Real Analysis

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Theorem 1. Let (a_i), (b_i) be sequences of real numbers such that

- $a_i \leq b_i \forall i \in \mathbb{N}$
- $a_i \leq a_{i+1}$ and $b_i \geq b_{i+1}$ $\forall i \in \mathbb{N}$.

Then

$$\bigcap_{i=0}^{\infty} [a_i, b_i] \neq \emptyset.$$

Proof. We first establish that $\{a_i\}$ (the set of images of the sequence) is bounded above, and $\{b_i\}$ is bounded below.

For an upper bound of $\{a_i\}$, consider b_1. We proceed by induction on i. The base case clearly holds, as this is the first condition. Now suppose we have $a_i \leq b_1$ for some i. Then $a_{i+1} \leq a_i$. By the induction step, we have $a_i \leq b_1$. Then $a_{i+1} \leq b_1$ by the transitivity of the relation $\leq$. Hence for all i, we have $a_i \leq b_1$.

We can similarly show that a_1 is a lower bound for $\{b_i\}$. Also, by induction, we can show that for all $i \in \mathbb{N}$, we have $b_i \leq b_1$. So by transitivity of the order relation, we have $a_i \leq b_1 \leq b_j$ for all $i, j \in \mathbb{N}$, since we have shown b_1 to be an upper bound for $\{a_i\}$.

Since $\{a_i\}$ is bounded above, $\{a_i\}$ must have a supremum; call it a. Moreover, since $\{b_i\}$ is bounded below, so it must have an infimum. Call it b. We now claim that $a \leq b$. For suppose it weren't, that is, that $a > b$. Since $a > b$, and since b is the greatest lower bound of the set $\{b_i\}$, it follows that a is not a lower bound for $\{b_i\}$. So there must exist some element of $\{b_i\}$, and consequently (since it is an indexed set) some $i \in \mathbb{N}$ such that

$b_i < a$. Moreover, since b_i is less than a, and since a is the supremum of the set $\{a_i\}$, it follows that b_i cannot be an upper bound of the set $\{b_i\}$. So there must exist some b_j, and consequently some $j \in \mathbb{N}$ such that $b_j < a_i$. But we have shown that for all $i, j \in \mathbb{N}$, $b_i \geq a_i$. So this is a contradiction.

The definition of the supremum gives us $a_i \leq a$ for all $i \in \mathbb{N}$. The definition of the infimum gives us $b \leq b_i$ for all $i \in \mathbb{N}$. We have shown that $a \leq b$. Hence by transitivity of the relation $\leq$, we have $a_i \leq a \leq b_i$ for all $i \in \mathbb{N}$. Hence $a \in [a_i, b_i]$ for all $i \in \mathbb{N}$. Hence

$$a \in \bigcap_{i=0}^{\infty} [a_i, b_i].$$

This, of course, shows the set is non-empty. □

Remark 1. I had tried to prove that the nested interval theorem worked in any metric space, but it actually fails in $\mathbb{Q}$, under the absolute value restricted to it as the metric. There is something special about $\mathbb{R}$ as a metric space. The nested interval theorem does not generalize easily.

Problem 1. (2.2.4)

Let $\mathbb{R}^{\mathbb{R}}$ be the set of all real valued functions. Consider some $d \in (\mathbb{R}^{\mathbb{R}})^{\mathbb{R}}$ (lol!) such that $d(f, g) = |f(0) - g(0)|$. Is d a metric?

It isn't!

Proof. Consider f defined $f(x) = x$ for all $x \in \mathbb{R}$, and g defined $g(x) = x^2$. Obviously $f(0) = g(0) = 0$, hence $|f(0) - g(0)| = 0$. This would violate positive definiteness, since f and g are as distinct as they can be! (well, that's a hyperbole of magnitude greater than x^2 !) □