HYPERBOLIC GROUPS

Essay
MASt in Pure Mathematics

Supervisor:
Henry Wilton

Cambridge, 4 May 2018

Contents

Introduction

The main goal of this work is to develop the basic theory of hyperbolic groups.

The matter is, as far as possible, self-contained, although the reader must have an elementary background in point-set topology and group theory. Familiarity with metric spaces, free groups and group presentations will be assumed.

Before going into detail, I would like to explain in few words the development of the theory written in the essay.
In 1912 Max Dehn was working on the basic problems of recognition and classification for low-dimensional manifolds. For example, he asked the question of when a curve on a compact orientable surface can be continuously shrunk to a point. In that setting, he realised that the fundamental group of that space could keep useful information. As a consequence, he formulated three different problems:

(1) *The Word Problem.* Given a finite presentation of a group, is there an algorithm that decides whether a word in the generators represents the identity of the group?

(2) *The Conjugacy Problem.* Given a finite presentation of a group, is there an algorithm that decides whether two elements are conjugate?

(3) *The Isomorphism Problem.* Given finite presentations of two groups, is there an algorithm that decides whether they are isomorphic?

Dehn showed that the Word Problem is solvable for a surface group and he gave such an algorithm, which is known as *Dehn's algorithm.* Later on, Gromov generalised it to hyperbolic groups.

The essay consists of proving that the Word Problem for hyperbolic groups is solvable. In the first three chapters, definitions and properties concerning to hyperbolic groups are introduced. Finally, in Chapter 4, the algorithmic problem is solved. I would also like to point out that in order not to exceed in length, I have not discussed the other two algorithmic problems. Nevertheless, on the one hand, the Conjugacy Problem is solvable for hyperbolic groups, as it can be read in [4]. On the other hand, it has been proved that the Isomorphism Problem is solvable in the case of torsion-free hyperbolic groups.

To sum up, the first chapter is drawn from [6]. The second one, however, is based on [7]. In this case, with the aim of completing the theory and facilitating understanding, the notes [1] have also been used. In the third chapter, two different topics are developed. While the first one (Section 3.1) is taken from [7], the second one (Section 3.2) is based on [5]. Finally, various sources have been considered for the last chapter: [1], [2] and [4].

Chapter 1

Cayley graphs

The main purpose of this chapter is to introduce the basic notions used in the essay.

Firstly, Section 1.1 will describe the length of an element. Secondly, in Section 1.2 graphs will be briefly recalled. Thirdly, in Section 1.4 Cayley graphs will be introduced, which are a way of constructing graphs by using groups. Finally, Section 1.5 will examine geodesic metric spaces and why Cayley graphs have such property.

1.1 Length of an element

From now on, let us assume that all the groups are finitely generated. Groups will be denoted by G and finite generating sets by S. Suppose also that S is symmetric; that is, for each s in S, s^{-1} also lies in S and $e \notin S$. Moreover, the natural surjection $F(S) \longrightarrow G$ will be μ.

Definition 1.1.1. The *length* of an element $g \in G$ with respect to S, $|g|_S$, is the minimum number of generators of S that are needed to represent G. That is,

$$|g|_S = \min\{l(w) \mid w \in F(S), \mu(w) = g\}.$$

The *distance* between two elements of G, h and h', is defined in this way:

$$d_S(h, h') = |h^{-1}h'|_S.$$

Proposition 1.1.1. *$d_S \colon G \times G \longrightarrow \mathbb{R}$ is a distance map.*

Proof. We need to show that the following three properties are satisfied:

(M1) $d_S(h_1, h_2) \geq 0$ for all $h_1, h_2 \in G$ and $d_S(h_1, h_2) = 0 \iff h_1 = h_2$;

(M2) $d_S(h_1, h_2) = d_S(h_2, h_1)$ for all $h_1, h_2 \in G$;

(M3) $d_S(h_1, h_2) \leq d_S(h_1, h_3) + d_S(h_3, h_2)$ for all $h_1, h_2, h_3 \in G$.

The condition (M1) is trivial. Now, suppose that

$$h_1^{-1}h_2 = x_1 \cdots x_n,$$

where $x_1 \cdots x_n$ is a reduced word in S. Hence,

$$h_2^{-1}h_1 = x_n^{-1} \cdots x_1^{-1}.$$

Therefore, $d_S(h_2, h_1) \leq n$. By arguing similarly the other way around, we obtain (M2). Finally, if

$$h_1^{-1}h_3 = x_1 \cdots x_n \quad \text{and} \quad h_3^{-1}h_2 = y_1 \cdots y_m,$$

with $x_1 \cdots x_n$ and $y_1 \cdots y_m$ reduced words in S, then

$$h_1^{-1}h_2 = h_1^{-1}h_3h_3^{-1}h_2 = x_1 \cdots x_n y_1 \cdots y_m,$$

so by definition, $d_S(h_1, h_2) \leq n + m$. □

1.2 Graphs

Definition 1.2.1. An *unoriented graph* Γ consists of two sets E (*set of edges*) and V (*set of vertices*) and a map ι (*incidence map*) defined on E and taking values in the set of subsets of V of cardinality one or two.

Two vertices u and v are called *adjacent* if $\iota(e) = \{u, v\}$ for some $e \in E$. In this case, u and v are the *endpoints* of e.

Remark 1.2.1. Unoriented graphs may be seen as 1-dimensional cell complexes, where the 0-skeleton is V and the 1-cells are labelled by elements of E. Therefore, they may be treated as topological spaces.

Remark 1.2.2. *Monogons* (edges connecting a vertex to itself) and *bigons* (distinct edges connecting the same vertices) are allowed in the definition.

Let us denote by $[u, v]$ an edge connecting the vertices u and v of Γ, although it may be ambiguous by the previous remark.

Definition 1.2.2. An *edge-path* in Γ is an ordered set $[v_1, v_2], [v_2, v_3], \cdots,$ $[v_n, v_{n+1}]$.

The number n is called the *combinatorial length* of such edge-path.

There is also the notion of oriented graph. Nevertheless, it will be explained after defining them that unoriented and oriented graphs are closely related.

Definition 1.2.3. An *oriented graph* $\overline{\Gamma}$ consists of two sets $\overline{E}$ (*set of edges*) and V (*set of vertices*) and two maps

$$o\colon \overline{E} \longrightarrow V \quad (\textit{origin map}),$$

$$t\colon \overline{E} \longrightarrow V \quad (\textit{tail map}).$$

In this case, for every $x, y \in V$ let us define $E_{(x,y)}$ to be

$$\{\bar{e} \in \overline{E} \mid (o(\bar{e}), t(\bar{e})) = (x, y)\}.$$

Definition 1.2.4. An oriented graph is *symmetric* if for every subset $\{x, y\}$ of V the sets $E_{(x,y)}$ and $E_{(y,x)}$ have the same cardinality.

Remark 1.2.3. A symmetric oriented graph $\overline{\Gamma}$ is equivalent to an unoriented graph Γ with the same vertex set, via the following procedure.

Let us define an involutive bijection $\beta\colon \overline{E} \longrightarrow \overline{E}$. For each $\overline{e_1} \in \overline{E}$, let $(x, y) = (o(\overline{e_1}), t(\overline{e_1}))$. Since $\overline{\Gamma}$ is symmetric, we can pick $\overline{e_2} \in \overline{E}$ such that $(y, x) = (o(\overline{e_2}), t(\overline{e_2}))$. Let $\beta(\overline{e_1})$ be $\overline{e_2}$. We want β to be bijective, so we have to assign a different edge of $E_{(y,x)}$ to each edge of $E_{(x,y)}$. Moreover, in order β to be involutive, $\beta(\overline{e_2})$ must be $\overline{e_1}$.

Let us consider the finest equivalence relation defined on $\overline{E}$ such that $\bar{e} \sim \beta(\bar{e})$ for $\bar{e} \in \overline{E}$. To sum up, let us define the unoriented graph Γ to have vertex set V, edge set $\overline{E}/\sim$ and incidence map $\iota([e]) = \{o(e), t(e)\}$.

The graph Γ is called the *underlying unoriented graph* of $\overline{\Gamma}$. Note that 'the' has been used instead of 'an' because it is routine to check that any two underlying unoriented graphs of the same oriented graph are isomorphic, in the sense that there is a homeomorphism from one graph to the other one so that the images of the edges are edges and the images of the vertices are vertices.

Conversely, by starting with an unoriented graph Γ, a symmetric oriented graph can be constructed in an analogous way, so that Γ is the underlying graph of it.

1.3 Length metric spaces

Suppose that X is a metric space with distance map d and let $p\colon [a, b] \longrightarrow X$ be a path.

A partition of $[a, b]$,

$$a = t_0 < t_1 < \cdots < t_n = b,$$

defines a collection of points $p(t_0), \cdots, p(t_n)$ in X.

Definition 1.3.1. The *length* of a path p is defined to be

$$\text{length}(p) = \sup_{a=t_0<\cdots<t_n=b} \sum_{i=0}^{n-1} d(p(t_i), p(t_{i+1})),$$

where the supremum is taken over all possible partitions of the interval $[a, b]$ and all natural numbers n.

If length(p) is finite, p is called *rectifiable*.

We are interested in the case of graphs. Therefore, assume that Γ is a connected graph. The notation $x \in \Gamma$ simply means that x is a point of Γ considered as a topological space.
Let us introduce a distance on Γ by declaring every edge of Γ to be isometric to the unit interval in $\mathbb{R}$. Points of the interiors of the edges are not connected by any edge-path. Thus, fractional edge-paths have to be considered, where in addition to the edges of Γ intervals contained in the edges are allowed. The length of such a fractional path is the sum of lengths of the intervals in the path.
Finally, for $x, y \in \Gamma$,

$$\operatorname{dist}(x, y) = \inf\{\operatorname{length}(p) \mid p \text{ fractional edge-path in } \Gamma \text{ connecting } x \text{ and } y\}.$$

Hence, the distance between any two vertices of Γ is the combinatorial length of the shortest edge-path connecting them.

Proposition 1.3.1. *The previous infimum equals the minimum.*

Proof. If x and y are two vertices of the graph Γ,

$$\{\operatorname{length}(p) \mid p \text{ fractional edge-path in } \Gamma \text{ connecting } x \text{ and } y\} \subseteq \mathbb{N},$$

so the infimum equals the minimum.

If x or y lie on the interiors of the edges e_1 and e_2 of Γ, suppose that

$$\iota(e_1) = \{u_1, v_1\} \quad \text{and} \quad \iota(e_2) = \{u_2, v_2\}.$$

Let us take the paths p_1, p_2, p_3 and p_4 minimizing the distances between u_1, u_2 and u_1, v_2 and v_1, u_2 and v_1, v_2, respectively. To connect x and y it is necessary to go from x to u_1 or v_1 and from y to u_2 or v_2, so we take the minimum of the four possible cases. $\square$

To sum up, for $x, y \in \Gamma$,

$$\operatorname{dist}(x, y) = \min\{\operatorname{length}(p) \mid p \text{ fractional edge-path in } \Gamma \text{ connecting } x \text{ and } y\}.$$

It is routine to check that dist is a metric on Γ, which is called the *standard length metric on* Γ.

1.4 Cayley graphs

Definition 1.4.1. The *Cayley graph* of G with respect to S, $\operatorname{Cayley}_{\mathrm{dir}}(G, S)$, is a symmetric oriented graph such that

- its set of vertices is G,
- its set of oriented edges consists of an edge from g to gs for $g \in G$, $s \in S$.

The underlying unoriented graph, Cayley(G, S), is also called the *Cayley graph* of G with respect to S. In this case, the set of edges consists of all pairs of elements in G, $\{g, h\}$, such that $h = gs$ for $s \in S$.

Remark 1.4.1. Notice that the Cayley graph of a group depends on the choice of the generators.

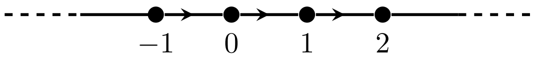


Figure 1.1: Cayley graph of $\mathbb{Z}$ with respect to $\{1\}$.

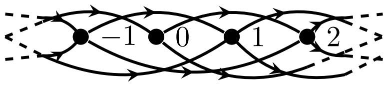


Figure 1.2: Cayley graph of $\mathbb{Z}$ with respect to $\{2, 3\}$.

It is possible to endow Cayley(G, S) with the standard length metric.

Definition 1.4.2. The restriction of the standard length metric to G is called *the word metric associated to* S and it is denoted by d_S.

Remark 1.4.2. The definition of d_S in Definition 1.1.1 and this definition are equivalent.

1.5 Geodesic metric spaces

Suppose, again, that X is a metric space with distance map d.

