Real Analysis

August Bergquist

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1 Theorem Establishing Equivalent Definitions for Continuity at a Point

Theorem 1. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $a \in X$, and let $f : X \to Y$. Then the following are equivalent:

- f is continuous at a.
- For every $\epsilon > 0$ we can produce a positive δ so that for any $x \in X$ we have that $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$.
- For any sequence $\sigma : \mathbb{N} \to X$ which converges to a, we have $f \circ \sigma$ converges to $f(a)$.

Proof.

- First suppose that f is continuous at a. Then there are two cases (indeed, whether or not f is continuous lol), either a is a limit point of X or it isn't. First suppose that it is a limit point. Then by definition of continuity, it follows that $\lim_{x \to a} f(x) = f(a)$, and by definition of a limit it follows that for all $x \in X$ there is some $\delta > 0$ such that $0 < d_X(x, a) < \delta$ implies that $d_Y(f(a), f(a))\epsilon$. Indeed, if $x = a$, by positive definiteness of d_X we have that the distance is zero, and $\epsilon > 0$, whence $d(x, a) < \epsilon$, so it follows that $d_X(a, a) < \delta$ only when $d_Y(f(a), f(x)) < \epsilon$. So then, in the case where a is a limit point, it follows that for all $\epsilon > 0$ there is some $\delta > 0$ such that $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$.

  Now suppose that a is not a limit point. Let $\epsilon > 0$. Negating one of the equivalent definitions for a limit point, we have some $\delta > 0$ such that
  $$B_\delta(a) \cap (X \setminus \{a\}).$$

From this it follows that for any $x \in X$, $x \in B_\delta(a)$ only when $x = a$. Then if we let $d_X(a, x) < \delta$ we have by definition of an open ball that $x \in B_\delta(a)$, so by the last remark we have that $x = a$. By well defindendess of a function, $f(a) = f(x)$, so by positive definiteness it follows that $d_Y(f(a), f(x)) = 0$, and since $\epsilon < 0$, obviously $d_Y(f(a), f(x)) < \epsilon$. Hence, in this case as well, (2) of the equivalences holds.

- Now suppose that for every $\epsilon > 0$ we can produce a positive δ so that for any $x \in X$ we have that $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$. Suppose we have a sequence $\sigma : \mathbb{N} \to X$ which converges to a. We want to show that $f\sigma$ converges to $f(a)$. Let $\epsilon > 0$. By supposition we have some $\delta > 0$ such that $d_X(x, a) < \delta$ only if $d_Y(f(x), f(a)) < \epsilon$. Since $\delta > 0$, by convergence of σ to a there must exist some $N \in \mathbb{N}$ such that $d_X(\sigma(n), a) < \delta$ for all $n > \mathbb{N}$. Now suppose we have $n > N$. So $d_X(\sigma(n), a) < \delta$, and by construction of δ it follows that $d_Y(f(\sigma(n)), f(a)) = d_Y(f\sigma(n), f(a)) < \epsilon$. So then for all $n > N$ it follows that $d_Y(f\sigma(n), f(a)) < \epsilon$. So then, since ϵ was arbitrary greater than zero, it follows that for all $\epsilon > 0$ there exists some $N \in \mathbb{N}$ such that for all $n > N$ $d_Y(f\sigma(n), f(a)) < \epsilon$. By definition of convergence, it follows that $f\sigma$ converges to $f(a)$. Since the sequence σ as arbitrary as a sequence which converges to a, it follows that for any sequence which converges to a, the composition of that sequence with f converges to $f(a)$. This is the statement (3) of the equivalence!

- Finally, suppose that for all sequences $\sigma : \mathbb{N} \to X$, convergence to a implies convergence of $f\sigma$ to $f(a)$. We desire to show that f is continuous. If a is not a limit, we are done. So suppose it is a limit point. Then we have a coherent notion of the limit of the function f to a. Suppose that there exist a distinct sequence σ which converges to a. Well a distinct sequence is a sequence just as any other, so by our supposition it follows that $f\sigma$ converges to $f(a)$. By Theorem 4.2.4 it follows that the limit of f as x approaches a is $f(a)$. Hence in this case as well, f is continuous at a. In either case, (1) of the equivalence holds as desired.

□

2 That The limit of a Function Is Unique if it Exists

Theorem 2. Suppose that X is a metric space, $K \subset X$, Y a metric space, and that $a \in X$ is a limit point of K, and that $f : K \to Y$ so that $\lim_{x\to a} f(x) = L, L'$. Instead of using contradiction, let us directly show that $L = L'$. Recall that a non-negative number $r = 0$ if and only for every $\epsilon < 0$, $r < \epsilon$.

Now let consider $d_Y(L, L')$. By the triangle inequality and symmetry, $d_Y(L, L') \leq d_Y(f(x), L) + d_Y(f(x), L')$. Now let $\epsilon > 0$. Then there is some δ such that for any $x \in K$, $0 < d_X(x, a) < \delta$ implies $d_Y(f(x), L) < /2$ and $d_Y(f(x), L') < \epsilon/2$. Now since a is a limit point, it follows by our previous results that $B_\delta(a) \cap K \setminus \{x\} \neq \emptyset$, so there is some $x \in B_\delta(a)$ with $x \neq a$. By definition of an open ball, $d_X(x, a) < \delta$, so by construction of δ it follows that $d_Y(f(x), L') < \epsilon/2$ and $d_Y(f(x), L) < \epsilon/2$, from which it follows that $d_Y(f(x), L') + d_Y(f(x), L) < \epsilon$. By transitivity of the relation $<$, it follows that $d_Y(L, L') < \epsilon$. But $\epsilon > 0$ was arbitrary, hence for all $\epsilon > 0$, $d_Y(L, L') < \epsilon$. Hence $d_Y(L, L') = 0$. By positive definiteness, $L = L'$. Hence the limit is unique.