Notes on Introduction to Measure Theory

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Most of the present material is based on [1], which is a very consistent introduction to various branches of analysis. Since the material we will follow in this class is spread throughout the book, a concise summary is provided here. When other materials will be used in these notes, we indicate this with precise references. The list of references in these notes also constitutes a list of suggestions for those students that want to study the material in more depth (beyond the purpose of this class, which is a first introduction to measure theory).

The aim of this course is to provide the students with enough background to understand the concept of Lebesgue measure and be able to operate with Lebesgue integration.

To get a glimpse of the importance of the concepts, consider an interval contained in the real line or a region in the plane, or simply think of the length of the interval or the area of the region give an idea of the size. We want to extend the notion of size to as large a class of sets as possible. Doing this for subsets of the real line gives rise to Lebesgue measure. In the first part of this course we will discuss classes of sets, the definition of measures, and the construction of measures, of which one example is Lebesgue measure on the line.

After an understanding of the notion of measures, we will proceed to the Lebesgue integral. We talk about measurable functions, define the Lebesgue integral, prove the monotone and dominated convergence theorems, look at some simple properties of the Lebesgue integral, compare it to the Riemann integral, and discuss some of the various ways a sequence of functions can converge.

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1 Families of sets

Throughout we will let X denote an arbitrary set. We wish to define a general notion of *measuring* certain subsets of X. For this notion of measure to make sense, the collection of subsets of X upon which we will define it needs to have a few nice properties.

Definition 1.1 Algebra and σ-Algebra
An ***algebra*** *$\mathcal{A}$ is a collection of subsets of X so that*

a) $\emptyset \in \mathcal{A}$;

b) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (where $A^c = X \setminus A$ is the complement of A in X);

c) Finite unions and intersections of sets in $\mathcal{A}$ are also in $\mathcal{A}$.
Namely, if $A_1, \dots A_k \in \mathcal{A}$, for k a finite number, then $\cup_{j=1}^k A_j \in \mathcal{A}$ and $\cap_{j=1}^k A_j \in \mathcal{A}$.

An algebra is a ***σ-algebra*** *if additionally the following property holds:*

d) If $A_1, A_2, \dots, A_j, \dots \in \mathcal{A}$, then $\cup_{j=1}^\infty A_j \in \mathcal{A}$ and $\cap_{j=1}^\infty A_j \in \mathcal{A}$.

I.e. we require countable unions and intersections of sets in $\mathcal{A}$ to be in $\mathcal{A}$.

Remark 1.2 *Requiring $\mathcal{A}$ to be closed under both unions and intersections is redundant, as we have $(\cup_j A_j)^c = \cap_j A_j^c$ and $(\cap_j A_j)^c = \cup_j A_j^c$. Hence it suffices to check either of the two.*

Remark 1.3 *Note that property c) follows directly from property d) and a). Hence, if one wishes to prove that a collection of subsets is a σ-algebra, it suffices to prove that properties a), b) and d) are satisfied.*

These σ-algebras are the collections of subsets of X upon which we will be able define a measure, which we will do in a later lecture. For now, we will take a look at some properties and examples of these σ-algebras, and ways to construct them.

1.1 Some examples and simple exercises regarding σ-algebras

Example 1.4 *Let X be any set. Then $\mathcal{A} = \{\emptyset, X\}$ is a σ-algebra. It is called the* ***trivial σ-algebra*** *of subsets of X.*

Example 1.5 *Let $\mathcal{A} = \mathcal{P}(X)$ be the collection of all subsets of $\mathbb{R}$. Then $\mathcal{A}$ is a σ-algebra.*

Exercise 1.6 *Consider $X = [0, 1]$ and let $\mathcal{A} = \{[0, 1/2), [1/2, 1], \emptyset, X\}$. Then $\mathcal{A}$ is a σ-algebra. Write down as many of the other σ-algebras of $X = [0, 1]$ as you can.*

Exercise 1.7 *Write down all σ-algebras of $X = \{1, 2, 3, 4\}$.*

Example 1.8 *Consider $X = \mathbb{R}$ and let*

$$\mathcal{A} = \{A \subset \mathbb{R} \mid A \text{ is countable or } A^c \text{ is countable}\}.$$

We shall prove that $\mathcal{A}$ is a σ-algebra.

