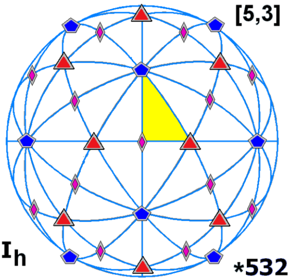
Complementation of Subquandles

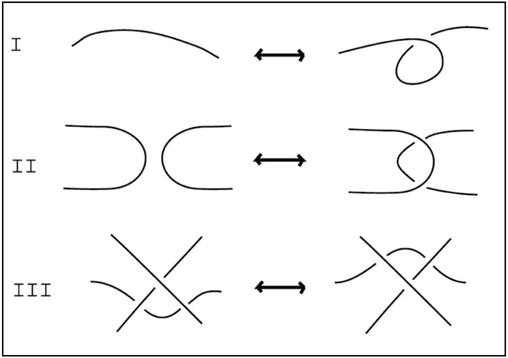
D.N. Yetter, K.J. Amsberry, M.H. Lee, T.A. Horstkamp, J.A. Bergquist

February 23, 2023

Introduction

Just as groups tell the algebraic story of symmetry, quandles tell the algebraic story of the Reidemeister moves.





Background: History

- Dr. Mizumitzu Takasaki was the first to investigate quandles. He was motivated by reflections in finite geometry.

Background: History

- Dr. Mizumitzu Takasaki was the first to investigate quandles. He was motivated by reflections in finite geometry.
- Dr. Matveev and Dr. David Joyce independently discovered the **knot quandle**: a complete invariant up to orientation.

Background: History

- Dr. Mizumitzu Takasaki was the first to investigate quandles. He was motivated by reflections in finite geometry.
- Dr. Matveev and Dr. David Joyce independently discovered the **knot quandle**: a complete invariant up to orientation.
- Matveev's formulation of the knot quandle was geometric, while David Joyce's was combinatorial.Both definitions are equivalent.

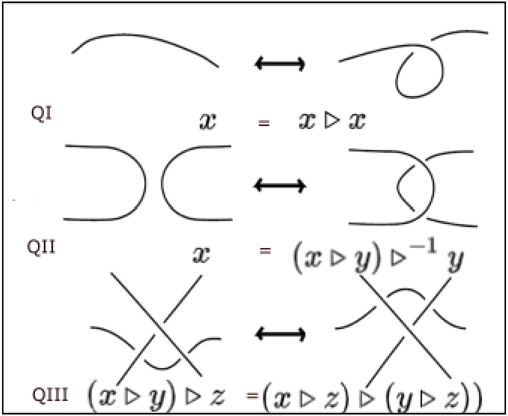
Background: History

- Dr. Mizumitzu Takasaki was the first to investigate quandles. He was motivated by reflections in finite geometry.
- Dr. Matveev and Dr. David Joyce independently discovered the **knot quandle**: a complete invariant up to orientation.
- Matveev's formulation of the knot quandle was geometric, while David Joyce's was combinatorial.Both definitions are equivalent.
- Quandles are a complete invariant, but hard to understand. This motivates the study of quandles as an algebraic theory in their own right. [1]

Background: Definition of a quandle

Definition

[1] [3] A quandle is a set Q, paired with operations $\triangleright$ and $\triangleright^{-1}$ such that the following axioms are satisfied:



Background: Quandles and Groups

- Quandles and groups are closely related.

Background: Quandles and Groups

- Quandles and groups are closely related.
- Quandles are the algebraic theory of conjugation in groups.

Background: Quandles and Groups

- Quandles and groups are closely related.
- Quandles are the algebraic theory of conjugation in groups.
- Every conjugacy class C_α is a subquandle of $\mathrm{Conj}(G)$.

Definition

Every group G gives rise to a quandle $\mathrm{Conj}(G)$, where the operation is $g \triangleright h = h^{-1}gh$.

- The correspondence $G \mapsto \mathrm{Conj}(G)$ is the object function of a functor,

$$\mathrm{Conj} : \mathbf{Quand} \to \mathbf{Grp}$$

Background: Adconj

Theorem

[3] The functor $\mathrm{Conj} : \mathbf{Grp} \to \mathbf{Quand}$ *has a left adjoint (David Joyce).*

Definition

The universal augmentation group, $\mathrm{Adconj}(Q)$, is the group defined $\mathrm{Adconj}(Q) = \langle x \in Q | x \triangleright y y^{-1} x^{-1} y^{-1}; x, y \in Q \rangle$ The correspondence $Q \mapsto \mathrm{Adconj}(Q)$ is the object function of a functor $\mathbf{Quand} \to \mathbf{Grp}$.

