Abstract Algebra

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5/9/2021

$\boxed{\text{Ch0, 12}}$ Show that 5n+3 and 7n+4 are relatively prime for all n.

$\boxed{\text{proof}}$ We know that, if a and b are relatively prime, the greatest common divisor of a and b, gcd(a,b), is 1. By applying the Euclidean Algorithm we get:

$\gcd(5n+3, 7n+4) = \gcd(2n+1, 5n+3) = \gcd(3n+2, 2n+1) = \gcd(n+1, 2n+1) = \gcd(n, n+1) = \gcd(1, n) = 1$

Hence, 5n+3 and 7n+4 are relatively prime for all n. Q.E.D.

$\boxed{\text{Ch1, 24}}$ For each design below, determine the symmetry group (ignore imperfections).

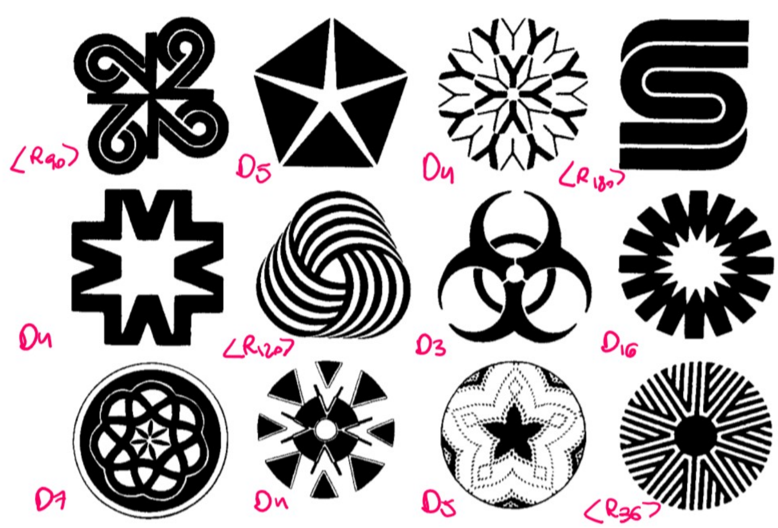

Figure 1:

$\boxed{\text{Ch2, 19}}$ Prove that the set of all 2x2 matrices with entries from R and determinant +1 is a group under matrix multiplication.

$\boxed{\text{proof}}$

1) The set is non-empty, since, for example, the identity matrix is one element of the set.
2) The matrix multiplication is a closed, binary operation on this set, since the product again is a 2x2 matrix with real entries. Also, if A and B are elements of the set, then $\det(A) = \det(B) = +1$. Furthermore, $\det(AB) = \det(A) \cdot \det(B) = +1$
3) The identity element e is the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, since $A \cdot I = I \cdot A = A$ for all matrices

A in the set.
4) Every element A of the set has an inverse A^{-1}, since every square matrix with determinant +1 is invertible.
5) Matrix multiplication is associative.

Hence, the set of all 2x2 matrices with real entries and determinant +1 is a group under matrix multiplication.

$\boxed{\text{Ch2, 47}}$ Show that if in a group there is a rule that any square of an element is the identity, then it follows that that group is Abelian.

$\boxed{\text{proof}}$ Let G be a group, and let a and b be elements of that group. Suppose that the square of any element in G is the identiy, e. Then by closure and the definition of G, behold, we have

$$(a * b)^2 = e.$$

Furthermore, $a * (a * b)^2 = a * e = a$. As groups are associative, we have

$$a * a * b * a * b = a = (a * a) * b * a * b = b * a * b.$$

Multiplying b on the right, we have

$$b * a * b * b = a * b = b * a * (b * b) = b * a * e = b * a.$$

Hence we have $b * a = a * b$. Q.E.D.

$\boxed{\text{Ch2, 37}}$ Let G be a finite group. Then the number of elements x such that $x^3 = e$ is odd, and the number of elements y such that $y^2 \neq e$ is even.

$\boxed{\text{proof}}$ Let $|G|$ be the order of G.

- Suppose that there is a subset of G, S, such that each element $x \in S$ has the property that $x^3 = e$. Clearly e is a member of this group, as e raised to any power must be e. Let x be an arbitrary element of S not equal to e. Since $x^3 = e$, by the associative law of groups, $x(x^2) = e$, hence $x^{-1} = x^2$. Furthermore, by the uniqueness property of the inverse in groups, $x^2 \neq x$. Furthermore, $(x^2)^3 = x^6 = x^3x^3 = 1$, hence x^2 is another element of S. Since for each element of S not e, there is a pair element of S corresponding to it, the order of S must be $2n + 1$, so the number of elements in a finite group x such that $x^3 = e$ is odd.
- Let S be a subset of G such that each element $x \in S$ has the property that $x^2 \neq e$. Note that S could be empty, in which case the proposition holds. Suppose then that S is not empty. Let x be an arbitrary element of S. By the associative, it is clear that $(x^{-1})^2x^2 = x^{-1}x^{-1}xx = x^{-1}(x^{-1}x = e)x = e$, hence $(x^{-1})^2 = (x^2)^-1$. Furthermore, since $x^2 \neq e$, it follows that $(x^{-1})^2x^2 \neq (x^{-1})^2e$, hence $e \neq (x^{-12})$. Furthermore, x^{-1} is distinct from x, as $x^2 \neq e$. Hence it follows that for each element S, there is exactly one more element in S. In other words, $|S|$ has cardinality $2n$, so there are an even number of elements x in G such that $x^2 \neq e$