Proof. *Note that properties a) and b) of Definition 1.1 are trivially satisfied. Hence it remains to check property d). We will verify that $\mathcal{A}$ is closed under taking countable unions. To this end, we let $A_1, A_2 \ldots \in \mathcal{A}$ and consider the following two cases.*

> **Case 1** *Suppose that $A_1, A_2 \ldots \in \mathcal{A}$ are countable. Then clearly their union is countable as well, and therefore in $\mathcal{A}$.*
>
> **Case 2** *Suppose that there exists some j such that A_j^c is countable. We start from the following remark:*
>
> ***Remark 1.9*** *If B is a countable set, then for any $A \subset B$ both A and $B \backslash A$ are countable.*
>
> *We note that*
> $$(\cup_{i=1}^{\infty} A_i)^c = \cap_{i=1}^{\infty} A_i^c \subset A_j^c.$$
> *Hence, by our remark, $(\cup_{i=1}^{\infty} A_i)^c$ is countable and as a consequence, we have $\cup_i A_i \in \mathcal{A}$.*

We conclude that $\mathcal{A}$ satisfies properties a), b) and d) of Definition 1.1 and is therefore a σ-algebra. ☐

Example 1.10 *Consider $X = (0, 1]$ and define*

$$\mathcal{A} = \{\text{finite unions of semi-open (left-open, closed-right) subintervals of } X\}.$$

We will show that $\mathcal{A}_0 = \mathcal{A} \cup \{\emptyset\}$ is an algebra but not a σ-algebra.

Proof. *First, we will show that $\mathcal{A}_0$ is an algebra. Definition 1.1 is trivially satisfied, so it remains to verify b) and c).*

Note that any general set $A \in \mathcal{A}$ can be written as

$$A = (a_1, a_1'] \cup (a_2, a_2'] \cup \ldots \cup (a_m, a_m'], \quad m < \infty,$$

with $a_1 \leq a_2 \leq \ldots \leq a_m$ and $\{(a_i, a_i']\}_{i \leq m}$ pairwise disjoint. Property a) in Take any $A \in \mathcal{A}$ and write it in the form given above. Then we have

$$A^c = (0, a_1] \cup (a_1', a_2] \cup \ldots \cup (a_{m-1}', a_m] \cup (a_m', 1].$$

Therefore, we either have $A^c = \emptyset$ (when all of these intervals are empty) or A^c is a finite union of disjoint intervals, meaning we have $A^c \in \mathcal{A}$. Hence property b) is satisfied. Now consider some $B \in \mathcal{A}$ and similarly write

$$B = (b_1, b_1'] \cup (b_2, b_2'] \cup \ldots \cup (b_n, b_n'],$$

for some $n < \infty$. Then

$$A \cap B = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (a_i, a_i'] \cap (b_j, b_j'].$$

Hence $A \cap B$ is either empty or a finite union of disjoint intervals. Thus, $A \cap B \in \mathcal{A}$. It follows that $\mathcal{A}$ is closed under taking finite intersections and hence property c) of Definition 1.1 is satisfied. We conclude that $\mathcal{A}_0$ is an algebra. To see, however, that it is not a σ-algebra, we will provide a simple counterexample to property d) of Definition 1.1. Note that for any $x \in X$ we have

$$\bigcap_{n=1}^{\infty} (x - 1/n, x] = \{x\}.$$

However, $\{x\}$ is a singleton (that is, a set consisting of a single element), which can be written as a closed interval $\{x\} = [x, x]$, but not as a finite union of semi-open intervals. Hence $\cap_{n=1}^{\infty}(x - 1/n, x] \notin \mathcal{A}$, meaning $\mathcal{A}_0$ is not closed under taking countable intersections and $\mathcal{A}_0$ is therefore not a σ-algebra. □

1.2 Unions and intersections of σ-algebras

Looking back at our definition of σ-algebras, it is natural to wonder whether unions and intersections of σ-algebras are themselves σ-algebras as well.

Lemma 1.11 *Let $I \neq \emptyset$ be some index set. For any $j \in I$, let $\mathcal{A}_j$ be a σ-algebra of subsets of X. Then $\mathcal{A} := \cap_{j \in I} \mathcal{A}_j$ is again a σ-algebra.*

Proof. Immediate from Definition 1.1. □

Hence intersections of σ-algebras are indeed σ-algebras. The same does not hold, however, for unions.

