Real Analysis

August Bergquist

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1 A Wild Generalization

I have the craziest generalization of little ell infinity. The exercise, as well as many other results, follow immediately as a corollary.

Let (X, d) be a metric space, and let A be any set non-empty set. We say that a function $f : A \to X$ all sets of the form $D_x(f) = \{d(f(a), x) : a \in A\}$ for $x \in X$ are bounded above in the reals (positiveness of a metric ensures it is bounded below). Let $\ell(A, X)$ be the set of all such bounded functions. Now consider the function $\delta : \ell(A, X) \to \mathbb{R}$ defined $\delta(f, g) = \sup\{d(f(x), g(x)) : x \in X\}$. This is a well-defined function, and a metric on the set $\ell(A, X)$. We'll need some lemmas before we prove this. For all the lemmas, let X, A, d, and δ be as above.

Lemma 1. Let $f, g \in \ell(A, X)$. Then the set $\{d(f(x), g(x)) : x \in X\}$ is bounded (above, though this is not ambiguous since all images under a metric must be positive or zero, and hence are bounded below by zero.)

Proof. Chose any x in X (it must be non-empty because it is a metric space). Since f and g are bounded, the sets $D_x(f) = \{d(f(a), x) : a \in A\}$ and $D_x(g) = \{d(g(a), x) : a \in A\}$ are bounded by definition of a function's boundedness, and let b_f and b_g upper bounds for these sets. Now let a be arbitrary within A. Since b_f and b_g are upper bounds, we have $d(f(a), x) \leq b_f$ and $d(g(a), x) \leq b_g$. By symmetry of metrics such as d, we have $d(g(a), x) = d(x, g(a))$, so $d(x, g(a)) \leq b_g$. By Theorem 1.3.6 of the textbook, it follows that $d(f(a), x) + d(x, g(a)) \leq b_f + b_g$. Since d is a metric on X, by the triangle inequality we have $d(f(a), g(a)) \leq d(f(a), x) + d(x, g(a))$. So by transitivity of the relation $\leq$, it follows that $d(f(a), g(a)) \leq b_f + b_g$. Moreover, a was arbitrary in A, hence for all elements $a \in A$, we have $d(f(a), g(a)) \leq b_f + b_g$. Hence $b_f + b_g$ is an upper bound for the set $\{d(f(a), g(a)) : a \in A\}$, and hence the set is bounded above as desired. $\square$

Lemma 2. The function $\delta : \ell(A, X)^2 \to \mathbb{R}$ is well defined.

Proof. Let $(f, g) \in \ell(A, X)^2$. We have shown that the sets of the form $\{d(f(a), g(a)) : a \in A\}$ bounded. While it may seem trivial, we must show that all of these sets are non-empty. This is ensured from the fact that A is non-empty, as is demanded in the definition of the set A as above. Hence this set is non-empty. Because the set is non-empty, there must exist some supremum (note that this is a set of real numbers). Moreover, we have shown already that the supremum, should exist, is unique. In other words, every two pair of elements $\ell(A, X)^2$ is mapped to a unique image under δ, namely $\sup\{d(f(a), g(a)) : a \in A\}$, which we have shown to exist uniquely. $\square$

Lemma 3. δ is positive (or zero)

Proof. This comes from the fact that, if we have a set of non-negative real numbers, a negative number could never be an upper bound. So the supremum must never be negative. Hence the images of δ are never negative. $\square$

Lemma 4. δ is positive definite.

Proof. First suppose that $f, g \in \ell(A, X)$ so that $\delta(f, g) = 0$. We aim to show that $f = g$ in this case. By definition of δ, we have $\sup\{d(f(a), g(a)) : a \in A\} = 0$. Chose arbitrary $a \in A$. Since 0 is the supremum, by defintion of a supremum we have $d(f(a), g(a)) \leq 0$. By positivity (as verified in the previous lemma), we have $0 \leq d(f(a), g(a))$. By antisymmetry of the relation $\leq$, it follows that $d(f(a), g(a)) = 0$. Moreover, since d is a metric, by positive definiteness it follows that $f(a) = g(a)$. But a was arbitrary in A. Hence for all $a \in A$, we have $f(a) = g(a)$. Since A is the domain of both f and g, it follows by definition of function equality that $f = g$. This verifies one direction of positive definiteness.

