Modern Geometry

August

October 25, 2022

1 Problem 3.29

Proposition 1. It is impossible for there to exist a 3-cycle subgraph of a flip graph.

Proof. Suppose we did have one. Chose any direction on the cycle, and number the edges of the graph 1, 2, 3. Let v_1 and v_2 be the vertices that are the endpoints of the diagonalization that is fliped in 1. In order for the cycle to be a cycle, we must eventually return to the original triangulation, in which case v_1 and v_2 must be re-connected by a diagonal by the remaining flips corresponding to the graph edges 2 and 3. Another observation is helpful: notice that the line segment between v_1 and v_2 is now blocked by the newly formed diagonal in T_1. As a result, it is also necessary to flip e at some point.

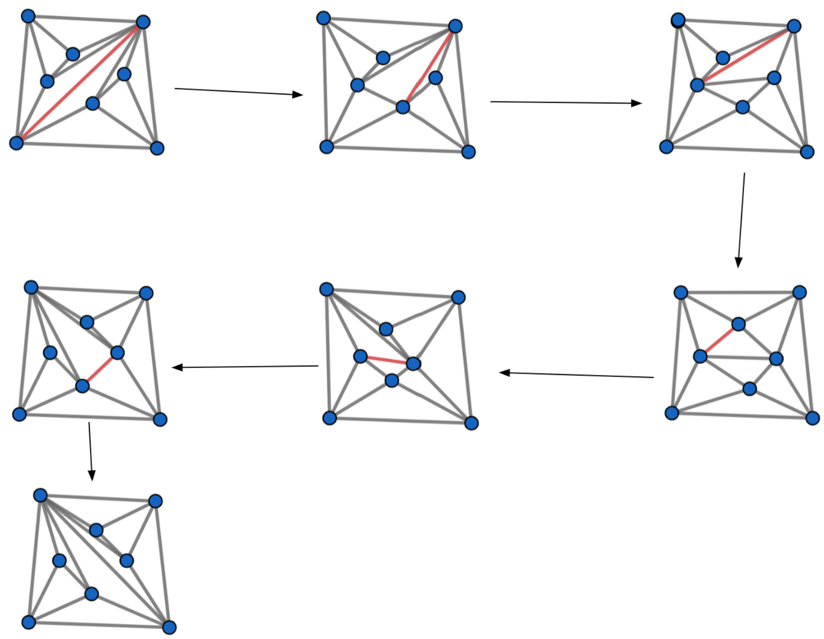
There are two cases: either 2 flips e or it doesn't. Suppose first that it flips e back to T_1. In this case, we have travelled back to the original node, and we do not have a cycle. Hence 2 cannot flip e back. Having concluded that 2 must flip an edge other than e, let that edge be f. Let w_1 and w_2 be the original end-points of f in T_2, so that they are not connected by an edge in T_3.

We now have two pairs of vertices which are no longer connected by a diagonal, but which were connected by a diagonal in T_1. Notice also that each flip draws a diagonal between exactly two edges. One flip is not sufficient to re-connect v_1 with v_2, as well as w_1 with w_2. Hence we do not have a cycle: a contradiction.

□

2 3.24

These triangulations are connected by no more than sixteen flips, because they are connected by six flips! See:



3 3.52

Proposition 2. Suppose that two triangles ABC and ADC belonging to a triangulation T share the edge AC. Then B and D lie on opposite sides of the line AC.

Proof. For suppose that they were on the same side of AC. Then there are two cases: either B is inside ACD or itsn't. If it is inside, then the triangle ACD is split by the segment BD, in which case ACD is not a triangle of the triangulation.

Moreover, if we suppose that B is outside, then the edges BA or CB (depending on which side of the line connecting the point D and the perpendicular bisector of AC lie on) must cross one of the edges of the triangle

ACD, hence it is not a diagonal, and hence ABD does not belong to a triangulation.

Q.E.D. $\square$

Proposition 3. Suppose that the triangle ABC and ADC share the edge AC, such that B and C lie on opposite sides of the line AC, and where $ABCD$ is a convex quadrilateral. Then D is exterior to the circumcircle of the triangle ABC if and only if B is exterior to the circumcircle of the triangle ADC.