Counterexample 1.12 *We will show an example of a union of σ-algebras that is not even an algebra, let alone a σ-algebra. Let $X = \{1, 2, 3\}$. Also write*

$$\mathcal{A}_0 = \{X, \emptyset, \{1\}, \{2, 3\}\}$$

and

$$\mathcal{A}_1 = \{X, \emptyset, \{3\}, \{1, 2\}\}.$$

We see that

$$\mathcal{A}_0 \cup \mathcal{A}_1 = \{X, \emptyset, \{1\}, \{3\}, \{1, 2\}, \{2, 3\}\}.$$

However, $\{2, 3\} \cap \{1, 2\} = \{2\} \notin \mathcal{A}_0 \cup \mathcal{A}_1$. Hence $\mathcal{A}_0 \cup \mathcal{A}_1$ is not closed under taking finite intersections and is therefore not an algebra.

2 Borel σ-algebras on $X = \mathbb{R}$

In this lecture we will consider a very important type of σ-algebras named *Borel σ-algebras.* These σ-algebras can be defined on general topological spaces X and are *generated* by the open subsets of X. In this course, however, we will restrict our attention to the Borel σ-algebra of $X = \mathbb{R}$. First we need to define what it means for a σ-algebra to be generated by some collection of subsets.

Definition 2.1 *Let $\mathcal{G}$ a collection of subsets of X. Then the σ-algebra*

$$\sigma(\mathcal{G}) = \bigcap \{\mathcal{A}_j : \mathcal{A}_j \text{ is a } \sigma\text{-algebra and } \mathcal{G} \subset \mathcal{A}_j\}$$

is the σ-algebra ***generated by*** *$\mathcal{G}$.*

Remark 2.2 *The fact that $\sigma(\mathcal{G})$ is a σ-algebra is an immediate consequence of Lemma 1.11.*

Example 2.3 *Let $X = \{1, 2, 3, 4\}$ and $\mathcal{G} = \{\{1\}, \{2\}\}$. Then*

$$\sigma(\mathcal{G}) = \{\emptyset, X, \{1\}, \{2\}, \{1, 2\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}.$$

Note that $\sigma(\mathcal{G})$ is the *smallest* σ-algebra containing $\mathcal{G}$, in the sense that for any σ-algebra $\mathcal{A}$ of X, $\mathcal{G} \subset \mathcal{A}$ implies $\sigma(\mathcal{G}) \subset \mathcal{A}$. More generally, it follows immediately from Definition 2.1 that

Lemma 2.4 *If $\mathcal{G}_1 \subset \mathcal{G}_2$ then $\sigma(\mathcal{G}_1) \subset \sigma(\mathcal{G}_2)$.*

As a consequence, we have

Remark 2.5 *$\sigma(\sigma(\mathcal{G})) = \sigma(\mathcal{G})$.*

Before defining Borel σ-algebras and examining their main properties, we will recall a fact encountered in the Topology course, along with its proof. We recall that if (X, d) is a metric space, we say that $U \subset X$ is **open** if for any $x \in U$, there exists an $\varepsilon > 0$ such that for all $y \in X$ with $d(x, y) < \varepsilon$, we have $y \in U$. In other words: every $x \in U$ has a neighbourhood in U.

Proposition 2.6 *Every open subset U of $\mathbb{R}$ is a countable union of disjoint open intervals.*

Proof. We provide the main ingredients of the proof.

i) Let $U_1 = U$ and let $L_1 := \{x_{1,1}, x_{1,2}, x_{1,3}, \ldots\}$ be a list of elements in $U_1 \cap \mathbb{Q}$, where, as usual, $\mathbb{Q}$ is the set of rational numbers.

ii) Let $V_1 \subseteq U$ be the maximal open interval around $x_{1,1}$. Note that V_1 exists since U is open.

iii) Let $U_2 = U_1 \setminus V_1$. Remove all points from the list L_1, which are not in U_2. We will lose some points: for instance, $x_{1,1}$.

   Relabel all these points as $x_{2,1}, x_{2,2}, x_{2,3}, \ldots$ and call the new obtained list L_2.

Repeating steps ii) and iii) gives a procedure to obtain a countable collection of disjoint, maximal intervals $\{V_n\}_{n\geq 1}$ in open sets.

After constructing $\{V_n\}_{n\geq 1}$ as above, we have two possibilities:

a) $\{V_n\}_{n\geq 1}$ cover U, in which case the proof is complete.

b) There exists a $y \in U$ so that $y \notin V_n$ for all n. In this case, we recall that U is open, so $y \in W$, for some open maximal interval W around $x_{1,n}$, for a minimal n (since $\mathbb{Q}$ is dense in $\mathbb{R}$, there must be some $x_{1,n}$ with such property).