Definition 1.5.1. A *geodesic path* joining two points x and y of X is a map $c\colon [0, l] \longrightarrow X$ such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, l]$.

The image of c in X is called a *geodesic segment* with endpoints x and y and it is denoted also by $[x, y]$.

Definition 1.5.2. (X, d) is a *geodesic metric space* if every two points in X are joined by a geodesic path.

Example 1.5.1. From Proposition 1.3.1 we deduce that Cayley(G, S) with the standard length metric is a geodesic metric space.

Example 1.5.2. $\mathbb{R}^2 \setminus \{(0, 0)\}$ is not a geodesic metric space with the metric induced by the plane.
Suppose, by contradiction, that it is. The distance between $(-1, 0)$ and $(1, 0)$ is 2. If α were a geodesic path joining $(-1, 0)$ and $(1, 0)$, there would exist $t \in [0, 2]$ with $\alpha(t) = (0, y)$ for some $y \in \mathbb{R} \setminus \{0\}$. Then, $t = d(\alpha(0), \alpha(t)) = d((-1, 0), (0, y)) = \sqrt{y^2 + 1} > 1$ and $2 - t = d(\alpha(t), \alpha(2)) = d((0, y), (1, 0)) = \sqrt{y^2 + 1} > 1$, which is a contradiction.

Remark 1.5.1. Geodesic paths may not be unique.

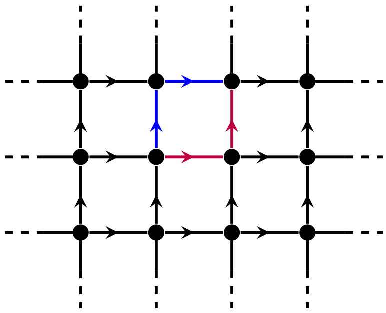


Figure 1.3: Cayley graph of $\mathbb{Z} \times \mathbb{Z}$ with respect to $\{(1,0),(0,1)\}$.

Chapter 2

Hyperbolic groups

This chapter will examine hyperbolic groups. Although the first definition will not be intuitive, later on it will be proven that it is equivalent to other four definitions which have a geometric intuition.

As always, we work in a metric space (X, d).

Definition 2.0.1. Given a basepoint $x_0 \in X$, the *Gromov product* of $y, z \in X$ is

$$(y.\, z)_{x_0} = \frac{1}{2}(d(y, x_0) + d(x_0, z) - d(y, z)).$$

Observe that $d(y, z) \leq d(y, x_0) + d(x_0, z)$, so

$$(y.\, z)_{x_0} \geq 0,$$

for all $x_0, y, z \in X$.

Moreover, $d(x_0, z) \leq d(x_0, y) + d(y, z)$, so $d(x_0, z) - d(y, z) \leq d(x_0, y)$ and

$$d(x_0, y) + d(x_0, z) - d(y, z) \leq 2d(x_0, y).$$

That is,

$$\frac{1}{2}(d(y, x_0) + d(x_0, z) - d(y, z)) \leq d(x_0, y).$$

In the same way, we obtain that $(y.\, z)_{x_0} \leq d(x_0, z)$. Therefore,

$$0 \leq (y.\, z)_{x_0} \leq \min\{d(x_0, y), d(x_0, z)\}.$$

Remark 2.0.1. If x_0 lies on a geodesic segment $[y, z]$, $(y.\, z)_{x_0} = 0$, since $d(y, z) = d(y, x_0) + d(x_0, z)$.

Definition 2.0.2. Let $\delta \in \mathbb{R}^+ \cup \{0\}$. (X, d) is called *δ-hyperbolic* with respect to x_0 if

$$(x.\, z)_{x_0} \geq \min\{(x.\, y)_{x_0}, (y.\, z)_{x_0}\} - \delta,$$

for all x, y, $z \in X$.

In general, (X, d) is *hyperbolic* if it is δ-hyperbolic for some $\delta \in \mathbb{R}^+ \cup \{0\}$.

First of all, let us show that the property of being hyperbolic is independent of the basepoint.

Proposition 2.0.1. *If (X,d) is δ-hyperbolic with respect to $x_0 \in X$, then it is 2δ-hyperbolic with respect to any point $t \in X$.*

In order to prove it, the following lemma will be used.

Lemma 2.0.2. *If (X,d) is δ-hyperbolic with respect to $x_0 \in X$, then*

$$(x.\, y)_{x_0} + (z.\, t)_{x_0} - \min\{(x.\, z)_{x_0} + (y.\, t)_{x_0}, (x.\, t)_{x_0} + (y.\, z)_{x_0}\} \geq -2\delta,$$

for any $t \in X$.

Proof. By definition and using basic properties of the minimum,

$$(x.\, y)_{x_0} + (z.\, t)_{x_0} \geq \min\{(x.\, t)_{x_0}, (t.\, y)_{x_0}\} - \delta + (z.\, t)_{x_0} \geq \min\big\{(x.\, t)_{x_0} + \min\{(z.\, y)_{x_0}, (y.\, t)_{x_0}\} - \delta, (t.\, y)_{x_0} + \min\{(z.\, x)_{x_0}, (x.\, t)_{x_0}\} - \delta\big\} - \delta = \min\big\{(x.\, t)_{x_0} + \min\{(z.\, y)_{x_0}\}, (y.\, t)_{x_0}, (t.\, y)_{x_0} + \min\{(z.\, x)_{x_0}, (x.\, t)_{x_0}\}\big\} - 2\delta = \min\{(x.\, t)_{x_0} + (z.\, y)_{x_0}, (x.\, t)_{x_0} + (y.\, t)_{x_0}, (t.\, y)_{x_0} + (z.\, x)_{x_0}\} - 2\delta.$$

Following the same procedure,

$$(x.\, y)_{x_0} + (z.\, t)_{x_0} \geq \min\{(x.\, z)_{x_0} + (z.\, y)_{x_0}, (x.\, z)_{x_0} + (y.\, t)_{x_0}, (z.\, y)_{x_0} + (x.\, t)_{x_0}\} - 2\delta.$$

Suppose that in the first case the minimum is achieved in $(x.\, t)_{x_0} + (y.\, t)_{x_0}$. Then,

$$(y.\, t)_{x_0} \leq (z.\, y)_{x_0} \quad \text{and} \quad (x.\, t)_{x_0} \leq (z.\, x)_{x_0}.$$

If the minimum in the second case is achieved in $(x.\, z)_{x_0} + (z.\, y)_{x_0}$,

$$(z.\, y)_{x_0} \leq (y.\, t)_{x_0} \quad \text{and} \quad (x.\, z)_{x_0} \leq (x.\, t)_{x_0}.$$

Both cases cannot happen if in both cases we have strict inequalities, so the result holds. $\square$

Proposition 2.0.3. *If (X,d) is δ-hyperbolic with respect to $x_0 \in X$, then it is 2δ-hyperbolic with respect to every point $t \in X$.*

Proof. By definition,

$$\min\{(x.\, z)_t, (z.\, y)_t\} - (x.\, y)_t = \min\big\{\tfrac{1}{2}(d(x,t) + d(z,t) - d(z,x)), \tfrac{1}{2}(d(z,t) + d(y,t) - d(z,y))\big\} - \tfrac{1}{2}(d(x,t) + d(y,t) - d(x,y)) = \min\big\{\tfrac{1}{2}(d(x,t) + d(z,t) - d(z,x)), \tfrac{1}{2}(d(z,t) + d(y,t) - d(z,y))\big\} + \tfrac{1}{2}(d(x,y) - d(x,t) - d(y,t)).$$

By adding and substracting $\frac{1}{2}(d(x,t) + d(y,t) + d(z,t))$, that is equal to

$$\min\big\{\tfrac{1}{2}(d(x,t) + d(z,t) - d(z,x)) - \tfrac{1}{2}d(x,t) - \tfrac{1}{2}d(y,t) - \tfrac{1}{2}d(z,t), \tfrac{1}{2}(d(z,t) + d(y,t) - d(y,z)) - \tfrac{1}{2}d(y,t) - \tfrac{1}{2}d(x,t) - \tfrac{1}{2}d(z,t)\big\} + \tfrac{1}{2}d(x,y) + \tfrac{1}{2}d(z,t) = \min\big\{-\tfrac{1}{2}(d(y,t) + d(z,x)), -\tfrac{1}{2}(d(x,t) + d(z,y))\big\} + \tfrac{1}{2}(d(x,y) + d(z,t)).$$

Again we add and substract $\frac{1}{2}(d(x,x_0)+d(y,x_0)+d(z,x_0)+d(t,x_0))$, so we obtain

$$\min\big\{-\tfrac{1}{2}d(y,t)-\tfrac{1}{2}d(z,x)+\tfrac{1}{2}d(x,x_0)+\tfrac{1}{2}d(y,x_0)+\tfrac{1}{2}d(z,x_0)+ \tfrac{1}{2}d(t,x_0), -\tfrac{1}{2}d(x,t)-\tfrac{1}{2}d(z,y)+\tfrac{1}{2}d(x,x_0)+\tfrac{1}{2}d(y,x_0)+\tfrac{1}{2}d(z,x_0)+ \tfrac{1}{2}d(t,x_0)\big\}+\tfrac{1}{2}d(x,y)+\tfrac{1}{2}d(z,t)-\tfrac{1}{2}d(x,x_0)-\tfrac{1}{2}d(y,x_0)-\tfrac{1}{2}d(z,x_0)-\tfrac{1}{2}d(t,x_0) = \min\{(y.t)_{x_0}+(x.z)_{x_0},(x.t)_{x_0}+(z.y)_{x_0}\}-(z.t)_{x_0}-(x.y)_{x_0}.$$

Applying Lemma 2.0.2, that is less or equal than 2δ. □

Proposition 2.0.4. *If $x,y,z,x_0 \in X$ and $\delta \in \mathbb{R}^+ \cup \{0\}$, then*

$$(x.z)_{x_0} \geq \min\{(x.y)_{x_0},(y.z)_{x_0}\}-\delta \iff$$

$$d(x,z)+d(y,x_0) \leq \max\{d(x,y)+d(z,x_0),d(x,x_0)+d(z,y)\}+2\delta.$$

Proof. We need to show when does the following inequality hold:

$$\tfrac{1}{2}(d(x,x_0)+d(z,x_0)-d(x,z)) \geq \min\big\{\tfrac{1}{2}(d(x,x_0)+d(y,x_0)-d(x,y)),\tfrac{1}{2}(d(y,x_0)+d(z,x_0)-d(y,z))\big\}-\delta.$$

Multiplying by 2,

$$d(x,x_0)+d(z,x_0)-d(x,z) \geq \min\{d(x,x_0)+d(y,x_0)-d(x,y),d(y,x_0)+d(z,x_0)-d(y,z)\}-2\delta.$$

Reordering,

$$-d(x,z)-d(y,x_0) \geq \min\{-d(x,y)-d(z,x_0),-d(y,z)-d(x,x_0)\}-2\delta.$$

To conclude with, this happens if an only if

$$d(x,z)+d(y,x_0) \leq -\min\{-d(x,y)-d(z,x_0),-d(y,z)-d(x,x_0)\}+2\delta = \max\{d(x,y)+d(z,x_0),d(y,z)+d(x,x_0)\}+2\delta.$$

□

Example 2.0.1. If (X,d) is a bounded space, that is, if

$$\operatorname{diam}(X) = \sup_{x,y\in X} d(x,y) \leq k < \infty,$$

then (X,d) is k-hyperbolic.

Proof. Let $x_0 \in X$. For all $x,y,z \in X$, we need to prove that

$$(x.y)_{x_0} \geq \min\{(x.z)_{x_0},(y.z)_{x_0}\}-k.$$

We know that $(x.z)_{x_0} \leq \min\{d(x,x_0),d(z,x_0)\}$, so $(x.z)_{x_0} \leq k$. Similarly, $(x.y)_{x_0} \leq k$.
Therefore,

$$\min\{(x.z)_{x_0},(y.z)_{x_0}\} \leq k \leq k+(x.y)_{x_0}.$$

□

Example 2.0.2. If (X, d) is a metric space, we can define another metric on X by

$$d'(x, y) = \ln(1 + d(x, y)) \quad \text{for} \quad x, y \in X.$$

By the triangle inequality, $d'(x, y) \leq \ln(1 + d(x, z) + d(z, y))$.

Let us assume that $d(x, z) = \max\{d(x, z), d(z, y)\}$. Then,

$$d'(x, y) \leq \ln(2(1 + d(x, z)) = \ln(2) + \ln(1 + d(x, z)).$$

In conclusion, $d'(x, y) \leq \ln(2) + \max\{d'(x, z), d'(z, y)\}$.