Theorem

Adconj *is the left adjoint to* Conj.

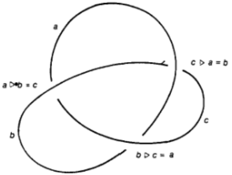
$$\begin{tikzcd} Q \arrow[r, "\eta"] \arrow[dr, "\phi"'] & \mathrm{Conj}(\mathrm{Adconj}(Q)) \arrow[d, dashed, "\mathrm{Conj}(\tilde{\phi})"] \\ & \mathrm{Conj}(G) \end{tikzcd}$$

Background: Significance of Adconj

There is geometric significance to the functor Adconj.

Theorem

Let K be a knot, and let $Q(K)$ be it's quandle. Then $\mathrm{Adconj}(Q(K)) \cong \pi_1(S^3 \setminus K)$. [1]

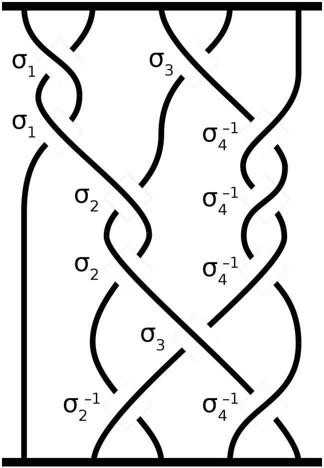


Example

As an example, $Q(K_3)$ is the quandle of transpositions of S_3 under conjugation. $\mathrm{Adconj}(Q(K_3))$ is the braid group on three strands.

[5]

Background: Significance of Adconj



Takasaki Keis

Definition

[3] A **kei** is a quandle K whose operation is involutory: $x \triangleright y \triangleright y = x$ for all $x, y \in K$.

Definition

Given any abelian group A, one can form a quandle, $\mathrm{Tak}(A)$, with operation $x \triangleright y = 2y - x$. This is the object function of a functor,

$$\mathbf{Kei} \to \mathbf{Ab}.$$

We'll show that this functor has an adjoint. The Takasaki functor provides a lot of useful (counter)examples.

Background: Semidisjoint Union

Definition

[2] Given a sequence of quandles $Q_1, \dots, Q_n$ and a *nxn* matrix of group homomorphisms $(M)_{ij} = g_{ij}$:, Ehrman et al. [2] defined the **semidisjoint union** as follows:

$$\#(Q_1, \dots, Q_n, M) = \left(\coprod_{i=1}^{n} Q_i, \triangleright \right).$$

Background: Semidisjoint Union

Definition

[2] Given a sequence of quandles $Q_1, \ldots, Q_n$ and a *nxn* matrix of group homomorphisms $(M)_{ij} = g_{ij}$:, Ehrman et al. [2] defined the **semidisjoint union** as follows:

$$\#(Q_1, \ldots, Q_n, M) = \Big(\coprod_{i=1}^{n} Q_i, \triangleright \Big).$$

- Each entry of the matrix $g_{ij} : \mathrm{Adconj}(Q_i) \to \mathrm{Aut}(Q_j)$ is a group homomorphism.
- $\triangleright$ is defined as $x \triangleright y = x \cdot g_{ij}(|y|_{Q_i})$ for $x \in Q_i$ and $y \in Q_j$.
- Note that we are not guaranteed that the semidisjoint union is a quandle. If the matrix M gives rise to a quandle, it is called a **mesh**.
- Ehrman et al. provided a necessary and sufficient condition for M to be a mesh.

Background: The Inner Automorphism Group

Probably the most important group (for our purposes) is the inner automorphism group.

Definition

An inner automorphism is a string of successive symmetries. The inner automorphism group is the group generated by the symmetries at elements, denoted $\mathrm{Inn}(Q) = < S(Q) >$.

Warning! While the symmetries at elements are inner automorphisms, not all inner automorphisms are symmetries at elements. This is due to the fact that the quandle operation is **not** associative in general.

