Subquandle Lattices

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Motivation

- Quandles are useful in knot theory, expressing a complete invariant of unoriented knots; they also have interesting properties as an algebraic structure.
- As with other algebraic structures such as groups and rings, quandles have sub-objects called subquandles.
- We consider inner automorphism groups, lattice structure, and complemented properties of subquandles.

Definitions

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Q3. Distributivity: $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

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Q3. Distributivity: $(x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$.

We say $Q' \subseteq Q$ is a **subquandle** of Q if it closed under $\triangleright$ and $\triangleright^{-1}$. Denote this by $Q' \preccurlyeq Q$, or $Q' \prec Q$ if $Q' \neq Q$.

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- If $(Q, \triangleright)$ and $(R, \triangleright_1)$ are quandles, a **quandle homomorphism** $f : Q \to R$ is a function satisfying $f(a \triangleright b) = f(a) \triangleright_1 f(b)$ for every $a, b \in Q$.

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Definition

Given a quandle Q and an element $y \in Q$, the **symmetry** at y is the automorphism of Q of the form $S_y : x \mapsto x \triangleright y$.

The **inner automorphism group** of Q is $\mathrm{Inn}(Q) = \langle \{S_q \mid q \in Q\} \rangle$. Note that $\mathrm{Inn}(Q) \trianglelefteq \mathrm{Aut}(Q)$ **?**.

In fact, if $Q' \preccurlyeq Q$, $\mathrm{Inn}(Q)$ acts on Q' by functional application. This action also induces an action on the subquandle lattice.

Multiplication Tables For Finite Subquandles

Example

The Tait quandle $(\mathbf{T}_3, \triangleright)$ with underlying set $\{1, 2, 3\}$ has operation multiplication table

$$\begin{array}{c|ccc} \triangleright & 1 & 2 & 3 \\ \hline 1 & 1 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 3 \end{array}$$

Subquotient Theorem

Theorem

Suppose $Q' \preccurlyeq Q$. Then, $\mathrm{Inn}(Q')$ is a subquotient of $\mathrm{Inn}(Q)$. That is, there is some subgroup $S_{Q'} \leq \mathrm{Inn}(Q)$ and some normal subgroup $K_{Q'} \trianglelefteq S_{Q'}$, such that $\mathrm{Inn}(Q') \cong S_{Q'}/K_{Q'}$.

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Proof.

Define $S_{Q'} = \langle S_x \mid x \in Q' \rangle \leq \mathrm{Inn}(Q)$. Note that $\tau : S_{Q'} \to \mathrm{Inn}(Q')$ via $\tau(f) = f|_{Q'}$ is a well-defined surjective homomorphism with image $\mathrm{Inn}(Q')$. Hence, $\mathrm{Inn}(Q') \cong S_{Q'}/\ker(\tau)$. $\square$

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Definition

A subquandle $Q_1 \preccurlyeq Q$ is **complemented** in Q if there is some $Q_2 \preccurlyeq Q$ such that $Q_1 \wedge Q_2 = \emptyset$, and $Q_1 \vee Q_2 = Q$. The subquandle lattice $\mathcal{L}(Q)$ is complemented if every subquandle is complemented.

Saki And Kiani's Paper

Saki and Kiani **?** showed that the subrack lattice of a finite rack is complemented. Since quandles are racks, this holds for quandles as a corollary.

Corollary

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Theorem (Saki and Kiani)

Consider the quandle $\mathbb{Q}$ *with the dihedral structure,* $x \triangleright^* y = 2x - y$, $\forall x, y \in \mathbb{Q}$. *The subquandle* $\{0\}$ *is not complemented.*

Ind-finite Quandles

Definition

A nonempty quandle Q is **ind-finite** if there is a family of finite subquandles $\{Q_i\}_{i\in\mathcal{I}}$ indexed by a directed set $\mathcal{I}$ such that each $Q_i \preccurlyeq Q$, $|Q_i| < \infty$, $Q_i \prec Q_{i+1}$, and $Q = \bigcup_{i\in\mathcal{I}} Q_i$.

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Lemma

Suppose Q is a quandle. Then, Q is ind-finite if and only if every finitely generated subquandle of Q is finite.

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Example

Let the quandle operation $\triangleright^*$ be given by $x \triangleright^* y = 2x - y$.

- Neither $(\mathbb{Q}, \triangleright^*)$ nor $(\mathbb{Q}/\mathbb{Z}, \triangleright^*)$ has a complemented sublattice.
- $(\mathbb{Q}/\mathbb{Z}, \triangleright^*)$ is ind-finite but $(\mathbb{Q}, \triangleright^*)$ is not.

Strongly Complemented Subquandles

Definition

A subquandle $Q' \preccurlyeq Q$ is **strongly complemented** if $Q \setminus Q' \preccurlyeq Q$.

Example

In the quandle Q represented by the matrix

$$M_Q = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 2 \\ 2 & 3 & 3 \end{bmatrix}.$$

The subquandles

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix}, [1]$$

are strongly complemented.

Semidisjoint Union

Definition (Ehrman et al.)