Using the same argument as in the proof of Lemma 2.0.2, it can be seen that

$$-d'(x, y) - d'(z, t) + \max\{d'(x, z) + d'(y, t), d'(x, t) + d'(y, z)\} \geq -2\ln(2).$$

As a result, by Proposition 2.0.4, (X, d') is δ-hyperbolic for $\delta = 2\ln(2)$.

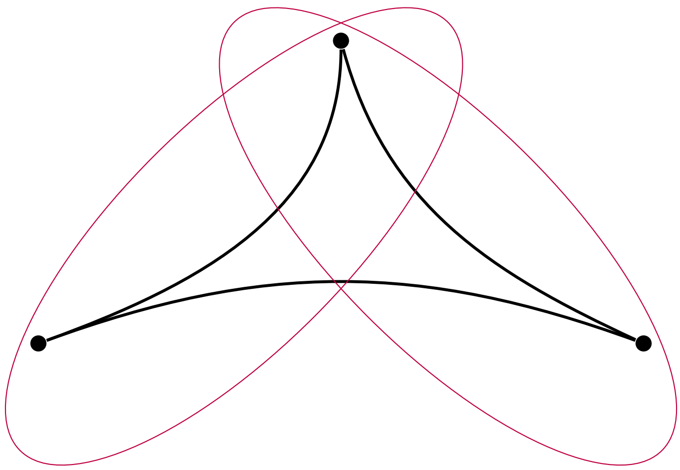
Now, another two definitions will be given and later on it will be proven that they are equivalent.

Suppose that (X, d) is a geodesic metric space.

Definition 2.0.3. A *geodesic triangle* of vertices x, y, z on X is the union of three geodesic segments joining those three points in pairs.

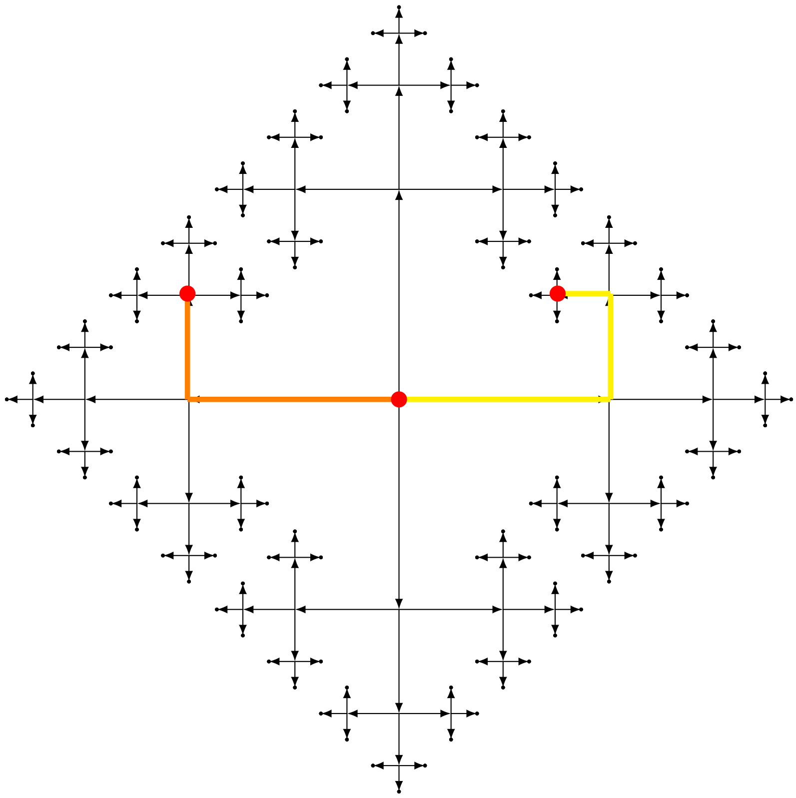
Definition 2.0.4. Let $\delta \in \mathbb{R}^+ \cup \{0\}$. A geodesic triangle of vertices x, y and z is *δ-slim* if for any point $w \in [x, y]$,

$$d(w, [x, z] \cup [y, z]) \leq \delta.$$



Definition 2.0.5. (X, d) satisfies *the Rips condition for δ* if all triangles are δ-slim.

Example 2.0.3. Trees endowed with the standard length metric satisfy the Rips condition for $\delta = 0$.



This follows from the fact that each side of the triangle is contained in the union of the other two sides.

Example 2.0.4. $\mathbb{R}^2$ does not satisfy the Rips condition for any $\delta \in \mathbb{R}^+ \cup \{0\}$, since for a fixed δ we can always construct an isosceles triangle not being δ-slim.

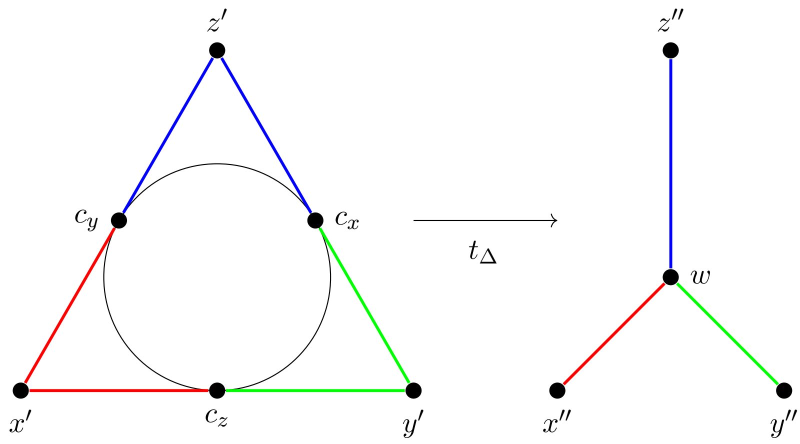
Nevertheless, $\mathbb{R}$ satisfies the Rips condition for $\delta = 0$ by using the same argument as with trees.

Let us continue with the second equivalent definition.
Let Δ be a geodesic triangle with vertices x, y and z. Let Δ' be an Euclidean comparison triangle with vertices x', y', z' such that

$$d_E(x', y') = d_X(x, y), \quad d_E(x', z') = d_X(x, z) \quad \text{and} \quad d_E(y', z') = d_X(y, z).$$

There is a natural map $f \colon \Delta \longrightarrow \Delta'$. Working with the Euclidean metric, there is also a maximum inscribed circle in Δ' which meets the side $[x', y']$ in c_z, $[x', z']$ in c_y and $[y', z']$ in c_x such that

$$d_E(x', c_y) = d_E(x', c_z), \quad d_E(y', c_x) = d_E(y', c_z) \quad \text{and} \quad d_E(z', c_y) = d_E(z', c_x).$$



There is also a unique isometry t_Δ from the triangle Δ' onto a tripod T_Δ (see the above figure) such that

$$d(w, z'') = d_E(z', c_y) = d_E(z', c_x),$$

$$d(w, x'') = d_E(x', c_y) = d_E(x', c_z),$$

$$d(w, y'') = d_E(y', c_x) = d_E(y', c_z).$$

Let $f_\Delta \colon t_\Delta \circ f \colon \Delta \longrightarrow T_\Delta$ and $\delta \in \mathbb{R}^+ \cup \{0\}$.

Definition 2.0.6. The geodesic triangle Δ is *δ-thin* if for all $p, q \in \Delta$ with $f_\Delta(p) = f_\Delta(q)$, then $d_X(p, q) \leq \delta$.

Definition 2.0.7. *Triangles are thin* in (X, d) if there exists $\delta \in \mathbb{R}^+ \cup \{0\}$ such that all geodesic triangles in X are δ-thin.

Let us prove that the stated definitions are equivalent.

Proposition 2.0.5. *(X, d) satisfies the Rips condition if and only if triangles are thin.*

Proof. Trivially, if triangles are thin, (X, d) satisfies the Rips condition.

For the other implication, suppose, by contradiction, that there exists a geodesic triangle $\Delta = [x, y] \cup [y, z] \cup [z, x]$ in X which is δ-slim and $u \in [x, y]$, $v \in [z, x]$ such that $d(u, x) = d(v, x)$ and $d(u, v) > 6\delta$.
Without loss of generality, we may suppose that $u \in [x, c_z]$ and $v \in [x, c_y]$. If we denote $d(x, u) = d(x, v)$ by t, $t \leq d(x, c_z) = (y.\, z)_x$. Then,

$$d(v, [x, y]) = \min\big\{d(v, [x, u]), d(v, [u, y])\big\} \geq \min\{(x.\, u)_v, (u.\, y)_v\}.$$

In the last inequality we have used that, in general, if $d(w, [x, y]) = d(t, w)$ for $t \in [x, y]$, then

$$d(w, t) + d(t, x) \geq d(w, x) \quad \text{and} \quad d(w, t) + d(t, y) \geq d(w, y),$$

but $d(x,t) + d(y,t) = d(x,y)$, so $d(w,[x,y]) \geq (x.\,y)_w$.

Since $d(x,v) = d(x,u)$,

$$2(x.\,u)_v = d(x,v) + d(u,v) - d(x,u) = d(u,v).$$

On the other hand, by using that $d(x,y) = d(x,u) + d(u,y)$,

$$2(u.\,y)_v = d(u,v) + d(y,v) - d(u,y) = d(u,v) + d(y,v) - (d(x,y) - d(x,u)) = d(u,v) + (d(y,v) + d(v,x) - d(x,y)) \geq d(u,v).$$

Then, $d(v,[x,y]) \geq \frac{1}{2}d(u,v) > 3\delta$. In particular, $d(v,x) \geq 3\delta$, so there exists a point $p \in [x,v]$ such that $d(p,v) = \delta + \epsilon$ where $\epsilon < \delta$.

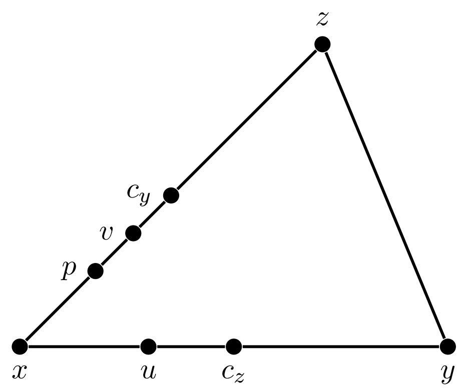
By the triangle inequality,

$$d(p,[x,y]) \geq d(v,[x,y]) - d(v,p) > 3\delta - \delta - \epsilon = 2\delta - \epsilon > \delta.$$

In addition,

$$d(p,[y,z]) \geq d(x,[y,z]) - d(x,p) \geq (y.\,z)_x - d(x,p) \geq t - d(x,p) = d(x,v) - d(x,p) = d(p,v) = \delta + \epsilon > \delta.$$

To sum up, $d(p,[x,y] \cup [y,z]) > \delta$ which contradicts the hypothesis.



□

To finish proving the equivalence of the definitions, it remains the following proposition.

Proposition 2.0.6. (X,d) *is hyperbolic if and only if triangles are thin.*

Proof. Let Δ be a geodesic triangle and $u,v \in \Delta$ distinct points with $f_\Delta(u) = f_\Delta(v)$. Without loss of generality, we may suppose that $u \in [x,c_z]$ and $v \in [x,c_y]$. Let us prove that $d(u,v) \leq 4\delta$.

Let $t = d(x,u) = d(x,v) \leq (y.\,z)_x$. By hypothesis,

$$(u.\,v)_x \geq \min\{(u.\,y)_x, (y.\,v)_x\} - \delta \text{ and } (y.\,v)_x \geq \min\{(y.\,z)_x, (z.\,v)_x\} - \delta.$$

Then,

$$(u.v)_x \geq \min\left\{(u.y)_x, \min\{(y.z)_x,(z.v)_x\}\right\} - 2\delta,$$

which is equal to $\min\{(u.y)_x,(y.z)_x,(z.v)_x\} - 2\delta$.
Observe that $(u.y)_x = \frac{1}{2}(d(u,x)+d(y,x)-d(u,y)) = d(u,x) = t$, and by the same reason, $(z.v)_x = t$. As a consequence,

$$(u.v)_x \geq t - 2\delta.$$

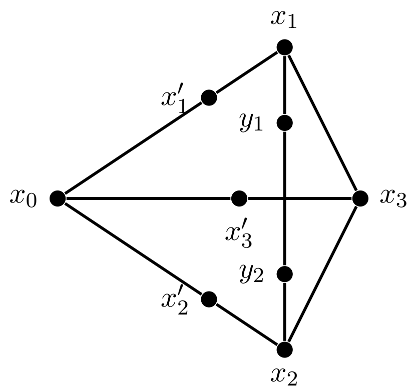
Moreover, $(u.v)_x = t - \frac{1}{2}d(u,v)$, so

$$d(u,v) = 2t - 2(u.v)_x \leq 2t - 2(t-2\delta) = 4\delta.$$

To prove the other implication, let $x_0, x_1, x_2, x_3 \in X$ and let us check that

$$(x_1.x_2)_{x_0} \geq t - 2\delta,$$

if $t = \min\{(x_1.x_3)_{x_0},(x_2.x_3)_{x_0}\}$.
Suppose that we have 6 geodesic segments in X joining the points by pairs.