Background: Orbit Decomposition

Theorem

[2] Let Q be a quandle, and let $Q_1, \ldots, Q_n$ be its orbits under the inner automorphism group. Then we can construct a mesh M such that

$$Q = \#(Q_1, \ldots, Q_n, M).$$

Background: Orbit Decomposition

Theorem

[2] Let Q be a quandle, and let $Q_1, \dots, Q_n$ be its orbits under the inner automorphism group. Then we can construct a mesh M such that

$$Q = \#(Q_1, \dots, Q_n, M).$$

- Note that the orbits need not be connected. Hence the orbits themselves may be decomposable via the previous theorem.

Subquandle Lattices

Definition

The set of subquandles of any quandle Q under inclusion forms a lattice [4], which we denote as $\mathcal{L}(Q)$.

Subquandle Lattices

Definition

The set of subquandles of any quandle Q under inclusion forms a lattice [4], which we denote as $\mathcal{L}(Q)$.

Definition

Given two subquandles $Q_1, Q_2 \preccurlyeq Q$, their **meet** is $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ and their **join** is $Q_1 \vee Q_2 = \langle\langle Q_1 \cup Q_2 \rangle\rangle$.

Subquandle Lattices

Definition

The set of subquandles of any quandle Q under inclusion forms a lattice [4], which we denote as $\mathcal{L}(Q)$.

Definition

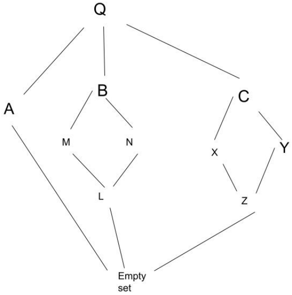
Given two subquandles $Q_1, Q_2 \preccurlyeq Q$, their **meet** is $Q_1 \wedge Q_2 = Q_1 \cap Q_2$ and their **join** is $Q_1 \vee Q_2 = \langle\langle Q_1 \cup Q_2 \rangle\rangle$.

Definition

A subquandle $Q_1 \preccurlyeq Q$ is **complemented** in Q if there is some $Q_2 \preccurlyeq Q$ such that $Q_1 \wedge Q_2 = \emptyset$, and $Q_1 \vee Q_2 = Q$. The subquandle lattice $\mathcal{L}(Q)$ is complemented if every subquandle is complemented.

Subquandle Lattices

Just as with groups and other algebraic structures, subquandle lattices can be visualized by an inclusion diagram.



Two Actions

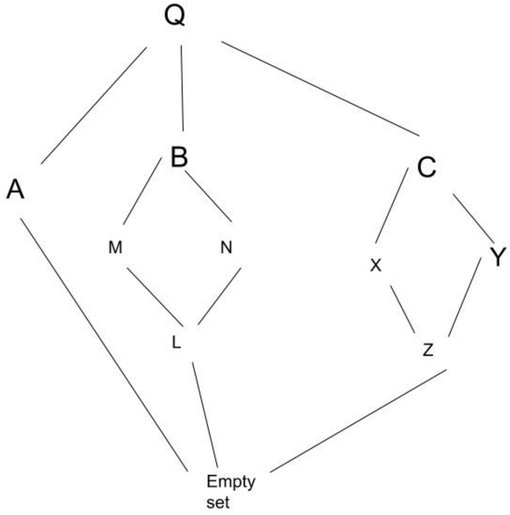
- We've shown how $\mathrm{Inn}(Q)$ acts on Q by functional application.
- This action allows us to construct an action of $\mathrm{Inn}(Q)$ upon $\mathcal{L}(Q)$.

Definition

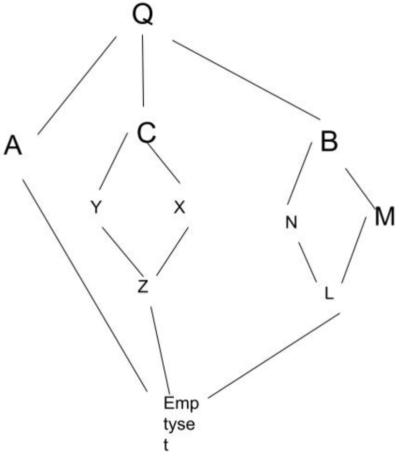
The action of $\mathrm{Inn}(Q)$ on Q' is also given by functional application, denoted $Q' \cdot \mathrm{Inn}(Q)$.

The action of $\mathrm{Inn}(Q)$ upon $\mathcal{L}(Q)$ is defined $Q'\sigma = \sigma(Q')$ for all $Q' \in \mathcal{L}(Q)$, and for all $\sigma \in \mathrm{Inn}(Q)$. The orbit of Q' under this action is denoted by $[Q'] \cdot \mathrm{Inn}(Q) = \{Q'\sigma : \sigma \in \mathrm{Inn}(Q)\}$.

