Aughdon Breslin

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I pledge my honor that I have abided by the Stevens Honor System.

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HW9

1) (PDE) $U_t = 2U_{xx}$ $\quad 0<x<2\pi \quad t>0$

(BC) $U_x(0,t)=0 \quad U(2\pi,t)=0 \quad t>0$

(IC) $U(x,0)=f(x)=\begin{cases} 1, & 0<x<\pi \\ 0, & \pi\le x<2\pi \end{cases}$

$U=XT \to X''+\lambda X=0,\ T'+2\lambda T=0$

$\boxed{\lambda=\left(\frac{2n-1}{4}\right)^2 \quad X_n(x)=A_n\cos\left(\frac{(2n-1)x}{4}\right)}$ for $n=1,2,3,\ldots$

$T'+2\lambda T=0 \to T=e^{rt} \to T'=re^{rt} \quad r=-2\lambda$ $\boxed{T_n(t)=B_n e^{-2\lambda t}}$

$U_n=X_nT_n=C_ne^{-2\lambda t}\cos\left(\frac{(2n-1)x}{4}\right) \to \boxed{U=\frac{C_0}{2}+\sum_{n=1}^{\infty} C_ne^{-2\left(\frac{2n-1}{4}\right)^2 t}\cos\left(\frac{(2n-1)x}{4}\right)}$

$C_0=\frac{2}{L}\int_0^L f(x)dx \quad C_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{\pi n x}{L}\right)dx$

$C_0=\frac{2}{2\pi}\int_0^\pi 1dx=\frac{2\pi}{2\pi}=\boxed{1} \quad C_n=\frac{2}{2\pi}\int_0^\pi 1\cos\left(\frac{\pi n x}{2\pi}\right)dx=\frac{1}{\pi}\left(\frac{2}{n}\left(\sin\left(\frac{n\pi}{2}\right)-0\right)\right)$

$C_n=\frac{2}{\pi^2}\sin\left(\frac{n\pi}{2}\right)$

$$\boxed{U=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi^2}\sin\left(\frac{n\pi}{2}\right)e^{-2\left(\frac{2n-1}{4}\right)^2 t}\cos\left(\frac{(2n-1)x}{4}\right)}$$

2) (PDE) $u_{tt} = 16 u_{xx}$, for $0 < x < 2$, $t > 0$

(BC) $u(0,t) = 0$, $u(2,t) = 0$, $t > 0$

(IC) $u(x,0) = \frac{1}{4}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{16}\sin\left(\frac{3\pi x}{2}\right)$, $0 < x < 2$

$u_t(x,0) = \frac{1}{4}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{20}\sin\left(\frac{5\pi x}{2}\right)$, $0 < x < 2$

$U = XT \to XT'' = 16X''T \to \frac{X''}{X} = \frac{T''}{16T} = -\lambda$

$X'' + \lambda X = 0 \to m^2 + \lambda = 0 \to m = \pm\sqrt{-\lambda}$, $\lambda > 0$ $\mu = \lambda^2 \to m = \pm \mu i$ $\alpha = 0$ $\beta = \mu$

$X_n(x) = A_n\cos(\mu x) + B_n\sin(\mu x)$

$0 = A_n(1) + 0 \to A_n = 0$, $0 = 0 + B_n\sin(\mu 2) \to 2\mu = \pi n \to \mu = \frac{\pi n}{2}$

$\boxed{\lambda = \left(\frac{\pi n}{2}\right)^2 \quad X_n(x) = B_n\sin\left(\frac{\pi n x}{2}\right)}$

$T'' + 16\lambda T = 0 \to m^2 + 16\lambda = 0 \to m = \pm\sqrt{-16\lambda}$, $\lambda > 0$, $\mu = \lambda^2 \to m = \pm 4\mu i$ $\alpha = 0$ $\beta = 4\mu$

$T_n(t) = C_n\cos(4\mu t) + D_n\sin(4\mu t)$

$\boxed{T_n(t) = C_n\cos(2\pi n t) + D_n\sin(2\pi n t)}$

$U_n = X_n T_n = (E_n\cos(2\pi n t) + F_n\sin(2\pi n t))\sin\left(\frac{\pi n x}{2}\right)$

$U_n(x,0) = E_n\sin\left(\frac{\pi n x}{2}\right) \to U(x,0) = \sum_{n=1}^{\infty} E_n\sin\left(\frac{\pi n x}{2}\right) = \frac{1}{4}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{16}\sin\left(\frac{3\pi x}{2}\right)$

$n = 1 \to E_1 = \frac{1}{4}$, $n = 3 \to E_3 = \frac{-1}{16}$, $E_n = 0$ for all other n

$U_{n_t} = (-2E_n\pi n\sin(2\pi n t) + 2F_n\pi n\cos(2\pi n t))\sin\left(\frac{\pi n x}{2}\right)$