Now to verify the other way. Suppose that $f = g$. Then $f(a) = g(a)$ for all $a \in A$ by definition of function equality. Hence by positive definiteness of d as a metric, it follows that $d(f(a), g(a)) = 0$ for all a, hence the set $\{d(f(a), g(a)) : a \in A\}$ is just the singleton $\{0\}$. Since the supremum of a singleton is just the one element, it follows that $\delta(f(a), g(a)) = \sup\{0\} = 0$, so this side of positive definateness works as well! $\square$

Lemma 5. δ is symmetric.

Proof. Consider any $f, g \in \ell(A, X)$. We know that $\delta(f, g)$ and $\delta(g, f)$ exists. Symmetry is easy to show. Consider the sets $P = \{d(f(a), g(a)) : a \in A\}$,

$Q = \{d(g(a), f(a)) : a \in A\}$. Now suppose we have $d(f(a), g(a))$. By symmetry of d, we have $d(f(a), g(a)) = d(g(a), f(a))$, which is in Q by construction, so $P \subset Q$. f and g were arbitrary, so the other direction follows from the first. Hence $P = Q$. Since $\delta(f, g) = \sup(P)$, and since $\delta(f, g) = \sup(Q = P)$, it follows that $\delta(f, g) = \delta(g, f)$ as desired. Hence δ is symmetric. □

We'll need a lemma to show the triangle inequality. The textbook actually has this as an exercise problem.

Lemma 6. Let $A = \{a_\lambda\}_{\lambda\in\Lambda}$ and $\{b_\lambda\}_{\lambda\in\Lambda}$ be sets of real numbers, indexed by a suitable set Λ, and let them both be bounded above. Then the following are true:

- $a_\lambda \leq b_\lambda \forall \lambda \in \Lambda$ implies $\sup A \leq \sup B$
- $\sup(A + B) = \sup A + \sup B$ where $A + B = \{a_\lambda + b_\lambda\}_{\lambda\in\Lambda}$

Lemma 7. δ satisfies the triangle inequality.

Proof. Suppose we have $f, g, h \in \ell(A, X)$. Consider $\delta(f, g)$. We must show that $\delta(f, g) \leq \delta(f, h) + \delta(h, g)$. Notice that the sets $S = \{d(f(a), g(a)) : a \in A\}, T = \{d(f(a), h(a)) : a \in A\}$ and $U = \{d(h(a), g(a)) : a \in A\}$ are sets of real numbers indexed over A which are bounded above. Moreover, $\sup(T + U) = \sup(T) + \sup(U)$ by the lemma. Now for any $a \in A$, notice $d(f(a), g(a)) \leq d(f(a), h(a)) + d(h(a), g(a))$ by the triangle inequality for d. Hence for each $a \in A$ (since a was arbitrary), we have $S_a = d(f(a), g(a)) \leq d(f(a), h(a)) + d(h(a), g(a)) = (T + U)_a$, so by the lemma it follows that $\sup S \leq \sup(T + U)$. But it also follows that $\sup(T + U) = \sup T + \sup U$. Also, $\delta(f, g) = \sup S$, $\delta(f, h) = \sup T$, and $\delta(h, g) = \sup U$. Hence $\delta(f, g) \leq \delta(f, h) + \delta(h, g)$ as desired. □

Theorem 1. $(\ell(A, X), \delta)$ is a metric space.

