Ma 221 - Differential Equations **Homework 9** **Due: Nov 22, 2021**

See the canvas assignment page for detailed instructions on submitting your work.

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1. Consider the following initial-boundary value problem (IBVP) modeling heat flow in a wire.

$$\text{(PDE)} \qquad \frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}, \qquad \text{for } 0 < x < 2\pi, \quad t > 0$$

$$\text{(BC)} \qquad u_x(0,t) = 0, \; u(2\pi, t) = 0, \quad t > 0$$

$$\text{(IC)} \qquad u(x,0) = f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi \le x < 2\pi \end{cases}$$

Assuming a product solution, $u(x,t) = X(x) \cdot T(t)$, separation of variables leads to an eigenvalue problem for $X(x)$ and a first-order ODE for $T(t)$.

$X''(x) + \lambda X(x) = 0, \;\; X'(0) = 0, \; X(2\pi) = 0 \quad \text{(EVP)}$

$T'(t) + 2\lambda T(t) = 0$

The eigenvalues are $\lambda_n = \left(\frac{2n-1}{4}\right)^2$ with solutions, $X_n(x) = A_n \cos\left(\frac{(2n-1)x}{4}\right)$, for $n = 1, 2, 3, \ldots$

Complete the solution to the IBVP as follows:

- For each eigenvalue, λ_n, determine $T_n(t)$.
- Form the general solution, $u(x,t)$, as an infinite series.
- Use the initial condition, $u(x,0) = f(x)$, to determine the values of the coefficients in the general solution. This will require the appropriate Fourier Series for $f(x)$.

2. Solve the following initial-boundary value problem.

$$\begin{array}{rl}
\text{(PDE)} & u_{tt} = 16\,u_{xx}\,, \quad \text{for } 0 < x < 2, \quad t > 0 \\
\text{(BC)} & u(0,t) = 0, \ \ u(2,t) = 0, \quad t > 0 \\
\text{(IC)} & u(x,0) = \dfrac{\sin(\pi x/2)}{4} - \dfrac{\sin(3\pi x/2)}{16}, \quad 0 < x < 2 \\
\text{(IC)} & u_t(x,0) = \dfrac{\sin(\pi x/2)}{4} - \dfrac{\sin(5\pi x/2)}{20}, \quad 0 < x < 2
\end{array}$$

Complete the solution to the IBVP as follows:

Step 1. Assuming a product solution, $u(x,t) = X(x) \cdot T(t)$, derive the eigenvalue problem (EVP) for $X(x)$ and a second order ODE for $T(t)$.

Step 2. Determine all eigenvalues (λ_n) and corresponding solutions $X_n(x)$.

Step 3. For each λ_n in Step 2, determine the corresponding solution, $T_n(t)$, to the ODE for $T(t)$.

Step 4. Form the general solution, $u(x,t)$, as an infinite series.

Step 5. Use the initial conditions, $u(x,0)$ and $u_t(x,0)$, to determine the (unique) values of the coefficients in the general solution.

3. Use *separation of variables* to derive the *general solution* to the following initial-boundary value problem.

$$\begin{array}{rl}
\text{(PDE)} & u_{tt} + 2u_t = 9\,u_{xx}\,, \quad \text{for } 0 < x < \pi, \quad t > 0 \\
\text{(BC)} & u(0,t) = 0, \ \ u(\pi,t) = 0, \quad t > 0 \\
\text{(IC)} & u(x,0) = f(x), \quad 0 < x < \pi \\
\text{(IC)} & u_t(x,0) = g(x), \quad 0 < x < \pi
\end{array}$$