Goh Soong Wen Ryan

## 2021/2022 Semester 2

## SC1004 Take Home Assignment 2 (50 marks)

**Please write out all workings in detail, writing down all the necessary steps.**

**Partial working will get little marks.**

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**Question 1: (15 marks)** Let

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(a) Show step by step: the characteristic polynomial of $A$ is 4-9λ+….(5M)
(b) Find the eigenvalues of $A$. (6M).
(c) Hence find a basis for each eigenspace of $A$. (4M)

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**Question 2a: (14 marks)**

Diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix},$$

and use the result to find $\mathbf{A}^5$.

You are given its eigenvalues are 2,1,-1 and their corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(Hint: Reduce (P|I) to RREF to compute 3x3 inverse. Check online.

**Question 2b: (4 marks)** Find the eigenvector(s) of the Reflection transformation matrix about the **line y = x** in the 2 dimensional real x-y plane.

**Question 3: (10 marks)**

Two parameters, ***x*1** and ***x*2**, are linearly related. Three samples are taken that lead to this system of equations

$$\begin{aligned} 2x_1 + x_2 &= 0 \\ x_1 + x_2 &= 0 \\ 2x_1 + x_2 &= 2. \end{aligned}$$

Find the least squares solution for this system $\boldsymbol{Ax} = \boldsymbol{b}$ by solving

(i) normal equation (5 marks)

(ii) approximate vector solution. (5 marks)

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**Question 4: (4 marks)**

Show that if $A$ is a $2 \times 2$ matrix such that $A^2 = I_2$, and if $\mathbf{x}$ is any vector in $\mathbb{R}^2$, then $\mathbf{y} = \mathbf{x} + A\mathbf{x}$ and $\mathbf{z} = \mathbf{x} - A\mathbf{x}$ are eigenvectors of $A$. Find their corresponding eigenvalues.

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**Question 5: (3 marks)**

Find a 4x4 matrix ***A*** (NON-ZERO ENTRIES ONLY) with **3 eigenvectors coming from a single eigenvalue** of your choice.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

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(a) Show step by step: the characteristic polynomial of $\boldsymbol{A}$ is 4-9λ+….(5M)
(b) Find the eigenvalues of $\boldsymbol{A}$. (6M).
(c) Hence find a basis for each eigenspace of $\boldsymbol{A}$. (4M)

a) $\det(A - \lambda I) = 0$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}\right) = -\lambda^3 + 6\lambda^2 - 9\lambda + 4 = 0 //$$

b) $-\lambda^3 + 6\lambda^2 - 9\lambda + 4 = (-1)(\lambda - 4)(\lambda - 1)^2$

$\therefore \lambda = 4, \quad \lambda = 1 //$

c) for $\lambda = 4$, $\begin{bmatrix} 2-4 & 1 & 1 \\ 1 & 2-4 & 1 \\ 1 & 1 & 2-4 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 - x_3 = 0, \quad x_3 = x_1$
$x_2 - x_3 = 0 \quad x_3 = x_2$

$\therefore V_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

for $\lambda = 1$, $\begin{bmatrix} 2-1 & 1 & 1 \\ 1 & 2-1 & 1 \\ 1 & 1 & 2-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 + x_3 = 0$

$\therefore V_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$V_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ Eigenspace $= \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} //$

**Question 2a: (14 marks)**

Diagonalize the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix},$$

and use the result to find $\mathbf{A}^5$.

You are given its eigenvalues are 2,1,-1 and their corresponding eigenvectors are

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

(Hint: Reduce (P|I) to RREF to compute 3x3 inverse. Check online.

$$P, \text{ Eigenspace} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$P|I = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$A = PDP^{-1}$$
$$AP = PD$$
$$P^{-1}AP = P^{-1}PD$$
$$P^{-1}AP = D$$

$$\therefore P^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\therefore D = P^{-1}AP = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & -3 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -2 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^5 = (PDP^{-1})^5 = PD^5P^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2^5 & 0 & 0 \\ 0 & 1^5 & 0 \\ 0 & 0 & -1^5 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 32 & 31 & -31 \\ 33 & 32 & -33 \\ 33 & 31 & -32 \end{bmatrix} \quad //$$

**Question 2b: (4 marks)** Find the eigenvector(s) of the Reflection transformation matrix about the **line y = x** in the 2 dimensional real x-y plane.