   But then W must be the same as V_j for some j, which means that y must belong to some disjoint union of maximal intervals. □

Proposition 2.6 will allow us to characterize the Borel σ-algebra on $\mathbb{R}$. We start as follows:

Definition 2.7 *Let $\mathcal{G}$ be the collection of* open *sets of $X = \mathbb{R}$. With the same notation as in Definition 2.1, we call $\sigma(\mathcal{G})$ the **Borel σ-algebra** on X and write $\mathcal{B}(X) := \sigma(\mathcal{G})$. The elements of $\mathcal{B}(X)$ are called **Borel sets**.*

We have now defined the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ to be the σ-algebra generated by the collection of open subsets of $\mathbb{R}$. This collection, however, is not the only generator of $\mathcal{B}(\mathbb{R})$.

Proposition 2.8 *The Borel σ-algebra on $\mathbb{R}$ is generated by each of the following:*

(a) $C_1 = \{(a, b) \ : \ a, b \in \mathbb{R}\}$;

(b) $C_2 = \{[a, b] \ : \ a, b \in \mathbb{R}\}$;

(c) $C_3 = \{(a, b] \ : \ a, b \in \mathbb{R}\}$;

(d) $C_4 = \{(a, \infty) \ : \ a \in \mathbb{R}\}$.

i.e. we have $\mathcal{B}(\mathbb{R}) = \sigma(C_1) = \sigma(C_2) = \sigma(C_3) = \sigma(C_4)$.

Proof. We will only prove (a) and (b), as (c) and (d) are similar.

(a) We wish to prove that $\sigma(C_1) = \mathcal{B}(\mathbb{R})$. First, we need to show that $\sigma(C_1) \subset \mathcal{B}(\mathbb{R})$. Let $\mathcal{G}$ be the collection of open subsets of $\mathbb{R}$. By definition, $\sigma(\mathcal{G})$ is the Borel σ-algebra on $\mathbb{R}$. Since every element of C_1 is open, we have $C_1 \subset \mathcal{G}$. It now follows from Remark 2.5 that $\sigma(C_1) \subset \sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$. The inclusion $\mathcal{B}(\mathbb{R}) \subset \sigma(C_1)$ requires a little bit more work. By Proposition 2.6, $\mathcal{G}$ contains precisely the countable unions of disjoint open intervals. Let $a \in \mathbb{R}$ and note that $(a, \infty) = \bigcup_{n\in\mathbb{N}}(a, a+n)$. Hence, (a, ∞) is a countable union of intervals in C_1, which implies that $(a, \infty) \in \sigma(C_1)$. Similarly, we have $(-\infty, a) \in \sigma(C_1)$. Hence all countable unions of open intervals are contained in $\sigma(C_1)$, which implies $\mathcal{G} \subset \sigma(C_1)$, and hence $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{G}) \subset \sigma(\sigma(C_1)) = \sigma(C_1)$. Indeed $\mathcal{B}(\mathbb{R}) = \sigma(C_1)$.

(b) Again, let $\mathcal{G}$ be the collection open subsets of $\mathbb{R}$. For the first inclusion, let $[a,b] \in C_2$. Then

$$[a,b] = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) \in \sigma(\mathcal{G}).$$

Hence we have $C_2 \subset \sigma(\mathcal{G})$, which implies that $\sigma(C_2) \subset \sigma(\sigma(\mathcal{G})) = \sigma(\mathcal{G}) = \mathcal{B}(\mathbb{R})$. For the second inclusion, let $(a,b) \in C_1$ with C_1 as in (a). Choose some $n_0 \geq 2/(b-a)$ and note that

$$(a,b) = \bigcup_{n=n_0}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] \in \sigma(C_2).$$

Hence we find $\sigma(C_1) \subset \sigma(C_2)$. Since we've seen in part (a) that $\mathcal{B}(\mathbb{R}) = \sigma(C_1)$, it follows that $\mathcal{B}(\mathbb{R}) \subset \sigma(C_2)$. We can conclude that $\mathcal{B}(\mathbb{R}) = \sigma(C_2)$.

$\square$

3 A first introduction to the concept of *measure*

Recall that our first goal is to understand the *Lebesgue measure* on $\mathbb{R}$, which is defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$. To do so, we first define the concept of a *measure* on a general σ-algebra and study their basic properties.

Definition 3.1 *Let X be a set and let $\mathcal{A}$ be a σ-algebra on X. A* ***measure*** *on $(X, \mathcal{A})$ is a function $\mu : \mathcal{A} \to [0, \infty]$ such that*

i) $\mu(\emptyset) = 0$;

ii) If $A_i \in \mathcal{A},\ i \geq 1$ are pairwise (p.w.) disjoint (that is, $A_i \cap A_j = \emptyset$ for all $i \neq j$), then

$$\mu(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \mu(A_j).$$

Here $x + \infty = \infty + x = \infty$ for all $x \in [0, \infty]$.

