Real Analysis

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1 A closed subspace of a complete space is complete

Proposition 1. Let X by complete, and let $Y \subset X$ be closed. Then Y is complete.

Proof. Suppose σ is a Cauchy sequence in Y. Then, regarding σ as a sequence in XS, σ is still Cauchy, and X is complete, therefore σ converges to some value $x \in X$. Suppose then that $x \notin Y$. Since σ is contained within Y, it follows that σ never assumes the value x to which it converges. By definition of a limit point, it follows that x is a limit point of Y. But Y is closed, so it contains all it's limit points, hence $x \in Y$, a contradiction. Hence $x \in Y$. Since $\sigma \to x \in Y$, σ converges in Y. Since σ was an arbitrary Cauchy sequence in Y, it follows that each Cauchy sequence in Y converges. Hence Y is complete.

□

2 Real-space is complete

Proposition 2. For $n \in \mathbb{N}$, $\mathbb{R}^n$ is complete.

Proof. Let σ be Cauchy in $\mathbb{R}^n$. Then we have n sequences in $\mathbb{R}$, formed by projecting $p_i : \mathbb{R}^n \to \mathbb{R}$, sending $\mathbf{x} \to x_i$. To see that each of the $p_i\sigma = \sigma_i$ is Cauchy, recall that for $\mathbf{y} = (y_1, \ldots, y_n)$ and $\mathbf{x} = (x_1, \ldots, x_n)$, and for all $i \in \mathbf{n}$, we have $|x_i - y_i| \leq d(\mathbf{x}, \mathbf{y})$. Chose any $i \in \mathbf{n}$. Now let $\epsilon > 0$. Then there is some $N \in \mathbb{N}$ so that for $n, m > N$, $d(\sigma(n), \sigma(m)) < \epsilon$. Hence for $m, n > N$, we also have $|\sigma_i(m), \sigma_i(n)| < d(\sigma(n), \sigma(m)) < \epsilon$. Hence σ_i is Cauchy for each of the σ_i. Since each of the σ_i are Cauchy, and sequences in $\mathbb{R}$ which is complete, it follows that for each i there is an $x_i \in \mathbb{R}$ such that

$\sigma_i \to x_i$. Consider the point $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$. We will see that $\sigma \to \mathbf{x}$.

To see this, let $\epsilon > 0$. Then $\epsilon/\sqrt{n} > 0$. Hence by convergence of each σ_i to x_i, it follows that for each i there is an N_i such that for $n > N_i$, $|\sigma_i(n) - x_i| < \epsilon/\sqrt{n}$. Let $N = \max\{N_i\}_{i \in \mathbf{n}}$. Suppose $n > N$. Then for each i, $n > N_i$, hence $|\sigma_i(n) - x_i| < \epsilon/\sqrt{n}$, so obviously $(\sigma_i(n) - x_i)^2 < \epsilon^2/n$. Hence by definition of the Euclidean metric on $\mathbb{R}^n$, we have

$$\begin{aligned} d(\sigma(n), \mathbf{x}) &= \sqrt{\textstyle\sum_{i=0}^{n} (\sigma_i(n) - x_i)^2} \\ &< \sqrt{\epsilon^2/n + \cdots + \epsilon^2/n} = \epsilon \end{aligned}$$

as desired. Hence for all $\epsilon > 0$, there is some $N \in \mathbb{N}$, such that for all $n > N$, $d(\sigma(n), \mathbf{x}) < \epsilon$. By definition of convergence, $\sigma \to \mathbf{x}$ in $\mathbb{R}^n$. Since σ was arbitrary as a Cauchy sequence in $\mathbb{R}^n$, it follows that all Cauchy sequences converge. Hence $\mathbb{R}^n$ is complete. $\square$

3 Arithmetic and Limits of Functions are Nice to Each Other

Lemma 1. Let $f : K \to Y$ be a function between metric spaces with $K \subset X$ a metric space, and let a be a limit point of K. Then the following are equivalent:

- $\lim_{x \to a} f(x) = L$
- for any sequence $\sigma \to a$, we have $f\sigma \to L$.
- If g non-zero on K and M is non-zero, then $\lim_{x \to a} f(x)/g(x) = L/M$.

