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I pledge my honor that I have abided by the Stevens Honor System.

HW3

1) a) Find a third order ODE w constant coefficients

$y_c = C_1 e^{-x}\cos(3x) + C_2 e^{-x}\sin(3x) + C_3 e^{-3x} \to m=-3,\ \overset{\alpha=-1}{\beta=3} \to m=-3, -1\pm 3i$

$\to (m+3)((m+1)^2+3^2)=0 \to (m+3)(m^2+2m+1+9) \to (m+3)(m^2+2m+10) \to m^3+5m^2+16m+30=0$

$\to \boxed{y''' + 5y'' + 16y' + 30y = 0}$

b) Fourth Order

$y_c = C_1\cos(2x) + C_2\sin(2x) + C_3 e^{-2x} + C_4 e^{2x} \to \overset{\alpha=0}{\beta=2} \to m=\pm 2i, \pm 2$

$\to (m^2+4)(m^2-4) \to m^4-16=0 \to \boxed{y'''' - 16y = 0}$

2) a) Find gen sol to $L[y] = y''' + y' + 10y = 0$; $y_1 = e^x\sin(2x) \to y_2 = e^x\cos(2x) \to \overset{\alpha=1}{\beta=2} \to m = 1\pm 2i$

$L[y]=0 \xrightarrow{y=e^{mx}} m^3+m+10=0$; $(m-1)^2+4=0 \to m^2-2m+5=0$

$$\begin{array}{r} m+2 \\ m^2-2m+5 \,\big)\, m^3+0m^2+m+10 \\ -\ (m^3-2m^2+5m) \\ 2m^2-4m+10 \\ -\ (2m^2-4m+10) \\ 0 \end{array}$$

$\to (m+2)((m-1)^2+4)=0 \to m=-2, 1\pm 2i \longrightarrow$

$\to \boxed{y = C_1 e^{-2x} + C_2 e^x\cos(2x) + C_3 e^x\sin(2x)}$

b) Find gen sol to $L[y] = y''' + 5y'' + 8y' + 4y = 0 \xrightarrow{y=e^{mx}} m^3+5m^2+8m+4=0 \quad m=-1$

$$\begin{array}{r} m^2+4m+4 \\ m+1 \,\big)\, m^3+5m^2+8m+4 \\ -\ (m^3+m^2) \\ 4m^2+8m \\ -\ (4m^2+4m) \\ 4m+4 \\ -\ (4m+4) \end{array}$$

$\to (m+1)(m^2+4m+4)=0 \to (m+1)(m+2)(m+2)=0 \to m=-2,-2,-1$

$\to \boxed{y = C_1 e^{-2x} + C_2 x e^{-2x} + C_3 e^{-x}}$

3) $\underbrace{L[y] = y'' - 4y' + 4y}_{L[y]} = \underbrace{6x^2e^{2x}}_{g(x)}$

a) $L[y] = 0 \xrightarrow{y=e^{mx}} m^2 - 4m + 4 = 0 \rightarrow (m-2)^2 = 0 \quad , m = 2, 2 \qquad y_c = \underbrace{c_1e^{2x}}_{y_1} + \underbrace{c_2xe^{2x}}_{y_2}$

$$W[y_1, y_2] = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} + 2xe^{4x} - 2xe^{4x} = \boxed{e^{4x}} \neq 0$$

$y_p = u_1y_1 + u_2y_2$

$u_1' = \frac{-y_2 g(x)}{W} = \frac{-xe^{2x}\,6x^2e^{2x}}{e^{4x}} = -6x^3 \qquad u_1 = -6\int x^3dx = -6\frac{x^4}{4} = \frac{-3}{2}x^4$

$u_2' = \frac{y_1 g(x)}{W} = \frac{e^{2x}\,6x^2e^{2x}}{e^{4x}} = 6x^2 \qquad u_2 = 6\int x^2dx = 6\frac{x^3}{3} = 2x^3$

$y_p = \frac{-3}{2}x^4e^{2x} + 2x^4e^{2x} = \boxed{\frac{1}{2}x^4e^{2x}}$

b) $y = y_c + y_p = \boxed{c_1e^{2x} + c_2xe^{2x} + \frac{1}{2}x^4e^{2x}}$

4) Solve IVP $L[y] = \overbrace{y''-2y'+y}^{L[y]} = \overbrace{4e^t \ln t}^{g(x)}$, $y(1)=0$, $y'(1)=0$

$L[y]=0 \xrightarrow{y=e^{mx}} m^2-2m+1=0 \rightarrow (m-1)^2=0 \rightarrow m=1,1 \rightarrow y_c = c_1 \overset{y_1}{e^t} + c_2 \overset{y_2}{te^t}$

$W[y_1,y_2] = \begin{vmatrix} e^t & te^t \\ e^t & e^t+te^t \end{vmatrix} = e^{2t}+te^{2t}-te^{2t} = \boxed{e^{2t}} \neq 0$

$y_p = u_1 y_1 + u_2 y_2$

$u_1' = \frac{-y_2\, g(x)}{W} = \frac{-te^t\, 4e^t \ln t}{e^{2t}} = -4t\ln t$ $\quad u_1 = -4\int t\ln t\, dt \rightarrow u=\ln t \quad dv = t\,dt \rightarrow$

$du = \frac{1}{t}dt \quad v = \frac{t^2}{2}$

$u_1 = -4\left(\ln t \frac{t^2}{2} - \frac{1}{2}\int t^2 \frac{1}{t} dt\right) = -2t^2\ln t + 2\int t\,dt = -2t^2 \ln t + 2\frac{t^2}{2} = \boxed{-2t^2\ln t + t^2}$

$u_2' = \frac{y_1\, g(x)}{W} = \frac{e^t\, 4e^t \ln t}{e^{2t}} = 4\ln t$ $\quad u_2 = 4\int \ln t\, dt \rightarrow u = \ln t \quad dv = dt \rightarrow$

$du = \frac{1}{t}dt \quad v = t$

$u_2 = 4\left(\ln t\, t - \int t \frac{1}{t} dt\right) = 4t\ln t - 4\int dt = \boxed{4t\ln t - 4t}$

$y_p = (-2t^2\ln t + t^2)e^t + (4t\ln t - 4t)te^t = -2t^2e^t\ln t + t^2e^t + 4t^2e^t\ln t - 4t^2e^t$

$y_p = 2t^2e^t\ln t - 3t^2e^t$

$y(1)=0$

$y = c_1e^t + c_2te^t + 2t^2e^t\ln t - 3t^2e^t$

$0 = c_1e + c_2e + 0 - 3e$

$c_1e + c_2e = 3e$

$c_1 + c_2 = 3 \rightarrow c_1 = 3 - c_2 \rightarrow c_1 = 3-4 = -1$

$y'(1)=0$

$y' = c_1e^t + c_2e^t + c_2te^t + 4te^t\ln t + 2t^2e^t\ln t - 4te^t - 3t^2e^t$

$0 = c_1e + c_2e + c_2e + 0 + 0 - 4e - 3e$

$c_1e + 2c_2e = 7e$

$c_1 + 2c_2 = 7 \rightarrow 3 + c_2 = 7 \rightarrow c_2 = 4$

$y = -e^t + 4te^t + 2t^2e^t\ln t - 3t^2e^t$

$\boxed{y = e^t(2t^2\ln t - 3t^2 + 4t - 1)}$