Real Analysis

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October 25, 2022

1 Problem 3.6.4

Proposition 1. Closed balls are closed.

Proof. Let X be a metric space, and let x be a point in it. Let $\epsilon > 0$. Suppose by way of contradiction that $C_\epsilon(x)$ is not closed. Then there exists some limit point which is not in it, let that limit point be p. By definition of an open ball, since $p \notin C_\epsilon(x)$, it follows that $d(x,p) > \epsilon$, hence $\epsilon' = d(x,p) - \epsilon > 0$. Moreover, since p is a limit point, it follows by definition of a limit point (one of them at least) that there exists a sequence $\sigma : \mathbb{N} \to C_\epsilon(x)$ which converges to p. Since $\epsilon' > 0$, it follows by definition of convergence that there exists some natural number N such that for all $n > N \in \mathbb{N}$, $d(\sigma(n), p) < \epsilon'$. By the triangle inequality it follows that $d(x,p) \leq d(x,\sigma(n)) + d(\sigma(n),p)$, whence by symmetry of a metric it follows that $d(x,p) - d(\sigma(n),p) \leq d(x,\sigma(n))$. Since $d(\sigma(n),p) < \epsilon'$, we have $\epsilon = d(x,p) - \epsilon' < d(x,p) - d(\sigma(n),p) \leq d(x,\sigma(n))$. But since $\sigma(n)$ is in the closed ball of radius ϵ, it follows that $d(\sigma(n),x) \leq \epsilon$: a contradiction. From this it follows that $C_\epsilon(x)$ contains all of it's limit points. So any closed ball is closed. $\square$

2 Theorem 3.7.10

Theorem 1. Let X be a metric space, and S a subset of it. The following hold:

- $\partial(S)$ is closed.
- For any $x \in X$, $x \in \partial(S)$ if and only if for all $r > 0$, $B_r(x) \cap S \neq \emptyset$ and $B_r(x) \cap X \setminus S \neq \emptyset$.
- S is closed if and only if $\partial(S) \subset S$.

- S is open if and only if S and $\partial(S)$ are disjoint.

Proof. Before jumping in, I shall point out that for any $S \subset X$ a metric space, $\partial(S) = \partial(X \setminus S)$. This follows immediately from the definition of the boundary, and the fact that intersection is a commutative operation on sets.

- Recall that the closure of a set is closed (this is exercise 3.7.4, but it also follows immediately from Theorem 3.7.2 and the definition of a closed set, since closedness is preserved under arbitrary intersection, and the closure is defined as an intersection over closed sets). Now since the boundary $\partial(S) = \overline{S} \cap \overline{X \setminus S}$, and since $\overline{S}$ and $\overline{X \setminus S}$ are closed, and since closure is preserved under intersection, it follows that $\partial(S)$ is closed.

- To make things simpler, notice 1) that $X \setminus (X \setminus S) = S$, and 2) that $x \in S$ if and only if $x \notin X \setminus S$. Let $x \in \partial(S)$.

  Without loss of generality (by 1 and 2), let us suppose that $x \in S$, and hence that $x \notin X \setminus S$. By definition of the closure and intersection, we have that $x \in \overline{S}$ and that $x \in \overline{X \setminus S}$. By Theorem 3.7.5, it follows that $x \in X \setminus S \cup \text{lp}(X \setminus S)$. Since $x \in S$, $x \notin X \setminus S$, hence $x \in \text{lp}(X \setminus S)$. Now let $r > 0$. Since $x \in B_r(x)$, and since $x \in S$, we have $x \in S \cap B_r(x)$, whence $S \cap B_r(x) \neq \emptyset$. Moreover, since x is a limit point of $X \setminus S$, it follows (Theorem 3.5.1) that $(X \setminus S) \cap B_r(x)$ is infinite, hence it is non-empty.

