1 Problem 12.5.5: an interesting function

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined

$$f(x) = \begin{cases} 0 \text{ if } x \text{ is irrational} \\ 1/q \text{ if } x \text{ is rational, where } q \text{ is the denominator in lowest terms.} \end{cases}$$

Its fairly easy to see that this function is discontinuous for all rational numbers. Recall one of the equivalent definitions of continuity, which states that a function $g : X \to Y$ is continuous at some $a \in X$ iff for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in X$ with $d(x, a) < \delta$, $d(g(x), g(a)) < \epsilon$. Negating this, we must show that there is some $\epsilon > 0$ such that whenever $\delta > 0$, there is some $x \in X$ with $d(x, a) < \delta$ and $d(g(x), g(a)) \geq \epsilon$.

Proposition 1. The function f, defined above, is discontinuous at all $s \in \mathbb{Q}$.

Proof. Back to our specific function, let $s \in \mathbb{Q}$. Then, by definition of a rational number $s = p/q$ (in lowest terms), with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Let $\delta > 0$. Consider the positive real number $1/q$. Using a basic property of the reals (which we proved!), there must exist some irrational r with $s < r < s + \delta$. From this it follows that $|r - s| < \delta$. Moreover, by definition of f, we have $f(s) = 1/q$, and $f(r) = 0$. So $|f(s) - f(r)| = |1/q - 0| = 1/q \geq 1/q$. But δ was arbitrary greater than zero, so for all δ, there exists some $r \in \mathbb{R}$ with $|r - s| < \delta$ and $|f(r) - f(s) \geq \epsilon$. Hence f is discontinuous at s! $\square$

Now we'll take it's integral. Unfortunately (or fortunately for coolness's sake), we can't take an antiderivative, for clearly none exists (it isn't continuous on any interval, since every interval contains a rational). Along the way, we'll prove the function is continuous at every irrational number.

Lemma 1. Let $t \in \mathbb{R}$. Let $r > 0$, and let $n \in \mathbb{N}$. Construct the set $S_n = \{s \in [t - r, t + r] : f(s) = 1/n\}$. Then S_n is finite.

Proof. Let $s \in S_n$. Then $f(s) = 1/n$. This isn't zero, hence s is rational, and by definition of f, $s = p/n$ for some $p \in \mathbb{Z}$, where the fraction p/n

is in reduced terms. But since $s \in [t-r, t+r]$, it must be the case that $t - r \leq s = p/n \leq t + r$, and since n is positive, this is equivalent to

$$p \in [n(t-r), n(t+r)].$$

But since p is an integer, only finitely many such p can fit within this interval. So only finitely many elements could be in S_n, for each would correspond to a distinct integer $p \in [n(t-r), n(t+r)]$. $\square$

This allows us to establish an important property on how "big" the function can be, and how often it can be that big.

Lemma 2. Let $K \subset \mathbb{R}$ be bounded. Then for all $\epsilon > 0$, the set $A = \{s \in K : f(s) > \epsilon\}$ is finite.

Proof. First, note that since K is bounded, it follows that it lives inside of some open interval of some radius, and hence inside the corresponding closed interval. Let that closed interval be $[t-r, t+r]$ for some $t \in r$, and for some $r > 0$. Note that the set $M_\epsilon = \{s \in [t-r, t+r] : f(s) > k\}$ is a subset of the set of all $s \in K$ with the same property. Therefore it suffices to prove that M_ϵ is finite.

By the archimedian principle, there must exist some natural number N such that $1/N < \epsilon$. Note that if $f(s) > \epsilon$, then $f(s) \geq 1/N$. Therefore, $M_\epsilon \subset \{s \in [t-r, t+r] : f(s) >\} = M_N$, and it suffices to show that M_N set is finite.

To show that M_N is finite, we note that the set $\{n \in \mathbb{N} : n < N\} = I$ is finite. Moreover, by the previous lemma, each set $S_n = \{s \in [t-r, t+r] : f(s) = 1/n\}$ is finite. We will show that $M_N \subset \cup_{i \in I} S_i$, which will prove that M_N is finite, since this union is a finite union of finite sets!