Proof. First suppose tha First suppose that D is exterior to the circumcircle of the triangle Draw the line segment BD. Then since $ABCD$ is a convex quadrilateral (a bit of hand-waviness here), the line segment BD intersects the line segment AC at a point, call it O. Since the angles $\angle ADB = \delta$ and $\angle ACB = \gamma$ have a common base, since D is exterior to ABC, and since C lies on the circle ABC, it follows by the correct version of what the textbook calls Thales's theorem that $\gamma > \delta$. Now let $\kappa = \angle AOD$. Since BOC is a vertical angle to $\angle AOD$, it follows that they are both equal to κ. Now let $\alpha = \angle CAD$, and let $\beta = \angle CBD$. Since the angles of a triangle sum to π, it follows that $\alpha = (\pi - \kappa) - \delta$, while $\beta = (\pi = \kappa) - \gamma$. But we have shown that $\gamma > \delta$, hence $\alpha > \beta$. To really drive down the point we can substitute back in: $\angle DBC < \angle CAD$. Moreover, since these angles have a common base, and since A lies on the circle ACD, it follows by "Thales'" "Theorem" that B is exterior to the circle ACD.

Moreover, for the converse, notice that everything about this proof is symmetric in B and D. So without loss of generality, we have proven both ways.

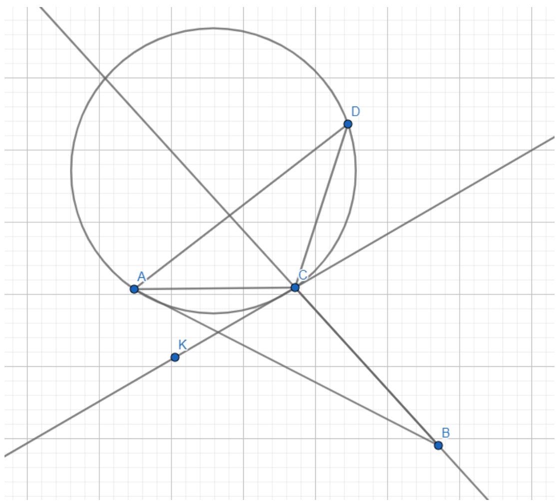
Q.E.D. $\square$

More is true in the non-convex case.

Proposition 4. Suppose that ABC and ADC are triangles sharing the edge AC, where B and D are on opposite sides of AC, and where $ABCD$ is not a convex quadrilateral. Then B is outside the circumcircle of ADC, and D is outside the circumcircle of ABC.

Proof. By symmetry, we need only prove that B is outside the circumcircle of ADC. Since the quadrilateral is non-convex, and since AC is a diagonal, it follows that either A or C is a reflex vertex. Without loss of generality, suppose that A is a reflex vertex. Since A is a reflex vertex, the angle $\angle ACB$

is greater than π, hence B is on the opposite side of A with respect to the line BC. Because the line KC is tangent to the cirle ACD at C, and since A and D are also part of the circle and on the other side than D, it follows that D is not in the circumcircle, since KC is a tangent and the circumcircle is a convex region. $\square$



4 3.61

Definition. A compatible map between two triangulations of polygons X and Y, T_X and T_Y respectively, is a bijective map from the vertices of X to the vertices Y which preserves triangles (abc a triangle of T_X implies $\phi(a)\phi(b)\phi(c)$.

(note that I have un-generalized to just polygons)

Lemma 1. The inverse of a compatible map from convex vertices is compatible, and compatible maps preserve degrees of vertices.

Proof. It is easy to see that ϕ induces a map $\phi' : T_X \to T_Y$, which maps triangle to triangles, and that this map is injective (since triangles are the same only if they have the same vertices). Since the number of triangles in a triangulation of a polygon is a function of the number of vertices, and since ϕ is a bijection from the vertices of X to the vertices of Y (hence they have the same number of elements), it follows that T_X and T_Y are both of the same size. So we have an injection from two finite sets of the same size, so

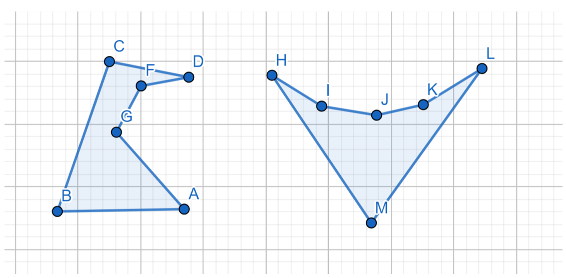
ϕ' is a bijection, and it must have an inverse. Now notice that this inverse is induced by a function from the vertices of Y to the vertices of X, and that this function, when composed with ϕ, is the identity, hence this function is the inverse of ϕ, and it maps triangles to triangles.

First, it's easy to see that $\phi : X \to Y$ respects diagonals and edges. If ab is a diagonal or edge of X, then it is part of some triangle abc, which is sent to the triangle $\phi(a)\phi(b)\phi(c)$. Hence $\phi(a)\phi(b)$ is either an edge or diagonal. Now let p be a point, and let pa be a diagonal or edge connecting to p. Then $\phi(p)\phi(a)$ is also a diagonal connecting to $\phi(p)$. Moreover, each of these is distinct, so it follows that this induced map from diagonals of to p to diagonals to $\phi(p)$ is injective, so $\deg p \leq \deg \phi(p)$. Applying the same argument for $\phi(p)$ and using the compatible map ϕ^{-1}, we find that $\deg p \geq \deg \phi(p)$. Hence $\deg p = \deg \phi(p)$.

