Real Analysis

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1 Problem D.2.4

Lemma 1. Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$. Then for each $i \in \mathbf{n}$, $|x_i - y_i| \leq d(\mathbf{x}, \mathbf{y})$, where d is the euclidean metric on $\mathbb{R}^n$.

Proof. This follows immediately from the fact (which I also proved) that the square root function is monotonic on the non-negative reals, and positive definiteness of a metric (and the fact that the euclidean metric is one). The details are in my second to last assignment. □

Proposition 1. Suppose that we have a sequence $\sigma : \mathbb{N} \to \mathbb{R}^m$ such that for each k, $\sigma(k) = (\sigma_1(k), \ldots, \sigma_m(k))$ for $\sigma_i : \mathbb{N} \to \mathbb{R}$ for all $i \in \mathbf{m}$. Then σ converges to $\mathbf{x} = (x_1, \ldots, x_m)$ if and only if σ_i converges to x_i for each x.

Proof. First suppose for contrapositive that for some $j \in \mathbf{m}$, we have σ_j not converging to x_j in $\mathbb{R}$. Then there exists some $\epsilon > 0$ so that for all $N \in \mathbb{N}$, there is at least one $n > N$ so that $|\sigma_j(n) - x_j| \geq \epsilon$. Now consider that very same ϵ, and let N be any natural number. Then for that N and ϵ, there must exist some $n > N$ so that $|\sigma_j(n) - x_j| \geq \epsilon$. But since by the first lemma we have $d(\sigma(n), \mathbf{x}) \geq |\sigma_j(n) - x_j| \geq \epsilon$. So for any positive ϵ, and for any natural number N, we can produce some natural number larger than it so that the distance between $\sigma(n)$ and $\mathbf{x}$ is not less than that ϵ, and σ fails to converge to $\mathbf{x}$.

Now suppose that each of the σ_j converge for $j \in \mathbf{m}$. Let $\epsilon > 0$. Since the square root function is monotonic, and since $\sqrt{0} = 0$ and $m > 0$, it follows that $\sqrt{m} > 0$. Hence $\epsilon/\sqrt{m} > 0$. From this, and the convergence of each of the σ_j, it follows that for each $j \in \mathbf{m}$ there exists some $N_j \in \mathbb{N}$, such that for each $n > N_j$, $|\sigma_j(n) - x_j| < \epsilon/\sqrt{m}$. Now let $N = \max N_1, \ldots, N_m$, and let $n > N$. Since N is the maximum, it follows that for each $j \in \mathbf{m}$,

$|\sigma_j(n) - x_j| < \epsilon/\sqrt{m}$.

Now by definition of the euclidean metric and since $\sqrt{\cdot}$ is monotonic, for this same n, we have

$$d(\sigma(n), \mathbf{x}) = \sqrt{\sum_{j\in\mathbf{m}} |\sigma_j(n) - x_j|^2} < \sqrt{\sum_{j\in\mathbf{m}} \epsilon^2/m} = \sqrt{\epsilon^2} = \epsilon.$$

To reiterate, we have shown that for all $\epsilon > 0$, there exists some natural number N, such that for all $n > N$, the distance between $\sigma(n)$ and $\mathbf{x}$ is less than ϵ. By definition of convergence, σ converges to $\mathbf{x}$.

Q.E.D. $\square$

2 Problem 3.4.9

Proposition 2. Suppose that a_n converges to a in $\mathbb{R}$, and let $k \in \mathbb{R}$. Then ka_n converges to ka in $\mathbb{R}$.

Proof. Notice that if $k = 0$, then each term in the sequence ka_n is zero, so the sequence is constant at zero. We have shown that constant sequences converge to their value, hence the sequence converges to $ka = 0$ in this case.

