Real Analysis

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February 16, 2023

Lemma 1. Let $k > 1$. Then the sequence $n \mapsto k^n$ has the property that for all real $M > 0$, there exists some natural number N such that if $n > N$, then $k^n > M$.

Proof. Recall the inequality from my last homework assignment (which I had proven in my last assignment), which states that if $h > 0$, then $(1+h)^n > 1 + nh$. So for any given $M > 0$, we have by the archimedean principle some $N > \frac{M-1}{h}$, with $N \in \mathbb{N}$. Now suppose $n > N$. Then $n > \frac{M-1}{h}$, whence $k^n > 1 + nh > M$. This concludes the proof. $\square$

Lemma 2. Let $0 \leq k < 1$ for $k \in \mathbb{R}$. Then the sequence $\sigma(n) = k^n$ converges to 0.

Proof. Now let $\epsilon > 0$. Notice that $\frac{1}{k} > 1$, so by the last proposition there exists a natural number N with the property that if $n > N$, then $\frac{1}{k^n} > \frac{1}{\epsilon}$. So for any $n > N$, we have $|k^n - 0|k^n < \epsilon$. This proves the limit. $\square$

For the next two theorems, let $U \subset \mathbb{R}$ be an open set, and let $f : U \to \mathbb{R}$ be differentiable on all of U, and let $p \in U$ such that f' is continuous at p.

Theorem 1. If $|f'(p)| < 1$, then p is an attractor.

Proof. Since f' is continuous at p, and since the absolute value function is continuous everywhere, it follows that the composition $|f'|$ is continuous at p. Moreover, since $|f'(p)| < 1$, there exists some $|f'(p)| < k < 1$, and $0 \geq |f'(p)|$ by definition of the absolute value. Since $\frac{k-|f'(p)|}{2}$ is positive, by continuity there exists some $\delta > 0$ such that if $x \in U$ such that $|x - p| < \delta$, then $\big||f'(x)| - |f'(p)|\big| < \frac{k-|f'(p)x|}{2}$. By openess of U, there exists an $\epsilon > 0$ such that if $|x-p|<\epsilon$, then $x \in U$. Set $\eta = \min\{\epsilon, \delta\}$, and construct the interval $I = (p-\eta, p+\eta)$.

First, note that for any $x \in I$, $|x-p| < \eta \leq \delta$, whence we have by construction of δ that

$$|f'(x)| < |f'(p)| - \frac{k - |f'(p)|}{2} = \frac{|f'(p)|}{2} + \frac{k}{2} < \frac{k}{2} + \frac{k}{2} = k.$$

From a previous result which we presented in class, it follows that f is Lipschitze with constant k. In other words, $|f(x) - f(p)| = |f(x) - p| \leq k|x-p|$.

I will use this to show that $|f^n(x) - p| \leq k^n|x-p|$ for all n, and that $f^n(x) \in I$ for all n. Let the previous observation serve as the base case. Moreover, note that since $k < 1$, $|f(x)-p| \leq k|x-p| < |x-p| < \eta \leq \delta$, so $|f(x) - p| \in I$.

Now suppose that for $n \in \mathbb{N}$, $|f^n(x) - p| \leq k^n|x-p|$, and that $f^n(x) \in I$. Then by Lipschitzeness, we have

$$|f(f^n(x)) - f(p)| = |f^{n+1}(x) - p| \leq k|f^n(x) - p| \leq k(k^n|x-p|) = k^{n+1}|x-p|.$$

It follows then that $|f^n(x) - p| \leq k^n|x-p|$ for all $n \in \mathbb{N}$.

If $x = p$, it immediately follows that $f^n(x)$ converges to p, for p is a fixed point. Assume then that $x \neq p$, from which it follows that $|x-p| > 0$.

Now let $\epsilon > 0$. Then $\epsilon/|x-p| > 0$. By Lemma2, there must exist some natural number N such that for $n > N$, $k^n < \epsilon/|x-p|$. For such an $n > N$, we also have $0 \leq |f^n(x) - p| < k^n|x-p|$, whence

$$||f^n(x) - p| - 0| < k^n|x-p| < (\epsilon/|x-p|)|x-p| = \epsilon| \,.$$

By definition of a limit, the sequence $|f^n(x) - p|$ converges to zero, so by a previous result which we presented in class (relating distance and convergence), it follows that $f^n(x) \to p$ as $n \to \infty$.

Our element x was arbitrary in I, so we have shown that there exists an open ball I about p such the iterated sequence of f based at $x \in I$ converge to p. This is the definition of an attractor.

$\square$

A quick remark should be made about the definition of a repellor. I think a repellor should be defined as

Definition. Given a functino $\phi : X \to X$ in a metric space, a fixed point $q \in X$ is a *repellor* if and only if there exists an open ball $B_\epsilon(q)$ such that for any point $x \in B_\epsilon(q)$, there exists a natrual number n such that $\phi^n(x) \notin B_\epsilon(q)$.