So, let $s \in M_N$. Then $f(s) > 1/N$. Moreover, by definition of f, $f(s) = 1/q$ for some $q \in \mathbb{N}$ (otherwise it would be zero, which it cannot be by assumption that $f(s) > 1/N$). Hence, for that q, we must have $1/q > 1/N$, by assumption that $f(s) < 1/N$ and by substitution. Hence, since both N and q are positive, we have $q < N$. Hence $q \in I$, by construction of the set I. Hence $s \in S_q$, for $q \in I$. By definition of the union of an indexed family of sets, $s \in \cup_{i \in I} S_i$.

Since s was arbitrary (the tutoring foundations has rubbed off on me here), all elements of M_N are also in $\cup_{i \in I} S_i$. Hence $M_N \subset \cup_{i \in I} S_i$.

$\square$

Proposition 2. The function f is continuous at all irrationals.

Proof. Let $r \in \mathbb{R}$ be irrational. Let $\epsilon > 0$. There exists some $\eta > 0$. Form the interval $[r - \eta, r + \eta]$. Since $\epsilon/2 > 0$, by the previous lemma there are only a finite number of points $s \in [r - \eta, r + \eta]$ with $f(s) > \epsilon/2$. Order and index each of these points to form an indexed set of finite points $\{s_i\}_{i \in \{1, \ldots, n\}}$, and form the partition

$$P = \{x_0 = r - \eta, x_1 = s_1, \ldots, x_n = s_n, x_{n+1} = r + \eta\}.$$

Since r is irrational, and each of the s_i are rational (otherwise they would have $f(s_i) = 0 < \epsilon/2!$), it follows that r is on the interior of one of the intervals of this partition. Let that interval be $[x_i, x_{i+1}]$. Since the interior of this interval does not contain any of the s_ks, for all $x \in (x_i, x_{i+1})$, we have $f(x) \leq \epsilon < \epsilon/2$.

Since r is in (x_i, x_{i-1}), and this interval is open, there exists some δ such that for all $x \in \mathbb{R}$ with $|x - r| < \delta$, we must have $x \in (x_i, x_{i-1})$.

Now chose any $x \in \mathbb{R}$ with $|x - r| < \delta$. As just stated, it follows that $x \in (x_i, x_{i-1})$. As noted, this implies that $f(x) < \epsilon$. Note that $f(x)$ is never neagtive by definition of f, hence $|f(x)| = f(x)$. Moreover, r is irrational, so by definition of f, we have $f(r) = 0$. So

$$|f(x) - f(r)| = |f(x) - 0| = f(x) < \epsilon.$$

But $\epsilon > 0$ was arbitrary, hence for all $\epsilon > 0$, there exists some $\delta > 0$ such that for all $x \in \mathbb{R}$ with $|x - r| < \delta$, we have $|f(x) - f(r)| < \epsilon$. This is one of the equivalent definitions of continuity at a point, hence f is continuous at r, as desired.

$\square$

Proposition 3. The function f is integrable, with $\int_0^1 f = 0$

Proof. Let $\epsilon > 0$. Then $\epsilon/2 > 0$. So by Lemma 2, we have only finitely many points $s \in [0,1]$ with $f(s) > \epsilon/2$. For ease of reference, let S be (finite) set of all such points. Let L be the sum of all such points. Consider

$$\delta = \frac{\epsilon}{2L}.$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[0,1]$ with $\|P\| < \delta$. Chose some collection of sample points x_i^* for P, to form the Reimann sum

$$\mathcal{R}(f,P) = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

Let A be the (possibly empty) set of all indices i of the partition, such $x_i^* \in S$, and let B be the rest of the indices. Certainly $\sum_{i \in A} f(x_i^*) \leq L$. Then by associativity,

$$\mathcal{R}(f,P) = \sum_{i \in A} f(x_i^*)(x_i - x_{i-1}) + \sum_{i \in B} f(x_i^*)(x_i - x_{i-1}).$$

Since $\|P\| < \delta$, $x_i - x_{i-1} < \delta$, hence

$$\sum_{i \in A} f(x_i^*)(x_i - x_{i-1}) < \sum_{i \in A} f(x_i)\delta \leq \delta L = \frac{\epsilon}{2L} L = \epsilon/2.$$

