Real Analysis

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1 Converse Part of the Cauchy Criterion Theorem

Theorem 1. Suppose $f : [a, b] \to \mathbb{R}$ is Reimann integrable. Then it satisfies the Cauchy Criterion.

Proof. Let $\epsilon > 0$. Then $\epsilon/2 > 0$. By integrability, it follows that there exists some $\delta > 0$ such that whenever $\mathcal{R}(f, P)$ is a Reimann sum for a partition P of mesh less than δ, we have

$$\left|\mathcal{R}(f, P) - \int_a^b f\right| < \epsilon/2.$$

Now consider any such partition P with $\|P\| < \delta$, and consider any two Reimann sums $\mathcal{R}_1(f, P)$ and $\mathcal{R}_2(f, P)$. By construction of δ and P, it follows that

$$\left|\mathcal{R}_{1,2}(f, P) - \int_a^b f\right| < \delta.$$

Then

$$|\mathcal{R}_1(f, P) - \mathcal{R}_2(f, P)|$$
$$= \left|\mathcal{R}_1(f, P) - \int_a^b f + (I - \mathcal{R}_2(f, P))\right| \leq \left|\mathcal{R}_1(f, P) - \int_a^b f\right| + \left|\mathcal{R}_2(f, P) - \int_a^b f\right| < \epsilon/2 + \epsilon/2 = \epsilon. \tag{1}$$

This proves that f on $[a, b]$ satisfies the Cauchy Criterion.

$\square$

2 Upper and Lower Sum Criterion

Proposition 1. Let $f : [a, b] \to \mathbb{R}$ be a bounded function (boundedness is needed to make sense of upper and lower sums). Then for all $\epsilon > 0$, there exists some $\delta > 0$ such that whenever P is a partition of $[a, b]$ with $\|P\| < \delta$, the inequality

$$|\mathcal{U}(f, P) - \mathcal{L}(f, P)| < \epsilon$$

holds, if and only if f is Reimann integrable on $[a, b]$.

Proof. First suppose that f is Reimann integrable. Let $\epsilon > 0$. Then $\epsilon/2 > 0$. By the converse to the Cauchy Criterion Theorem, there exists some δ such that for all P with $\|P\| < \delta$, and for any Reimann sums $\mathcal{R}_1(f, P)$ and $\mathcal{R}_2(f, P)$, we have

$$|\mathbb{R}_2 U(f, P) - \mathbb{R}_1 L(f, P)| < \epsilon/2.$$

Chose any such P. Then the above inequality holds for any Reimann sums of the above form. By Lemma 11.4.7 of the textbook, we have

$$|\mathcal{U}(f, P) - \mathcal{L}(f, P)| \leq \epsilon/2 < \epsilon.$$

The partition P was arbitrary with mesh size less than δ, hence for any partition, the above inequality would hold. This proves the converse.

Now suppose that for all $\epsilon > 0$, there exists some $\delta > 0$ such that whenever P is a partition of $[a, b]$ with $\|P\| < \delta$, the inequality

$$|\mathcal{U}(f, P) - \mathcal{L}(f, P)| = \mathcal{U}(f, P) - \mathcal{L}(f, P) < \epsilon.$$

For ϵ fixed, we have one such δ. Suppose we have some parition P with mesh size less that δ. Let $\mathcal{R}_1(f, P)$ and $\mathcal{R}_2(f, P)$ be Reimann sums for this partition. Since the inequality with the upper and lower sums hold for this particular δ, and this particular ϵ, and this particular partition, we have $\mathcal{L}(f,P) \leq \mathcal{R}_1(f,P), \mathcal{R}_2(f,P) \leq \mathcal{U}(f,P)$. Without loss of generality, suppose that $\mathcal{R}_1(f,P) \geq \mathcal{R}_2(f,P)$, whence $|\mathcal{R}_1(f,P) - \mathcal{R}_2(f,P)| = \mathcal{R}_1(f,P) - \mathcal{R}_2(f,P)$. Since $\mathcal{U}(f,P) \geq \mathcal{R}_1(f,P)$ and $\mathcal{L} \leq \mathcal{R}_2(f,P)$, we have

$$| \, |\mathcal{R}_1(f,P) - \mathcal{R}_2(f,P)| = \mathcal{R}_1(f,P) - \mathcal{R}_2(f,P) \leq \mathcal{U}(f,P) - \mathcal{L}(f,P) < \epsilon.$$

Since ϵ was arbitrary, and P was as well, and so were the Reimann sums, it follows that f satisfies the Cauchy Criterion on $[a,b]$. From this and the Caucy Criterion Theorem, it follows that f is Riemann integrable on $[a,b]$. $\square$