We may assume that $t > (x_1.x_2)_{x_0}$. Otherwise, the proof would be finished. For $j \in \{1,2,3\}$, let x_j' be the point that lies in $[x_0,x_j]$ such that $d(x_0,x_j') = t$ (recall that $t \leq d(x_0,x_i)$ for $i \in \{1,2,3\}$). For $j \in \{1,2\}$, let $(f_\Delta)_{0j3}$ be the map of the Definition 2.0.6 for the triangle $[x_0,x_j]\cup[x_j,x_3]\cup[x_3,x_0]$.
Since $d(x_j',x_0) = d(x_3',x_0) \leq (x_j.x_3)_{x_0}$,

$$(f_\Delta)_{0j3}(x_j') = (f_\Delta)_{0j3}(x_3'),$$

so by hypothesis, $d(x_j',x_3') \leq \delta$. Therefore, $d(x_1',x_2') \leq 2\delta$.

Since $t > (x_1.x_2)_{x_0}$, there exists $y_j \in [x_1,x_2]$ such that $(f_\Delta)_{012}(x_j') = (f_\Delta)_{012}(y_j)$, so by hypothesis, $d(x_j',y_j) \leq \delta$.
By the triangle inequality,

$$2\delta \geq d(x_1',x_2') \geq d(y_1,y_2) - 2\delta = d(x_1,x_2) - d(x_1,y_1) - d(x_2,y_2) - 2\delta.$$

Now, $d(x_1,y_1) = d(x_1',x_1)$ and $d(x_2,y_2) = d(x_2,x_2')$, so that equals

$$d(x_1,x_2) - (d(x_1,x_0) - d(x_1',x_0)) - (d(x_2,x_0) - d(x_2',x_0)) - 2\delta = \\ 2t - 2(x_1.\,x_2)_{x_0} - 2\delta.$$

In conclusion, $(x_1.\,x_2)_{x_0} \geq t - 2\delta$. □

Definition 2.0.8. Let $\Delta = [x,y] \cup [y,z] \cup [x,z]$ be a geodesic triangle and let x', y' and z' be points in Δ where $x' \in [y,z]$, $y' \in [x,z]$ and $z' \in [x,y]$. The *insize* of the triangle Δ is defined to be

$$\operatorname{minsize}(\Delta) = \inf \operatorname{diam}\{x', y', z'\},$$

where the infimum is taken over all triples of points $\{x', y', z'\}$.

Observe that if Δ is a slim δ-triangle and we define the following sets,

$$N^+ = \{p \in [x,z] \mid d(p,[x,y]) \leq \delta\},$$

$$N^- = \{p \in [x,z] \mid d(p,[z,y]) \leq \delta\},$$

then $[x,z] = N^+ \cup N^-$ and N^+ and N^- are closed sets, so there exists a point $y' \in N^+ \cap N^-$. Therefore,

$$d(y',z') \leq \delta \quad \text{and} \quad d(y',x') \leq \delta,$$

for some $z' \in [x,y]$ and $x' \in [y,z]$. Then, $\operatorname{diam}\{x',y',z'\} \leq 2\delta$, so $\operatorname{minsize}(\Delta) \leq 2\delta$.

A similar notion can be defined but only taking into account the points c_x, c_y and c_z.

Definition 2.0.9. Let $\Delta = [x,y] \cup [y,z] \cup [x,z]$ be a geodesic triangle. Then,

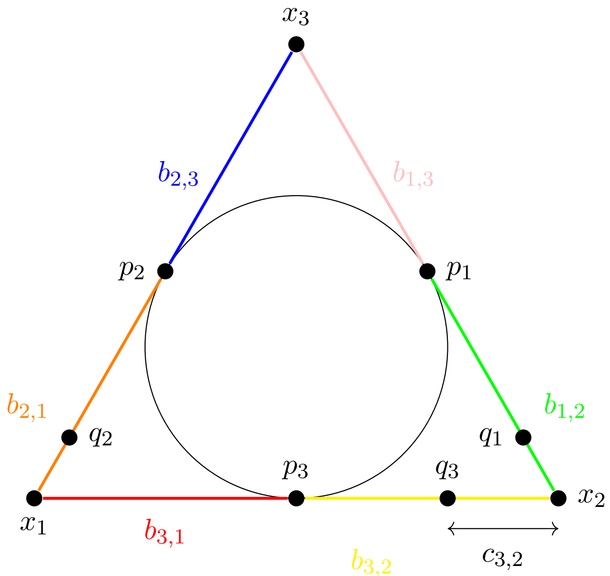
$$\operatorname{insize}(\Delta) = \operatorname{diam}\{c_x, c_y, c_z\}.$$

Remark 2.0.2. $\operatorname{minsize}(\Delta) \leq \operatorname{insize}(\Delta)$.

Lemma 2.0.7. *Let $\Delta = [x_1,x_2] \cup [x_2,x_3] \cup [x_3,x_1]$ be a geodesic triangle such that $\delta' = \operatorname{insize}(\Delta)$ and $\delta = \operatorname{minsize}(\Delta)$. Then,*

$$\delta \leq \delta' \leq 4\delta.$$

Proof. We only need to prove that $\delta' \leq 4\delta$.



Let $q_1 \in [x_2, x_3]$, $q_2 \in [x_1, x_3]$ and $q_3 \in [x_1, x_2]$ such that $\delta = \operatorname{diam}\{q_1, q_2, q_3\}$. Let p_1, p_2 and p_3 be the points of the inscribed circle that meet the geodesic triangle.

Let $a_1 = d(x_2, x_3)$, $a_2 = d(x_1, x_3)$, $a_3 = d(x_1, x_2)$, $b_{i,j}$ as drawn on the picture, and $c_{i,j} = d(q_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Then, by the properties of the inscribed circle,

$$b_{i,k} = b_{j,k} = \frac{1}{2}(a_i + a_j - a_k) \quad \text{and} \quad b_{i,j} + b_{i,k} = c_{i,j} + c_{i,k} = a_i.$$

Since $d(q_i, q_j) \leq \delta$,

$$d(q_i, x_k) - d(q_j, x_k) \leq d(q_i, q_j) \leq \delta \quad \text{and} \quad d(q_j, x_k) - d(q_i, x_k) \leq \delta.$$

That is, $|c_{i,k} - c_{j,k}| \leq \delta$. As a consequence,

$$\begin{gathered} 2b_{i,k} = a_i + a_j - a_k = c_{i,j} + c_{i,k} + b_{j,i} + b_{j,k} - c_{k,i} - c_{k,j}, \\ c_{i,j} + c_{i,k} + b_{k,i} - b_{i,k} - c_{k,i} - c_{k,j} = 0, \\ |c_{i,k} - b_{i,k} - c_{k,i} + b_{k,i}| = |c_{i,j} - c_{k,j}| \leq \delta. \end{gathered}$$

If we denote $c_{1,2} - b_{1,2}$ by d_1, we have that

$$d_1 = c_{1,2} - b_{1,2} = d(q_1, x_2) - d(p_1, x_2) = d(p_1, x_3) - d(q_1, x_3) = -(c_{1,3} - b_{1,3}).$$

In the same way,

$$d_2 = c_{2,3} - b_{2,3} = -(c_{2,1} - b_{2,1}) \quad \text{and} \quad d_3 = c_{3,1} - c_{3,1} = -(c_{3,2} - b_{3,2}).$$

Hence,

$$|d_1| = d(p_1, q_1), \quad |d_2| = d(p_2, q_2) \quad \text{and} \quad |d_3| = d(p_3, q_3).$$

In addition, $|d_1 + d_2| = |c_{1,2} - b_{1,2} - c_{2,1} + b_{2,1}| \leq \delta$, and in the same fashion, $|d_i + d_j| \leq \delta$, for $i, j \in \{1, 2, 3\}$.
As a result,

$$|d_i| = \frac{1}{2}|d_i + d_j + d_i + d_k - d_j - d_k| \leq \frac{3}{2}\delta.$$

In conclusion, $d(p_j, p_k) \leq d(p_j, q_j) + d(q_j, q_k) + d(q_k, p_k) \leq \frac{3}{2}\delta + \delta + \frac{3}{2}\delta = 4\delta$. □

Proposition 2.0.8. *The following are equivalent:*

(1) Triangles are slim;

(2) There is a global bound on the insize of geodesic triangles;

(3) There is a global bound on the minsize of geodesic triangles.

Proof. Let us start proving (1) $\implies$ (2). Let $\Delta = [x, y] \cup [y, z] \cup [z, x]$ be a geodesic triangle in X, and let c_x, c_y and c_z be the internal points.
By hypothesis, there exists $t \in [x, z] \cup [y, z]$ such that $d(c_z, t) \leq \delta$. Without loss of generality, suppose that $t \in [x, z]$. On the one hand,

$$d(c_y, x) = d(c_z, x) \leq d(x, t) + d(t, c_z) \leq d(x, t) + \delta.$$

On the other hand,

$$d(x, t) \leq d(x, c_z) + d(c_z, t) \leq d(x, c_z) + \delta.$$

Then, $d(t, c_y) = d(x, c_y) - d(x, t) \leq d(x, t) + \delta - d(x, t) = \delta$ and $d(c_y, c_z) \leq d(c_y, t) + d(t, c_z) \leq 2\delta$.
Similarly, we can prove that

$$d(c_x, c_z) \leq 2\delta \quad \text{and} \quad d(c_x, c_y) \leq 2\delta.$$

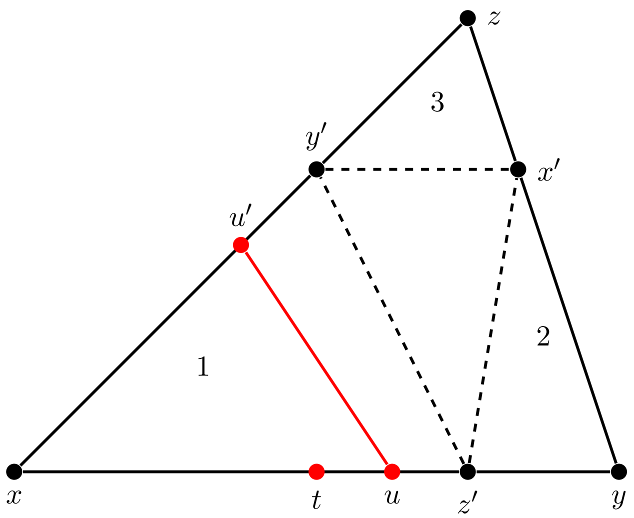
In conclusion, $\operatorname{diam}\{c_x, c_y, c_z\} \leq 4\delta$.

That (2) implies (3) is proved in Lemma 2.0.7.

Finally, we need to show (3) $\implies$ (1).

Let $\Delta = [x, y] \cup [y, z] \cup [z, x]$ be a geodesic triangle and $x' \in [z, y]$, $y' \in [x, z]$

and $z' \in [x, y]$ such that $\operatorname{diam}\{x', y', z'\} \leq \delta$.



The problem is therefore reduced to studying three geodesic triangles. Without loss of generality, assume that we work in 1.

Suppose that there is a point $t \in [x, z']$ such that $d(t, [x, y']) > 2\delta$. Otherwise, the proof would be finished. Since $d(z', y') \leq \delta$, we can take $u \in [t, z']$ the nearest point to t such that $d(u, u') \leq 2\delta$ for some $u' \in [x, y']$.

By hypothesis, there exist $a \in [x, u]$, $b \in [u, u']$ and $c \in [x, u']$ such that $\operatorname{diam}\{a, b, c\} \leq \delta$. In addition, a does not lie in $[t, u]$; otherwise, u would not be the nearest point to t satisfying $d(u, u') \leq 2\delta$ for $u' \in [x, y']$.

