Computer Vision: Assignment #1

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Problem 1

To prove that the set of matrices $\{M_i\}$, representing Euclidean transformations in 3D space, forms a group, we need to verify the four group properties: closure, associativity, identity element, and inverse element.

Definitions

The Euclidean transformation matrix M_i is given by:

$$M_i = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix}$$

where:

- $R_i \in \mathbb{R}^{3\times 3}$ is an orthonormal matrix with $\det(R_i) = 1$, which represents a 3D rotation.
- $\mathbf{t}_i \in \mathbb{R}^{3\times 1}$ is a translation vector.
- $\mathbf{0}^T$ is a row vector of zeros.

Closure

We must show that the product of any two matrices M_i and M_j in the set results in a matrix that is also in the set.
Let M_i and M_j be two Euclidean transformation matrices:

$$M_i = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad M_j = \begin{bmatrix} R_j & \mathbf{t}_j \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

The matrix product M_iM_j is:

$$M_iM_j = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_j & \mathbf{t}_j \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_iR_j & R_i\mathbf{t}_j + \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

Since R_i and R_j are both orthonormal matrices (rotations), their product R_iR_j is also an orthonormal matrix with $\det(R_iR_j) = \det(R_i)\det(R_j) = 1$. Additionally, the translation part $R_i\mathbf{t}_j + \mathbf{t}_i$ is still a 3D vector. Thus, the product M_iM_j has the same form as the original matrices, meaning the set is closed under matrix multiplication.

Associativity

Matrix multiplication is always associative. For any three matrices M_i, M_j, and M_k in the set, we have:

$$M_i(M_jM_k) = (M_iM_j)M_k.$$

Since this property holds for all matrices, associativity is satisfied.

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Identity Element

The identity element M_{id} must satisfy $M_{\text{id}}M_i = M_iM_{\text{id}} = M_i$ for all M_i in the set.
The identity element in the set of Euclidean transformations is:

$$M_{\text{id}} = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix},$$

where I_3 is the 3x3 identity matrix and $\mathbf{0}$ is the zero vector.
Multiplying any matrix M_i by M_{id} gives:

$$M_iM_{\text{id}} = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} = M_i.$$

Similarly, $M_{\text{id}}M_i = M_i$, so M_{id} acts as the identity element in the set.

Inverse Element

We need to show that for every matrix M_i, there exists an inverse matrix M_i^{-1} such that $M_iM_i^{-1} = M_i^{-1}M_i = M_{\text{id}}$.
The inverse of M_i is:

$$M_i^{-1} = \begin{bmatrix} R_i^{-1} & -R_i^{-1}\mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix},$$

where $R_i^{-1} = R_i^T$ because R_i is orthonormal (i.e., $R_i^{-1} = R_i^T$ and $R_i^TR_i = I_3$).
Now, multiplying M_i by M_i^{-1}:

$$M_iM_i^{-1} = \begin{bmatrix} R_i & \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} R_i^{-1} & -R_i^{-1}\mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} R_iR_i^{-1} & R_i(-R_i^{-1}\mathbf{t}_i) + \mathbf{t}_i \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} I_3 & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} = M_{\text{id}}.$$

Thus, each matrix M_i has an inverse, which is also in the set.

Conclusion

Since the set of matrices $\{M_i\}$ satisfies closure, associativity, contains an identity element, and has an inverse for each element, it forms a group under matrix multiplication.

Problem 2

When deriving the Harris corner detector, we get the following matrix M composed of first-order partial derivatives in a local image patch w:

$$M = \begin{bmatrix} \sum_w I_x^2 & \sum_w I_xI_y \\ \sum_w I_xI_y & \sum_w I_y^2 \end{bmatrix}$$

where I_x and I_y are the partial derivatives of the image intensity in the x and y directions, respectively.

Problem 2 continued on next page...

Prove that M is positive semi-definite

To prove that M is positive semi-definite, we need to show that for any vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, the quadratic form $\mathbf{v}^T M \mathbf{v} \geq 0$.

The quadratic form is:

$$\mathbf{v}^T M \mathbf{v} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} \sum_w I_x^2 & \sum_w I_x I_y \\ \sum_w I_x I_y & \sum_w I_y^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Expanding this:

$$\mathbf{v}^T M \mathbf{v} = v_1^2 \sum_w I_x^2 + 2 v_1 v_2 \sum_w I_x I_y + v_2^2 \sum_w I_y^2$$

Since $I_x^2 \geq 0$, $I_y^2 \geq 0$, and the cross terms don’t introduce negative contributions, the sum is always non-negative. Thus, $\mathbf{v}^T M \mathbf{v} \geq 0$, proving that M is positive semi-definite.