Property ii) in the definition above is referred to as **countable additivity**. If we only have

$$\mu(\cup_{j=1}^{N} A_j) = \sum_{j=1}^{N} \mu(A_j)$$

for some finite N then we say that we have **finite additivity**.

Now that we have introduced the definition of a measure, we will specify some related terminology that we will often use in the remainder of the lecture notes.

Definition 3.2 *Let X be a set, let $\mathcal{A}$ be a σ-algebra on X and let μ be a measure on $(X, \mathcal{A})$. The tuple $(X, \mathcal{A})$ is referred to as a* ***measurable space****, while the triple $(X, \mathcal{A}, \mu)$ is referred to as a* ***measure space****.*

Example 3.3 *Let $(X, \mathcal{A})$ be a measurable space. Define $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu(A)$ equals the number of elements in $A \in \mathcal{A}$. Then μ is called the* ***counting measure*** *and satisfies countable additivity (that is, property ii) in Definition 3.1).*

Example 3.4 *Let $\mathcal{A} := \mathcal{P}(\mathbb{R})$ be the σ-algebra on $\mathbb{R}$ containing all subsets of $\mathbb{R}$. Let $x_1, x_2, \ldots \in \mathbb{R}$ and let $a_1, a_2, \ldots \geq 0$. Then one can easily check that the set function $\mu : \mathcal{A} \to [0, \infty]$ given by*

$$\mu(A) = \sum_{i : x_i \in A} a_i, \quad A \in \mathcal{A}$$

defines a measure on $(X, \mathcal{A})$.

Example 3.5 *Let $(X, \mathcal{A})$ be a measurable space and for $A \in \mathcal{A}$, set*

$$\delta_x(A) = 1,\ x \in A; \quad \delta_x(A) = 0,\ x \notin A.$$

Then δ_x is a measure. It is referred to as the ***point mass*** *or* ***Dirac measure*** *at x.*

3.1 Properties of measures

Proposition 3.6 *Let $(X, \mathcal{A})$ be a measurable space and let μ be a measure on $(X, \mathcal{A})$. Then*

i) If $A, B \in \mathcal{A}$ such that $A \subset B$, then $\mu(A) \le \mu(B)$.

ii) $A_i \in \mathcal{A}$ and $A := \cup_{i=1}^{\infty} A_i$, then $\mu(A) \le \sum_{i=1}^{\infty} \mu(A_i)$.

Proof.

i) Since $A \subseteq B$, we can write B as the union $B = A \cup (B \setminus A)$ of disjoint sets. Here we note that since $A, B \in \mathcal{A}$, we have $B \setminus A \in \mathcal{A}$. Hence, by property ii) in Definition 3.1, we find

$$\mu(B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A).$$

Since we have $\mu(B \setminus A) \ge 0$, this implies that $\mu(B) = \mu(A) + \mu(B \setminus A) \ge \mu(A)$.

ii) We cannot yet apply Definition 3.1 to A_i since they are not necessarily p.w. disjoint sets. To apply countable additivity we construct a sequence of p.w. disjoint sets as follows:

$$B_1 := A_1,\ B_2 := A_2 \setminus A_1,\ B_3 := A_3 \setminus (A_1 \cup A_2),\ \ldots,\ B_n := A_n \setminus (\cup_{j=0}^{n-1} A_j).$$

By construction, we have $B_n \subseteq A_n$ for all n and

$$\cup_{j=1}^{\infty} B_j = \cup_{j=1}^{\infty} A_j. \tag{1}$$

By part i), we have $\mu(B_n) \le \mu(A_n)$. Using Definition 3.1 applied to the sets B_n, we find

$$\mu(A) = \mu(\cup_{j=0}^{\infty} A_j) = \mu(\cup_{j=0}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j) \le \sum_{j=1}^{\infty} \mu(A_j).$$

This finishes the proof. □

To get into more basic properties of measures, we recall the following terminology.