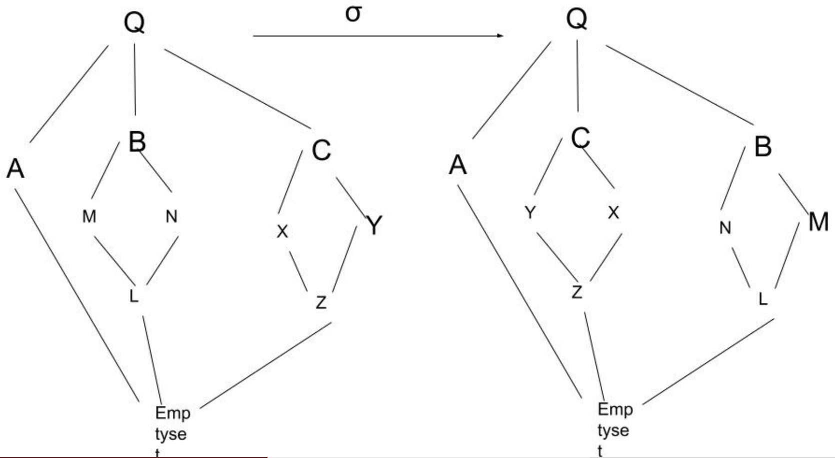
Two Actions



Acting on the subquandle lattice



Acting on the subquandle lattice



Background: Saki And Kiani's Paper

Saki and Kiani showed that the subrack lattice of a finite rack is complemented. Since quandles are racks, this holds for quandles as a corollary.

Corollary

The subquandle lattice of every finite quandle is complemented.

Background: Saki And Kiani's Paper

Saki and Kiani showed that the subrack lattice of a finite rack is complemented. Since quandles are racks, this holds for quandles as a corollary.

Corollary

The subquandle lattice of every finite quandle is complemented.

Theorem (Saki and Kiani)

Consider the quandle $\mathbb{Q}$ with the dihedral structure, $x \triangleright^* y = 2x - y$, $\forall x, y \in \mathbb{Q}$. The subquandle $\{0\}$ is not complemented.

Equivalence Theorem

Theorem

Let Q be a quandle, and let $Q' \preccurlyeq Q$. Denote the subquandle lattice of Q by $\mathcal{L}(Q)$. The following are equivalent:

1. $Q \setminus Q' \preccurlyeq Q$,
2. Q' *is a union of orbits under the action of* $\mathrm{Inn}(Q)$ *on* Q,
3. Q' *is a fixed point of the action of* $\mathrm{Inn}(Q)$ *on* $\mathcal{L}(Q)$,
4. $Q = \#(Q', Q \setminus Q', M)$ *for a mesh* M.

Strongly Complemented Within Strongly Complemented

Lemma (1)

Let $Q'' \preccurlyeq Q' \preccurlyeq Q$ such that Q'' is strongly complemented within Q', and Q' is strongly complemented within Q. Then for any $\gamma \in \mathrm{Inn}(Q)$, $Q''\gamma$ is strongly complemented within Q'.

Strongly Complemented Within Strongly Complemented

Lemma (1)

Let $Q'' \preccurlyeq Q' \preccurlyeq Q$ such that Q'' is strongly complemented within Q', and Q' is strongly complemented within Q. Then for any $\gamma \in \mathrm{Inn}(Q)$, $Q''\gamma$ is strongly complemented within Q'.

Lemma (2)

Let $Q'' \preccurlyeq Q' \preccurlyeq Q$ be as in Lemma (1). Then for all $\sigma \in \mathrm{Inn}(Q)$, there exists some $\sigma' \in \langle S(Q \setminus Q') \rangle$ such that $Q''\sigma = Q''\sigma'$.

Proof.

- Suppose $\sigma = S_{x_1}^{\epsilon_1} \dots S_{x_n}^{\epsilon_n}$ for each $x_j \in Q$, $\epsilon_j \in \{-1, 1\}$.
- Suppose $\exists x_i \in Q'$ whose symmetry appears in σ which we cannot remove without changing the action.
- By Lemma 1, Q'' acted upon by the string of σ up until x_i, denote it by γ, is strongly complemented in Q'.

□

Strongly Complimented Subquandles

Our research centered around revealing finer structure to the ways in which subquandle fit within the subquandle lattice.