By the triangle inequality, we obtain these two inequalities:

$$d(u, a) \leq d(a, b) + d(b, u) \leq d(a, b) + d(u, u') \leq 3\delta,$$

$$d(t, u') \leq d(t, u) + d(u, u') \leq 2\delta + d(t, u).$$

On the one hand, if $d(t, u) \leq \delta$, then $d(t, u') \leq 3\delta$. On the other hand, observe that $d(u, a) = d(u, t) + d(t, a)$, so if $d(t, u) > \delta$, $d(t, a) \leq 2\delta$. As a consequence, $d(t, c) \leq d(t, a) + d(a, c) \leq 3\delta$.
To sum up, $d(t, [x, y']) \leq 3\delta$. □

Corollary 2.0.9. *The Poincaré disc is a hyperbolic space.*

Proof. By the Gauss-Bonnet formula, if Δ is a hyperbolic triangle with interior angles α, β and γ, then

$$\operatorname{area}(\Delta) = \pi - (\alpha + \beta + \gamma) \leq \pi.$$

Therefore, the area of the inscribed circle is at most π, so the diameter has an upper bound. □

Chapter 3

Construction of the Rips Complex, quasi-geodesics and quasi-isometries

Let us fix the group G and a finite generating set S. Moreover, the word metric associated to S will be denoted by d instead of d_S.

3.1 The Rips Complex

Definition 3.1.1. Let $n \in \mathbb{N}$. The *Rips complex* $P_n(G,S)$ is the simplicial complex where the 0-simplices are the elements of the group G and the k-simplices are $(k+1)$-tuples $(g_0, \cdots, g_k)$ of distinct elements of G such that $d(g_i, g_j) \leq n$ for all i, $j \in \{0, \cdots, k\}$.

Remark 3.1.1. Let $b = |B(e,n)| = |\{g \in G \mid d(e,g) \leq n\}|$. Note that b is finite since S is finite. Furthermore, the dimension of $P_n(G,S)$ is at most $b-1$. Suppose, by contradiction, that there exist $g_0, \cdots, g_b \in G$ such that

$$d(g_0, g_i) \leq n \quad \text{for} \quad i \in \{0, \cdots, b\}.$$

Then, $g_0^{-1}g_0, \cdots, g_0^{-1}g_b \in B(e,n)$ and they are all distinct.

Moreover, every 0-simplex is the boundary of exactly $b-1$ 1-simplices, since if $d(g_1, g_2) \leq n$ for some g_1, $g_2 \in G$, then $g_1^{-1}g_2 \in B(e,n)$ and this happens exactly b times, but one case is $g_1^{-1}g_1$.

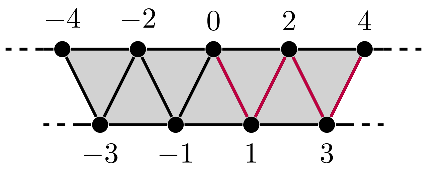
Remark 3.1.2. The 1-skeleton of $P_n(G,S)$ is the Cayley graph of G with respect to the set $B(e,n) \setminus \{e\}$, which is in fact a generating set because $S \subseteq B(e,n) \setminus \{e\}$.

Hence, two 0-simplices g_1 and g_2 can be joined by a geodesic segment which lies in Cayley(G,S). For $g_1^{-1}g_2$ there is a minimum $l \in \mathbb{N}$ such that $g_1^{-1}g_2 =$

$\mu(s_1 s_2 \cdots s_l)$ where $s_i \in S$, $i \in \{1, \cdots, l\}$. Thus, there is a geodesic segment $[g_1, g_2]$ in Cayley(G, S) which is the union of the 1-simplices

$$(g_1, g_1 s_1), \quad (g_1 s_1, g_1 s_1 s_2), \quad (g_1 s_1 \cdots s_{l-1}, g_2).$$

Example 3.1.1. Let $G = \mathbb{Z}$ and $S = \{1, -1\}$. The Rips Complex $P_2(G, S)$ can be represented as follows.



In this case, $b = 5$ and the dimension is 2. The geodesic segment $[0, 4]$ is drawn in purple.

Observe that G acts on $P_n(G, S)$:

$$G \times P_n(G, S) \to P_n(G, S)$$
$$(g, (g_0, g_1, \cdots, g_k)) \mapsto (gg_0, gg_1, \cdots, gg_k).$$

Proposition 3.1.1. *This action has the following properties:*

(i) It is faithful;

(ii) The stabilizer of all the simplices is finite;

(iii) It is properly discontinuous;

(iv) If G is torsion free, it is free;

(v) $G/P_n(G, S)$ is compact.

Proof. (i) The action is free in the set of 0-simplices, so it is faithful.

(ii) If $g \in G$ leaves invariant the k-simplex $(g_0, \cdots, g_k)$, then g permutes the $k+1$ elements $g_0, \cdots, g_k$. Suppose, without loss of generality, that

$$gg_i = g_{i+1} \quad \text{for} \quad i \in \{0, \cdots, k-1\} \quad \text{and} \quad gg_k = g_0.$$

If the number of elements that permute $g_0, \cdots, g_k$ is greater than or equal to k, then $gg_0 = hg_0$ for another $h \in G$ which leaves invariant the k-simplex. Hence, $g = h$.

(iii) If σ, σ' are two simplices,

$$\{g \in G \mid g\sigma \cap \sigma' \neq \varnothing\}$$

is finite. Since every compact subset of $P_n(G, S)$ can be covered by a finite number of simplices, the claim is proved.

(iv) Suppose that $g \in G$ leaves invariant the k-simplex $(g_0, \cdots, g_k)$. Then, for $n \in \mathbb{N}$, g^n also has that property. Nevertheless, the stabilizer is finite, so $g^{n_0} = 1$ for some $n_0 \in \mathbb{N}$. In conclusion, if G is torsion free, $g = 1$.

(v) Let A be the finite union of all the simplices such that e is one of the vertices. A is compact, so G/A is a compact space. Let us define φ as follows:

$$\varphi \colon G/A = \big\{\mathrm{Orb}(x) \mid x \in A\big\} \to G/P_n(G,S) = \big\{\mathrm{Orb}(y) \mid y \in P_n(G,S)\big\}$$
$$\mathrm{Orb}(x) \mapsto \mathrm{Orb}(x).$$

Firstly, φ is continuous because $\varphi \circ \pi_A = \pi_{P_n(G,S)} \circ \iota$, where π_A and $\pi_{P_n(G,S)}$ are the quotient maps and ι is the inclusion map. Secondly, φ is clearly injective. Thirdly, let us check that φ is surjective. Let $(g_0, \cdots, g_k)$ be a k-simplex. Then, $(g_0^{-1}g_0, \cdots, g_0^{-1}g_k)$ lies in A, and

$$\varphi\big(\mathrm{Orb}((g_0^{-1}g_0, \cdots, g_0^{-1}g_k))\big) = \mathrm{Orb}((g_0, \cdots, g_k)).$$

Finally, it can be shown that φ^{-1} is continuous by using the same argument as above. □

Definition 3.1.2. The group G is *hyperbolic* (*δ-hyperbolic*) for S if it is hyperbolic (δ-hyperbolic) as a metric space with the word metric associated to S.
Equivalently, G is hyperbolic (δ-hyperbolic) if Cayley(G,S) is hyperbolic (δ-hyperbolic) with the standard length metric.

From now on, G will be assumed to be δ-hyperbolic for S with $\delta \in \mathbb{R}^+ \cup \{0\}$.

Theorem 3.1.2 (Rips' Theorem). *If $n \geq 4\delta + 2$, $P_n(G,S)$ is contractible.*

Two statements are needed before proving the theorem.

Proposition 3.1.3. *Let $n \in \mathbb{N}$ such that $n \geq 4\delta+2$ and let g_0 be a 0-simplex of $P_n(G,S)$. If g is a 0-simplex of $P_n(G,S)$ such that $d(g_0, g) > \lfloor \frac{n}{2} \rfloor$, there exists a 0-simplex g' of $P_n(G,S)$ such that*

(i) $d(g_0, g') = d(g_0, g) - d(g', g)$;

(ii) $d(g', g) = \lfloor \frac{n}{2} \rfloor$;

(iii) For all 0-simplex g'' of $P_n(G,S)$, $d(g', g'') \leq \max\{\lfloor \frac{n}{2} \rfloor + d(g_0, g'') - d(g_0, g), d(g, g'') - \lfloor \frac{n}{2} \rfloor\} + 2\delta$.

Proof. Let $[g_0, g]$ be a geodesic segment as in Remark 3.1.2. Then, we take g' to be the 0-simplex of that segment with

$$d(g', g) = \lfloor \frac{n}{2} \rfloor,$$

so (i) and (ii) hold.
Since G is δ-hyperbolic, by Proposition 2.0.4,

$$d(g',g'')+d(g_0,g)\leq \max\{d(g',g)+d(g_0,g''),d(g_0,g')+d(g'',g)\}+2\delta.$$

Then,

$$d(g',g'')\leq \max\{d(g',g)+d(g_0,g'')-d(g_0,g),d(g_0,g')+d(g'',g)-d(g_0,g)\}$$

$$+2\delta,$$

which is equal to

$$\max\Big\{\lfloor\frac{n}{2}\rfloor+d(g_0,g'')-d(g_0,g),d(g'',g)-d(g',g)\Big\}+2\delta=$$

$$\max\Big\{\lfloor\frac{n}{2}\rfloor+d(g_0,g'')-d(g_0,g),d(g,g'')-\lfloor\frac{n}{2}\rfloor\Big\}+2\delta.$$

□

Proposition 3.1.4. *Let $n\in\mathbb{N}$ such that $n\geq 4\delta+2$, K be a finite simplicial complex with vertices $\{p_0,\cdots,p_k\}$ and f be a simplicial map $K\longrightarrow P_n(G,S)$. Then, there exists a homotopy $h\colon K\times[0,1]\longrightarrow P_n(G,S)$ from f to a simplicial map $f'\colon K\longrightarrow P_n(G,S)$ such that*

$$d(f'(p_0),f'(p_j))\leq\frac{n}{2}\quad \textit{for all}\quad j\in\{0,\cdots,k\}.$$

Proof. Let $i\in\{0,\cdots,k\}$ such that

$$d(f(p_0),f(p_i))=\sup_{0\leq j\leq k}d(f(p_0),f(p_j)).$$

If $d(f(p_0),f(p_i))\leq\lfloor\frac{n}{2}\rfloor$, the statement holds.
Otherwise, applying Proposition 3.1.3 to $g_0=f(p_0)$ and $g=f(p_i)$, there exists g' in $P_n(G,S)$ such that

$$d(f(p_0),g')=d(f(p_0),f(p_i))-\lfloor\frac{n}{2}\rfloor,$$

and for $f(p_0),\cdots,f(p_k)$,

$$d(g',f(p_j))\leq\max\Big\{\lfloor\frac{n}{2}\rfloor+d(f(p_0),f(p_j))-d(f(p_0),f(p_i))+2\delta,$$

$$d(f(p_i),f(p_j))-\lfloor\frac{n}{2}\rfloor+2\delta\Big\}\leq$$

$$\max\Big\{\lfloor\frac{n}{2}\rfloor+2\delta,d(f(p_i),f(p_j))-\lfloor\frac{n}{2}\rfloor+2\delta\Big\}.$$

Moreover, $\lfloor \frac{n}{2} \rfloor + 2\delta \leq \frac{n}{2} + 2\delta \leq n$ and $2\delta + 1 \leq \frac{n}{2}$, so $2\delta \leq \lfloor \frac{n}{2} \rfloor$.
Hence, for $j \in \{0, \cdots, k\}$,

$$d(g', f(p_j)) \leq \max\{n, d(f(p_i), f(p_j))\}. \tag{3.1}$$

Let $h_1 \colon K \times [0,1] \longrightarrow P_n(G,S)$ be defined in the vertices of K as follows:

$$h_1(p_j, t) = \begin{cases} f(p_j), & f(p_j) \neq f(p_i), \\ tg' + (1-t)f(p_i) & f(p_j) = f(p_i). \end{cases}$$

We extend the definition linearly to all the simplices of K:

$$h_1\left(\sum_{j=0}^{k} \lambda_j p_j, t\right) = \sum_{j=0}^{k} \lambda_j h_1(p_j, t).$$

Because of (3.1), h_1 is a homotopy from f to a simplicial map f_1 verifying the following properties:

(i) If p_j is a vertex of K such that $d(f(p_0), f(p_j)) \leq \lfloor \frac{n}{2} \rfloor$, then $f_1(p_j) = f(p_j)$. This follows from the fact that

$$f_1(p_j) = h_1(p_j, 1) = \begin{cases} f(p_j), & f(p_j) \neq f(p_i), \\ g' & f(p_j) = f(p_i), \end{cases}$$

and $f(p_j) \neq f(p_i)$.