  Now for the converse, suppose by way of contrapositive that $x \notin \partial(S)$. Then there are two cases: either $x \notin \overline{S}$ or $x \notin \overline{X \setminus S}$. Without loss of generality (by 2, and since clearly $\partial(S) = \partial(X \setminus S)$, as intersection is commutative), let us suppose that $x \notin \overline{S}$. Then (by Theorem 3.7.5 [and DeMorgan and intersection if absolutely necessary]), it follows that $x \notin S$, and that $x \notin \text{lp}(S)$. Since x is not a limit point of S, it follows (by the converse of Theorem 3.5.1) that there exists some $r > 0$ such that $B_r(x) \cap (S \setminus \{x\}) = \emptyset$. Since $x \notin S$ either, it follows that $B_r(x) \cap S = \emptyset$. So the converse is also true.

- Now suppose that S is closed. Then $S = \overline{S}$ (part 4 of Theorem 3.7.4), and since the intersection is contained within all of it's intersectees (recall from foundations), it follows that $\partial(S) = \overline{S} \cap \overline{X \setminus S} = S \cap \overline{X \setminus S} \subset S$. For the converse, suppose that S is not closed. We shall construct boundary point which is not contained therein. Since S is not closed, there exists a limit point of S which is not in S, and call it p. Since $p \notin S$, we have $p \in X \setminus S$. By Theorem 3.7.5, we have $p \in \overline{X \setminus S}$. Moreover, by the same theorem, since p is a limit point of S, it follows that $p \in \overline{S}$. So $p \in \overline{S} \cap \overline{X \setminus S} = \partial(S)$. Hence there exists a point (namely p), in the boundary of S, which is not contained within S.

- Now suppose that S is open. Then by Theorem 3.7.1 it follows that $X \setminus S$ is closed. From this it follows that $\partial(X \setminus S) \subset X \setminus S$, hence no points of S are shared with $\partial(X \setminus S)$, that is, $S \cap \partial(X \setminus S) = \emptyset$. But as previously remarked, $\partial(X \setminus S) = \partial(S)$, hence it follows that $S \cap \partial(S) = \emptyset$.

  For the converse, suppose that $S \cap \partial(S) = \emptyset$. Then no points of $\partial(S)$ are contained within S (otherwise they would be in the intersection, which is empty). So $\partial(S) \subset X \setminus S$. But as previously remarked, $\partial(S) = \partial(X \setminus S)$, so $\partial(X \setminus S) \subset X \setminus S$. As shown in an earlier part of this proof, it follows from this that $X \setminus S$ is closed. So by Theorem 3.7.1 it follows that S is open as desired.

Q.E.D. □

3 3.5.1

Theorem 2. Let X a metric space, and let $S \subset X$. let $x \in X$. The following are equivalent.

- There exists a sequence $\sigma : \mathbb{N} \to X$, whose range is contained entirely within S, which converges to x.
- All open balls around x contain points of S.
- For each open $U \subset X$ with $x \in U$, $U \cap S \neq \emptyset$.

Proof.
- First we show that the first condition implies the second. Suppose that there exists some $\sigma : \mathbb{N} \to X$ whose range is entirely contained within x, and such that σ converges to x. Now let $\epsilon > 0$, and we wish to show that $B_\epsilon(x) \cap S \neq \emptyset$. Since σ converges to x, it follows that there exists a natural number N, such that for all $n > N$ in the natural numbers, $d(\sigma(n), x) < \epsilon$. Since $N + 1 > N$, it follows that $d(\sigma(N + 1), x) < \epsilon$, so by definition of an open ball it follows that

$\sigma(N+1) \in B_\epsilon(x)$. Moreover, by construction of the sequence σ, we have that $\sigma(N+1) \in S$. So $\sigma(N+1) \in B_\epsilon(x) \cap S$, hence $B_\epsilon(x) \cap S \neq \emptyset$. Since ϵ was arbitrary greater than 0, it follows that for all radii about x, the open ball around x shares some points with S, which is what we set out to show.