This gives us the following theorem:

Theorem 1. Let X be a metric space, and let a be a limit point of X. Suppose we have $f, g : X \to \mathbb{R}$ such that $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then the following hold:

- $\lim_{x \to a} f(x) + g(x) = L + M$
- $\lim_{x \to a} f(x)g(x) = LM$

Proof. First, suppose we have any $\sigma \to a$. Since $\sigma \to a$, and since $f(x) \to L$ as $x \to a$, by the above lemma we have $f\sigma \to L$. By an identical argument, we have $g\sigma \to L$.

- First, that $g\sigma + f\sigma \to L + M$ comes immediately from the parallel result about limits of sequences.

- Second, that $(g\sigma)(f\sigma) \to LM$ comes immediately from the parallel result about limits of sequences.

- Now declare that g is non-zero on K, and that M is non-zero. Well then, the sequence $g\sigma$ is non-zero, for each $\sigma(n) \in K$ whence $g\sigma(n) \neq 0$, since g is non-zero on K. It immediately follows from the parallel result about limits of sequences that $f\sigma/g\sigma \to L/M$.

Generalizing, since σ was an arbitrary sequence which converged to a, the composition of any such and since a is a limit point of K, it follows that $\lim_{x\to a} f(x)+g(x) = L+M$, $\lim_{x\to a} f(x)g(x) = LM$, and $\lim_{x\to a} f(x)/g(x) = L/M$ in the case where g is non-zero on K and M is non-zero. This is what we set out to show.

□

Trivially, since the function $x \to -x$ is continuous on the reals, and since composition of functions is continuous, we can show that $f-g$ is continuous.

Corollary 1. Let $f : X \to \mathbb{R}$ and $g : X \to \mathbb{R}$ be continuous. Then the point-wise defined functions $f + g$, fg are continuous. Moreover, if g is non-zero on X, then f/g is continuous thereon.

Proof. Let $a \in X$. If a isn't a limit point, we are done in each case. Now suppose that it is. Well then, we better show that $f + g \to f(a) + g(a)$, $fg \to f(a)g(a)$ as $x \to a$. Since f and g are continuous, and since a is a limit point, we have $f \to f(a)$ and $g \to g(a)$ as $x \to a$. So we have by the above theorem that $f+g \to f(a)+g(a)$. Now let g be non-zero on X. Then $g(a)$ is non-zero. So then, by the theorem, $f/g \to f(a)/g(a)$.

This was sufficient in proving that each of these functions is continuous.

□

4 Dense sets and continuous functions

Lemma 2. Let $f : D \to Y$ be uniformly continuous, and let $x \in X \supset D$, D and Y be metric spaces, and let Y be complete. Then if we have two $\sigma, \sigma' \to x$, then $f\sigma$ and $f\sigma'$ converge to the same value $y \in \mathbb{R}$.

Proof. First that they converge. Notice that σ and σ' are both convergent in X, and therefore Cauchy therein. Moreover, f is uniformly continuous, therefore $f\sigma$ and $f\sigma'$ are Cauchy. But $\mathbb{R}$ is complete, hence $f\sigma$ and $f\sigma'$ converge to some y and y' in $\mathbb{R}$ respectively. It suffices to show that $f\sigma' \to y$.

Let $\epsilon > 0$. Then $\epsilon/2 > 0$. Hence there exists some $\delta > 0$ such that for $a, b \in D$, $d(a, b) < \delta$ means $d(f(a), f(b)) < \epsilon/2$. Moreover, since $f\sigma \to y$, there must exist some $N' \in \mathbb{N}$ so that for $n > N'$, $d(f\sigma(n), y) < \epsilon/2$.

Since $\delta/2 > 0$, by convergence of σ and σ' to x, there must exist some $N'' \in \mathbb{N}$ such that for $n > N$, $d(\sigma(n), x) < \delta/2$ and $d(\sigma'(n), x) < \delta/2$. Now suppose $n > N''$. Then $d(\sigma'(n), x), d(\sigma(n), x) < \delta/2$. By the triangle inequality we have $d(\sigma(n), \sigma'(n)) < d(\sigma'(n), x) + d(\sigma(n), x) < \delta/2 + \delta/2 = \delta$. But since $\sigma(n), \sigma'(n) \in D$, by construction of δ it follows that $d(f\sigma(n), f\sigma'(n)) < \epsilon/2$. Generalizing, it follows that for $n > N'$, we have $d(f\sigma(n), f\sigma'(n)) < \epsilon/2$.