Given a sequence of quandles $Q_1, \ldots, Q_n$ and a *nxn* matrix of group homomorphisms $(M)_{ij} = g_{ij}$:, Ehrman et al. **?** defined the **semidisjoint union** as follows:

$$\#(Q_1, \ldots, Q_n, M) = \Big(\coprod_{i=1}^{n} Q_i, \triangleright\Big).$$

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$$\#(Q_1, \ldots, Q_n, M) = \Big(\coprod_{i=1}^{n} Q_i, \triangleright \Big).$$

- Each entry of the matrix $g_{ij} : \mathrm{Adconj}(Q_i) \to \mathrm{Aut}(Q_j)$ is a group homomorphism.
- $\triangleright$ is defined as $x \triangleright y = x \cdot g_{ij}(|y|_{Q_i})$ for $x \in Q_i$ and $y \in Q_j$.
- Note that we are not guaranteed that the semidisjoint union is a quandle. If the matrix M gives rise to a quandle, it is called a **mesh**.
- Ehrman et al. provided a necessary and sufficient condition for M to be a mesh.

Orbit Decomposition Theorem

Theorem (Ehrman et al.)

Let Q be a quandle, and let $Q_1, \ldots, Q_n$ be its orbits under the inner automorphism group. Then we can construct a mesh M such that

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- Note that the orbits need not be connected. Hence the orbits themselves may be decomposable via the previous theorem.

Two Actions

- We've shown how $\mathrm{Inn}(Q)$ acts on Q by functional application.
- This action allows us to construct an action of $\mathrm{Inn}(Q)$ upon $\mathcal{L}(Q)$.

Definition

The action of $\mathrm{Inn}(Q)$ on Q' is also given by functional application, denoted $Q' \cdot \mathrm{Inn}(Q)$.

The action of $\mathrm{Inn}(Q)$ upon $\mathcal{L}(Q)$ is defined $Q'\sigma = \sigma(Q')$ for all $Q' \in \mathcal{L}(Q)$, and for all $\sigma \in \mathrm{Inn}(Q)$. The orbit of Q' under this action is denoted by $[Q'] \cdot \mathrm{Inn}(Q) = \{Q'\sigma : \sigma \in \mathrm{Inn}(Q)\}$.

Strongly Complemented Classification

Using both of these group actions, we managed to classify strongly complemented subquandles using the following theorem:

Theorem

Let Q be a quandle, and let $Q' \preccurlyeq Q$. Denote the subquandle lattice of Q by $\mathcal{L}(Q)$. The following are equivalent:

- $Q \setminus Q' \preccurlyeq Q$,
- *Q' is a union of orbits under the action of $\mathrm{Inn}(Q)$ on Q,*
- *Q' is a fixed point of the action of $\mathrm{Inn}(Q)$ on $\mathcal{L}(Q)$,*
- *$Q = \#(Q', Q \setminus Q', M)$ for a mesh M as constructed in the orbit decomposition theorem of Ehrman et al.*

Strongly Complemented Within Strongly Complemented

Lemma (1)

Let $Q'' \preccurlyeq Q' \preccurlyeq Q$ such that Q'' is strongly complemented within Q', and Q' is strongly complemented within Q. Then for any $\gamma \in \mathrm{Inn}(Q)$, $Q''\gamma$ is strongly complemented within Q'.

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Lemma (2)

Let $Q'' \preccurlyeq Q' \preccurlyeq Q$ be as in Lemma (1). Then for all $\sigma \in \mathrm{Inn}(Q)$, there exists some $\sigma' \in \langle S(Q \setminus Q')\rangle$ such that $Q''\sigma = Q''\sigma'$.

Proof.

- Suppose $\sigma = S_{x_1}^{\epsilon_1} \dots S_{x_n}^{\epsilon_n}$ for each $x_j \in Q$, $\epsilon_j \in \{-1, 1\}$.
- Suppose $\exists x_i \in Q'$ whose symmetry appears in σ which we cannot remove without changing the action.
- By Lemma 1, Q'' acted upon by the string of σ up until x_i, denote it by γ, is strongly complemented in Q'.

$\square$

Strongly Complemented Within Strongly Complemented

Proof (*continued*).

- Since $x_i \in Q'$, $S_{x_i}|_{Q'}$ acts the same upon Q' as S_{x_i}
- Then, $Q''\gamma S_{x_i} = Q''\gamma$ by the equivalence theorem. Hence, removing S_{x_i} won't change anything, so there never was a counterexample.
- Thus, we can construct σ' by removing symmetries represented by elements in Q'.

$\square$

Lemma (3)

Suppose that Q'' is strongly complemented within Q, while $Q'' \preccurlyeq Q' \preccurlyeq Q$. Then, Q'' is strongly complemented within Q'.

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Theorem

Suppose Q is a quandle, with subquandles $Q'' \preccurlyeq Q' \preccurlyeq Q$, such that Q'' is strongly complemented within Q', while Q' is strongly complemented within Q. Then Q'' is complemented within Q by the subquandle $Q \setminus Q'' \cdot \mathrm{Inn}(Q)$.

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Proof.

- By Lemma 3 and the equivalence theorem, Q'' is strongly complemented within $Q'' \cdot \mathrm{Inn}(Q)$, which in turn is strongly complemented within Q.
- $Q'' \wedge Q \setminus Q'' \cdot \mathrm{Inn}(Q) = \emptyset$
- Using Lemmas 2 and 1, we show that $Q'' \cdot \mathrm{Inn}(Q) \subset Q'' \vee Q \setminus Q'' \cdot \mathrm{Inn}(Q)$, hence $Q'' \vee Q \setminus Q'' \cdot \mathrm{Inn}(Q) = Q$.
- Hence Q'' has $Q \setminus Q'' \cdot \mathrm{Inn}(Q)$ as a complement.

□

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   - Proved profinite abelian groups are profinite Takasaki quandles ($x \triangleright y = 2y - x$) under coordinatewise operations.
3. Are the subquandle lattices of profinite quandles complemented?

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Questions?