HOMEWORK 1: Exercises for Monte Carlo Methods

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Exercise 1.

The Monte Carlo method can be used to generate an approximate value of pi. The figure below shows a unit square with a quarter of a circle inscribed. The area of the square is 1 and the area of the quarter circle is pi/4. Write a script to generate random points that are distributed uniformly in the unit square. The ratio between the number of points that fall inside the circle (red points) and the total number of points thrown (red and green points) gives an approximation to the value of pi/4. This process is a Monte Carlo simulation approximating pi. Let N be the total number of points thrown. When N=50, 100, 200, 300, 500, 1000, 5000, what are the estimated pi values, respectively? For each N, repeat the throwing process 100 times, and report the mean and variance. Record the means and the corresponding variances in a table.

蒙特卡洛方法可以用于产生接近 pi 的近似值。图 1 显示了一个带有 1/4 内切圆在内的边长为 1 的正方形。正方形的面积是 1，该 1/4 圆的面积为 pi/4。通过编程实现在这个正方形中产生均匀分布的点。落在圈内（红点）的点和总的投在正方形（红和绿点）上的点的比率给出了 pi/4 的近似值。这一过程称为使用蒙特卡洛方法来仿真逼近 pi 实际值。令 N 表示总的投在正方形的点。当投点个数分别是 20, 50, 100, 200, 300, 500, 1000, 5000 时，pi 值分别是多少？对于每个 N，每次实验算出 pi 值，重复这个过程 100 次，并在表中记下均值和方差。

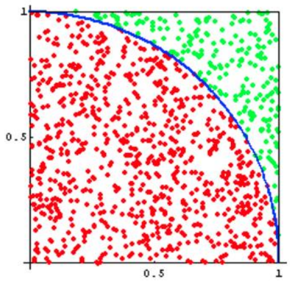

Figure 1 蒙特卡洛方法求解 pi

Exercise 2.

We are now trying to integrate the another function by Monte Carlo method:

$$\int_0^1 x^3$$

A simple analytic solution exists here: $\int_{x=0}^{1} x^3 = 1/4$. If you compute this integration using Monte Carlo method, what distribution do you use to sample x? How good do you get when N = 5, 10, 20, 30, 40, 50, 60, 70, 80, 100, respectively? For each N, repeat the Monte Carlo process 100 times, and report the mean and variance of the integrate in a table.

我们现在尝试通过蒙特卡洛的方法求解如下的积分：

$$\int_0^1 x^3$$

该积分的求解我们可以直接求解，即有$\int_{x=0}^{1} x^3 = 1/4$。如果你用蒙特卡洛的方法求解该积分，你认为 x 可以通过什么分布采样获得？如果采样次数是分别是 N = 5, 10, 20, 30, 40, 50, 60, 70, 80, 100，积分结果有多好？对于每个采样次数 N，重复蒙特卡洛过程 100 次，求出均值和方差，然后在表格中记录对应的均值和方差。

Exercise 3.

We are now trying to integrate a more difficult function by Monte Carlo method that may not be analytically computed:

$$\int_{x=2}^{4}\int_{y=-1}^{1} f(x,y) = \frac{y^2 * e^{-y^2} + x^4 * e^{-x^2}}{x * e^{-x^2}}$$

Can you compute the above integration analytically? If you compute this integration using Monte Carlo method, what distribution do you use to sample (x,y)? How good do you get when the sample sizes are N = 5, 10, 20, 30, 40, 50, 60, 70, 80, 100, 200 respectively? For each N, repeat the Monte Carlo process 100 times, and report the mean and variance of the integrate.

我们现在尝试通过蒙特卡洛的方法求解如下的更复杂的积分：

$$\int_{x=2}^{4}\int_{y=-1}^{1} f(x,y) = \frac{y^2 * e^{-y^2} + x^4 * e^{-x^2}}{x * e^{-x^2}}$$

你能够通过公式直接求解上述的积分吗？如果你用蒙特卡洛的方法求解该积分，你认为(x, y)可以通过什么分布采样获得？如果点（x, y）的采样次数是分别是 N = 10, 20, 30, 40, 50, 60, 70, 80, 100, 200, 500, 积分结果有多好？对于每个采样次数 N，重复蒙特卡洛过程 100 次，求出均值和方差，然后在表格中记录对应的均值和方差。

Exercise 4.

An ant is trying to get from point A to point B in a grid. The coordinates of point A is (1,1) (this is top left corner), and the coordinates of point B is (n,n) (this is bottom right corner, n is the size of the grid).

Once the ant starts moving, there are four options, it can go left, right, up or down (no diagonal movement allowed). If any of these four options satisfy the following:

(a) The new point should still be within the boundaries of the n×n grid

(b) Only the center point (4, 4) is allowed to be visited zero, one or two times, while the remainder points should not be visited previously (are allowed to be visited zero or one time).

If P is the probability of the ant reaching point B for a 7×7 grid, use Monte Carlo simulation to compute P. Pick the answer closest to P in value (assume 20,000 simulations are sufficient enough to compute P).

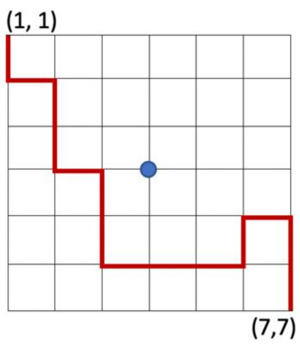

Figure 2 An ant is trying to get from point A (1,1) to point B (7,7) in a grid.

Exercise 5.

Given a system made of discrete components with known reliability, what is the reliability of the overall system? For example, suppose we have a system that can be described with the following high-level diagram:

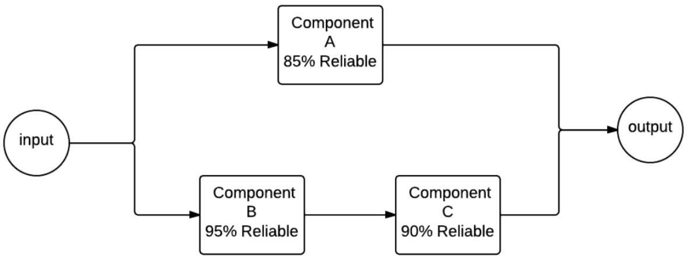

Figure 3 A system made of discrete components.

When given an input to the system, that input flows through component A or through components B and C, each of which has a certain reliability of correctness. Probability theory tells us the following:

$$reliability_{BC} = 0.95 * 0.90 = 0.855$$

$$reliability_A = 0.85$$

And the overall reliability of the system is:

$$\begin{aligned} reliability_{sys} &= 1.0 - [(1.0 - 0.85) * (1.0 - 0.855)] \\ &= 0.97825 \end{aligned}$$

Create a simulation of this system where half the time the input travels through component A. To simulate its reliability, generate a number between 0 and 1. If the number is 0.85 or below, component A succeeded, and the system works. The other half of the time, the input would travel on the lower half of the diagram. To simulate this, you will generate two numbers between 0 and 1. If the number for component B is less than 0.95 and the number for component C is less than 0.90, then the system also succeeds. Run many trials to see if you converge on the same reliability as predicted by probability theory.