Proof. The set $\ell(A, X)$ is non-empty. Consider the constant map $f(a) = x$ for some $x \in X$, and for all $a \in A$ (such a map must exist, for both A and X are supposed non-empty. Now chose $y \in X$. Then certainly the distance $d(x, y)$ exists in the reals. Moreover, for all $a \in A$, we have $d(f(a), y) = d(x, y)$. So the set $D_x(f) = \{d(x, y)\}$ for all $y \in X$. Each such set is finite, and hence bounded above. So f is bounded, and hence $f \in \ell(A, X)$, and so $\ell(A, X) \neq \emptyset$.

We have shown that we have a non-empty set $\ell(A, X)$, along with a function $\delta : \ell(A, X)^2 \to \mathbb{R}$ satisfying the positivity, positive definiteness, symmetry, and triangle inequality. It follows then that $(\ell(A, X), \delta)$ is a metric space. $\square$

2 Problem 2.2.9

Now we can use this generalization for the homework. First, I have a cute little lemma that will help to make one of my answers rigorous.

Lemma 8. Let $q, p \in \mathbb{R}^+$ such that $p < q$, and let $\epsilon \in \mathbb{R}^+$. Then $\frac{p}{q} < \frac{p+\epsilon}{q+\epsilon}$. Moreover, the set $\{\frac{n-1}{n+1} : n \in \mathbb{N}\}$ has 1 as it's supremum.

Proof. Chose any natural n. Let p, q be as stated in the lemma. Since $n > 0$, it follows that $p + n < q + n$, hence $\frac{p+n}{q+n} < 1$. So 1 is certainly an upper bound for the set described above. It remains to show that it is the least such. First we must make do a sidequest: proving the first half of the lemma.

Let $\epsilon > 0$. Then $p\epsilon < q\epsilon$. Hence $p(q + \epsilon) = pq + p\epsilon < pq + q\epsilon = q(p + \epsilon)$. By closure of the positives, it follows that $q(q + \epsilon) > 0$, hence, dividing by this and applying previous results, we obtain

$$\frac{p}{q} < \frac{p + \epsilon}{q + \epsilon}$$

as we claimed.

Now suppose by way of contradictino that the $r < 1$ is an upper bound. Since each $(n - 1)/(n + 1) \geq 0$, and some are positive, we must also have $r > 0$. But since the rationals are dense within the reals, it follows that there exists some $s \in \mathbb{Q}$ so that $r < s < q$, so we must also have s as an upper bound. Moreover, since s is rational and positive, there must exist natural m, n so that $s = m/n$. But since $s < 1$, it follows that $m < n$. Hence $m + 1 \leq n$. From this and the irst half of the proof, it follows that $s = \frac{m}{n} \leq \frac{m}{m+1} < \frac{m}{}$

$\square$

Consider the set S of all bounded sequences of real numbers. For each pair $\mathbf{x}, \mathbf{y} \in S$ the distance between them is defined $d(\mathbf{x}, \mathbf{y}) = \sup\{|x_i - y_i| : i \in \mathbb{N}\}$.

- The distance between the sequences $\mathbf{x} = (1, 0, \dots)$ and $\mathbf{y} = (0, 1, 0, \dots)$ is 1. This is because for all $i > 2$, we have $x_i = y_i = 0$ so $|x_i - y_i| = |0 - 0| = 0$. For $i = 1$, we have $x_1 = 1$, $y_1 = 0$, so $|x_1 - y_1| = |1 - 0| = |1| = 1$. Similarly, we find $|x_2 - y_2| = |0 - 1| = |-1| = 1$. So we only have two elements in our set of distances between terms, in which case the supremum is just the maximum of them: not very interesting. $d(\mathbf{x}, \mathbf{y}) = 1$ in this case.

- Now for the sequences $(1, 0, 1, 0, \dots)$ and $(0, -1, 0, -1, \dots)$, we have the distance between them as 1. This is because the elements of the set of distances between terms is even smaller! For the even terms of the first sequence are all 0, and the even terms of the second are all -1, so the distance between these terms is $|0 - (-1)| = 1$. Similarly, for the odd terms of the first sequence are all 1, and the odd terms of the second are all 0, so the distance between each of these terms is $|1 - 0| = 1$. So we have the set of distances between terms as $\{1\}$. There is only one element; obviously the supremum is 1. These sequences might be even less interesting than the last, because the set of distances is as small as it can be.

