \frac{1/4}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right)$$

$$\approx \frac{1}{12}\left(0 + 4y_1 + 2y_2 + 4y_3 + 0\right) \approx -0.27410740383.$$

Example

Example

Use Simpson's Rule, with $n = 4$ to approximate $\int_{-1}^{1} (x^2 + 1)\, dx$.

Solution

With $n = 4$, *we have* $h = \frac{2}{4} = \frac{1}{2}$, *and*

k	x_k	$f(x_k) = x_k^2 + 1$
0	-1	2
1	-0.5	1.25
2	0	1
3	0.5	1.25
4	1	2

Solution

By Simpson's Rule, we have

$$\int_{-1}^{1} (x^2+1)\,dx \approx S_4$$

$$= \frac{1}{6}\left(1(2)+4(1.25)+2(1)+4(1.25)+1(2)\right) = \frac{16}{6} = \frac{8}{3}.$$

Solution

Now, integrating directly, we have
$\int_{-1}^{1} (x^2 + 1)\, dx = \left(\frac{x^3}{3} + x \right) \Big|_{-1}^{1} = \frac{8}{3}$ which coincides with S_4.

This is not surprising as $f(x) = x^2 + 1$, is a quadratic function. Hence the approximating function $Q(x)$ is $f(x)$.