Now suppose that $k \neq 0$. By properties of absolute value (or positive definateness when viewed as a distance function, where $|k| = |k - 0|$), it follows that $|k| > 0$, hence by a previous exercise $1/|k| > 0$. Let $\epsilon > 0$. Then $\epsilon/|k| > 0$. By definition of convergence, and since a_n converges to a, it follows that there exists some $N \in \mathbb{N}$ such that for all $n > N$, we have $|a_n - a| < \epsilon/|k|$, and let n be any such $n > N$. Then it follows that

$$|k||a_n - a| = |k(a_n - a)| = |ka_n - ka| < \epsilon$$

. So then, for all $\epsilon > 0$ there is some $N \in \mathbb{N}$ such that for all $n > N$, $|ka_n - ka| < \epsilon$. From this it follows by definition of convergence that ka_n converges to ka. $\square$

Proposition 3. Suppose that b_n is a non-zero real sequence which converges to a non-zero real number b, and that a_n is a real valued sequence which converges to a. Then the sequence of quotients a_n/b_n converges to a/b.

Proof. We begin by finding a positive lower bound for terms of the sequence $|b_n|$, which we will use. Since $b \neq 0$, it follows that $|b| > 0$. Hence $|b|/2 > 0$.

Since (as we have shown in a previous exercise) the convergence of b_n to b implies the convergence of $|b_n|$ to $|b|$, it follows by definition of convergence that there must exist some natural number N, such that for all natural numbers n greater than it, $||b_n| - b| < |b|/2$. But recall from exercise () that this implies $|b_n| \in (|b|/2, 3|b|/2)$ for all $n > N$. By definition of an open interval, it follows that $0 < |b|/2 < |b_n|$. Moreover, the set of terms of $|b_n|$ of index not exceeding N is finite, hence it must have a minimum element; call it $|b_m|$ for some $m \leq N$. Moreover, since each term of b_n is non-zero, it follows that $|b_n| > 0$. Let L be the smaller of $|b_m|$ and $|b|/2$. So then, we have that all terms of $|b_n|$ are greater than or equal to L, and L is greater than 0. So we have a positive lower bound, L, for the terms of the sequence $|b_n|$.

Now let $\epsilon > 0$. Since L is positive, and since $1 + \frac{|a|}{|b|}$ is positive, it follows that $\epsilon' = \frac{\epsilon L}{1+\frac{|a|}{|b|}} > 0$. By definition of convergence, it follows that there exists $N \in \mathbb{N}$ such that for all natural $n > N'$, $|a_n - a| < \epsilon'$, as well as $N'' \in \mathbb{N}$ such that for all $n > N''$, $|b_n - b| < \epsilon'$. So let $N = \max N', N''$, and let $n > N$ be arbitrary.

Now we examine $|\frac{a_n}{b_n} - \frac{a}{b}|$, and algebrize the hell out of it (using the established relationships between division, absolute value, and inequality):

$$\begin{aligned}
&|\tfrac{a_n}{b_n} - \tfrac{a}{b}| \\
&= |\tfrac{a_n b - a b_n}{b_n b}| \\
&= \tfrac{|a_n b - a b_n|}{|b_n||b|} \\
&= \tfrac{|(a_n b - ab) + (ab - a b_n)|}{|b_n||b|} \\
&\leq \tfrac{|a_n b - ab|}{|b_n|b} + \tfrac{|ab - b_n a|}{|b_n|b} \\
&= \tfrac{|b(a_n - a)|}{|b_n|b} + \tfrac{|a(b - b_n)|}{|b_n|b} \\
&= \tfrac{|b||a_n - a|}{|b_n|b} + \tfrac{|a||b - b_n|}{|b_n|b} \\
&< \tfrac{1}{L}(|a_n - a| + \tfrac{|a|}{|b|}|b_n - b|) \\
&< \tfrac{\epsilon'}{L}(1 + \tfrac{|a|}{|b|}) = \tfrac{\epsilon L}{1+\frac{|a|}{|b|}} \tfrac{1+\frac{|a|}{|b|}}{L} = \epsilon
\end{aligned}$$

So $|\frac{a_n}{b_n} - \frac{a}{b}| < \epsilon$. Since n was arbitrary greater than N, it follows that this inequality holds for all such n greater than N. So then, for all $\epsilon > 0$, there exists a natural number N, such that for all $n > N$, $|\frac{a_n}{b_n} - \frac{a}{b}| < \epsilon$. By definition of a limit, it follows that $\frac{a_n}{b_n}$ converges to $\frac{a}{b}$. $\square$