Aughdon Breslin

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HW4

1) Find gen sol to $L[y] = y'' - 4y' + 3y = 2te^t$ using undet. coeff.

$L[y] = 0 \xrightarrow{y=e^{mx}} m^2 - 4m + 3 = 0 \rightarrow (m-3)(m-1) = 0 \rightarrow m = 1, 3 \quad y_c = C_1e^t + C_2e^{3t}$

$(At + B)e^t$ $\qquad g(x) = 2te^t$

$y_p = At^2e^t + Bte^t \qquad y_p' = 2Ate^t + At^2e^t + Be^t + Bte^t$

$y_p'' = 2Ae^t + 2Ate^t + 2Ate^t + At^2e^t + Be^t + Be^t + Bte^t = At^2e^t + 4Ate^t + 2Ae^t + 2Be^t + Bte^t$

$At^2e^t + (4A+B)te^t + (2A+2B)e^t - 4(At^2e^t + (2A+B)te^t + Be^t) + 3(At^2e^t + Bte^t) = 2te^t$

$(A - 4A + 3A)t^2e^t + (4A + B + 8A - 4B + 3B)te^t + (2A + 2B - 4B)e^t = 2te^t$

$0 + (-4A)te^t + (2A - 2B)e^t = 2te^t$

$-4A = 2 \rightarrow A = \frac{-1}{2} \qquad 2A - 2B = 0 \rightarrow 2A = 2B \rightarrow A = B \rightarrow B = \frac{-1}{2}$

$y_p = \frac{-1}{2}t^2e^t - \frac{1}{2}te^t$

$\boxed{y = C_1e^t + C_2e^{3t} - \frac{1}{2}t^2e^t - \frac{1}{2}te^t}$

2) Find gen sol to $L[y] = x^2y'' + xy' - y = \underbrace{\frac{4\ln x}{x^3}}_{g(x)}$, $y = x^m$, $y' = mx^{m-1}$, $y'' = m(m-1)x^{m-2}$

$L[y] = 0 \xrightarrow{y=x^m} m^2 + (1-1)m - 1 = 0 \to m^2 - 1 = 0 \to m = \pm 1 \to y = C_1x^{-1} + C_2x$

$\underline{y_1 = x^{-1}}$ $\underline{y_2 = x}$ $W[y_1,y_2] = \begin{vmatrix} x^{-1} & x \\ -x^{-2} & 1 \end{vmatrix} = x^{-1} + x(+x^{-2}) = \underline{2x^{-1}} \neq 0$

$y_p = u_1y_1 + u_2y_2$

$u_1' = \frac{-y_2 g(x)}{a_2 W} = \frac{-x \frac{4\ln x}{x^3}}{x^2\, 2x^{-1}} = -2\frac{\ln x}{x^3} = -2x^{-3}\ln x$ $u_1 = -\int 2x^{-3}\ln x\,dx \to u = \ln x$ $dv = 2x^{-3}dx \to$

$du = \frac{1}{x}dx$ $v = -x^{-2}$

$\to -x^{-2}\ln x - \int -x^{-2}\frac{1}{x}dx \to -x^{-2}\ln x + \int x^{-3}dx \to -x^{-2}\ln x + \frac{x^{-2}}{-2} \to -(-x^{-2}\ln x - \frac{1}{2}x^{-2}) \to \underline{x^{-2}\ln x + \frac{1}{2}x^{-2}}$

$u_2' = \frac{y_1 g(x)}{a_2 W} = \frac{x^{-1}\frac{4\ln x}{x^3}}{x^2\, 2x^{-1}} = \frac{4\ln x}{2x^5} = 2x^{-5}\ln x$ $u_2 = \int 2x^{-5}\ln x\,dx \to u = \ln x$ $dv = 2x^{-5}dx \to$

$du = \frac{1}{x}dx$ $v = \frac{-1}{2}x^{-4}$

$\to \frac{-1}{2}x^{-4}\ln x - \int \frac{-1}{2}x^{-4}\frac{1}{x}dx \to \frac{-1}{2}x^{-4}\ln x + \frac{1}{2}\int x^{-5}dx \to \frac{-1}{2}x^{-4}\ln x + \frac{1}{2}\left(\frac{x^{-4}}{-4}\right) \to$

$u_2 = \underline{\frac{-1}{2}x^{-4}\ln x - \frac{1}{8}x^{-4}}$

$y_p = (x^{-2}\ln x + \frac{1}{2}x^{-2})x^{-1} + (\frac{-1}{2}x^{-4}\ln x - \frac{1}{8}x^{-4})x$

$y = C_1x^{-1} + C_2x + x^{-3}\ln x + \frac{1}{2}x^{-3} - \frac{1}{2}x^{-3}\ln x - \frac{1}{8}x^{-3}$