(ii) $d(f_1(p_0), f_1(p_i)) = d(f(p_0), f(p_i)) - \lfloor \frac{n}{2} \rfloor$.

If $d(f_1(p_0), f_1(p_j)) \leq \frac{n}{2}$ for all $j \in \{0, \cdots, k\}$, we obtain what we want to prove. Otherwise, we choose p_i' in K such that

$$d(f_1(p_0), f_1(p_i')) = \sup_{0 \leq j \leq k} d(f_1(p_0), f_1(p_j)),$$

and we repeat the same procedure.
In that way, we obtain a finite number of homotopies and by composing them we verify the claim. □

At this stage, the proof of Rips' Theorem turns out to be easy.

Proof of Rips' Theorem. If every finite subcomplex is contractible, then the simplicial complex is contractible. Therefore, we can just consider a subcomplex K of $P_n(G,S)$ with 0-simplices $\{p_0, \cdots, p_k\}$. By the previous proposition, the inclusion map ι is homotopic to a map f' where its image is contained in the ball, which will be denoted by B,

$$\left\{ g \in G \mid d(\iota(p_0), g) \leq \frac{n}{2} \right\}.$$

Since $\max\{d(g', g'') \mid g', g'' \in B\} \leq n$, the image of f' is contained in a simplex of $P_n(G,S)$, so K is contractible. □

3.2 Quasi-geodesics and quasi-isometries

Note that until this point, hyperbolicity has depended on the generating set. As a consequence, the next step will be to prove that the property is independent from the generating set. In order to obtain that goal, some technical definitions and properties are necessary.

Definition 3.2.1. Let (X, d) be a metric space and $f\colon [a, b] \longrightarrow X$ be a path. f is (λ, k)-*quasi-geodesic* (with $\lambda \geq 1$ and $k \geq 0$) if for all $[a', b'] \subseteq [a, b]$,

$$\operatorname{length} f([a', b']) \leq \lambda d(f(a'), f(b')) + k.$$

f is (λ, k, L)-*local quasi-geodesic* (with $\lambda \geq 1$, $k \geq 0$ and $L > 0$) if for all $[a', b'] \subseteq [a, b]$ and $\operatorname{length} f([a', b']) \leq L$,

$$\operatorname{length} f([a', b']) \leq \lambda d(f(a'), f(b')) + k.$$

The following theorem and lemma will not be proved due to the lack of interest of the proofs, which can be found in [5].

Theorem 3.2.1. *Let (X, d) be a geodesic δ-hyperbolic metric space and $\lambda, k \in \mathbb{R}$, $\lambda \geq 1$, $k \geq 0$. Then, there exist L, $\epsilon \in \mathbb{R}$ depending only on δ, λ and k such that for all bounded interval $[a, b]$ of $\mathbb{R}$, if $f\colon [a, b] \longrightarrow X$ is a local quasi-geodesic, $f([a, b])$ is contained in the ϵ-neighbourhood of every geodesic segment joining $f(a)$ and $f(b)$. Moreover, the converse also holds: every geodesic segment joining $f(a)$ and $f(b)$ is contained in some neighbourhood of $f([a, b])$.*

Lemma 3.2.2. *Let (X, d) be a metric space, $\gamma\colon [a, b] \longrightarrow X$ be a path and $[x, y]$ be a geodesic segment with $x = \gamma(a)$ and $y = \gamma(b)$. Suppose that $\gamma([a, b])$ is contained in the k-neighbourhood of $[x, y]$ for some $k \in \mathbb{R}$. Then, $[x, y]$ is contained in the $2k$-neighbourhood of $\gamma([a, b])$.*

Definition 3.2.2. Let (X_1, d_1) and (X_2, d_2) be two metric spaces, λ, $k \in \mathbb{R}$, $\lambda > 0$ and $f\colon X_1 \longrightarrow X_2$. f is a (λ, k)-*quasi-isometry* if for all x,$y \in X_1$,

$$\lambda^{-1} d_1(x, y) - k \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y) + k.$$

f is a (λ, k)-*strong quasi-isometry* if for all x, $y \in X_1$,

$$\lambda^{-1} d_1(x, y) \leq d_2(f(x), f(y)) \leq \lambda d_1(x, y).$$

Lemma 3.2.3. *Let (X_1, d_1) and (X_2, d_2) be two metric spaces and f be a (λ, k)-strong quasi-isometry between them. Then, for all geodesic segments $[x, y]$ in X_1, $f([x, y])$ is $(\lambda^2, \lambda^2 k)$-quasi-geodesic.*

Proof. Suppose that $\varphi\colon [a,b] \longrightarrow X_1$ is geodesic, $\varphi(a) = x$, $\varphi(b) = y$ and $g = f \circ \varphi$. Let $[a', b'] \subseteq [a,b]$. By definition,

$$\text{length}\, g([a',b']) = \sup \sum d_2(g(a_i), g(a_{i+1})) =$$

$$\sup \sum d_2(f(\varphi(a_i)), f(\varphi(a_{i+1}))).$$

f is a (λ, k)-strong quasi-isometry, so the above line is less than or equal to

$$\lambda \sup \sum d_1(\varphi(a_i), \varphi(a_{i+1})) = \lambda \text{length}\, [x', y'].$$

Again, by the same reason, it is less than or equal to

$$\lambda^2 \big(d_2(f(x'), f(y')) + k\big) = \lambda^2 d_2(f(x'), f(y')) + \lambda^2 k.$$

□

Lemma 3.2.4. *Let (X_1, d_1) and (X_2, d_2) be two geodesic metric spaces, X_2 be δ-hyperbolic and $f\colon X_1 \longrightarrow X_2$ be a (λ, k)-strong quasi-isometry. Then, there exists a constant $A \in \mathbb{R}$ which depends only on δ, k and λ such that for every geodesic segment $[f(x), f(y)]$ and every point z in $f([x,y])$,*

$$d_2(z, [f(x), f(y)]) \leq A.$$

Proof. By the previous lemma, $f([x,y])$ is $(\lambda^2, \lambda^2 k)$-quasi-geodesic. Finally, applying Theorem 3.2.1, there exists a constant $A \in \mathbb{R}$ which depends only on δ, k and λ such that $f([x,y])$ is contained in the A-neighbourhood of $[f(x), f(y)]$. □

The following corollary is just a consequence of Lemma 3.2.2 and Lemma 3.2.4.

Corollary 3.2.5. *Under the same hypothesis as above, there exists a constant $c \in \mathbb{R}$ such that for all geodesic segments $[f(x), f(y)]$ and z in $[f(x), f(y)]$,*

$$d_2(z, f([x,y])) \leq c.$$

The fact that being hyperbolic is independent from the generating set will be proven by using the previous results, as follows. Firstly, it will be shown that if there is a strong quasi-isometry between two geodesic metric spaces and one of them is hyperbolic, then the other one is too. Secondly, if S_1 and S_2 are two generating sets of a group G, a strong quasi-isometry will be defined between $\text{Cayley}(G, S_1)$ and $\text{Cayley}(G, S_2)$.

Theorem 3.2.6. *Let (X_1, d_1) and (X_2, d_2) be two geodesic metric spaces with X_2 hyperbolic and a strong quasi-isometry $f\colon X_1 \longrightarrow X_2$. Then, X_1 is also hyperbolic.*

Proof. Let $\Delta_1 = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_1]$ be a geodesic triangle and suppose that f is a (λ, k)-strong quasi-isometry. Let us consider a geodesic triangle

$$\Delta_2 = [f(x_1), f(x_2)] \cup [f(x_2), f(x_3)] \cup [f(x_3), f(x_1)].$$

X_2 is δ-hyperbolic, so the minsize of Δ_2 is less than or equal to 4δ. Therefore, there exist $z_1 \in [f(x_2), f(x_3)]$, $z_2 \in [f(x_1), f(x_3)]$ and $z_3 \in [f(x_2), f(x_1)]$ such that $d_2(z_i, z_j) \leq 4\delta$ for $i, j \in \{1, 2, 3\}$.

By the second part of Theorem 3.2.1, there exists $c \in \mathbb{R}$ such that

$$d_2(z_i, f(c_i)) \leq c,$$

where $c_1 \in [x_2, x_3]$, $c_2 \in [x_1, x_3]$ and $c_3 \in [x_1, x_2]$.
Hence,

$$d_2(f(c_i), f(c_j)) \leq d_2(f(c_i), z_i) + d_2(z_i, z_j) + d_2(f(c_j), z_j) \leq 2c + 4\delta.$$

Therefore,

$$d_1(c_i, c_j) \leq \lambda d_2(f(c_i), f(c_j)) + k \leq \lambda(2c + 4\delta) + k.$$

That is, minsize $(\Delta_1) \leq 4\delta'$ with $\delta' = \lambda(2c + 4\delta) + k$, so X_1 is δ'-hyperbolic. □

From now on, assume that S_1 and S_2 are two finite generating sets of a group G.

Lemma 3.2.7. *There exists $c \in \mathbb{R}$ such that*

$$c^{-1} d_{S_2} \leq d_{S_1} \leq c d_{S_2}.$$

Proof. Suppose that every element of S_1 can be written with at most c_1 elements of S_2 and that every element of S_2 can be written with at most c_2 elements of S_1. Let $c = \max\{c_1, c_2\}$ and $g \in G$.
Suppose that $|g|_{S_1} = n$ and $g = s_1 \cdots s_n$ in S_1. Then, by hypothesis, g can be written with at most $c_1 n$ elements of S_2, so

$$|g|_{S_2} \leq c_1 n = c|g|_{S_1}.$$

The other inequality may be proven in the same fashion. □

Finally, let us define a map f from Cayley(G, S_1) to Cayley(G, S_2) such that the restriction of f to G is the identity and every edge is sent, by linearity, to a geodesic segment.

Lemma 3.2.8. *f is a strong quasi-isometry.*

Proof. Observe that for all x_1, $x_2 \in \text{Cayley}(G, S_1)$,

$$d_{S_2}(f(x_1), f(x_2)) \leq \text{length}\, f([x_1, x_2]) \leq c d_{S_1}(x_1, x_2),$$

where c is the constant of Lemma 3.2.7. Let g_1 and g_2 be the vertices of $\text{Cayley}(G, S_1)$ such that

$$d_{S_1}(x_1, g_1) \leq \frac{1}{2} \quad \text{and} \quad d_{S_1}(x_2, g_2) \leq \frac{1}{2}.$$

Since the restriction of f to G is the identity map, $d_{S_1}(g_1, g_2) \leq c d_{S_2}(f(g_1), f(g_2))$. By the triangle inequality,

$$c d_{S_2}(f(g_1), f(g_2)) \leq c\big(d_{S_2}(f(g_1), f(x_1)) + d_{S_2}(f(x_1), f(x_2)) + d_{S_2}(f(x_2), f(g_2))\big).$$

Again by Lemma 3.2.7, the right hand side of the above inequality is less than or equal to

$$c\big(d_{S_2}(f(x_1), f(x_2)) + c d_{S_1}(g_1, x_1) + c d_{S_1}(g_2, x_2)\big),$$

which is less than or equal to

$$c\Big(d_{S_2}(f(x_1), f(x_2)) + \frac{c}{2} + \frac{c}{2}\Big).$$

Then, $d_{S_1}(x_1, x_2) \leq d_{S_1}(x_1, g_1) + d_{S_1}(g_1, g_2) + d_{S_1}(x_2, g_2) \leq 1 + d_{S_1}(g_1, g_2) \leq c d_{S_2}(f(x_1), f(x_2)) + c^2 + 1$.
As a consequence,

$$c^{-1} d_{S_1}(x_1, x_2) - c - \frac{1}{c} \leq d_{S_2}(f(x_1), f(x_2)) \leq c d_{S_1}(x_1, x_2),$$

so f is a $(c, c + \frac{1}{c})$-strong quasi-isometry. □

In conclusion, the desired result has been shown:

Theorem 3.2.9. *G is hyperbolic for S_1 if and only if G is hyperbolic for S_2.*

Example 3.2.1. Every free group is hyperbolic. By taking a free generating set, the Cayley graph is a tree.

Example 3.2.2. $\mathbb{Z}^2$ is not hyperbolic. Let us consider $S = \{(1, 0), (0, 1)\}$. For any $n \in \mathbb{N}$, the insize of the geodesic triangle

$$\Delta_n = [(0, 0), (0, n)] \cup [(0, 0), (n, 0)] \cup [(0, n), (n, 0)]$$

is $2n$ and this tends to infinity when n goes to infinity.