- Now how about the sequences $(1, 2, 3, 1, 2, 3, \dots)$ and $(-1, -2, -3, -1, -2, -3)$. Here, we only have a finite number of differences between terms, and the greatest of these distances is 6. Hence the supremum of these distances is 6.

- How about $(1/2.1/3, 1/4, \dots)$, $(1/2, 2/3, 3/4, \dots)$? This one is nice. We show that $\sup\{|1/n - ((n-1)/n\} = 1$. This could be done by a proof by contradiction, which I am currently too tired to complete.

- Since a sequence is a function $\mathbb{N} \to \mathbb{R}$, and since ℓ_∞ is all bounded such sequences, by the previous lemmas it follows that ℓ_∞ is just $\ell(\mathbb{N}, \mathbb{R})$ as defined above. Hence it is a metric space, in the positivity, positive definiteness, symmetry, and the triangle inequality all hold.

3 Some more generalizations and remarks

I wasn't feeling so great about the square root function. I won't prove it's well defined (on the positive reals, it definately isn't well defined on the complex numbers). I will however show it's monotonic, as well as quickly generalize what monotonic means.

Definition. The square root function

Lemma 9. Let $(X, \leq_1), (Y, \leq_2)$ be a poset. A morphism of order (which could be called a monotonic function from X to Y) could be defined as a map $f : X \to Y$ so that $x \leq_1 y$ implies $f(x) \leq_2 f(y)$ for all $x, y \in X$. (The reals are a poset, and a totally ordered set, under the relation $\leq$, as is every subset of $\mathbb{R}$ under the same restricted order relation). Then $\sqrt{\cdot} : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ is a morphism of order $(\mathbb{R}^+ \cup \{0\}, \leq') \to (\mathbb{R}, \leq)$, where $\leq'$ is the restriction relation of $\leq$ to $(\mathbb{R} \cup \{0\}^2$.

Proof. Suppose by way of contradiction that there exist some $x, y \in \mathbb{R}^+ \cup \{0\}$ so that $x \leq y$ while $\sqrt{x} > \sqrt{y}$. Then (by theorem 1.3.6 of the textbook that multiplication respects the order relation), we have $\sqrt{x}^2 > \sqrt{y}^2$. By definition of the square root, we have $x > y$, contradicting our assumption that $x \leq y$. Hence $\sqrt{\cdot}$ is a morphism of order. $\square$

Lemma 10. Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then for $i \in \mathbf{n}$, $|x_i - y_i| \leq d(\mathbf{x}, \mathbf{y})$, where d is the euclidean metric on $\mathbb{R}^n$

Proof. By definition of the euclidean metric, we have $d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{k=1}^n (x_k - y_k)^2}$. Hence, by defintion of the square root, we have $d(\mathbf{x}, \mathbf{y})^2 = \sum_{k=1}^n (x_k - y_k)^2$, hence $(x_i - y_i)^2 = d(\mathbf{x}, \mathbf{y})^2 - \sum_{k \neq i} (x_k - y_k)^2$. Now since $\sum_{k \neq i} (x_k - y_k)^2$ is at least zero, and since it is at most equal to $d(\mathbf{x}, \mathbf{y})^2$, it follows that $0 \leq d(\mathbf{x}, \mathbf{y})^2 - \sum_{k \neq i} (x_k - y_k)^2 \leq d(\mathbf{x}, \mathbf{y})^2$. By positive or zero-ness, it follows that $\sqrt{d(\mathbf{x}, \mathbf{y})^2 - \sum_{k \neq i} (x_k - y_k)^2}$ is defined. Since we have shown that the square root function is monotonic, it follows that

$$\begin{gathered} |x_i - y_i| = \sqrt{(x_i - y_i)^2} \\ = \sqrt{d(\mathbf{x}, \mathbf{y})^2 - \sum_{k \neq i} (x_k - y_k)^2} \\ \leq \sqrt{d(\mathbf{x}, \mathbf{y})^2} \\ = d(\mathbf{x}, \mathbf{y}) \end{gathered}$$

as desired. $\square$

This should be enough for a clean and tidy proof for the next exercise.

