Real Analysis

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1 Converse Part of the Cauchy Criterion Theorem

Theorem 1. Suppose $f : [a, b] \to \mathbb{R}$ is Reimann integrable. Then it satisfies the Cauchy Criterion.

Proof. Let $\epsilon > 0$. Then $\epsilon/2 > 0$. By integrability, it follows that there exists some $\delta > 0$ such that whenever $\mathcal{R}(f, P)$ is a Reimann sum for a partition P of mesh less than δ, we have

$$\left|\mathcal{R}(f, P) - \int_a^b f\right| < \epsilon/2.$$

Now consider any such partition P with $\|P\| < \delta$, and consider any two Reimann sums $\mathcal{R}_1(f, P)$ and $\mathcal{R}_2(f, P)$. By construction of δ and P, it follows that

$$\left|\mathcal{R}_{1,2}(f, P) - \int_a^b f\right| < \delta.$$

Then

$$\begin{aligned} &|\mathcal{R}_1(f, P) - \mathcal{R}_2(f, P)| \\ &= \left|\mathcal{R}_1(f, P) - \int_a^b f + (I - \mathcal{R}_2(f, P))\right| \\ &\leq \left|\mathcal{R}_1(f, P) - \int_a^b f\right| + \left|\mathcal{R}_2(f, P) - \int_a^b f\right| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned} \tag{1}$$

This proves that f on $[a, b]$ satisfies the Cauchy Criterion.

$\square$

2 Upper and Lower Sum Criterion

Proposition 1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function (boundedness is needed to make sense of upper and lower sums). Then for all $\epsilon > 0$, there exists some $\delta > 0$ such that whenever P is a partition of $[a, b]$ with $\|P\| < \delta$, the inequality

$$|\mathcal{U}(f, P) - \mathcal{L}(f, P)| < \epsilon$$

holds, if and only if f is Reimann integrable on $[a, b]$.

Proof. First suppose that f is Reimann integrable. Let $\epsilon > 0$. Then $\epsilon/2 > 0$. By the converse to the Cauchy Criterion Theorem, there exists some δ such that for all P with $\|P\| < \delta$, and for any Reimann sums $\mathcal{R}_1(f, P)$ and $\mathcal{R}_2(f, P)$, we have

$$|\mathbb{R}_2 U(f, P) - \mathbb{R}_1 L(f, P)| < \epsilon/2.$$

Chose any such P. Then the above inequality holds for any Reimann sums of the above form. By Lemma 11.4.7 of the textbook, we have

$$|\mathcal{U}(f, P) - \mathcal{L}(f, P)| \leq \epsilon/2 < \epsilon.$$

The partition P was arbitrary with mesh size less than δ, hence for any partition, the above inequality would hold. This proves the converse.

Now suppose that for all $\epsilon > 0$, there exists some $\delta > 0$ such that whenever P is a partition of $[a, b]$ with $\|P\| < \delta$, the inequality

$$|\mathcal{U}(f, P) - \mathcal{L}(f, P)| = \mathcal{U}(f, P) - \mathcal{L}(f, P) < \epsilon.$$

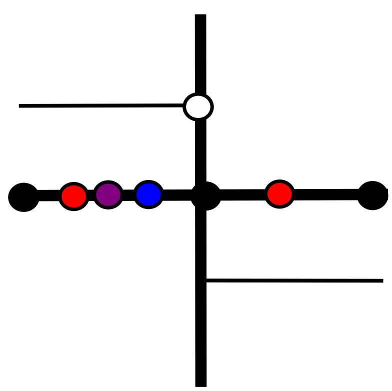
For ϵ fixed, we have one such δ. Suppose we have some parition P with mesh size less that δ. Let $\mathcal{R}_1(f, P)$ and $\mathcal{R}_2(f, P)$ be Reimann sums for this partition. Since the inequality with the upper and lower sums hold for this particular δ, and this particular ϵ, and this particular partition, we have $\mathcal{L}(f,P) \leq \mathcal{R}_1(f,P), \mathcal{R}_2(f,P) \leq \mathcal{U}(f,P)$. Without loss of generality, suppose that $\mathcal{R}_1(f,P) \geq \mathcal{R}_2(f,P)$, whence $|\mathcal{R}_1(f,P) - \mathcal{R}_2(f,P)| = \mathcal{R}_1(f,P) - \mathcal{R}_2(f,P)$. Since $\mathcal{U}(f,P) \geq \mathcal{R}_1(f,P)$ and $\mathcal{L} \leq \mathcal{R}_2(f,P)$, we have

$$| \, |\mathcal{R}_1(f,P) - \mathcal{R}_2(f,P)| = \mathcal{R}_1(f,P) - \mathcal{R}_2(f,P) \leq \mathcal{U}(f,P) - \mathcal{L}(f,P) < \epsilon.$$

Since ϵ was arbitrary, and P was as well, and so were the Reimann sums, it follows that f satisfies the Cauchy Criterion on $[a,b]$. From this and the Caucy Criterion Theorem, it follows that f is Riemann integrable on $[a,b]$. $\square$

3 A Refinement, and Reimann sum, that isn't better

Consider the function $f(x)$ which is defined as -1 if $x > 0$ and 1 otherwise. This is Reimann integrable on $[-1,1]$, and it's integral is zero. To see this, consider the diagram shown below.



Here, the black dots denote the actual points of our partition P. A refinement Q is created by adding in the purple dot.

First, let $\mathcal{R}(f,P)$ denote the Reimann sum with the sample points chosen in red. There is one on each side, and the black dots are placed uniformly, hence their contributions will cancel out and $\mathcal{R}(f,P) = 0$, which is exactly the integral. Now use the old sample points from before, but also add in the blue point. The resulting sum will no longer cancel out, which means our "finer" partition actually gave us a worse approximation.