$U_{n_t}(x,0) = 2F_n\pi n\sin\left(\frac{\pi n x}{2}\right) \to U_t(x,0) = \sum_{n=1}^{\infty} F_n\pi n\sin\left(\frac{\pi n x}{2}\right) = \frac{1}{4}\sin\left(\frac{\pi x}{2}\right) - \frac{1}{20}\sin\left(\frac{5\pi x}{2}\right)$

$n = 1 \to F_1\pi(1) = \frac{1}{4} \to F_1 = \frac{1}{4\pi}$, $n = 5 \to F_5\pi(5) = \frac{-1}{20} \to F_5 = \frac{-1}{100\pi}$, $F_n = 0$ for all other n

$$\boxed{U = \sum_{n=1}^{\infty}\left(\frac{1}{4}\cos(2\pi n t) + \frac{1}{4\pi}\sin(2\pi n t)\right)\sin\left(\frac{\pi n x}{2}\right) - \frac{1}{16}\cos(2\pi n t)\sin\left(\frac{\pi n x}{2}\right) - \frac{1}{100\pi}\sin(2\pi n t)\sin\left(\frac{\pi n x}{2}\right)}$$

3) (PDE) $u_{tt} + 2u_t = 9u_{xx}$ for $0 < x < \pi$, $t > 0$

(BC) $u(0,t) = 0$, $u(\pi, t) = 0$, $t > 0$

(IC) $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $0 < x < \pi$

$U = XT \rightarrow XT'' + 2XT' = 9X''T \rightarrow \frac{X''}{X} = \frac{T'' + 2T'}{9T} = -\lambda$

$X'' + \lambda X = 0 \rightarrow m^2 + \lambda = 0 \rightarrow m = \pm\sqrt{-\lambda}$ $\lambda > 0$, $\mu = \lambda^2$, $m = \pm \mu i$ $\begin{matrix}\alpha = 0 \\ \beta = \mu\end{matrix}$

$X_n(x) = A_n \cos(\mu x) + B_n \sin(\mu x)$

$0 = A_n(1) + 0 \rightarrow A_n = 0$, $0 = 0 + B_n \sin(\mu\pi) \rightarrow \mu\pi = \pi n \rightarrow \mu = n$

$\lambda = n^2$ $\underline{X_n(x) = B_n \sin(nx)}$

$T'' + 2T' + 9\lambda T = 0 \rightarrow m^2 + 2m + 9\lambda = 0$ $m = \frac{-2 \pm \sqrt{4 - 36\lambda}}{2} = \frac{-2 \pm 2\sqrt{1 - 9\lambda}}{2} = -1 \pm \sqrt{1 - 9\lambda} = -1 \pm \sqrt{-(9\lambda - 1)}$

$(9\lambda - 1) > 0$ $\mu = (9\lambda - 1)^2$ $m = -1 \pm \mu i$ $\begin{matrix}\alpha = -1 \\ \beta = \mu\end{matrix}$

$T_n(t) = C_n e^{-t} \cos(\mu t) + D_n e^{-t} \sin(\mu t) = C_n e^{-t} \cos((9n-1)^2 t) + D_n e^{-t} \sin((9n-1)^2 t)$

$T_n(t) = C_n e^{-t} \cos((9n-1)^2 t) + D_n e^{-t} \sin((9n-1)^2 t)$

$U_n = X_n T_n = (E_n e^{-t} \cos((9n-1)^2 t) + F_n e^{-t} \sin((9n-1)^2 t)) \sin(nx)$

$U_n(x,0) = E_n \sin(nx) = f(x) \rightarrow E_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\,dx$

$U_{n_t} = (E_n(-e^{-t} \cos((9n-1)^2 t) - e^{-t} \sin((9n-1)^2 t)(9n-1)^2) + F_n(-e^{-t} \sin((9n-1)^2 t) + e^{-t}(9n-1)^2 \cos((9n-1)^2 t)) \sin(nx)$

$U_{n_t}(x,0) = (E_n(-1 - 0) + F_n(0 + (9n-1)^2)) \sin(nx)$

$U_{n_t}(x,0) = (\frac{-2}{\pi} \int_0^\pi f(x) \sin(nx)\,dx + F_n(9n-1)^2) \sin(nx)$

$F_n = \frac{2}{\pi} \int_0^\pi g(x) \sin(nx)\,dx + \frac{2}{\pi} \int_0^\pi f(x) \sin(nx)\,dx$

$$\boxed{\begin{aligned} U(x,t) = \sum_{n=1}^{\infty} \Big(&\Big(\frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx\Big) e^{-t} \cos((9n-1)^2 t) \\ &+ \Big(\frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\,dx + \frac{2}{\pi}\int_0^\pi f(x)\sin(nx)\,dx\Big) e^{-t} \sin((9n-1)^2 t)\Big) \sin(nx) \end{aligned}}$$