Now consider $N = \max\{N', N''\}$. Suppose we have $n > N$. Then $n > N'$ and $n > N''$, so, as we have shown, $d(f\sigma'(n), f\sigma(n)) < \epsilon/2$ and $d(f\sigma(n), y) < \epsilon/2$. By the triangle inequality it follows that $d(f\sigma'(n), y) < d(f\sigma'(n), f\sigma(n)) + d(f\sigma(n), y) < \epsilon/2 + \epsilon/2 = \epsilon$

But ϵ was arbitrary, hence for $\epsilon > 0$, there is some $N \in \mathbb{N}$ so that for $n > N$, $d(f\sigma'(n), y) < \epsilon$. So by definition of the limit of a sequence, $f\sigma' \to y$. Since $f\sigma' \to y'$, it follows by uniqueness of the limit of a sequence (should it exist, which it does), that $y = y'$. $\square$

Theorem 2. Let D be a dense set in a metric space X. Let $f : D \to Y$ be uniformly continuous on any bounded subset of D, and let Y be complete. There exists a unique continuous extension of f to X.

Proof. We shall define $F : X \to Y$ as such. Let $x \in X$. Since D is dense, it follows that there exists some sequence $\sigma_x : \mathbb{N} \to D$ such that $\sigma_x \to x$ in X. Moreover, since σ converges, it is bounded, therefore there exists some $r > 0$ so that $D' = B_r(\sigma(1)) \supset \sigma(\mathbb{N})$. Then $f|_{D'}$ is uniformly continuous, hence by the lemma $f\sigma$ converges to some y. Moreover, for any other sequence σ' converging to x, we define r' such that $\sigma'(\mathbb{N}) \subset B_{r'}(\sigma(1))$, so choosing the maximum radius between r and r', we find that σ' and σ both converge to y, since f is uniformly convergent on this ball as well (and Y is complete), and the lemma applies. Setting $F(x) = y$, we note that the definition of $F(x)$ is independent of the sequence used to define $F(x)$.

To see that F is a restriction of f, suppose $d \in D$. Select a ball of radius r around d, for some r. Clearly σ_d, the constant sequence at d, converges to D, and stays within $B_r(d) \cap D$. Then $B_r(d) \cap D$ is a bounded subset of D. Clearly $f\sigma \to f(d)$, for this sequence too is constant. Moreover, by our previous discussion, this is independent of our choice of sequence in D. Hence $F(d) = f(d)$. So F is an extension of f.

Now we show that F is uniformly continuous on any bounded set of X. Let $\epsilon > 0$. Then $\epsilon/3 > 0$. Let X' be any such bounded subset of X. Since X' is bounded, there must exist some $r > 0$ and $x \in X$ $X' \subset B_r(x)$. Moreover, $D' = D \cap B_r(x)$ is bounded, from which it follows that f is uniformly continuous on D'. Since f is uniformly continuous on D', there exists some $\delta > 0$ such that for $a, b \in D'$, $d(a, b) < \delta$ implies $d(f(a), f(b)) < \epsilon/3$.

Now set $\delta' = \delta/3$. Suppose we have $d(a, b) < \delta'$ for some $a, b \in X'$. We aim to show that $d(F(a), F(b)) < \epsilon$. By density of D, there exist sequences σ_a, σ_b which converge in X to a and b respectively. By the generalized triangle inequality, for all $n \in \mathbb{N}$, we have $d(F(a), F(b)) < d(F(a), f\sigma_a(n)) + d(f\sigma_a(n), f\sigma_b(n)) + d(f\sigma_b(n), F(b))$. We will construct some natural number n such that 1) $\sigma_a(n), \sigma_b(n) \in D'$, 2) such that $d(F(a), f\sigma_a(n)) < \epsilon/3$ and $d(f\sigma_b(n), F(b)) < \epsilon/3$, and finally 3) such that $d(\sigma_a(n), \sigma_b(n)) < \delta$. To keep things tidy, let us keep a bag, $\mathcal{N}$, of natural numbers handy. We shall be adding to it, and taking the largest of them afterwards.