Strongly Complemented Within Strongly Complemented

Proof (*continued*).

- Since $x_i \in Q'$, $S_{x_i}|_{Q'}$ acts the same upon Q' as S_{x_i}
- Then, $Q''\gamma S_{x_i} = Q''\gamma$ by the equivalence theorem. Hence, removing S_{x_i} won't change anything, so there never was a counterexample.
- Thus, we can construct σ' by removing symmetries represented by elements in Q'.

$\square$

Lemma (3)

Suppose that Q'' is strongly complemented within Q, while $Q'' \preccurlyeq Q' \preccurlyeq Q$. Then, Q'' is strongly complemented within Q'.

Strongly Complemented Within Strongly Complemented

Theorem

Suppose Q is a quandle, with subquandles $Q'' \preccurlyeq Q' \preccurlyeq Q$, such that Q'' is strongly complemented within Q', while Q' is strongly complemented within Q. Then Q'' is complemented within Q by the subquandle $Q \setminus Q'' \cdot \mathrm{Inn}(Q)$.

Strongly Complemented Within Strongly Complemented

Theorem

Suppose Q is a quandle, with subquandles $Q'' \preccurlyeq Q' \preccurlyeq Q$, such that Q'' is strongly complemented within Q', while Q' is strongly complemented within Q. Then Q'' is complemented within Q by the subquandle $Q \setminus Q'' \cdot \mathrm{Inn}(Q)$.

Proof.

- By Lemma 3 and the equivalence theorem, Q'' is strongly complemented within $Q'' \cdot \mathrm{Inn}(Q)$, which in turn is strongly complemented within Q.
- $Q'' \wedge Q \setminus Q'' \cdot \mathrm{Inn}(Q) = \emptyset$
- Using Lemmas 2 and 1, we show that $Q'' \cdot \mathrm{Inn}(Q) \subset Q'' \vee Q \setminus Q'' \cdot \mathrm{Inn}(Q)$, hence $Q'' \vee Q \setminus Q'' \cdot \mathrm{Inn}(Q) = Q$.
- Hence Q'' has $Q \setminus Q'' \cdot \mathrm{Inn}(Q)$ as a complement.

□

possible extensions of Saki and Kiani's results

Question

Do ind-finite quandles have complimented sublattices?

possible extensions of Saki and Kiani's results

Question

Do ind-finite quandles have complimented sublattices?

-

Question

Do pro-finite quandles have complimented sublattices?

-

Ind-finite Quandles

Although ind-finite objects are generally defined in terms of direct limits, a less abstract definition is available with quandles.

Definition

A nonempty quandle Q is **ind-finite** if there is a family of finite subquandles $\{Q_i\}_{i\in\mathcal{I}}$ indexed by a directed set $\mathcal{I}$ such that each $Q_i \preccurlyeq Q$, $|Q_i| < \infty$, $Q_i \prec Q_{i+1}$, and $Q = \bigcup_{i\in\mathcal{I}} Q_i$.

Ind-finite Quandles

Although ind-finite objects are generally defined in terms of direct limits, a less abstract definition is available with quandles.

Definition

A nonempty quandle Q is **ind-finite** if there is a family of finite subquandles $\{Q_i\}_{i\in\mathcal{I}}$ indexed by a directed set $\mathcal{I}$ such that each $Q_i \preccurlyeq Q$, $|Q_i| < \infty$, $Q_i \prec Q_{i+1}$, and $Q = \bigcup_{i\in\mathcal{I}} Q_i$.

Lemma

Suppose Q is a quandle. Then, Q is ind-finite if and only if every finitely generated subquandle of Q is finite.

Ind-finite Quandles

Although ind-finite objects are generally defined in terms of direct limits, a less abstract definition is available with quandles.

Definition

A nonempty quandle Q is **ind-finite** if there is a family of finite subquandles $\{Q_i\}_{i\in\mathcal{I}}$ indexed by a directed set $\mathcal{I}$ such that each $Q_i \preccurlyeq Q$, $|Q_i| < \infty$, $Q_i \prec Q_{i+1}$, and $Q = \bigcup_{i\in\mathcal{I}} Q_i$.

Lemma

Suppose Q is a quandle. Then, Q is ind-finite if and only if every finitely generated subquandle of Q is finite.

Example

Let the quandle operation $\triangleright^*$ be given by $x \triangleright^* y = 2x - y$.