$\boxed{\text{b}}$

Proposition 1. Let $n \in \mathbb{N}$. Let $r > 0$. Let $i \in \mathbf{n}$. Let $\mathbf{x} \in \mathbb{R}^n$. Let $K_{i,r}(\mathbf{x}) = \{\mathbf{a} = (a_1, \ldots, a_i, \ldots, a_n) : |x_i - a_i| < r\} \subset \mathbb{R}^n$. Then $K_{i,r}(\mathbf{x})$ is open.

Proof. Instantiate all variables as in the statement of the proposition. Consider any $\mathbf{a} = (a_1, \dots, a_n) \in K_{i,r}(\mathbf{x})$. Fix $\epsilon = r - |x_i - a_i|$. Select $\mathbf{y} = (y_1, \dots, y_n) \in B_\epsilon(\mathbf{a})$. By definition of an open ball, we have $d(a, y) < \epsilon$Since $|\cdot|$, the distance function on $\mathbb{R}$, is a metric, we have by the triangle inequality and symmetry, and by the previous lemma,

$$|x_i - y_i| \leq |x_i - a_i| + |a_i - y_i| < |x_i - a_i| + \epsilon = |a_i - y_i| + d - |a_i - y_i| = d.$$

So by definition of the set $K_{i,r}(\mathbf{x})$, we have $\mathbf{y} \in K_{i,r}(\mathbf{x})$. But $\mathbf{y}$ was arbitrary in $B_\epsilon(\mathbf{a})$, so $B_\epsilon(\mathbf{a}) \subset K_{i,r}(\mathbf{x})$. But $\mathbf{a}$ was arbitrary, so it is possible for each element of $K_{i,r}(\mathbf{x})$ to construct a ball around it which sits entirely within the set. By a previous theorem, it follows that $K_{i,r}(\mathbf{x})$ is open. □

$\boxed{\text{a.i}}$ I can't really draw this in LaTeX (except for using tedious and awful graphics from paint). What the i-th r-cell of radius 1/2 about a point in $\mathbb{R}^2$ would look like would be an infinite with either horizontal or vertical borders (depnding on whether $i = 1, 2$), and centered at the horizontal/vertical line crossing through the point a, where the boundaries are the horizontal lines crossing through $a + (1/2, 0)$, $a - (1/2, 0)$ in the case where $i = 1$, and $a + (0, 1/2)$, $a - (0, 1/2)$.

$\boxed{\text{a.ii}}$

Proposition 2. Let the r-cell about $\mathbf{x} \in \mathbb{R}^n$ be the set $K_r(\mathbf{x}) = \{\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}^n : |x_i - a_i| < r \forall i \in \mathbf{n}\}$. Then

$$K_r(\mathbf{x}) = \cap_{i \in \mathbf{n}} K_{i,r}(\mathbf{x})$$

Proof. This is pretty much trivial from the way the sets are constructed. Anyway, suppose first that we have $\mathbf{y} \in K_r(\mathbf{x})$. Then for each $i \in \mathbf{n}$, we have $|y_i - x_i| < r$, so for each such i, we have $\mathbf{y} \in K_{r,i}(\mathbf{x})$, so it must be in the intersection of all of them. Simiarly, if $\mathbf{y}$ is in the intersection of all of them, it must be the case that $|y_i - x_i| < r$ for each i. Hence $\mathbf{y} \in K_r(\mathbf{x})$, for that is how the set is defined.