Chapter 4

Isoperimetric inequality and Dehn presentation

In this final chapter, Dehn presentations and isoperimetric inequalities will be discussed. In Section 4.1, Section 4.2 and Section 4.3 it will be proven that for a finitely presented group hyperbolicity, having a Dehn presentation or satisfying an isoperimetric inequality are equivalent. The last section, Section 4.4, will be focused on solving one algorithmic property in the case of hyperbolic groups: the Word Problem. Moreover, it will be also shown that in a hyperbolic group, there are only finitely many conjugacy classes of elements of finite order (Theorem 4.4.1).

From now on, $G = \langle S \mid R \rangle$ will be a finitely presented group. Instead of denoting the length of a word by $|\,|_S$, $|\,|$ will be used. Moreover, δ will be assumed to be a natural number. In addition, there will not be any distinction between paths in the Cayley graph and elements of the group, although thanks to the context it will be clear which is the used notion. Paths will be supposed to be injective.

4.1 Hyperbolicity implies the isoperimetric inequality

If w is a word in $F(S)$ such that $\mu(w) = 1$, it can be written as follows:

$$w = \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1},$$

where $u_i \in F(S)$ and $r_i \in R$, for $i \in \{1, \cdots, n\}$.

Definition 4.1.1. Under the above conditions, the *area* of w is defined to be

$$A(w) = \min \Big\{ n \mid w = \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1} \Big\}.$$

Definition 4.1.2. G satisfies a *linear isoperimetric inequality* if there exists $k \in \mathbb{R}$ such that $A(w) \leq k|w|$ for all the words $w \in F(S)$ with $\mu(w) = 1$.

Definition 4.1.3. The *symmetric closure of* R, denoted by R_*, is the set of elements of R, their cyclic permutations and the cyclic permutations of the inverses.

Definition 4.1.4. A *Van Kampen diagram over* a presentation $\langle S \mid R \rangle$ is a planar finite cell complex $\mathcal{D}$ satisfying the following properties:

(i) $\mathcal{D}$ is connected and simply connected;

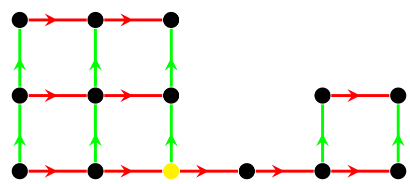
(ii) Each 1-cell of $\mathcal{D}$ is labelled by an arrow and a letter $s \in S$;

(iii) Some vertex which belongs to the topological boundary of $\mathcal{D} \subseteq \mathbb{R}^2$ is a base-vertex;

(iv) For each 2-cell of $\mathcal{D}$ and for each vertex of the boundary cycle of the 2-cell, the label of the boundary cycle of the region, if read from that vertex in both directions, is a reduced word in $F(S)$ that belongs to R_*.

The diagram also has a *boundary cycle*, denoted by $\partial\mathcal{D}$, which is an edge-loop in the 1-skeleton starting and ending at the base-vertex of $\mathcal{D}$ and going around $\mathcal{D}$ in the clockwise direction along the boundary of the unbounded complementary region. The label of the boundary cycle is a word in S that is called the *boundary label of* $\mathcal{D}$.

Example 4.1.1. Let $G = \langle a, b \mid aba^{-1}b^{-1} \rangle$.



The 1-cells labelled by a are drawn in red and the ones labelled by b in green. The base-vertex is drawn in yellow. The boundary label is

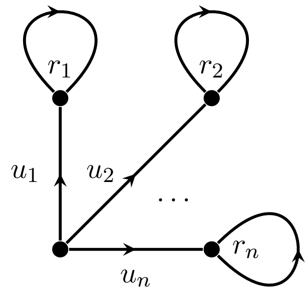
$$a^3ba^{-1}b^{-1}a^{-2}b^2a^{-2}b^{-2}a^2.$$

Proposition 4.1.1. *Let $w \in F(S)$. Then, $\mu(w) = 1$ if and only if there exists a Van Kampen diagram over G with boundary label w.*

Proof. Suppose that $\mu(w) = 1$ and that

$$w = \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1},$$

where $u_i \in F(S)$ and $r_i \in R$, for $i \in \{1, \cdots, n\}$. Then, we can consider the following Van Kampen diagram (the arrow of r_i depends on its power):



For the other implication, let k be the number of 2-cells in the Van Kampen diagram $\mathcal{D}$ over G. Let us prove the statement by induction on k.
If $k = 1$, it is obvious. Now, suppose that the statement holds for all the Van Kampen diagrams with k 2-cells and suppose that $\mathcal{D}$ has $k + 1$ 2-cells. Let F be a 2-cell containing an edge t on $\partial\mathcal{D}$. Then, $w = utv = utss^{-1}v = (utsu^{-1})(us^{-1}v)$, where u, v are words on $\partial\mathcal{D}$ and ts is the boundary label of F. Hence, since $us^{-1}v$ is the boundary label of another Van Kampen diagram that has k 2-cells, $\mu(us^{-1}v) = 1$. Moreover, $\mu(utsu^{-1}) = 1$ because $ts \in R$. In conclusion, $\mu(w) = 1$. □

Definition 4.1.5. A *minimal Van Kampen diagram* for w is a Van Kampen diagram for w with the minimum number of 2-cells.

Hence, the area of a word that is the identity in G is equal to the number of 2-cells of the minimal Van Kampen diagram and the length of w is equal to the length of $\partial\mathcal{D}$ endowed with the standard length metric.

Theorem 4.1.2. *If G is hyperbolic, it satisfies a linear isoperimetric inequality.*

Proof. By hypothesis, geodesic triangles of Cayley(G, S) are δ-thin. Let

$$k = \max\{A(w) \mid |w| \leq 10\delta\}.$$

Let us prove, by induction on $|w|$, that $A(w) \leq k|w|$ for all the words $w \in F(S)$ with $\mu(w) = 1$.

If $|w| \leq 10\delta$, it is obvious. Now, suppose that the statement is true for all the words with length smaller than or equal to n and let $|w| = n + 1$. We need to distinguish the following three cases:

(i) Suppose that for all vertices $w(i)$ of w, $|w(i)| < 5\delta$. Let p be an edge-path joining e and $w(5\delta)$. Then, supposing that $w = w_1 w_2$, where w_1 is the word that goes from e to $w(5\delta)$ and w_2 from $w(5\delta)$ to e,

$$w = w_1 p^{-1} p w_2.$$

Hence, $A(w) \leq A(w_1 p^{-1}) + A(p w_2)$. Firstly,

$$|w_1 p^{-1}| \leq |w_1| + |p| < 5\delta + 5\delta = 10\delta,$$

so $A(w_1 p^{-1}) \leq k$. Secondly, since

$$|p w_2| < 5\delta + |w_2| = 5\delta + |w| - 5\delta = |w|,$$

then

$$|p w_2| \leq |w| - 1 = n,$$

so by inductive hypothesis, $A(p w_2) \leq k|p w_2| \leq kn$. In conclusion,

$$A(w) \leq k(n+1) = k|w|.$$

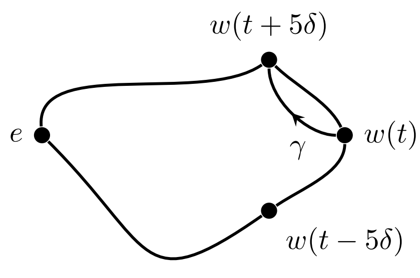
(ii) Suppose that there exists a vertex $w(t)$ such that $|w(t)| \geq 5\delta$ and

$$d(w(t), w(t+5\delta)) < 5\delta \quad \text{or} \quad d(w(t), w(t-5\delta)) < 5\delta.$$

We take the furthest vertex from e with those properties. Let us examine the first case. The other one is similar.
Let us consider a geodesic segment, denoted by γ, $[w(t), w(t+5\delta)]$. Let

$$w' = w|_{[t,t+5\delta]}\gamma^{-1}, \quad w'' = w|_{[0,t]}\gamma w|_{[t+5\delta,n]}.$$

Then, $A(w) = A(w') + A(w'')$.



By using the same arguments as in the first case, $A(w') \leq k$ and $A(w'') \leq k|n|$, so we obtain what we want to prove.

(iii) If neither of the two cases of above hold, consider these two geodesic triangles:

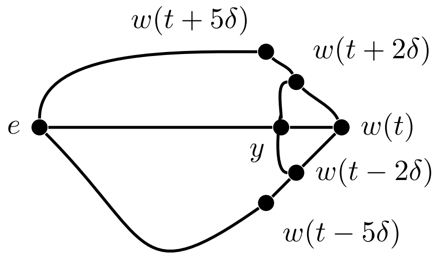
$$\Delta_1 = [e, w(t-5\delta)] \cup [w(t-5\delta), w(t)] \cup [e, w(t)],$$

$$\Delta_2 = [e, w(t)] \cup [w(t), w(t+5\delta)] \cup [w(t+5\delta), e].$$

Since triangles are δ-thin, there exists $y \in [e, w(t)]$ with

$$d(w(t-2\delta), y) \leq \delta \quad \text{and} \quad d(w(t+2\delta), y) \leq \delta.$$

Observe that the y is the same since $w(t+2\delta)$ and $w(t-2\delta)$ have the same image under the map of Definition 2.0.6. Then, $d(w(t-2\delta), w(t+2\delta)) \leq 2\delta$ and we can use the same arguments as in the previous two cases.



□

4.2 Hyperbolicity implies having a Dehn presentation

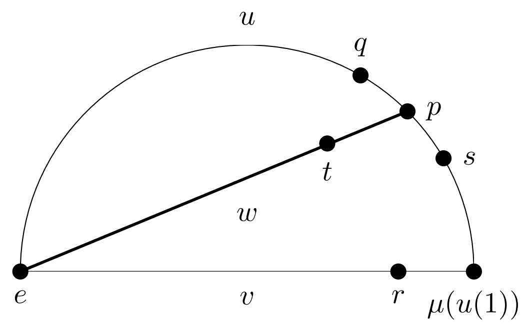
Definition 4.2.1. A *Dehn presentation* for G is a finite presentation $\langle S \mid R \rangle$ such that for all $w \in F(S)$ with $\mu(w) = 1$ there is a relation $r_1 r_2 \in R$ with $l(r_1) > l(r_2)$ and $w = w_1 r_1 w_2$, where w_1, $w_2 \in F(S)$.

Definition 4.2.2. A path is *k-local geodesic* if each sub-path of length at most k is geodesic.

Lemma 4.2.1. *Suppose that G is δ-hyperbolic. Let $k = 4\delta$, u be a k-local geodesic path in Cayley(G, S) and v be a geodesic path in Cayley(G, S) with $\mu(u(1)) = \mu(v(1))$. Suppose that v has length greater than or equal to 2δ and let r and s be points in u and v, respectively, at distance 2δ from $\mu(u(1)) = \mu(v(1))$. Then $d(r, s) \leq \delta$.*

Proof. Let us prove it by induction on the length of u.
If the length of u equals 4δ, it is an inmediate consequence of the fact that geodesic triangles are δ-thin. Now, suppose that the statement holds for paths with length smaller than or equal to N and assume that the length of u is between N and $N + k$. Let p be the vertex at distance $k = 4\delta$ from

$\mu(u(1))$ along u and w be a geodesic path from e to p.



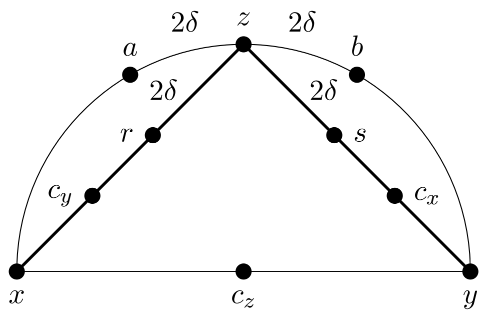
Let p, t and s be points on u and w at distance 2δ from p as in the picture. The segment joining q and s has length k, so it is geodesic. Hence, $d(q,s) = 4\delta$. Therefore, by inductive hypothesis, $d(q,t) \leq \delta$.
Observe that $d(t,s) \geq 3\delta$. Otherwise,

$$4\delta = d(q,s) \leq d(q,t) + d(t,s) < 4\delta.$$

Finally, the triangle with vertices e, $\mu(u(1))$ and p is geodesic. Thus, by hypothesis, it is δ-thin and r and s are at distance 2δ from $\mu(u(1))$, so $d(r,s) \leq \delta$. ☐

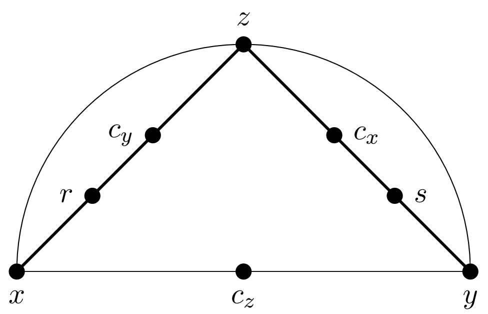
Lemma 4.2.2. *Suppose that G is δ-hyperbolic. If u is a k-local geodesic path and v is a geodesic path in Cayley(G,S) with $\mu(u(1)) = \mu(v(1))$, then u lies in a 3δ-neighbourhood of v.*

Proof. Let z be a point on u that is at distance at least 2δ from v. If z does not exist, the lemma is already proved.
Otherwise, let F be a geodesic path from x to z and G from y to z. Consider the points in the following picture:



By the previous lemma, $d(a,r) \leq \delta$ and $d(b,s) \leq \delta$. Moreover, $d(a,b) = 4\delta$.