$$\text{Matrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{Eigen value} = \det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix}$$

$$\lambda = 1, \quad \lambda = -1$$

For $\lambda = 1$, $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, eigenvector $= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ //

$\lambda = -1$, $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, eigenvector $= \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ //

**Question 3: (10 marks)**

Two parameters, ***x1*** and ***x2***, are linearly related. Three samples are taken that lead to this system of equations

$$2x_1 + x_2 = 0$$
$$x_1 + x_2 = 0$$
$$2x_1 + x_2 = 2.$$

Find the least squares solution for this system $\boldsymbol{Ax} = \boldsymbol{b}$ by solving

(i) normal equation (5 marks)

(ii) approximate vector solution. (5 marks)

i) normal : $A^T A \hat{x} = A^T b$

$$\hat{x} = (A^T A)^{-1} A^T b$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 5 \\ 5 & 3 \end{bmatrix}$$

$$[A^T A \mid I] = \left[\begin{array}{cc|cc} 9 & 5 & 1 & 0 \\ 5 & 3 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc|cc} 1 & 0 & 1.5 & -2.5 \\ 0 & 1 & -2.5 & 4.5 \end{array}\right]$$

$$A^T b = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\therefore \hat{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} 1.5 & -2.5 \\ -2.5 & 4.5 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} //$$

ii) Approx theorem : $\hat{x} = \text{Proj}_A b$

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$$V_1 = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

$$V2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{5/3}{1} \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 4/9 \\ -1/9 \end{bmatrix}$$

$$x = \frac{b \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{b \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= \frac{4/3}{1} V_1 + \frac{-2/9}{2/9} V_2 = \begin{bmatrix} 8/9 \\ 4/9 \\ 8/9 \end{bmatrix} - \begin{bmatrix} -1/9 \\ 4/9 \\ -1/9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$2x_1 + x_2 = 1$$
$$x_1 + x_2 = 0$$

$$\therefore \hat{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} //$$

**EXAMPLE 4** The distance from a point $\mathbf{y}$ in $\mathbb{R}^n$ to a subspace $W$ is defined as the distance from $\mathbf{y}$ to the nearest point in $W$. Find the distance from $\mathbf{y}$ to $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

**SOLUTION** By the Best Approximation Theorem, the distance from $\mathbf{y}$ to $W$ is $\|\mathbf{y} - \hat{\mathbf{y}}\|$, where $\hat{\mathbf{y}} = \text{proj}_W \mathbf{y}$. Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W$,

$$\hat{\mathbf{y}} = \frac{15}{30}\mathbf{u}_1 + \frac{-21}{6}\mathbf{u}_2 = \frac{1}{2}\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} - \frac{7}{2}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$\mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

$$\|\mathbf{y} - \hat{\mathbf{y}}\|^2 = 3^2 + 6^2 = 45$$

The distance from $\mathbf{y}$ to $W$ is $\sqrt{45} = 3\sqrt{5}$. ■

**Question 4: (4 marks)**

Show that if $A$ is a $2 \times 2$ matrix such that $A^2 = I_2$, and if $\mathbf{x}$ is any vector in $\mathbb{R}^2$, then $\mathbf{y} = \mathbf{x} + A\mathbf{x}$ and $\mathbf{z} = \mathbf{x} - A\mathbf{x}$ are eigenvectors of $A$. Find their corresponding eigenvalues.

$A = PDP^{-1}$, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = PD^2P^{-1}$

$Ay = A(x + Ax) = Ax + (A^2)x = Ax + Ix = Ax + x = y$

$Ax = \lambda$ for any eignvector $x$

$A^2 = I$

$Ay = 1y$, $Az = -1z$

$\therefore \lambda = 1$, $\lambda = -1$

**Question 5: (3 marks)**

Find a 4x4 matrix ***A*** (NON-ZERO ENTRIES ONLY) with **3 eigenvectors coming from a single eigenvalue** of your choice.

for single eigenvalue for 3 eigenvector
multiplicity of $\underline{\underline{3}}$
Requires something like $a\lambda^4 + b\lambda^3 = 0$

$$\lambda^3(a\lambda + b) = 0$$

Can $a = 1$ ?

$$\lambda^3(\lambda + b) = 0$$

Perhaps try $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, $\lambda^3(\lambda - 4) = 0$

$\lambda = 0$, $\lambda = 4$

for $\lambda = 0$, $\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$v_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ //

A works !