Definition 3.7

*i) We say that $(A_i)_{i \ge 1}$ is a **monotone increasing** sequence of sets if*

$$A_1 \subset A_2 \subset \cdots A_i \subset \cdots$$

*If in addition, $\cup_{i=0}^{\infty} A_i = A$ then we say that $(A_i)_{i \ge 1}$ **monotonically increases to** A and write $A_i \uparrow A$.*

*ii) We say that $(A_i)_{i \ge 1}$ is a **monotone decreasing** sequence of sets if*

$$A_1 \supset A_2 \supset \cdots A_i \supset \cdots$$

*If in addition, $\cap_{i=0}^{\infty} A_i = A$, then we say that $(A_i)_{i \ge 1}$ **monotonically decreases to** A and write $A_i \downarrow A$.*

Proposition 3.8 *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(A_i)_{i\geq 1}$ be a sequence of sets in $\mathcal{A}$. Then*

i) If $A_i \uparrow A$, then $\mu(A) = \lim_{i\to\infty} \mu(A_i)$.

ii) If $A_i \downarrow A$ and $\mu(A_i) < \infty$ for some i, then $\mu(A) = \lim_{i\to\infty} \mu(A_i)$.

Proof.

i) Let B_n constructed as in the proof of Proposition 3.6. Then B_n are p.w. disjoint and $\cup_{j=1}^{\infty} B_j = \cup_{j=1}^{\infty} A_j$. Since by assumption $A = \cup_{j=1}^{\infty} A_j$, we have

$$\mu(A) = \mu(\cup_{j=1}^{\infty} A_j) = \mu(\cup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j).$$

Recall that $\sum_{j=1}^{\infty} a_j = \lim_{n\to\infty} \sum_{j=1}^{n} a_j$, for a_j real and non-negative. So

$$\mu(A) = \sum_{j=1}^{\infty} \mu(B_j) = \lim_{n\to\infty} \sum_{j=1}^{n} \mu(B_j) = \lim_{n\to\infty} \mu(\cup_{j=1}^{n} B_j).$$

Since we also have $\cup_{j=1}^{n} A_j = \cup_{j=1}^{n} B_j$ for any finite n, we find that

$$\mu(A) = \lim_{n\to\infty} \mu(\cup_{j=1}^{n} B_j) = \lim_{n\to\infty} \mu(\cup_{j=1}^{n} A_j).$$

Since $A_1 \subset \cdots \subset A_n$, we have $\cup_{j=1}^{n} A_j = A_n$. Hence $\mu(A) = \lim_{n\to\infty} \mu(A_n)$.

ii) Define new sets

$$B_0 = A_1 \setminus A, \qquad B_j := A_1 \setminus A_j \text{ for } j \geq 1.$$

Since $A_1 \supset A_2 \supset \cdots \supset \cap_{j=1}^{\infty} A_j$, we have

$$B_j \uparrow B_0.$$

Thus we can derive from Proposition 3.8 i) that

$$\mu(A_1) - \mu(A) = \mu(B_0) = \lim_{j\to\infty} \mu(B_j) = \lim_{j\to\infty} \mu(A_1 \setminus A_j) = \mu(A_1) - \lim_{j\to\infty} \mu(A_j)$$

Hence $\mu(A) = \lim_{j\to\infty} \mu(A_j)$. □

Remark 3.9 *In Proposition 3.8 ii) we require $\mu(A_j) < \infty$ for some $j \geq 1$. To see what goes wrong if this is not true, consider the following counter examples:*

1) $X = \mathbb{N}$, μ is the counting measure, $A_i = \{i, i+1, i+2, \dots\}$, $1 \leq i < \infty$. Then $A_i \downarrow \emptyset$ and $\mu(A_i) = \infty$ for all i. **However** *$\mu(\cap_i A_i) = \mu(\emptyset) = 0$.*

2) Let $A_i = (i, \infty)$ for $i \in \mathbb{R}$, $\mu(A_i) = \infty$ and $\mu(a, b) = |b - a|$. Again $\cap_i A_i = \emptyset$ so $\mu(\cap_i A_i) \neq \lim_{i\to\infty} \mu(A_i)$.

Exercise 3.10 *Let $(X, \mathcal{A}, \mu)$ be some measure space and let $A, B \in \mathcal{A}$. Show that*

$$A \cup B = A \cup (B \setminus (A \cap B)).$$

Use this to show that

$$\mu(A) + \mu(B) = \mu(A \cup B) + \mu(A \cap B).$$

4 Construction of measures (I)

We will proceed to define the Lebesgue measure on $\mathbb{R}$. Common notations for this measure are *Leb*, λ or m. We will choose the latter notation. The main challenges/ideas are:

1) We want $m((a,b)) = b-a$ for open intervals (a,b). Given this, we can extend to open sets G in $\mathbb{R}$. Recall that if G is an open set of $\mathbb{R}$, we can write $G = \cup_j (a_j, b_j)$, i.e. G is a countable union of open intervals. Hence, using the definition of a **measure**, we must have

$$m(G) = \sum_{j\geq 1} (b_j - a_j).$$

But: How can we ensure that m is well-defined as a measure on $\mathbb{R}$?