If this is what a repellor is, then I can prove the following theorem.

Theorem 2. If $|f'(p)| > 1$, then p is a repellor.

Proof. By basic properties of the reals, there is some $K \in \mathbb{R}$ such that $|f'(p)| > k > 1$.

By the same argument for continuiuty, $|f'|$ is continuous at p. Since $\frac{|f'(p)|-k}{2} > 0$, by continuity of $|f'|$ there must exist some $\delta > 0$ such that if $x \in U$ such that $|x-p| < \delta$, then $||f'(x)| - |f'(p)|| < \frac{k-|f'(p)|}{2}$. By openness of U, there exists some $\epsilon > 0$ such that $x \in U$ whenever $|x-p| < \epsilon$. Let $\eta = \min\{\epsilon, \delta\}$, and construct the interval $I = (p-\eta, p+\eta)$.

Let $x \in I$. Then $|x-p| < \eta \leq \delta$, hence by construction of δ we have $||f'(x)| - |f'(p)|| < \frac{|f'(p)|-k}{2}$. Hence,

$$k = \frac{k}{2} + \frac{k}{2} < \frac{k}{2} + \frac{|f'(p)|}{2} = |f'(p)| - \frac{|f'(p)|-k}{2} < |f'(x)|.$$

So $k < |f'(x)|$ for any $x \in I$.

Suppose now that there existed some $x \neq p$ in I such that $|f(x)-f(p)| \leq k|x-p|$. Then by the mean value theorem, there exists some $c \in I$ such that $\frac{f(x)-f(p)}{x-p} = f'(c)$. Hence

$$|f'(c)| = \frac{|f(x)-f(p)|}{|x-p|} \leq k.$$

But $c \in I$, so this is a contradiction.

From this we obtain the inequality *

$$|f(x) - f(p)| = |f(x) - p| > k|x-p|,$$

for all $x \in I$.

Using this inequality, we aim to obtain a new inequality

$$|f^n(x) - p| > k^n|x-p|,$$

under the condition that $f^m(x) \in I$ for $m < n$.

The base case has already been established. Now suppose that $|f^n(x) - p| > k^n|x-p|$ with $n \in \mathbb{N}$ such that for $m < n$, $f^m(x) \in I$. For $n+1$, there are two possibilities. Either $f^n(x)$ is not in I or it is. If it isn't, the claim is satisfied. So suppose $f^n(x) \in I$. In this case, we have by the inequality * and the induction hypothesis that $|f(f^n(x)) - f(p)| > k|f^n(x) - p| > k^{n+1}|x - p|$. This establishes the claim. (Note we can't just assume that this inequality always holds; it only holds so long as we've "been in I the whole time.")

Now we suppose by way of contradiction that $f^n(x) \in I$ for all $n \in \mathbb{N}$.

Now chose any $x \in I$ distinct from p. We aim to construct a natural number N such that $f^N(x) \notin I$.

Suppose by way of contradiction that $f^n(x) \in I$ for all natural numbers n. By Lemma and the fact that $k > 1$, it follows that there exists a natural number N such that $k^N > \frac{\eta}{|x-p|}$. Since by assumption $f^n(x) \in I$ for all $n < N$ (indeed, for all $n \in \mathbb{N}$),

$$|f^N(x) - p| > k^n|x - p| > |x - p|\frac{\eta}{|x - p|} = \eta,$$

hence $f^N(x) \notin I$, which is a contradiction.

So then, it follows that there exists an N such that $f^N(x) \notin I$, (since supposing otherwise gave us a contradiction). But $x \in I$ was arbitrary, and so regardless of base point, we'll eventually leave (though we may come back to) I (which is an open interval about p). From this it follows that p is a repellor. $\square$

This can be used to determine when a fixed point is a repeollor or an attractor.

Proposition 1. For the family of functions $\mathbb{R} \to \mathbb{R}$ defined $f_k : x \mapsto kx(1 - x) = kx - kx^2$, the fixed points $x_k = \frac{k-1}{k}$ are attractors if $1 < k < 3$ and rellors when $3 < k$.

Proof. First, note that f_k is infinitely differentiable, so it's derivative is continuous. Upon taking the derivative, we find $f_k'(x) = k - 2kx$. Evaluating at x_k, we get $f_k(x_k) = k - 2k\frac{k-1}{k} = k - 2k + 2 = 2 - k$. The condition that $|f'(x_k)| = |2 - k| < 1$ is equivalent to the condition that $k \in (2 - 1, 2 + 1) = (1, 3)$, so x_k is an attractor when k is in this range. Moreover, for $k > 3$, we have $2 - k < 2 - 3 = -1$, from which it follows that $|f'(x_k)| > 1$ for $k > 3$, hence f_k is a repellor when this is the case. $\square$