1. Since $X' \subset B_r(x)$, and $a, b \in B_r(x)$, we have $d(a, x) < r$ and $d(b, x) < r$, whence $r - d(a, x) > 0$ and $d(b, x) - r > 0$, and let r' be the smaller of them. Since $r' > 0$, by convergence of σ_a and σ_b to a, b respectively, there must exist some N such that for $n > M$, $d(a, \sigma_a(n)) < r'$ and $d(b, \sigma_b(n)) < r'$. Since a and b are arbitrary, we need only address σ_a. Let $n > M$. Then $d(a, \sigma_a(n)) < r'$. By the triangle inequality we have $d(\sigma_a(n), x) \leq d(\sigma_a(n), a) + d(a, x) < r' + d(a, x) \leq r - d(a, x) + d(a, x) = r$. So for $n > N$, $\sigma_a(n) \in B_r(x)$. Since $\sigma_a(n) \in D$, we have $\sigma_a(n) \in B_r(x) \cap D = D'$. The same argument applies for $\sigma_b(n)$. Add this to the bag $\mathcal{N}$ of natural numbers.

2. Since we have in fact shown that $f\sigma_b \to F(b)$ and $f\sigma_a \to F(a)$, we have by convergence of sequences some $N' \in \mathbb{N}$ such that for $n > N$, $d(f\sigma_a(n), F(a)), d(f\sigma_b(n), F(b)) < \epsilon/3$ Append this N' to the bag $\mathcal{N}$ of natural numbers!

3. Finally, we must find a natural number threshold N such that for n beyond it $d(\sigma_b(n), \sigma_a(n)) < \delta$. By the generalized triangle inequality, we have $d(\sigma_b(n), \sigma_a(n)) \leq d(\sigma_b(n)b) + d(a, b) + d(\sigma_a(n), a)$. We have

supposed that $d(a,b) < \delta/3$. Moreover, by convergence of σ_a and σ_b to a and b respectively in X, it follows that we can find a threshold N'' such that for n beyond it, $d(\sigma_a(n), a), d(\sigma_b(n), b) < \delta/3$. Then if $n > N''$, we $d(\sigma_a(n), \sigma_b(n)) \leq d(\sigma_b(n)b) + d(a,b) + d(\sigma_a(n), a) < \delta/3 + \delta/3 + \delta/3 = \delta$. So for $n > N''$, we have $d(\sigma_a(n), \sigma_b(n)) < \delta$. Append N'' to $\mathcal{N}$.

So then, we have $\mathcal{N} = \{N, N', N''\}$. Let $M = \max \mathcal{N}$. By the archimedian principle, there must exist some natural number $n > M$. This is the natural number which I set out to construct. Now I will show that it does it's job.

Since $n > N$, we have by (1) that $\sigma_a(n), \sigma_b(n) \in D'$. Since $n > N''$, we have that $d(\sigma_a(n), \sigma_b(n)) < \delta$. By construction of δ, and by uniform continuity of f on D', it follows that $d(f\sigma_a(n), f\sigma_b(n)) < \epsilon/3$.

Sine $n > N'$, we have by (2) that $d(f\sigma_a(n), F(a)), d(f\sigma_b(n), F(b)) < \epsilon/3$.

By the triangle inequality, it follows that $d(F(a), F(b)) < d(F(a), f\sigma_a(n)) + d(f\sigma_a(n), f\sigma_b(n)) + d(f\sigma_b(n), F(b)) < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. So whenever $a, b \in X'$ such that $d(a,b) < \delta'$, we have $d(F(a), F(b)) < \epsilon$. But a, b were arbitrary in X', and ϵ was arbitrary greater than 0 in $\mathbb{R}$, so it follows that for any $\epsilon > 0$, there exists some δ such that whenever $a, b \in X'$ such that $d(a,b) < \delta$, $d(F(a), F(b)) < \epsilon$. Hence F is uniformly continuous on X'. But X' was just any bounded subset of X, so it follows that F is uniformly continuous on any bounded subset of X.