- Neither $(\mathbb{Q}, \triangleright^*)$ nor $(\mathbb{Q}/\mathbb{Z}, \triangleright^*)$ has a complemented sublattice.
- $(\mathbb{Q}/\mathbb{Z}, \triangleright^*)$ is ind-finite but $(\mathbb{Q}, \triangleright^*)$ is not.

This answers our first question: no!

Background: Adconj

The functor $\mathrm{Conj} : \mathbf{Grp} \to \mathbf{Quand}$ has a left adjoint (David Joyce).

Definition

Given a directed set $\mathcal{I}$, an inverse system of quandles is a family of quandles Q_i for $i \in \mathcal{I}$, and a family of quandle homomorphisms $\phi_{ij} : Q_i \to Q_j$ for $i \leq j$, such that for all $i \leq j \leq k$, we have a commutative diagram
The limit of this diagram (which always exists for quandles), is the inverse limit of such an inverse system.

$$\begin{tikzcd} Q_i \arrow[r, "\phi_{ij}"] \arrow[dr, "\phi_{ik}"'] & Q_j \arrow[d, "\phi_{jk}"] \\ & Q_k \end{tikzcd} .$$

A **profinite quandle** is the limit of an inverse system.

Background: Adconj

The functor $\mathrm{Conj} : \mathbf{Grp} \to \mathbf{Quand}$ has a left adjoint (David Joyce).

Definition

Given a directed set $\mathcal{I}$, an inverse system of quandles is a family of quandles Q_i for $i \in \mathcal{I}$, and a family of quandle homomorphisms $\phi_{ij} : Q_i \to Q_j$ for $i \leq j$, such that for all $i \leq j \leq k$, we have a commutative diagram
The limit of this diagram (which always exists for quandles), is the inverse limit of such an inverse system.

$$\begin{tikzcd} Q_i \arrow[r, "\phi_{ij}"] \arrow[dr, "\phi_{ik}"'] & Q_j \arrow[d, "\phi_{jk}"] \\ & Q_k \end{tikzcd} .$$

A **profinite quandle** is the limit of an inverse system.

Remark

As you might guess, these are a lot harder to think about.

Limit Preserving Functors

One useful way of finding functors that 1) preserve limits, and 2) send finite objects to finite objects.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is a right adjoint if there is another functor $G : \mathcal{D} \to \mathcal{C}$ such that for any object $c \in \mathcal{C}$, there is a morphism $\eta : c \to G(F(c))$

Theorem

A functor $F : \mathcal{C} \to \mathcal{D}$ (where $\mathcal{C}$ and $\mathcal{D}$ are complete) preserves limits if it has a right adjoint.

Profinite Quandles and the Takasaki Functor

Theorem

The functor $\mathrm{Tak} : \mathbf{Ab} \to \mathbf{Kei}$ *is a right adjoint, with left adjoint* $\mathrm{Adtak} : \mathbf{Kei} \to \mathbf{Ab}$, *with object function*

$$K \mapsto \langle x \in K \rangle_{\mathbf{Ab}} / \langle (x \triangleright y - 2y + x : x, y \in K) \rangle_{\mathbf{Ab}}$$

Corollary

For any profinite abelian group A, $\mathrm{Adtak}(A)$ *is a profinite quandle.*

Some Examples of Profinite Quandles

Example

$Tak(A)$ for any profinite abelian group. In particular, $\mathrm{Tak}(\mathbb{Z}_p)$ for any prime p.

Example

$\mathrm{Conj}(G)$ for any profninite group G (we already know Conj is a right adjiont).

Open Questions

Question

Do profinite quandles have complimented sublattices?

Question

Can the partial transitivity condition be extended?

Question

Can we find an example in which the partial transitivity condition holds non-trivially?

bib I

[1] Eleanor Birrell. "The knot quandle". In: *Instructions for Authors. All submissions should* (2007), p. 33.

[2] Geoff Ehrman et al. "Toward a classification of finite quandles". In: *Journal of Knot Theory and Its Ramifications* 17.04 (2008), pp. 511–520.

[3] David Joyce. "A classifying invariant of knots, the knot quandle". In: *Journal of Pure and Applied Algebra* 23.1 (1982), pp. 37–65.

[4] A Saki and D Kiani. "Complemented lattices of subracks". In: *Journal of Algebraic Combinatorics* 53.2 (2021), pp. 455–468.

[5] "Wikipedia: Knot Groups". In: ().