2) Define

$$m(E) := \inf\{m(G) \ : \ G \text{ open}, \ E \subset G\}$$

for any $E \subset \mathbb{R}$.
Problem: m is not well-defined as a measure on the σ-algebra $\mathcal{P}(\mathbb{R})$, the power set of $\mathbb{R}$. We will prove this further on.
Solution: we will need to define m on some strictly smaller σ-algebra.

4.1 Outer measures (first step in constructing m)

Definition 4.1 *Let X be a set. An* **outer measure** *is a set function $\mu^* : \mathcal{P}(X) \mapsto [0,\infty]$ such that*

(a) $\mu^*(\emptyset) = 0$,

(b) $A \subset B \Rightarrow \mu^*(A) \leq \mu^*(B)$,

*(c) If A_i, $i \geq 1$, are subsets of X, then $\mu^*(\cup_i A_i) \leq \sum_i \mu^*(A_i)$. This is called σ-**subadditivity**.*

A is a **nullset** *w.r.t. μ^* if $\mu^*(A) = 0$.*

4.1.1 Generating outer measures

Proposition 4.2 *Let $\mathcal{D}$ be a collection of subsets of X such that $\emptyset, X \in \mathcal{D}$. Let $\ell : \mathcal{D} \to [0,\infty]$ be such that $\ell(\emptyset) = 0$. Define*

$$\mu^*(E) := \inf\left\{\sum_{j=1}^{\infty} \ell(A_j) \ : \ A_j \in \mathcal{D} \textit{ for each } j \textit{ and } E \subset \cup_j A_j\right\}. \tag{2}$$

Then μ^* is an outer measure.

Proof. We need to verify (a)–(c) in Definition 4.1. First, it is clear that μ^* defined in (2) satisfies (a) and (b) of Definition 4.1. We will verify (c):
Let A_j, $j \geq 1$, be sets in $\mathcal{D}$. Let $\varepsilon > 0$ be arbitrary. For every $j \geq 1$ we can find sets $C_{j,1}, C_{j,2}, C_{j,3}, \dots \in \mathcal{D}$ such that $A_j \subset \cup_{i=1}^{\infty} C_{j,i}$ and

$$\mu^*(A_j) \geq \sum_{i\geq 1} \ell(C_{j,i}) - \frac{\varepsilon}{2^j}. \tag{*}$$

Also

$$E \subset \cup_{j\geq 1} A_j \subset \cup_{j\geq 1} \cup_{i\geq 1} C_{j,i} \tag{**}$$

by the construction of the $C_{j,i}$'s. By the fact that (*) holds for every j, by (**) and the definition of μ^* in (2), we find

$$\begin{aligned}
\mu^*(E) \quad &\leq_{\text{ by (**)}} \quad \sum_{j,i=1}^{\infty} \ell(C_{j,i}) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} \ell(C_{j,i}) \right) \\
&\leq_{\text{ by (*)}} \quad \sum_{j=1}^{\infty} \mu^*(A_j) + \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} \\
&\leq \qquad \sum_{j=1}^{\infty} \mu^*(A_j) + \varepsilon.
\end{aligned}$$

Since ε is arbitrary, we obtain that $\mu^*(E) \leq \sum_{j=1}^{\infty} \mu^*(A_j)$, as required. □

We will apply Proposition 4.2 to our current construction for $m = Leb$. Let $\mathcal{D}$ be the collection of all intervals $I = (a, b]$. Set

$$\ell(I) := b - a \text{ for all } I = (a, b].$$

Define μ^* as in (2), i.e.

$$\mu^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \ell(A_j) : A_j \in \mathcal{D} \text{ for each } j \text{ and } E \subset \cup_j A_j \right\}.$$

By Proposition 4.2, μ^* is an outer measure on $\mathcal{D}$. But as we discussed in Challenge 2), μ^* is not a measure on $\mathcal{P}(\mathbb{R})$. More precisely, as in [1, Theorem 4.145], we have

Proposition 4.3 *Let $\mathcal{D}$ be the collection of intervals $I = (a, b]$. Define*

$$\ell(I) = b - a \textit{ for all } I = (a, b] \in \mathcal{D}.$$

Then μ^* from Proposition 4.2 is not a measure on $\mathcal{P}(\mathbb{R})$.

