Real Analysis

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1 The Uniformly Convergent Limit of Continuous Functions is Continuous

Proposition 1. Let X and Y be metric spaces, and let $f_n : X \to Y$ be a uniformly convergent sequence of continuous functions, which converge to some function $f : X \to Y$. Then f is continuous.

Proof. Let everything be instantiated as above. Suppose for the sake of contradiction that f is not continuous. Then there exists some point of discontinuity $a \in X$. Hence there exists some $\epsilon > 0$ such that for all $\delta > 0$, there exists some $x \in X$ with $d_X(x, a) < \delta$ and $\epsilon \leq d_Y(f(x), f(a))$.

By uniform convergence of f_n, there exists natural number N such that for all $n > N$, and for all $x \in X$, $d_Y(f(x), f_n(x)) < \epsilon/3$. In particular, $d_Y(f_n(a), f(a)) \leq \epsilon/3$.

Moreover, by assumption that each f_n is continuous, it is continuous in particular at a. Hence there exists some $\delta > 0$ such that for any $x \in X$ with $d_X(x, a) < \delta$, we must have $d_Y(f_n(x), f_n(a)) < \epsilon/3$.

But by construction of ϵ, there exists some $b \in X$ such that $d_X(b, a) < \delta$ and $< \epsilon \leq d_Y(f(a), f(b))$. But $d_Y(f(a), f_n(a)) < \epsilon/3$, $d_Y(f_n(a), f_n(x)) < \epsilon/3$ and

$d_Y(f_n(x), f(x)) < \epsilon/3$, Hence by the triangle inequality we have

$$\begin{aligned} \epsilon &\leq d_Y(f(a), f_n(a)) \\ &\leq d_Y(f(a), f_n(a)) + d_Y(f_n(a), f_n(x)) + d_Y(f_n(x), f(x)) \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned} \tag{1}$$

Hence $\epsilon < \epsilon$, which is a contradiction. □

2 Uniform Convergence and Arithmetic

Lemma 1. Let X be a metric space, and let $f_n : X \to \mathbb{R}$ be a uniformly convergent sequence of functions, which converge to $f : X \to \mathbb{R}$. Suppose f is bounded by $M \in \mathbb{R}$, such that for all $x \in X$,

$$|f(x)| \leq M.$$

Then for all $\epsilon > 0$ there exists a natural number $N \in \mathbb{N}$ such that for all $n > N$, and for all $x \in X$,

$$|f_n(x)| \leq M + \epsilon.$$

Proof. Let everything be instantiated as above. Let $\epsilon > 0$. By uniform convergence of (f_n), we have some natural number N such that for all $n > N$, $|f(x) - f_n(x)| < \epsilon$. Take any $n > N$. Then by the Triangle Inequality, we have

$$\begin{aligned} |f_n(x)| &= |f_n(x) - 0| \\ |f_n(x) - f(x)| + |f(x) - 0| &= |f_n(x) - f(x)| + |f(x)|. \end{aligned} \tag{2}$$

But $x \in X$, so $|f(x)| \leq M$. Moreover, since $n > N$, by construction of N we have $|f_n(x) - f(x) < \epsilon$, hence

$$|f(x)| < M + \epsilon.$$

□

For what follows, let X be a metric space, and let f_n and g_n be sequences of functions $X \to \mathbb{R}$, which uniformly converge to functions $f, g : X \to \mathbb{R}$ respectively.

Proposition 2. The sequence of pointwise defined sums, $f_n + g_n$, converges uniformly to the pointwise sum $f + g$.

Proof. For let $\epsilon > 0$. Then $\epsilon/2 > 0$. By uniform convergence of f_n to f, there exists some natural number N_1 such that for all $n > N_1$, and for all $x \in X$, $|f_n(x) - f(x)| < \epsilon/2$. Similarly, there exists some natural number N_2 such that for all $n > N_2$, and for all $x \in X$, $|g_n(x) + g(x)| < \epsilon/2$.

Consider $N = \max(N_1, N_2)$. Let $n > N$, and let $x \in X$. Since $n > N$, $n > N_1, N_2$, hence $|f_n(x) - f(x)| < \epsilon/2$ and $|g_n(x) - (x)| < \epsilon/2$. From this it follows that

$$\begin{aligned} |(f_n + g_n)(x) - (f + g)(x)| &= |f_n(x) + g_n(x) - (f(x) + g(x))| \\ &= |(f_n(x) - f(x) + (g_n(x) - g(x))| \qquad (3) \\ &\leq |f_n(x) - f(x)| + |g_n(x) - g(x)| < \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

This concludes the proof. □

Proposition 3. If, in addition, f and g are both bounded, then the pointwise product $f_n g_n$ converges uniformly to the pointwise product fg.

Proof. Let $\epsilon > 0$. Since f and g are bounded, there exist real numbers $F, G > 0$ such that for all $x \in X$, $|f(x)| \leq F$ and $|g(x)| \leq G$. Hence

$$\eta = \frac{\epsilon}{F + G} > 0.$$

By the above lemma, it follows that there exists some natural N_1 such that for all $n > N_1$, and for all $x \in X$, $|f_n(x)| \leq F + \eta$. By the same lemma, there exists some natural number N_2 such that for all $n > N_2$, and for all $x \in X$, $|g_n(x)| < G + \eta$. Moreover, by uniform convergence of f_n to f, there exists some natural number N_3 such that for all $n > N_3$, and for all $x \in X$, $|f_n(x) - f(x)| < \eta$. By uniform convergence of g_n to g, there exists some natural number N_4 such that for all $n > N_4$, and for all $x \in X$,

$|g_n(x) - g(x)| < \eta$. Let $N = \max\{N_1, N_2, N_3, N_4\}$, and suppose $n > N$ and $x \in X$. Then

$$\begin{aligned}
|(f_n g_n)(x) - fg(x)| &= |f_n(x)g_n(x) - f(x)g(x)| \\
&= |f_n(x)g_n(x) - f_n(x)g(x) + f_n(x)g(x) - f(x)g(x)| \\
&\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f(x)g(x) - f_n(x)g(x)| \\
&= |f_n(x)||g_n(x) - g(x)| + |g(x)||f(x) - f_n(x)| \\
&< (F + \eta)\eta + G\eta = (\eta + F + G)\eta < (F + G)\eta \\
&= (F + G)\frac{\epsilon}{F + G} = \epsilon.
\end{aligned} \tag{4}$$

Generalizing and stuff, by definition of uniform convergence, $f_n g_n$ converges uniformly to fg as desired.

$\square$