Hence, $d(r,s) \geq 2\delta$. In conclusion, we would have a situation like this:



Then, there exists t in $[x, c_z]$ with $d(x,t) = d(x,r)$ and $d(r,t) \leq \delta$. To sum up, $d(z,t) \leq d(z,r) + d(r,t) \leq 3\delta$. □

It is inmediate to prove that if a group has a Dehn presentation, then it satisfies a linear isoperimetric inequality. In addition, at this point another implication can be proved.

Theorem 4.2.3. *If G is δ-hyperbolic, it has a Dehn presentation.*

Proof. Let $R = \{w \in F(S) \mid |w| \leq 8\delta \text{ and } \mu(w) = 1\}$. Let us show that $\langle R \mid S \rangle$ is a Dehn presentation for G.
It only remains to check that if $w \in F(S)$ with $\mu(w) = 1$ and $|w| > 8\delta$, then w contains a subword r_1 and R contains a relator $r_1 r_2$ with $l(r_1) > l(r_2)$.
Let us pick $w \in F(S)$ with $\mu(w) = 1$. If w is not 4δ-local geodesic, by definition, w has a sub-path of length at most 4δ which is not geodesic. Let us denote by u such sub-path and suppose that the initial and terminal vertices are u_1 and u_2, respectively. Let v be a geodesic path from u_2 to u_1. Then, $uv \in R$ since $\mu(uv) = 1$ and $|uv| \leq |u| + |v| \leq 2|u| \leq 8\delta$.
If w is 4δ-local geodesic, by Lemma 4.2.2, w lies in a 3δ-neigbourhood of any geodesic path for w. In particular, observe that a geodesic path for $\mu(w) = 1$ is the identity vertex. Therefore, it is not possible for w to have length greater than 3δ; otherwise, w would have a geodesic sub-path of length smaller than or equal to 4δ. As a consequence, $w \in R$. □

4.3 The isoperimetric inequality implies hyperbolicity

Theorem 4.3.1. *If G satisfies a linear isoperimetric inequality, then G satisfies the Rips condition.*

Proof. By hypothesis, there exists $k \in \mathbb{R}^+$ such that $A(w) \leq k|w|$ for all the words $w \in F(S)$ with $\mu(w) = 1$. Let us bound the size of natural numbers

for which there is a geodesic triangle in Cayley(G, S) which is not $(n+1)$-slim.
Fix n where there exists a geodesic triangle

$$\Delta = [p,q] \cup [q,r] \cup [p,r]$$

in Cayley(G, S) and a point $a \in [p,q]$ so that

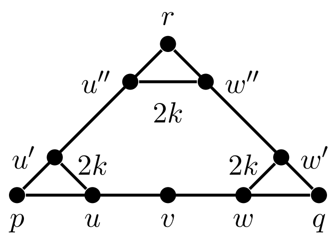
$$d(a, [q,r] \cup [p,r]) > n+1.$$

Let v be a vertex in Cayley(G, S) such that $d(a,v) \leq \frac{1}{2}$. Then,

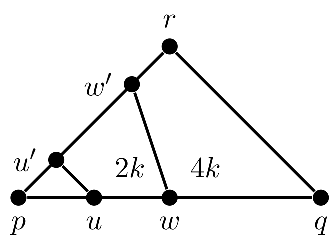
$$d(v, [q,r] \cup [p,r]) > n.$$

Let p be the maximum length of a relator in R, $K = \frac{k}{p^2}$, $m = pK$ and $n > 6k$. Reversing the roles of p and q if necessary, we have to consider only two cases:

(i) $[p,v]$ is disjoint from the $4k$-neighbourhood of $[r,q]$ and $[v,q]$ is disjoint from the $4k$-neighbourhood of $[p,r]$.



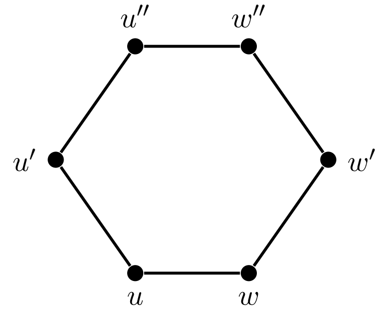
(ii) There exists $w \in [p,q]$ and $w' \in [p,r]$ such that $d(w,w') = 4k$.



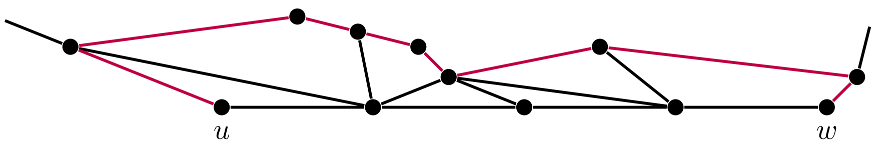
Consider points in the geodesic triangles as shown in both pictures. We are going to deal with the first case. The second one is similar.

Let H be the hexagon which is on the middle of the first triangle and let $\mathcal{D}$ be the minimal Van Kampen diagram for the word represented by the hexagon H. We want to estimate the number of 2-cells in $\mathcal{D}$.

Since $n > 6k$, the k-neighbourhoods of the segments $[u, w]$, $[u', u'']$ and $[w', w'']$ are disjoint.



First, consider the union of the 2-cells in $\mathcal{D}$ which intersect $[u, w]$ and let $\mathcal{D}_1$ be the minimal disc complex containing them. We iterate the process m times. Hence, the boundary of $\mathcal{D}_m$ lies in the $k = pm$ neighbourhood of $[u, w]$.
Let us estimate how many 2-cells are added at each stage. The perimeter of each 2-cell of $\mathcal{D}$ has length at most p and $d(u, w) = \alpha$, so there are at least $\frac{\alpha}{p}$ faces in $\mathcal{D}_1$. For $i \in \{1, \cdots, m-1\}$, there is an injective edge-path from u to w in $\partial\mathcal{D}_i$ with no edge in $[u, w]$.



This edge-path has length at least $\alpha = d(u, w)$. If any of the edges lie in H, they lie on $[u, u']$ or $[w, w']$, so there are at most $2k$. Now, every 2-cell has at most p edges, so there are at least $\frac{\alpha-2k}{p}$ 2-cells in $\mathcal{D} \setminus \mathcal{D}_i$ that have a 1-cell in such edge-path. In particular, there are at least this many faces in $\mathcal{D}_{i+1} \setminus \mathcal{D}_i$. Therefore, we get the following lower bound for the number of 2-cells in the k-neighbourhood of $[u, w]$ in $\mathcal{D}$:

$$\frac{m}{p}(\alpha - 2k) = K(\alpha - 2k).$$

Similarly, we get $K(\beta - 2k)$ and $K(\gamma - 2k)$ on the minimum number of 2-cells in the k-neighbourhood of $[u', u'']$ and $[w', w'']$, respectively. We know that such neighbourhoods are disjoint, so the number of 2-cells in $\mathcal{D}$ is greater than or equal to $K(\alpha + \beta + \gamma) - 6kK$. That is, if w is the boundary label of $\mathcal{D}$,

$$A(w) \geq K(\alpha + \beta + \gamma) - 6kK.$$

Note that $d(v, H \setminus [u, w]) > n - 2k$. Then, there is an arc A of length at least $n - 3k$ in $\partial\mathcal{D}_m \setminus H$. As a consequence, there are at least $\frac{n-3k}{p}$ 2-cells of $\mathcal{D}$ which do not lie in the neighbourhoods of $[u, w]$, $[u', u'']$ and $[w'', w']$. Then,

$$A(w) \geq K(\alpha + \beta + \gamma) - 6kK + \frac{n - 3k}{p}.$$

By the isoperimetric inequality, since the length of w is $\alpha + \beta + \gamma + 6k$,

$$A(w) \leq K(\alpha + \beta + \gamma) + 6kK.$$

In conclusion,

$$\frac{n - 3k}{p} \leq 12kK,$$

so n has been bounded. □

4.4 Consequences of Dehn presentation

The Word Problem for a finitely generated group G is the algorithmic problem of deciding whether a word in the generators represents the identity element. For example, given a word w on the alphabet $S = \{a, b, c, d\}$, there is the following algorithm that determines whether or not the word w is trivial in $F(S)$:

(i) In order a word w to be trivial in $F(S)$, the number of times a letter a, b, c or d appears in the word must be equal to the number of times the letter a^{-1}, b^{-1}, c^{-1} or d^{-1} appears, respectively. If this does not hold, the word is not trivial.

(ii) If it holds, it has to be checked if there exists a pair of letters together such that one is the inverse of the other one. If it happens, they can be erased and an equivalent word is obtained. Proceeding in this way, if the word is trivial, at some point an equivalent word of length 0 will be obtained. If not, the word is not trivial.

In general, hyperbolic groups (in particular free groups) admit a Dehn presentation. In this case, the Word Problem is also solvable:
Suppose that w is a word in $F(S)$. If it does not contain half of a relator, it is not equal to the identity. If it does, that is, if $w = ar_1b$ where $r_1r_2^{-1} \in R$ and $l(r_1) > l(r_2)$, then $w = ar_2b$ in G and it is a word of strictly shorter length. This process can be repeated finitely many times. If w is reduced to a word which does not contain more than half of a relator, it is not trivial in G; otherwise, it is trivial.

Example 4.4.1. It can be proved that the fundamental group of a closed surface of negative Euler characteristic is hyperbolic. For example, the fundamental group of the g-genus surface with $g \geq 2$. In particular, they have solvable Word Problem.

Theorem 4.4.1. *If G is hyperbolic, there are only finitely many conjugacy classes of elements of finite order.*

Proof. Suppose that $\langle S \mid R \rangle$ is a Dehn presentation for G. Let g be an element of finite order and $Cl(g)$ be the set of all conjugates of g in G. Let us pick w the shortest word with $\mu(w) \in Cl(g)$. If n is the order of $\mu(w)$ in G, $\mu(w^n) = 1$. Therefore, w^n has a subword which is more than half of a relator. Let us denote by r_1 and r the previous subword and word, respectively. If $l(w) \geq l(r)$, w or some cyclic permutation of w would have r_1 as a subword, so it could be shortened. Hence, since cyclic permutations of w also lie in $Cl(g)$, w would not be the shortest word with $\mu(w) \in Cl(g)$. In conclusion, the number of conjugacy classes of elements of finite order is at most the number of words of length at most $\max_{r\in R} l(r)$. $\square$

Bibliography

[1] J.M. Alonso, T. Brady, D. Cooper, V. Ferlini, M. Lustig, M. Mihalik, M. Shapiro and H. Short, Notes on word hyperbolic groups, World Scientific, Singapore, 1991, 3–63.

[2] M. Batty after P. Papasoglu, University of Newcastle upon Tyne, 2003. https://www.math.ucdavis.edu/~kapovich/280-2009/hyplectures_papasoglu.pdf

[3] N. Brady, T. Riley and H. Short, The Geometry of the Word Problem for Finitely Generated Groups, Birkhäuser Basel, 2007.

[4] M.R. Bridson and A. Häfliger, Metric spaces of non-positive curvature, Springer-Verlag, Berlin-Heidelberg, 1999.

[5] M. Coonaert, T. Delzant and A. Papadopoulos, Géometrie et théorie des groupes, Les groupes hyperboliques de Gromov, Springer-Verlag, Berlin-Heidelberg, 1990.

[6] C. Druţu and M. Kapovich, Geometric Group Theory, 2017. https://www.math.ucdavis.edu/~kapovich/EPR/ggt.pdf

[7] E. Ghys and P. de la Harpe, Sur les Groupes Hyperboliques d'après Mikhael Gromov, Birkhäuser Basel, 1990.

[8] M. Gromov, Hyperbolic groups, Essays in group theory, Mathematical Sciences Research Institute, Springer, New York, 1987, 75–263.