Having shown that F is uniformly continuous on every bounded subset of X, we now proceed to show that F is continuous on X. Let $a \in X$ be arbitrary. Let $\epsilon > 0$. Chose any $r > 0$. Clearly $B_r(x)$ is a bounded subset of X, hence F is uniformly continuous on $B_r(x)$. Hence there exists some $\delta' > 0$ such that for any $a, b \in B_r(x)$, $d(a,b) < \delta'$ implies that $d(F(a), F(b)) < \epsilon$. Set $\delta = \min\{r, \delta'\}$. Suppose we have some $y \in X$ such that $d(x,y) < \delta$. Then since $\delta \leq r$, we have $d(x,y) < r$, hence $y \in B_r(x)$. Moreover, since $\delta \leq \delta'$, we have $d(x,y) < \delta'$, and that $x, y \in B_r(x)$, so by construction of δ' we have that $d(F(x), F(y)) < \epsilon$. So for all $y \in X$, if $d(x,y) < \delta$, then $d(f(x), f(y)) < \epsilon$. We have shown before that this is an equivalent condition for continuity of F at x on X. Since x was arbitrary in X, it follows that for all $x \in X$, F is continuous there. So F is continuous on X as desired.

Thus far, we have constructed a continuous extension of f on X, namely F. It remains to show that this extension is unique. Suppose we have F' a function which extends f, and which differs from F. Since F' differs from F, and since both agree with f on D, we must have some $a \in X \setminus D$ such that $F(a) \neq F'(a)$. We can show that F' is in fact not continuous. By density of

D, it follows that we have a sequence in D, σ_a, which converges to a, and such that $f\sigma \to F(a)$. Moreover, for each term $\sigma(n)$, we have $\sigma(n) \in D$, so since F' extends f whose domain is D, it follows that $f\sigma(n) = F'\sigma(n)$. So $F\sigma(n) \to F'(a)$. Since $F'(a) \neq F(a)$, and the limit of a sequence is unique, it follows that $F'\sigma$ does not converge to $F'(a)$. By Theorem 4.3.3 of the textbook, it follows that $F'(a)$ is not continuous at a. Therefore $F'(a)$ is not continuous.

After all of this, we have shown that there exists a unique continuous extension of f to X. In my opinion, this is actually quite remarkable.

□

Lemma 3. $\mathbb{Q}$ is dense within $\mathbb{R}$.

Proof. Let $s \in \mathbb{R}$. We must construct a sequence of rationals which converges to s. Pick any $r > 0$. We shall define $\sigma_s : \mathbb{N} \to \mathbb{Q}$ inductively, and show that it converges to s. First, since $s + r > s$, as we have proven before, there must exist a rational $q_1 \in \mathbb{Q}$ such that $s < q_1 < s + r$. So set $\sigma_s(1) = q_1$.

Now suppose that, up until N, we have defined σ_s such that for all $n < N$, we have $\sigma_s(n) \in \mathbb{Q}$ such that $s < \sigma_s(n) < s + r/n$. Notice that since $N \in \mathbb{N} \subset \mathbb{R}^+$, $r/N > 0$, so $s < s + r/N$, hence there must exist some $q_N \in \mathbb{Q}$ such that $s < q_N < s + r/N$. Set this to be $\sigma_s(N)$.

So we have inductively defined a sequence of rational numbers such that for all $n \in \mathbb{N}$, $s < \sigma_s(n) < s + r/n$. It remains to show that $\sigma_s \to s$. This is easy, for set $\epsilon > 0$. Then clearly $\epsilon/r > 0$. By a corollary to the Archimedian principle there must exist some $N \in \mathbb{N}$ such that $1/N < \epsilon/r$. Moreover, for $n > N$, it clearly follows that $1/n < 1/N$, whence $1/n < \epsilon/r$, so $r/n < \epsilon$. So pick $n > N$. Then $\epsilon < s - r/n < s < \sigma_s(n) <$

□

5 Miscellaneous Conjectures about Complete Metric Spaces and Density

Conjecture 1. Density is a transitive relation. If $D_n \subset \cdots \subset D_1 \subset D_0$ is a chain of dense subspaces, then D_0 is dense within D_n.

Conjecture 2. If Y is complete, and Y is dense within Z, then $Y = Z$.

Conjecture 3. Given a metric space X, there exists a poset of dense subspaces, which we shall call $\mathcal{D}(X)$. If X is complete, this poset is not a sub-poset for any other metric space. There exists some complete Y such that $\mathcal{D}(X) \leq \mathcal{D}(Y)$. Moreover, $\mathcal{D}(X)$ is bounded below if and only if X is trivial and $\mathcal{D}(X)$ has only one element.