Real Analysis

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1 Problem 9.7.3: Taylor Remainders (what a catchy phrase)

Lemma 1. Let $n > 1$, and let $h > 0$. Then $(1+h)^n < 1 + nh$.

Proof. The proof is by induction. First, let $n = 2$. Then $(1+h)^2 = 1+2h+h^2$. Since $h > 0$, so is h^2, hence $1 + 2h > (1+h)^2$.

Now suppose the inequality holds for any $n > 1$ and less than $N \in \mathbb{N}$. Then by the induction step we have

$$\begin{aligned}(1+h)^N &= (1+h)^{N-1}(1+h)\\ &= (1+h)^{N-1} + h(1+h)^{N-1}\\ &> 1 + (N-1)h + h + (N-1)h^2\\ &> 1 + (N-1)h + h = 1 + Nh.\end{aligned} \tag{1}$$

Hence, by induction, the inequality holds for all N as desired. □

I got the idea for the proof from the textbook outline, as well as the idea to prove the above lemma.

Proposition 1. Let x be a real number. Then

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0.$$

Proof. It suffices to show that $\lim_{n\to\infty} \frac{|x|^n}{n!} = 0$. Moreover, if $x = 0$, then the limit is constant at 0, so suppose $x \neq 0$.

Let $\epsilon > 0$. By the archimedian principle, there exists $N \in \mathbb{N}$ such that $N > |x|/2$. From this it follows that $|x|/N < 1/2$. Hence for $n > N$, we have $n = N + m$ for $m > 0$. So we must must have

$$\frac{|x|^n}{n!} < \frac{|x|^{N-1}}{(N-1)!}\frac{1}{2^{m+1}} < \frac{C}{m}.$$

where $C = (|x|^{N-1})((N-1)!)$, and the last inequality was obtained from Lemma 1, and the fact that $2 = 1 + 1$, and $1 + (m+1) > m$. Since $C > 0$, by the archimedian principle there must exist some natural number M such that $1/M < \epsilon/C$. Consider the natural number $N + M$, and suppose that $n > N + M$. Then there exists $m > M$ such that $n = N + m$, hence $\frac{|x|^n}{n!} < C/m < C/M < C(\epsilon/C) = \epsilon$.

So, for all $\epsilon > 0$, there is some natural number N (which was $N + M$ in our consruction) such that whenever $n > N$, we have $|x|^n/(n!) < \epsilon$. This proves the limit, since we know a sequence converges to zero if and only if its absolute value does.

□

Remark 1. Not only did this proof prove the limit, but it allowed us a general method for constructing natural numbers which are sufficiently large for a given error ϵ. First, chose some $N > |x|/2$, and then chose some $M > \frac{|x|^{N-1}}{(N-1)!\epsilon}$. I've used this to construct a very goofy formula, in which I've replaced x with $|x|$ under the assumption x is positive. This was purely for aesthetic reasons.

$$M(\epsilon) = \left\lceil \frac{x^{\lceil \frac{x}{2} \rceil}}{(\lceil \frac{x}{2} \rceil)!\epsilon} \right\rceil + \lceil \frac{x}{2} \rceil + 2 \tag{2}$$

This formula always generates a "safe" natural number

In chosing M in the method, we could have adjusted to compensate for a positive constant factor of K. Instead of chosing $M > C/\epsilon$, we could have chosen $M > KC/\epsilon$. An adjusted formula is

$$M(\epsilon) = \left\lceil \frac{x^{\lceil \frac{x}{2} \rceil}K}{(\lceil \frac{x}{2} \rceil)!\epsilon} \right\rceil + \lceil \frac{x}{2} \rceil + 2. \tag{3}$$

Anyway, now let's move on. Suppose I have the function $\sin x = f(x)$ on the interval $[0, 3]$. Since $\sin x$ is infinitely differentiable, with all of its derivatives as $\pm \sin x$ or $\pm \cos x$, we know that $|f^(n)(x)| \leq 1$ on $[0, 3]$, as this is a property of both sin and cos as functions.

By the corollary to Taylor's Theorem in the textbook, it follows that for $3 \in [0, 3]$, we have the inequality $|R_n(x)| \leq 3^{n+1}$ as the tailor remainder based at 0. To compensate for adding 1 to n. Our formula gives us the number 3004 for $|x| = 3$. Just add one to it, to get a safe number of 3005.