This concludes the proof, and shows that degrees are invariant under compatible maps. Q.E.D.

□

Now consider this pair of polygons:



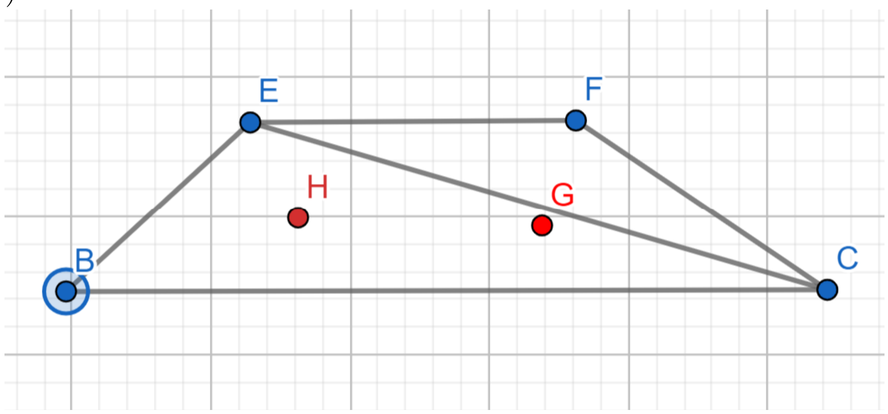
As I showed in a previous assignment, the one on the right has only one triangulation, and in that triangulation the vertex M has degree 5. Now notice that it is only possible for a vertex to have degree 5 among six vertices if an edge connects to all the rest. In the polygon on the left, no vertex can see all the others, hence do vertex has degree 5 in any triangulation. Since, as I have shown, compatible maps preserve degrees, it follows that these shapes are not compatible.

Q.E.D.

5 Problem b

Proposition 5. There exists a point set with four vertices which has no Pittaway triangulation.

Proof. Consider this point set (where the red points are not part of the point set.)



Recall that a convex quadrilateral has exactly two triangulations. Since this one is symmetric, without loss of generality suppose we have the triangulation above, where the diagonal EC is drawn. Then the point G is closest to F, which is not a vertex of the triangle EBC, which contains it. A similar thing happens if the diagonal FB is drawn, in which case H is closest to E. □

I think I have a rigorous that such a construction actually works. Using this proof, I think I can show how this construction can be extended to provide a point set with no pittaway triangulation for $n > 3$.

6 Problem a

Proposition 6. The only two dimensional facets of an n-dimensional associahedron are 4-cycles and 5-cycles.

Proof. Recall that the diagonalizations with all but two diagonals of a triangulation are in one-to-one correspondence with the two dimensional facets, and the number of nodes of those facets are the number of trianguliations corresponding to this diagonalization. So for any face, the corresponding diagonalization has two cases: either the two missing diagonals are adjacent to each other or they are not. If they are not adjacent, we have two quadrilaterals, each having two triangulations. Since each of these have two triangulations, in all we have four of them, hence a 4-cycle. If the two missing diagonals are adjacent, we have ourselves a convex pentagon. Since the flip graph of the convex pentagon is the two dimensional associahedron, which is a five cycle, we have ourselves a five cycle. In either case, we have 4-cycles and 5-cycles. Hence the only two dimensional facets of an n-dimensional associahedron are 4-cycles and 5-cyles.

Q.E.D.

□

Proposition 7. For all n-dimensional associahedra, n of the $n-1$ dimensional facets are $n-1$-dimensional associahedra. Moreover, every node belongs to one such facet, and is connected to a distinct other facet by an edge in the flip graph.

Proof. We proceed by induction. For the base case, consider the line. Clearly it's zero dimensional facets are points, which are zero dimensional associahedra, and they are connected by the single edge. Suppose the proposition holds for all $n-1$ dimensional associahedra. Now suppose that we have some n-dimensional associahedron, corresponding to the flip graph of an $n+3$ vertex convex polygon P. Since P is a convex polygon, it follows that each of the $n+3$ vertices is an ear. Hence at least one vertex is triangled off, and each triangulation corresponds to some such ear, leaving us with a convex $n+2$-gon, whose flip graph is the $n-1$ dimensional associahedron. Since there will always be one more adjacent edge next to the one triangling off our ear, it these nodes are all connected to a distinct flip, which would make the nieghboring ear triangled off. This neighboring ear corresponds to yet another $n-1$ dimsenional associahedron. So the proposition holds in the inductions step as well. Q.E.D. □