Aughdon Breslin

MA 221 - Patrick Miller

I pledge my honor that I have abided by the Stevens Honor System.
Aughdon Breslin

Homework 7

1) $u = XY \quad U_x = U_y + U \to X'Y = XY' + XY \to \frac{X'}{X} = \frac{Y'+Y}{Y} \to \frac{X'}{X} = \frac{Y'}{Y} + 1 = -\lambda$

$\int \frac{X'}{X} = \int -\lambda \to \ln|X| = -\lambda x + c \to X = ce^{-\lambda x}$

$\int \frac{Y'}{Y} = \int -\lambda - 1 \to \ln|Y| = (-\lambda-1)y + c \to Y = ce^{(-\lambda-1)y}, \quad u = XY = \boxed{ce^{-\lambda x} e^{(-\lambda-1)y}}$

2) For each $g(x)$, find all solutions to BVP. $4y'' + \pi^2 y = g(x)$ for $0<x<2$, $y'(0)=0$, $y'(2)=0$

$L[y]=0 \; y=e^{mx} \; 4m^2+\pi^2=0 \to m^2 = \frac{-\pi^2}{4} \to m = \pm\frac{\pi}{2}i \quad \alpha=0, \beta=\frac{\pi}{2} \quad , y_c = c_1\cos(\frac{\pi}{2}x) + c_2\sin(\frac{\pi}{2}x)$

a) $y = c_1\cos(\frac{\pi}{2}x) + c_2\sin(\frac{\pi}{2}x) \to y' = -c_1\frac{\pi}{2}\sin(\frac{\pi}{2}x) + c_2\frac{\pi}{2}\cos(\frac{\pi}{2}x)$

$y'(0)=0 \to 0 = -c_1\frac{\pi}{2}0 + c_2\frac{\pi}{2}(1) \to c_2 = 0 \qquad \boxed{y = c_1\cos(\frac{\pi}{2}x)}$

$y'(2)=0 \to 0 = -c_1\frac{\pi}{2}0 + c_2\frac{\pi}{2}(-1) \to c_2 = 0$

b) $y_p = Ax+B \qquad g(x) = x$

$y_p' = A \qquad 4(0) + \pi^2(Ax+B) = x \to A\pi^2 = 1 \to A = \frac{1}{\pi^2}, \; B\pi^2 = 0 \to B = 0$

$y_p'' = 0 \qquad y_p = \frac{x}{\pi^2}$

$y = c_1\cos(\frac{\pi}{2}x) + c_2\sin(\frac{\pi}{2}x) + \frac{1}{\pi^2}x \to y' = -c_1\frac{\pi}{2}\sin(\frac{\pi}{2}x) + c_2\frac{\pi}{2}\cos(\frac{\pi}{2}x) + \frac{1}{\pi^2}$

$y'(0)=0 \to 0 = -c_1\frac{\pi}{2}0 + c_2\frac{\pi}{2}(1) + \frac{1}{\pi^2} \to c_2 = -\frac{2}{\pi^3}$ $\quad c_2$ must be $\frac{-2}{\pi^3}$ and $\frac{2}{\pi^3}$ to satisfy BVP.

$y'(2)=0 \to 0 = -c_1\frac{\pi}{2}0 + c_2\frac{\pi}{2}(-1) + \frac{1}{\pi^2} \to c_2 = \frac{2}{\pi^3}$ $\quad$ Contradiction $\to$ $\boxed{\text{no solution.}}$

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3) $y'' + (\lambda-1)y = 0$ for $0 < x < \pi$, $y(0)=0$, $y'(\pi)=0$

a) $L[y]=0 \overset{y=e^{mx}}{\Rightarrow} m^2 + (\lambda-1) = 0 \rightarrow m^2 = -(\lambda-1) \rightarrow m = \pm\sqrt{-(\lambda-1)}$

