Real Analysis

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1 Problem 3.3.5

Proposition 1. Let X be a metric space, and let $x \in X$ be an isolated point. Then any sequence that eventually converges to x is constant.

Proof. Since x is isolated, it follows that $\{x\}$ is an open set. Hence there exists some open ball (Theorem 3.1.7) around x, corresponding to some real radius $r > 0$, such that $B_r(x) \subset \{x\}$. Since open balls are non-empty (it's an easy proof, following immediately from the positive definiteness property of a metric and the definition of an open ball), it follows that $B_r(x) = \{x\}$. Now suppose that $\sigma : \mathbb{N} \to X$ is a sequence converging to X. Then since $r > 0$, there exists some $N \in \mathbb{N}$ such that for all $n > N \in \mathbb{N}$ $d(\sigma(n), x) < r$. In other words, that $\sigma(n) \in B_r(x) = \{x\}$ for all $n > N$. In other words, $\sigma(n) = x$ for all $n > N$. In other words, σ is eventually constant. Q.E.D. $\square$

Proposition 2. Let X be a discrete metric space. Then the only convergent sequences are eventually constant.

Proof. Let σ be any sequence, and suppose that it converges. Then there exists some point $x \in X$ to which it converges. But x is in a discrete metric space, hence x is isolated, from which it follows by the last proposition that σ is eventually constant. But σ was an arbitrary convergent sequence in X, hence any convergent sequence in X must be eventually constant. Q.E.D. $\square$

Remark 1. This is another reason why discrete metric spaces are lame! :)

2 Problem 3.3.6

Proposition 3. The following are equivalent for a sequence $\sigma : \mathbb{N} \to X$, with X a metric space, and x a point within it.

- σ converges to x
- Every subsequence of σ converges to x.
- Every subsequence of σ has a subsequence which converges to x.

Proof.

- First suppose that σ converges to x, and let $\tau : \mathbb{N} \to \mathbb{N}$ be a strictly increasing natural number sequence. We need only show that $\sigma \circ \tau$ converges to x.

  Let $\epsilon > 0$. Then since σ is convergent to x, there exists some $N \in \mathbb{N}$ such that for any $n > N$, $d(\sigma(n), x) < \epsilon$. Now let n be any such n. Recall from a previous exercise that, for a strictly increasing sequence such as τ on the natural numbers, it's term cannot exceed it's index, hence $\tau(n) > n$. But then $\tau(n) > N$ by transitivity. Hence by definition of function composition it follows that $\sigma \circ \tau(n) = \sigma(\tau(n))$ so by construction of N it follows that $d(\sigma \circ \tau(n), x) < \epsilon$. So then, for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n > N \in \mathbb{N}$ $d(\sigma \circ \tau(n), x) < \epsilon$. BY definition of convergence, $\sigma \circ \tau$ converges to x. But $\sigma \circ \tau$ was an arbitrary subsequence, hence any subsequence converges. This fulfills the first chain of implication needed to prove the proposed equivalence.

- Now we suppose that every subsequence of σ converges to x, and need to show that every subsequence of x has a convergent subsequence. But recall from the random generalization from my last assignment that the subsequence relation is transitive. Hence any subsequence of a subsequence of σ is also a subsequence of σ, and therefore must converge to x by assumptino.

- Finally, for our last implication, we proceed by contrapositive. Suppose that σ does not converge to x. It suffices to construct a subsequence of σ which does not have a subsequence which converges to x.

  Since σ does not converge to x, it follows that there exists some $\epsilon > 0$ such that for all natural number N there exists some $n > N$ such that $d(x, \sigma(n)) \geq \epsilon$. We shall construct a sequence with no converging subsequence inductively, with the property that each of it's terms stays ϵ away from x.

- As our base case, let us consider the $\tau(1) = n_1$ where n_1 is that natural number whose existence is guaranteed to us such that $d(x, \sigma(n_1)) \geq \epsilon$. Clearly then $\sigma \circ \tau(1) = \sigma(n_1)$, so that it is ϵ away from x.
- Now suppose that we have defined τ thus far, so that $d(\sigma \circ \tau(n), x) \geq \epsilon$.
- Now define $\tau(n+1) = m_{\tau(n)}$ where $m_{\tau(n)}$ is that natural number greater than $\tau(n)$ such that $d(\sigma(m_{\tau(n)}), x) \geq \epsilon$. This construction fulfills the induction step.

We have constructed a subsequence $\sigma \circ \tau$ with the property that each of it's terms is ϵ away from x. Consider any subsequence. This subsequence's terms must also stay ϵ away to x, hence by definition of convergence, it does not converge to x. Hence no subsequence of $\sigma \circ \tau$ converges to x.
This fulfills the third implication, that non-converrgence to x means that there is a subsequence which has no subsequence which converges to x.

Having shown each of the necessary implications, it follows that that these properties are equivalent.

Q.E.D. $\square$