

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER II EXAMINATION 2016–2017

MH1812 – Discrete Mathematics

April 2017

TIME ALLOWED: 2 HOURS

---

INSTRUCTIONS TO CANDIDATES

1. This examination paper contains **FIVE (5)** questions and comprises **THREE (3)** printed pages.
2. Answer **ALL** questions. The marks for each question are indicated at the beginning of each question.
3. Answer each question beginning on a **FRESH** page of the answer book.
4. This **IS NOT** an **OPEN BOOK** exam.
5. Calculators are allowed.
6. Candidates should clearly explain their reasoning used in each of their answers.

**QUESTION 1.****(15 marks)**

Decide whether or not the following argument is valid:

$$\begin{aligned}
 &p \wedge q; \\
 &r \rightarrow s; \\
 &\neg r \rightarrow q; \\
 &p \vee r; \\
 &\therefore (p \vee q) \wedge r
 \end{aligned}$$

Briefly justify your answer.

**Solution:** The argument is invalid. Indeed, set  $p = T, q = T, r = F$ . Then the assumptions are all true and the conclusion is false.

**QUESTION 2.****(15 marks)**

Consider three sets  $S$ ,  $T$ , and  $U$  where  $S$  is defined to be the set of all even integers,  $T = \{n \in \mathbb{Z} : 3 \mid n\}$ , and  $U = \{n \in \mathbb{Z} : n \equiv 0 \pmod{6}\}$ .

(a) Prove the set equality  $S \cap T = U$ . **(8 marks)**

(b) Determine the truth value of the following proposition **(7 marks)**

$$\neg (\forall x \in U, \exists y \in T, x \cdot y \notin S),$$

where  $\cdot$  denotes multiplication. Justify your answer.

**Solution:**

(a)  $S \cap T \subseteq U$ : Let  $x \in S \cap T$ . Then  $x$  is both a multiple of 2 and 3. Hence  $x$  is a multiple of 6 and therefore  $x \in U$ .

$U \subseteq S \cap T$ : Let  $x \in U$ . Then  $x = 6k$  for some  $k \in \mathbb{Z}$ . Hence  $x = 2(3k)$  and  $x = 3(2k)$ . Therefore  $x$  is both a multiple of 2 and 3 as required.

(b) First note that

$$\neg (\forall x \in U, \exists y \in T, x \cdot y \notin S) \equiv \exists x \in U, \forall y \in T, x \cdot y \in S$$

This statement is true. Indeed,  $6 \in U$  and  $6 \cdot y$  is even for all  $y \in T$ .

**QUESTION 3.****(30 marks)**

- (a) Using the characteristic equation, solve the recurrence relation
- (10 marks)**

$$a_0 = 2, a_1 = 3, \quad a_n = 7a_{n-1} - 12a_{n-2}.$$

- (b) Consider the recurrence relation given by the initial conditions
- $D_0 = 1, D_1 = 0$
- , and
- $D_n = (n-1)(D_{n-1} + D_{n-2})$
- for all
- $n \geq 2$
- . Prove the equality

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

(Hint: use induction.)

**(20 marks)****Solution:**

- (a) We have the characteristic equation

$$x^2 - 7x + 12 = (x-3)(x-4).$$

Hence  $a_n = u3^n + v4^n$ . Since  $a_0 = u + v = 2$  and  $a_1 = 3u + 4v = 3$ . Therefore  $u = 5$  and  $v = -3$ .

- (b) Basis cases are OK. Let
- $P(n)$
- be our inductive hypothesis
- $D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$
- . Suppose that
- $P(k)$
- is true for all
- $k \leq n$
- . Consider
- $P(n+1)$
- . We have

$$\begin{aligned} D_{n+1} &= n(D_n + D_{n-1}) \\ &= n \left( n! \sum_{k=0}^n \frac{(-1)^k}{k!} + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) \\ &= n \left( n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + (-1)^n + (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) \\ &= n(-1)^n + n \left( (n+1)(n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \right) \\ &= n(-1)^n + (n+1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}. \end{aligned}$$

It suffices to show that  $n(-1)^n = (n+1)! \left( \frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!} \right)$ . This follows since

$$(n+1)! \left( \frac{(-1)^n}{n!} + \frac{(-1)^{n+1}}{(n+1)!} \right) = (n+1)(-1)^n + (-1)^{n+1}$$

and  $(-1)^n + (-1)^{n+1} = 0$  for all  $n \in \mathbb{Z}$ .

**QUESTION 4.****(25 marks)**

- (a) Consider the relation  $R$  on the set of integers  $\mathbb{Z}$  given by

$$aRb \iff b \equiv a^3 - a \pmod{3}.$$

- (i) Is  $R$  reflexive? **(5 marks)**
- (ii) Is  $R$  symmetric? **(5 marks)**
- (iii) Is  $R$  transitive? **(5 marks)**

Justify your answers.

- (b) Let  $S = \{1, 2, \dots, n\}$ . Determine

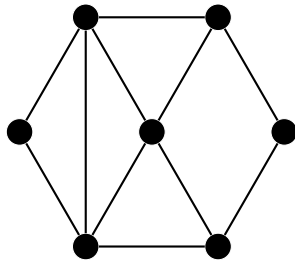
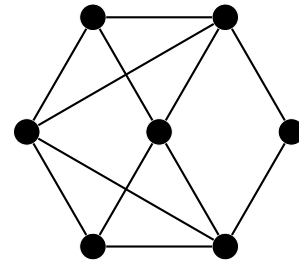
- (i) the cardinality of the set  $T$  of all functions  $f : S \rightarrow S$ ? **(5 marks)**
- (ii) the cardinality of the set  $U = \{f \in T \mid f \text{ is invertible}\}$ ? **(5 marks)**

**Solution:**

- (a)
  - (i) No.  $(1, 1) \notin R$ .
  - (ii) No.  $(1, 0) \in R$  but  $(0, 1) \notin R$ .
  - (iii) Yes. Suppose  $(x, y) \in R$  and  $(y, z) \in R$ . Then  $z$  is a multiple of 3. Now  $(x, w) \in R$  for all  $w$  that is divisible by 3. So  $(x, z)$  is also in  $R$ .
- (b)
  - (i) Each function  $f : S \rightarrow S$  must map each element of  $S$  to precisely one element in  $S$ . So each element can be mapped to  $n$  elements. Thus the cardinality is  $n^n$ .
  - (ii) Each invertible function  $f : S \rightarrow S$  must map each element of  $S$  to a unique element in  $S$ . So the first element can be mapped to  $n$  elements; the second to  $(n - 1)$  elements, and so on. Thus the cardinality is  $n!$ .

**QUESTION 5.**

Consider the two graphs in Figure 1.

 $X$  $Y$ Figure 1: The graphs  $X$  and  $Y$ .

- (a) For each of the graphs  $X$  and  $Y$
- (i) determine whether or not it has an Euler path; **(4 marks)**
  - (ii) determine whether or not it has an Euler circuit; **(4 marks)**
  - (iii) determine whether or not it has an Hamilton circuit. **(4 marks)**
- (b) Are the graphs  $X$  and  $Y$  are isomorphic? Justify your answer. **(3 marks)**

**Solution:**

- (a) (i) Both  $X$  and  $Y$  have precisely two vertices of odd degree. Hence both have Euler paths.
- (ii) Both  $X$  and  $Y$  have precisely two vertices of odd degree. Hence both do not have Euler circuits.
- (iii) Both  $X$  and  $Y$  have Hamilton circuits.
- (b) The graphs are not isomorphic. They have different degree sequences.

**END OF PAPER**