$\boxed{y = C_1x^{-1} + C_2x + \frac{1}{2}x^{-3}\ln x + \frac{3}{8}x^{-3}}$

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3) $m = M\,kg$, $\beta = 4$, $k = 8\frac{N}{m}$, $f(t) = 5\sin(4t)$ $x(0) = -.25m$, $x'(0) = 2m/s$

$mx'' + \beta x' + kx = f(t) \rightarrow \boxed{Mx'' + 4x' + 8x = 5\sin(4t)}$

b) Overdamped: $\beta^2 - 4mk > 0 \rightarrow 4^2 - 4M(8) > 0 \rightarrow 32M < 16 \rightarrow \boxed{M < \frac{1}{2}kg}$

Critically Damped: $\beta^2 - 4mk = 0 \rightarrow 4^2 - 4M(8) = 0 \rightarrow 32M = 16 \rightarrow \boxed{M = \frac{1}{2}kg}$

Underdamped: $\beta^2 - 4mk < 0 \rightarrow 4^2 - 4M(8) < 0 \rightarrow 32M > 16 \rightarrow \boxed{M > \frac{1}{2}kg}$

4) $L[y] = y'' + 4y' + 4y = 13\cos(3t)$, $y(0) = -1.0$, $y'(0) = 2$

a) $L[y] = 0 \xrightarrow{y = e^{mx}} m^2 + 4m + 4 = 0 \rightarrow (m+2)^2 = 0 \rightarrow m = -2, -2$ $y_c = c_1 e^{-2t} + c_2 t e^{-2t}$

$y_p = A\cos(3t) + B\sin(3t)$ $g(x) = 13\cos(3t)$

$y_p' = -3A\sin(3t) + 3B\cos(3t)$

$y_p'' = -9A\cos(3t) - 9B\sin(3t)$

$(-9A\cos(3t) - 9B\sin(3t)) + 4(-3A\sin(3t) + 3B\cos(3t)) + 4(A\cos(3t) + B\sin(3t)) = 13\cos(3t)$

$-9A\cos(3t) - 9B\sin(3t) - 12A\sin(3t) + 12B\cos(3t) + 4A\cos(3t) + 4B\sin(3t) = 13\cos(3t)$

$(-9A + 12B + 4A)\cos(3t) + (-9B - 12A + 4B)\sin(3t) = 13\cos(3t) + 0\sin(3t)$

$-5A + 12B = 13$ $-5B - 12A = 0 \rightarrow -5B = 12A \rightarrow B = \frac{-12}{5}A \rightarrow B = \frac{-12}{5}\left(\frac{-5}{13}\right) = \frac{12}{13}$

$-5A + 12\left(\frac{-12}{5}A\right) = 13 \rightarrow \frac{-144-25}{5}A = 13 \rightarrow \frac{-169}{5}A = 13 \rightarrow A = \frac{-5}{13}$

$y_p = \frac{12}{13}\sin(3t) - \frac{5}{13}\cos(3t)$

$y = c_1 e^{-2t} + c_2 t e^{-2t} + \frac{12}{13}\sin(3t) - \frac{5}{13}\cos(3t)$

$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t} + \frac{36}{13}\cos(3t) + \frac{15}{13}\sin(3t)$

$y(0) = -1.0$

$-1 = c_1 + 0 + \frac{12}{13}\sin(0) - \frac{5}{13}\cos(0)$

$c_1 = \frac{-8}{13}$

$y'(0) = 2$

$2 = -2c_1 + c_2 - 0 + \frac{36}{13}\cos(0) + \frac{15}{13}\sin(0)$

$2 = \frac{16}{13} + c_2 + \frac{36}{13}$

$2 = c_2 + \frac{52}{13} \rightarrow 2 = c_2 + 4 \rightarrow c_2 = -2$

$$\boxed{y = \frac{-8}{13}e^{-2t} - 2te^{-2t} + \frac{12}{13}\sin(3t) - \frac{5}{13}\cos(3t)}$$

$\tan^{-1}\left(\frac{c_1}{c_2}\right)$

4b) $c_1\cos(bt) + c_2\sin(bt) = \sqrt{c_1^2+c_2^2}\,\sin(bt+\phi)$

$\frac{-5}{13}\cos(3t) + \frac{12}{13}\sin(3t) = \sqrt{\left(\frac{-5}{13}\right)^2+\left(\frac{12}{13}\right)^2}\,\sin\left(3t + \tan^{-1}\left(\frac{-5}{12}\right)\right)$

$= \sqrt{\frac{169}{169}}\,\sin(3t - 0.395) = \sin(3t - 0.395)$

$\boxed{A=1,\ \phi \approx -0.395,\ y_p(t) = \sin(3t - 0.395)}$