

1 Problem 12.5.5: an interesting function

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1/q & \text{if } x \text{ is rational, where } q \text{ is the denominator in lowest terms.} \end{cases}$$

It's fairly easy to see that this function is discontinuous for all rational numbers. Recall one of the equivalent definitions of continuity, which states that a function $g : X \rightarrow Y$ is continuous at some $a \in X$ iff for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in X$ with $d(x, a) < \delta$, $d(g(x), g(a)) < \epsilon$. Negating this, we must show that there is some $\epsilon > 0$ such that whenever $\delta > 0$, there is some $x \in X$ with $d(x, a) < \delta$ and $d(g(x), g(a)) \geq \epsilon$.

Proposition 1. The function f , defined above, is discontinuous at all $s \in \mathbb{Q}$.

Proof. Back to our specific function, let $s \in \mathbb{Q}$. Then, by definition of a rational number $s = p/q$ (in lowest terms), with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $\delta > 0$. Consider the positive real number $1/q$. Using a basic property of the reals (which we proved!), there must exist some irrational r with $s < r < s + \delta$. From this it follows that $|r - s| < \delta$. Moreover, by definition of f , we have $f(s) = 1/q$, and $f(r) = 0$. So $|f(s) - f(r)| = |1/q - 0| = 1/q \geq 1/q$. But δ was arbitrary greater than zero, so for all δ , there exists some $r \in \mathbb{R}$ with $|r - s| < \delta$ and $|f(r) - f(s)| \geq 1/q$. Hence f is discontinuous at s ! \square

Now we'll take its integral. Unfortunately (or fortunately for coolness's sake), we can't take an antiderivative, for clearly none exists (it isn't continuous on any interval, since every interval contains a rational). Along the way, we'll prove the function is continuous at every irrational number.

Lemma 1. Let $t \in \mathbb{R}$. Let $r > 0$, and let $n \in \mathbb{N}$. Construct the set $S_n = \{s \in [t - r, t + r] : f(s) = 1/n\}$. Then S_n is finite.

Proof. Let $s \in S_n$. Then $f(s) = 1/n$. This isn't zero, hence s is rational, and by definition of f , $s = p/n$ for some $p \in \mathbb{Z}$, where the fraction p/n

is in reduced terms. But since $s \in [t - r, t + r]$, it must be the case that $t - r \leq s = p/n \leq t + r$, and since n is positive, this is equivalent to

$$p \in [n(t - r), n(t + r)].$$

But since p is an integer, only finitely many such p can fit within this interval. So only finitely many elements could be in S_n , for each would correspond to a distinct integer $p \in [n(t - r), n(t + r)]$. \square

This allows us to establish an important property on how "big" the function can be, and how often it can be that big.

Lemma 2. Let $K \subset \mathbb{R}$ be bounded. Then for all $\epsilon > 0$, the set $A = \{s \in K : f(s) > \epsilon\}$ is finite.

Proof. First, note that since K is bounded, it follows that it lives inside of some open interval of some radius, and hence inside the corresponding closed interval. Let that closed interval be $[t - r, t + r]$ for some $t \in \mathbb{R}$, and for some $r > 0$. Note that the set $S_\epsilon = \{s \in [t - r, t + r] : f(s) < 1/n\}$

By the archimedian principle, there must exist some natural number N such that $1/N < \epsilon$. Note that if \square

Proposition 2. The function f is continuous at all irrationals.

Proof. Let $r \in \mathbb{R}$ be irrational. Let $\epsilon > 0$. There exists some $\eta > 0$. Form the interval $[r - \eta, r + \eta]$. Since $\epsilon/2 > 0$, by the previous lemma there are only a finite number of points $s \in [r - \eta, r + \eta]$ with $f(s) > \epsilon/2$. Order and index each of these points to form an indexed set of finite points $\{s_i\}_{i \in \{1, \dots, n\}}$, and form the partition

$$P = \{x_0 = r - \eta, x_1 = s_1, \dots, x_n = s_n, x_{n+1} = r + \eta\}.$$

Since r is irrational, and each of the s_i are rational (otherwise they would have $f(s_i) = 0 < \epsilon/2$), it follows that r is on the interior of one of the intervals of this partition. Let that interval be $[x_i, x_{i+1}]$. Since the interior of this interval does not contain any of the s_k s, for all $x \in (x_i, x_{i+1})$, we have $f(x) \leq \epsilon < \epsilon/2$.

Since r is in (x_i, x_{i-1}) , and this interval is open, there exists some δ such that for all $x \in \mathbb{R}$ with $|x - r| < \delta$, we must have $x \in (x_i, x_{i-1})$.

Now chose any $x \in \mathbb{R}$ with $|x - r| < \delta$. As just stated, it follows that $x \in (x_i, x_{i-1})$. As noted, this implies that $f(x) < \epsilon$. Note that $f(x)$ is never neagtive by definition of f , hence $|f(x)| = f(x)$. Moreover, r is irrational, so by definition of f , we have $f(r) = 0$. So

$$|f(x) - f(r)| = |f(x) - 0| = f(x) < \epsilon.$$

But $\epsilon > 0$ was arbitrary, hence for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - r| < \delta$, we have $|f(x) - f(r)| < \epsilon$. This is one of the equivalent definitions of continuity at a point, hence f is continuous at r , as desired.