Prove that if M is positive definite, it represents an ellipse

If M is positive definite, its eigenvalues λ_1 and λ_2 are both positive, and the matrix represents a quadratic form in Cartesian coordinates:

$$\mathbf{v}^T M \mathbf{v} = 1$$

where $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$.

This represents a conic section, and since M is symmetric and positive definite, the conic is an ellipse.

Diagonalizing M:

$$M = Q \Lambda Q^T$$

where Q is the orthogonal matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues λ_1 and λ_2. In the new coordinates, this becomes:

$$\lambda_1 x'^2 + \lambda_2 y'^2 = 1$$

This is the standard form of an ellipse, proving that M represents an ellipse.

Prove that the lengths of the semi-major and semi-minor axes are $\frac{1}{\sqrt{\lambda_1}}$ and $\frac{1}{\sqrt{\lambda_2}}$

Given the ellipse equation:

$$\lambda_1 x'^2 + \lambda_2 y'^2 = 1$$

The lengths of the semi-major and semi-minor axes correspond to the directions of the eigenvalues. The semi-major axis corresponds to the smaller eigenvalue λ_2, and the semi-minor axis corresponds to the larger eigenvalue λ_1.

Thus, the lengths of the semi-major and semi-minor axes are:

$$\text{Semi-major axis length} = \frac{1}{\sqrt{\lambda_2}}, \quad \text{Semi-minor axis length} = \frac{1}{\sqrt{\lambda_1}}.$$

Problem 3

In the lecture, we talked about the least square method to solve an over-determined linear system, $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^{m\times 1}$, with $m > n$ and $\text{rank}(A) = n$. The closed-form solution is given by:

$$\mathbf{x} = (A^T A)^{-1} A^T b.$$

To prove that $A^T A$ is non-singular (i.e., invertible), we need to show that for a full-rank matrix A, the matrix $A^T A$ is invertible.

Matrix Definitions and Properties

- $A \in \mathbb{R}^{m\times n}$, where $m > n$ (over-determined system). - A is of full rank, meaning $\text{rank}(A) = n$. - The matrix $A^T A \in \mathbb{R}^{n\times n}$ is a square matrix.
We want to show that $A^T A$ is invertible, i.e., that $A^T A$ is non-singular. This can be achieved by proving that if $\mathbf{v}^T(A^T A)\mathbf{v} = 0$, it implies that $\mathbf{v} = 0$.

Positive Semi-Definiteness of $A^T A$

For any vector $\mathbf{v} \in \mathbb{R}^n$, consider the quadratic form:

$$\mathbf{v}^T(A^T A)\mathbf{v} = (A\mathbf{v})^T(A\mathbf{v}).$$

The expression $(A\mathbf{v})^T(A\mathbf{v})$ is the dot product of $A\mathbf{v}$ with itself, which is the squared norm of $A\mathbf{v}$:

$$\mathbf{v}^T(A^T A)\mathbf{v} = \|A\mathbf{v}\|^2.$$

Since $\|A\mathbf{v}\|^2 \geq 0$ for all $\mathbf{v}$, we conclude that $A^T A$ is positive semi-definite.

Full Rank Implies Positive Definiteness

Now, we need to show that $A^T A$ is positive definite. This means that $\mathbf{v}^T(A^T A)\mathbf{v} = 0$ implies $\mathbf{v} = 0$. Assume that $\mathbf{v}^T(A^T A)\mathbf{v} = 0$. This implies:

$$\|A\mathbf{v}\|^2 = 0.$$

Since the squared norm of a vector is zero if and only if the vector itself is zero, we have:

$$A\mathbf{v} = 0.$$

Since A is full-rank and $\text{rank}(A) = n$, the only solution to $A\mathbf{v} = 0$ is $\mathbf{v} = 0$.

Conclusion

Because the only solution to $A\mathbf{v} = 0$ is $\mathbf{v} = 0$, we conclude that $A^T A$ is positive definite and therefore invertible. In other words, $\det(A^T A) \neq 0$, and $A^T A$ is non-singular.
Thus, $A^T A$ is invertible.