- Now suppose that all open balls about x contain points of S. Suppose we have an open $U \subset X$ such that $x \in U$. Since U is open, and x a point within it, it follows by a previous theorem that there exists some $\epsilon > 0$, such that $x \in B_\epsilon(x) \subset U$. By assumption, it follows that there exists some point $p \in B_\epsilon(x)$ such that $p \in S$. Moreover, since $B_\epsilon(x) \subset U$, we have $p \in U$. So $p \in U \cap S$, hence $U \cap S \neq \emptyset$ as desired.

- Finally, let's suppose that for all open $U \subset X$ which contain x, $U \cap S \neq \emptyset$. Let $r > 0$. Then for all $n \in \mathbb{N}$, $r/n > 0$, hence we can construct the open ball $B_{r/n}(x)$ about x. I shall now construct by induction a sequence $\sigma : \mathbb{N} \to X$, whose range is entirely contained within S, and which converges to x.

  - Since $r > 0$, we have the open ball $B_r(x)$, which contains x. Since open balls are open, our assumption tells us that $B_r(x) \cap S \neq \emptyset$. So there must exist some $p \in B_r(x) \cap S$, so $p \in B_r(x)$, and $p \in S$. Let $\sigma(1) = p$.
  - Now suppose that $\sigma(n)$ has been defined for all $n < N$ for some natural N, and such that $\sigma(n) \in B_{r/n}(x)$ for all such n. Since $\frac{r}{N+1} > 0$, we can construct the open ball $B_{\frac{r}{N+1}}(x)$, which clearly contains x, so by our assumption $B_{\frac{r}{N+1}}(x) \cap S \neq \emptyset$, so there must exist some $p \in B_{\frac{r}{N+1}}(x) \cap S$, hence $p \in B_{\frac{r}{N+1}}(x)$ and $p \in S$. Define $\sigma(N+1) = p$. This is already shown to be in $B_{\frac{r}{N+1}}(x)$, and it is clearly in p.

Hence by induction we have a sequence $\sigma : \mathbb{N} \to X$ such that each of it's terms are contained within S, and such that for all $n \in \mathbb{N}$, $\sigma(n) \in B_{r/n}(x)$. We now show that this sequence converges to x. Let $\epsilon > 0$. Since $\epsilon > 0$, and since $r > 0$, it follows that $r/\epsilon \in \mathbb{R}$ (we aren't dividing by 0. By the archimedian principle, it follows that there exists some natural number N such that $r/\epsilon < N$, hence by previously shown results about the reals we have that $r/N < \epsilon$. Since for $n > N$, we have $r/n < r/N$, it follows by transitivity that for all natural $n > N$, we have $r/n < \epsilon$. Let n be any such $n > N$. By construction of the sequence σ, it follows that $\sigma(n) \in B_{r/n}(x)$. Hence $d(x, \sigma(n)) < r/n$. Since $r/n < \epsilon$, we have $d(x, \sigma(n) < \epsilon$. So for all natural numbers $n > N$, $d(x, \sigma(n)) < \epsilon$. Since ϵ was arbitrary greater than 0, it follows that for all $\epsilon > 0$, there exists a natural number N such that for all $n > N$ in the naturals, $d(x, \sigma(n)) < \epsilon$. By definition of convergence, it follows that σ converges to x. So we have constructed a sequence of points, whose range is entirely contained within S, which converges to x.

Having shown that each of these properties implies the other in a cycle, it follows that each of them is equivalent.

Note that this does not necessarily mean that x is a limit point. In fact, suppose that $\{x\} = S$. In the last assignment, it was shown that singletons have no limit points (in showing that singletons are closed). Hence x is not a limit point of S. However, the constant sequence converges to x, and is contained within S. By the last theorem, all of these equivalent properties follow immediately. But since it is not a limit point, by Theorem 3.5.1 (it's an equivalence) it follows that there must exist some $r > 0$ such that $B_r(x) \cap S$ contains only finitely many points, and open balls are open. Indeed, the intersection of any set with $S = \{x\}$ can have at most one point, so we didn't even need to go this far. $\square$