Moreover, note that for each $i \in B$, $f(x_i^*) \leq \epsilon/2$, and note also that $\sum_{i \in B}(x_i - x_{i-1}) \leq \sum_{i=1}^{n}(x_i - x_{i-1}) = 1$, since the last sum of our inequality was telescoping with $x_n = 1$ and $x_0 = 0$. Hence

$$\sum_{i \in B} f(x_i^*)(x_i - x_{i-1}) \leq \sum_{i \in B} \frac{\epsilon}{2}(x_i - x_{i-1}) = \frac{\epsilon}{2} \sum_{i \in B}(x_i - x_{i-1}) \leq \frac{\epsilon}{2}.$$

Hence

$$|\mathcal{R}(f,P) - 0| = \mathcal{R}(f,P) < \frac{\epsilon}{2} + \sum_{i \in B} f(x_i^*)(x_i - x_{i-1}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since the mesh and Reimann sum were arbitrary, and since ϵ was also arbtirary, it follows that for any $\epsilon > 0$, there is some $\delta > 0$ such that for any partition P with $\|P\| < \delta$, and for any Reimann sum $\mathcal{R}(f, P)$ with that partition, we have

$$|\mathcal{R}(f, P) - 0| < \epsilon.$$

By definition of the Reimann integral,

$$\int_0^1 f = 0$$

as desired. $\square$

2 Products of integrable functions

First, I should state the obvious lemma,

Lemma 3. Let $K \subset \mathbb{R}$ be compact, and let $S \subset K$. For any continuous function $f : K \to \mathbb{R}$, the restriction $f|_S : S \to \mathbb{R}$ is uniformly continuous.

This lemma is obvious because for any restriction, and for any two points $x, y \in S$ with $d(x, y) < \delta$, I would necessarily have $|(f(x) - f(y)| < \epsilon$. (Where ϵ and δ would be what they usually are in a proof.)

Theorem 1. Let $f, g : [a, b] \to \mathbb{R}$ be Reimann integrable. Then fg is Reimann integrable.

Proof. The key is to recognize the purely algebraic identity $\frac{1}{4}[(f + g)^2 - (f - g)^2] = fg$, where scalar multiplication and function addition are defined pointwise. This is obvious yet slightly tedious, so I won't write out all the steps. Once the terms are all foiled correctly, and canceled out, we find that the identity holds.

Because f and g are continuous, by a previous result (which we proved), $f + g$ and $f - g$ are both Reimann integrable. We aim to show that $(f + g)^2$ and $(f - g)^2$ are Reimann integrable. We will do this by recognizing these as compositions $h \circ (f + g)$ and $h \circ (f - g)$, $h : \mathbb{R} \to \mathbb{R}$ is the function which maps $x \mapsto x^2$ for all $x \in \mathbb{R}$.

We have actually proven that this function is not uniformly continuous in general. However, since continous functions from compact sets are uniformly continuous, the map h will be uniformly continuous if restricted to some such compact set. By our Lemma, it will also be compact if this domain is further restricted from such a compact set.

Since $f+g$ and $f-g$ are Reimann integrable functions $[a,b] \to \mathbb{R}$, both are bounded. Hence the sets $(f+g)([a,b])$ and $(f-g)([a,b])$ are bounded. By the theorem which proves that a the closure of a set is closed, the closures of both sets are closed. Since both are bounded as well, by the Heine Borel theorem, both are compact, hence h restricted to either of the closures of these sets is uniformly continuous. By the above lemma, the further restriction of h to the range of either $f+g$ or $f-g$ is also uniformly continuous.

So then, by Theorem 11.5.7 of the textbook, the compositions $h \circ (f+g) = (f+g)^2$ and $h \circ (f-g) = (f-g)^2$ are both Reimann integral. By Part (3) of Theorem 11.3.1 of the textbook, their difference

$$(f+g)^2 - (f-g)^2$$

is Reimann integrable on $[a,b]$. Moreover, by the second part of the same theorem, the scalar multiple $\frac{1}{4}[(f+g)^2 - (f-g)^2] = fg$ is Reimann integrable on the interval $[a,b]$, which is what we set out to prove. $\square$