Problem 4

Question 4 is in the folder **problem 4**,you can run **problem 4.m**
Results are as follows:

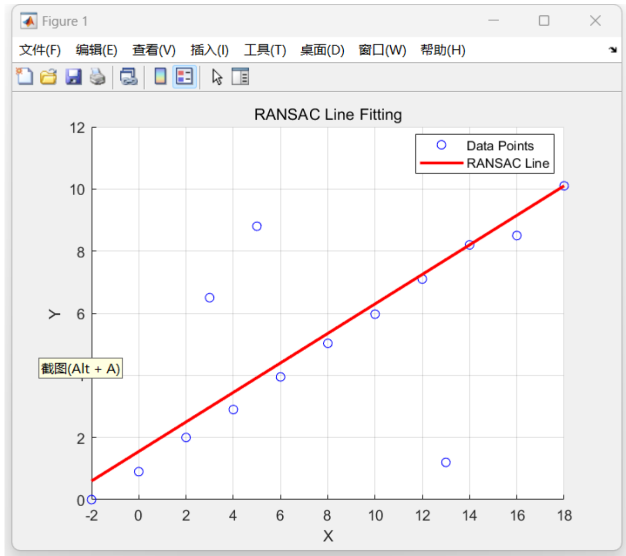

图 1: Fitted Line

Problem 5

Question 5 is in the folder **problem 5**,you can run **problem 5.m**
Results are as follows:

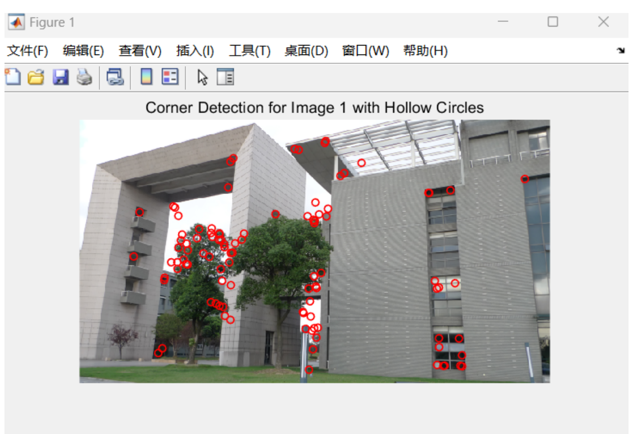

图 2: Corner detection for picture 1

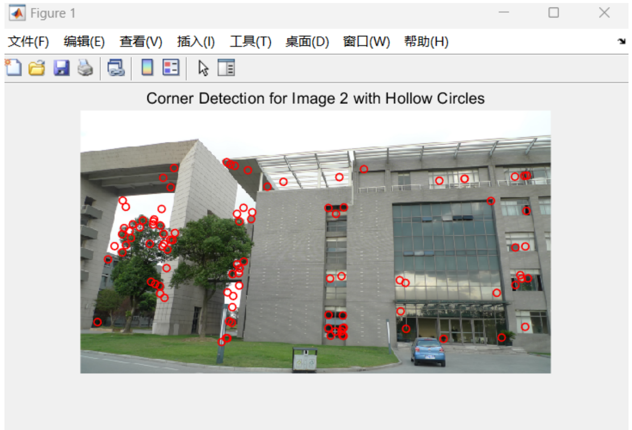

图 3: Corner detection for picture 2

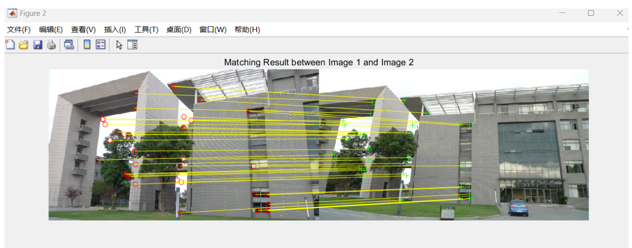

图 4: Corresponding point matching

图 5: Matching result

Problem 6

Question 6 is in the folder **problem 6**,you can run **problem 6.sln**
Results are as follows:

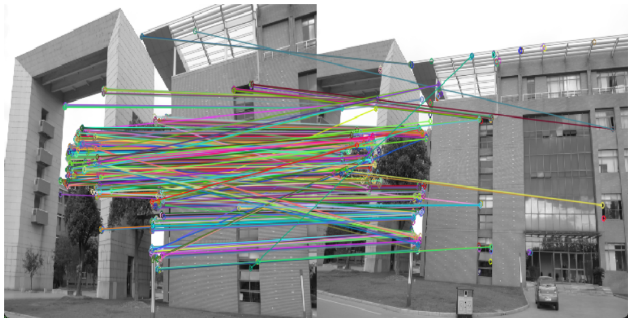

图 6: Matching result