□

Proposition 3. The function f is integrable, with $\int_0^1 f = 0$

Proof. Let $\epsilon > 0$. Then $\epsilon/2 > 0$. So by Lemma 2, we have only finitely many points $s \in [0, 1]$ with $f(s) > \epsilon/2$. For ease of reference, let S be (finite) set of all such points. Let L be the sum of all such points. Consider

$$\delta = \frac{\epsilon}{2L}.$$

Let $P = \{x_0, \dots, x_n\}$ be a partition of $[0, 1]$ with $\|P\| < \delta$. Chose some collection of sample points x_i^* for P , to form the Reimann sum

$$\mathcal{R}(f, P) = \sum_{i=1}^n f(x_i^*)(x_i - x_{i-1}).$$

Let A be the (possibly empty) set of all indices i of the partition, such $x_i^* \in S$, and let B be the rest of the indices. Certainly $\sum_{i \in A} f(x_i^*) \leq L$. Then by associativity,

$$\mathcal{R}(f, P) = \sum_{i \in A} f(x_i^*)(x_i - x_{i-1}) + \sum_{i \in B} f(x_i^*)(x_i - x_{i-1}).$$

Since $\|P\| < \delta$, $x_i - x_{i-1} < \delta$, hence

$$\sum_{i \in A} f(x_i^*)(x_i - x_{i-1}) < \sum_{i \in A} f(x_i)\delta \leq \delta L = \frac{\epsilon}{2L}L = \epsilon/2.$$

Moreover, note that for each $i \in B$, $f(x_i^*) \leq \epsilon/2$, and note also that $\sum_{i \in B} (x_i - x_{i-1}) \leq \sum_{i=1}^n (x_i - x_{i-1}) = 1$, since the last sum of our inequality was telescoping with $x_n = 1$ and $x_0 = 0$. Hence

$$\sum_{i \in B} f(x_i^*)(x_i - x_{i-1}) \leq \sum_{i \in B} \frac{\epsilon}{2}(x_i - x_{i-1}) = \frac{\epsilon}{2} \sum_{i \in B} (x_i - x_{i-1}) \leq \frac{\epsilon}{2}.$$

Hence

$$|\mathcal{R}(f, P) - 0| = \mathcal{R}(f, P) < \frac{\epsilon}{2} + \sum_{i \in B} f(x_i^*)(x_i - x_{i-1}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since the mesh and Reimann sum were arbitrary, and since ϵ was also arbitrary, it follows that for any $\epsilon > 0$, there is some $\delta > 0$ such that for any partition P with $\|P\| < \delta$, and for any Reimann sum $\mathcal{R}(f, P)$ with that partition, we have

$$|\mathcal{R}(f, P) - 0| < \epsilon.$$

By definition of the Reimann integral,

$$\int_0^1 f = 0$$

as desired. □

2 Products of integrable functions

First, I should state the obvious lemma,

Lemma 3. Let $K \subset \mathbb{R}$ be compact, and let $S \subset K$. For any continuous function $f : K \rightarrow \mathbb{R}$, the restriction $f|_S : S \rightarrow \mathbb{R}$ is uniformly continuous.

This lemma is obvious because for any restriction, and for any two points $x, y \in S$ with $d(x, y) < \delta$, I would necessarily have $|(f(x) - f(y))| < \epsilon$. (Where ϵ and δ would be what they usually are in a proof.)

Theorem 1. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be Riemann integrable. Then fg is Riemann integrable.

Proof. The key is to recognize the purely algebraic identity $\frac{1}{4}[(f + g)^2 - (f - g)^2] = fg$, where scalar multiplication and function addition are defined pointwise. This is obvious yet slightly tedious, so I won't write out all the steps. Once the terms are all foiled correctly, and canceled out, we find that the identity holds.

Because f and g are continuous, by a previous result (which we proved), $f + g$ and $f - g$ are both Riemann integrable. We aim to show that $(f + g)^2$ and $(f - g)^2$ are Riemann integrable. We will do this by recognizing these as compositions $h \circ (f + g)$ and $h \circ (f - g)$, $h : \mathbb{R} \rightarrow \mathbb{R}$ is the function which maps $x \mapsto x^2$ for all $x \in \mathbb{R}$.

We have actually proven that this function is not uniformly continuous in general. However, since continuous functions from compact sets are uniformly continuous, the map h will be uniformly continuous if restricted to some such compact set. By our Lemma, it will also be compact if this domain is further restricted from such a compact set.

Since $f + g$ and $f - g$ are Riemann integrable functions $[a, b] \rightarrow \mathbb{R}$, both are bounded. Hence the sets $(f + g)([a, b])$ and $(f - g)([a, b])$ are bounded. By the theorem which proves that a the closure of a set is closed, the closures of both sets are closed. Since both are bounded as well, by the Heine Borel theorem, both are compact, hence h restricted to either of the closures of these sets is uniformly continuous. By the above lemma, the further restriction of h to the range of either $f + g$ or $f - g$ is also uniformly continuous.

So then, by Theorem 11.5.7 of the textbook, the compositions $h \circ (f + g) = (f + g)^2$ and $h \circ (f - g) = (f - g)^2$ are both Riemann integral. By Part (3) of Theorem 11.3.1 of the textbook, their difference

$$(f + g)^2 - (f - g)^2$$

is Riemann integrable on $[a, b]$. Moreover, by the second part of the same

theorem, the scalar multiple $\frac{1}{4}[(f+g)^2 - (f-g)^2] = fg$ is Riemann integrable on the interval $[a, b]$, which is what we set out to prove. \square