□

$\boxed{\text{c}}$

Proposition 3. Any cell $K_r(\mathbf{a})$ is open in $\mathbb{R}^n$, for any $\mathbf{a} \in \mathbb{R}^n$

Proof. Instantiate $n \in \mathbb{N}$, $a \in \mathbb{R}^n$ and $r \in \mathbb{R}^+$. We have shown that the sets $K_{i,r}(\mathbf{a})$ intersect over $i \in \mathbf{n}$ to form the set $K_r(\mathbf{a})$, and that each of them is open. Since $\mathbf{n}$ is a finite set, it follows by 3.1.8 of the textbook that the intersection $\cap_{i \in \mathbf{n}} K_{i,r}(\mathbf{a}) = K_r(\mathbf{a})$ is open as desired. □

Proposition 4. Let $B_r(\mathbf{a})$ be open around $\mathbf{a}$ in $\mathbb{R}^n$ for some $r > 0$. Then there exists some $s > 0$ so that $K_s(a) \subset B_r(\mathbf{a})$. There also exists a $t > 0$ so that $B_t(a) \subset K_r(\mathbf{a})$.

Proof. Consider $s = r/n$, and the cell $K_s(\mathbf{a} = (a_1, \dots, a_n))$. Suppose $y = (y_1, \dots, y_n) \in K_s(\mathbf{s})$. Then we must have $|y_i - a_i| < r/n$ for each $i \in \mathbf{n}$. Then we must have

$d^2(\mathbf{y}, \mathbf{a}) = \sum_{i \in \mathbf{n}} |y_i - a_i|^2 < n(r/n)^2 = r^2/n$. Now since $1/n \leq 1$, it follows that $r^2/n \leq r^2$. Hence $d^2(\mathbf{y}, \mathbf{a}) < r^2$. Thus $d(\mathbf{y}, \mathbf{a}) < r$, hence $\mathbf{y} \in B_r(a)$ as desired. Since $\mathbf{y}$ was arbitrary, we have $B_r(a) \subset K_r(\mathbf{a})$.

Now consider the open cell $K_r(\mathbf{a})$. We claim that $B_r(\mathbf{a}) \subset K_r(\mathbf{a})$. For suppose $\mathbf{y} = (y_1, \dots, y_n) \in B_r(\mathbf{a})$, and let $i \in \mathbf{n}$. Then $d(\mathbf{y}, \mathbf{y}) < r$Then by Lemma 9 and transitivity of $<$, it follows that $|y_i - a_i| < r$. So by definition of $K_{i,r}(\mathbf{a})$, it follows that $\mathbf{y}$ is a member therein. Since $i \in \mathbf{n}$ was arbitrary, a , it follows that $\mathbf{y} \in \cap_{i \in \mathbf{n}} K_{i,r}(\mathbf{a}) = K_r(\mathbf{a})$. So since $\mathbf{y}$ was arbitrary, we have $B_r(\mathbf{a}) \subset K_r(\mathbf{a})$, which is what we wanted to show. □

Proposition 5. A set $U \subset \mathbb{R}^n$ is open if and only if for each $\mathbf{u} \in U$ there exists some $r > 0$ such that $K_r(\mathbf{x}) \subset U$.

Proof. Suppose first that U is open, and let $\mathbf{x} \in U$. Well then, by Theorem 3.1.7 of the textbook, it follows that there must exist some $t > 0$ such that $B_t(\mathbf{x}) \subset U$. But by the previous proposition there exists some $r > 0$ such that $K_r(\mathbf{x}) \subset B_t(\mathbf{x})$. By the transitivity of the subset relation, it follows that $K_r(\mathbf{x}) \subset U$. Hence for any $\mathbf{x} \in U$, we can find $r > 0$ so that $K_r(\mathbf{x}) \subset U$.

Now suppose that for each $\mathbf{x} \in U$ there exists some $r > 0$ so that $K_r(\mathbf{x}) \subset U$. By the previous result, it follows that there exists some $s > 0$ (in fact r will do) so that $B_s(\mathbf{x}) \subset U$. Hence for all $\mathbf{x} \in U$ we can find an open ball around $\mathbf{x}$ contained therein. So by Theorem 3.1.7 it follows that U is open. □