Similarly, notice that since $e < 3$, hence $e^x < 3^x$ and since $\exp : \mathbb{R} \to \mathbb{R}$ is monotonic, and it's derivative is itself, it follows that on the interval $[0, 3]$, e^x stays below 27. So, adjusting for $K = 27$, and applying the good ol' corollary, we get a safe number of 81005.

The above proposition allows us to prove a very interesting result. This, it could be argued, is why Taylor's theorem is actually worth paying attention to.

Theorem 1. Let $f : I \to \mathbb{R}$ (I an interval), such that f is infinitely differentiable on I, and let each of the derivatives of f be bounded on I. Then for any $s \in I$, the pointwise sequences based at $x \in K$ converge to 0 (where the remainder is based at s),

$$\lim_{n \to \infty} R_n(t) = 0.$$

Proof. Since the derivatives of f are bounded, and exist, there exists a real number $M > 0$, such that $|f^{(n)}(x)| \leq M$ for all $x \in I$, and for all $x \in I$. Let $t \in I$. So then, by the textbook's corollary to Taylor's theorem, we have the inequality.

$$|R_n(t)| \leq \frac{M}{(n+1)!}|s - t|^{n+1}.$$

By definition of the absolute value, $|R_n(t)| > 0$. Moreover, by Proposition 1, $\frac{M}{(n+1)!}|s - t|^{n+1}$ converges to 0 as $n \to \infty$. By the Squeeze Theorem, for which I have a proof, it follows that $\lim_{n \to \infty} |R_n(t)| = 0$. We have a theorem which states that this is implies $R_n(t)$ itself converges to 0, which was the desired result.

$\square$

But just how good is this theorem? Most of the infinitely differentiable functions we care about aren't bounded; nor are their othe derivatives (consider e^x!) The solution to this dilema is to first note that their derivatives tend to be continuous (we have counterexamples, but those are all rather pathological). If the function has continuous derivatives, chose K to be a closed interval. Continuous functions on closed intervals are bounded. An interesting downfal, however, is that if the domain K is not bounded, then R_n need not converge uniformly! This means that there isn't a "one size fits all" natural number, in general, for a given margin of error. This depends on the choice of domain.

Speaking of awful functions, we modern mathematicians love 'em, but not this guy:

"Logic sometimes makes monsters. For half a century we have seen a mass of bizarre functions which appear to be forced to resemble as little as possible honest functions which serve some purpose. More of continuity, or less of continuity, more derivatives, and so forth. Indeed, from the point of view of logic, these strange functions are the most general; on the other hand those which one meets without searching for them, and which follow simple laws appear as a particular case which does not amount to more than a small corner.

In former times when one invented a new function it was for a practical purpose; today one invents them purposely to show up defects in the reasoning of our fathers and one will deduce from them only that." –Poincare (the quote found on Wolfram Mathworld).

Naw, he's just gotta chill. But anyway, it's interesting how in analysis, we try to formally understand the functions that are familiar and useful, while in the process come up with ones that aren't!

2 Probelem 10.2.1

Proposition 2. Let A be a set, and let $f : A \to A$ be injective yet not surjective. Then there exists an injection $\mathbb{N} \to A$.

Proof. The map $\sigma : \mathbb{N} \to A$ will be constructed inductively. First, since f is not surjective, there is $a \in A$ such that $f(x) \neq a$ for all $x \in A$. Now inductively construct the iterated map $\sigma : n \mapsto f^n(a)$, based at this particular element $a \in A$. To see that σ is an injection, suppose we had $m, n \in \mathbb{N}$ such that $\sigma(n) = \sigma(m)$, while $m \neq n$. Then $f^n(a) = f^m(a)$. Without loss of generality suppose $n > m$, hence $m = n + r$ for some $r \in \mathbb{N}$. Then $f^n(a) = f^{n+r}(a) = f^n(f^r(a))$. It can be verified by an easy inductive argument that the n-fold composition of injections is injective. Hence $a = f^r(a) = f(f^{r-1}(a))$. So $f^{r-1}(a)$ is an element of A which maps to a.

By construction of a, this cannot be the case, so we arrive at a constradiction. $\square$

Remark 2. The converse is also true. These can actually be used as equivalent defintions of an infinite set. Most importantly, the Peano axioms imply that $s : N \to N$, the successor function which is such an important sequence that it doesn't get a greek letter, is defined to be an injection which is not a surjection. Using the principle of mathematical induction (another one of our friend Peano's axioms), we can show that s is almost a surjection (just not for the initial element). Iterated maps, in a sense, are used to **define** the natural numbers, which in turn are used to define infinity in a rigorous way.