(i) $-(\lambda-1) < 0 \rightarrow \lambda - 1 > 0 \rightarrow \boxed{\lambda > 1}$ (ii) $-(\lambda-1) = 0 \rightarrow \boxed{\lambda = 1}$ (iii) $-(\lambda-1) > 0 \rightarrow \boxed{\lambda < 1}$

$m = \pm\sqrt{\lambda-1}\, i \rightarrow$ b) $y = c_1\cos(\sqrt{(\lambda-1)}x) + c_2\sin(\sqrt{(\lambda-1)}x)$ $y' = -c_1\sqrt{(\lambda-1)}\sin(\sqrt{(\lambda-1)}x) + c_2\sqrt{(\lambda-1)}\cos(\sqrt{(\lambda-1)}x)$

$y(0)=0 \rightarrow 0 = c_1(1) + c_2 0 \rightarrow c_1 = 0$ $y'(\pi)=0 \rightarrow 0 = -0\sqrt{(\lambda-1)}\sin(\sqrt{\lambda-1}\,\pi) + c_2\sqrt{(\lambda-1)}\cos(\sqrt{(\lambda-1)}\,\pi)$

$0 = c_2\sqrt{(\lambda-1)}\cos(\sqrt{(\lambda-1)}\,\pi)$

(i) $c_2 = 0 \rightarrow$ Trivial Sol'n X (ii) $\sqrt{\lambda-1} = 0$ X $\rightarrow$ can't by definition (iii) $\cos(\sqrt{(\lambda-1)}\,\pi) = 0 \rightarrow \sqrt{(\lambda-1)}\,\pi = (2n-1)\frac{\pi}{2}$ for $n=1,2,3,\ldots$

$\rightarrow \sqrt{(\lambda-1)} = \frac{2n-1}{2} \rightarrow \boxed{\lambda = \left(\frac{2n-1}{2}\right)^2 + 1 \text{ for } n=1,2,3,\ldots}$ $\boxed{y = c_2\sin\left(\frac{2n-1}{2}x\right) \text{ for } n=1,2,3,\ldots}$

c) (ii) $\underline{\lambda=1} \rightarrow m = \pm 0 \rightarrow y = c_1 + c_2 x$ $y' = c_2$

$y(0)=0 \rightarrow 0 = c_1 + 0 \rightarrow c_1 = 0$ $y'(\pi) = 0 \rightarrow 0 = c_2 \rightarrow c_1, c_2 = 0 \rightarrow$ Trivial Solution

Conclusion: $\lambda=1 \rightarrow$ Trivial Solution, so $\lambda=1$ is not an eigenvalue

(iii) $\lambda < 1 \rightarrow m = \pm\sqrt{\lambda-1} \rightarrow y = c_1\cosh(\sqrt{\lambda-1}\,x) + c_2\sinh(\sqrt{\lambda-1}\,x)$ $y' = c_1\sqrt{\lambda-1}\sinh(\sqrt{\lambda-1}\,x) + c_2\sqrt{\lambda-1}\cosh(\sqrt{\lambda-1}\,x)$

$y(0)=0 \rightarrow 0 = c_1(1) + c_2(0) \rightarrow c_1 = 0$ $y'(\pi)=0 \rightarrow 0 = 0 + c_2\sqrt{\lambda-1}\cosh(\sqrt{\lambda-1}\,\pi) \rightarrow c_2\sqrt{\lambda-1}\cosh(\sqrt{\lambda-1}\,\pi) = 0$

(i) $c_2 = 0 \rightarrow$ Trivial Solution (ii) $\sqrt{\lambda-1} = 0 \rightarrow$ can't by definition X (iii) $\cosh(\sqrt{\lambda-1}\,\pi) = 0 \rightarrow$ impossible X

Therefore c_2 must equal 0. Thus $c_1 = 0$ and $c_2 = 0$ and so $\lambda < 1 \rightarrow$ Trivial Solution, so $\lambda < 1$ cannot be eigenvalues.