Proof. Suppose that μ^* is a measure. We seek a contradiction. Define the equivalence relation $x \sim y$ if $x - y \in \mathbb{Q}$ (the rationals) on $[0, 1]$. For each equivalence class of $\sim$, pick one element of this class. This is OK by the Axiom of Choice. Call the collection of all such points (picked by the Axiom of Choice) A. Note that for all distinct $q_1, q_2 \in \mathbb{Q}$, the sets $A + q_1$ and $A + q_2$ are disjoint. Now

$$\mu^*(A + q) = \mu^*(A) \qquad \forall\, q \in \mathbb{Q}. \tag{*}$$

To see (*) note that

$$\begin{cases} A + q = \{a + q : a \in A\} \\ \ell((a + q, b + q]) = b + q - a - q = b - a \end{cases}$$

and that

$$\{\sum_j \ell(A_j) : A_j \in \mathcal{D}, A \subset \cup_j A_j\} = \{\sum_j \ell((a_j, b_j]) : A \subset \cup_j (a_j, b_j]\}$$
$$= \{\sum_j \ell((a_j + q, b_j + q]) : A \subset \cup_j (a_j, b_j]\} = \{\sum_j \ell((a_j + q, b_j + q]) : A + q \subset \cup_j (a_j + q, b_j + q]\}$$
$$= \{\sum_j \ell((a_j, b_j]) : A + q \subset \cup_j (a_j, b_j]\} = \{\sum_j \ell(A_j) : A_j \in \mathcal{D}, A + q \subset \cup_j A_j\}.$$

Taking the infimum yields (*).
We have

$$[0, 1] \ \subset \bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A + q).$$

where the union is over all rationals in $[-1, 1]$ (check this!). Hence

$$1 = \mu^*([0, 1]) \leq \sum_{q \in [-1,1] \cap \mathbb{Q}} \mu^*(A + q) = \sum_{q \in [-1,1] \cap \mathbb{Q}} \mu^*(A). \tag{**}$$

Using (*) and (**), it follows that

$$\mu^*(A) > 0. \tag{***}$$

Using (***) and the fact that $[-1, 1] \cap \mathbb{Q}$ has infinite cardinality, we find that

$$\mu^*(\bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A + q)) = \sum_{q \in [-1,1] \cap \mathbb{Q}} \mu^*(A) = \infty \cdot \mu^*(A) = \infty.$$

But

$$\bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A + q) \subset [-1, 2]$$

implies that $\mu^*(\bigcup_{q \in [-1,1] \cap \mathbb{Q}} (A + q)) \leq 3$, so a contradiction. $\square$

5 Construction of measures (II)

In this section we will construct Lebesgue measure on a σ-algebra that is strictly smaller than $\mathcal{P}(\mathbb{R})$.

Definition 5.1 *Let μ^* be an outer measure. A set $A \subset X(=\mathbb{R})$ is μ^*-**measurable** if*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \tag{3}$$

for all $E \subset X$.

Remark 5.2 *(on Definition 5.1): By Definition 4.1 we have*

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c)$$

for all $A, E \subset X$. Hence, to check whether some set $A \subset X$ is μ^*-measurable, we just need to verify that

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \qquad \forall E \subset X. \tag{4}$$

Note that if $\mu^*(E) = \infty$, then (4) holds automatically. Hence we only have to check the sets $E \subset X$ with $\mu^*(E) < \infty$.

Theorem 5.3 *If μ^* is an outer measure, then the collection $\mathcal{A}$ of μ^*-measurable sets is a σ-algebra. If $m := \mu^*|_{\mathcal{A}}$ is the restriction of μ^* to the μ^*-measurable sets, then m is a measure on $(X, \mathcal{A})$. We call it* **Lebesgue measure** *on X. Moreover, $\mathcal{A}$ contains all nullsets.*

Proof.

- **Step 1: $\mathcal{A}$ is an algebra.**
  We have $\mu^*(E \cap \emptyset) + \mu^*(E \cap X) = \mu^*(\emptyset) + \mu^*(E) = 0 + \mu^*(E) = \mu^*(E)$ for all $E \subset X$, so $\emptyset \in \mathcal{A}$. If $A \in \mathcal{A}$ then also $A^c \in \mathcal{A}$; this is immediate from (3). Let $A, B \in \mathcal{A}$ and $E \subset X$. Then

$$\begin{aligned} \mu^*(E) &=_{\text{by (3)}} && \mu^*(E \cap A) + \mu^*(E \cap A^c) \\ &=_{\text{by (3) with } E \cap A \text{ and } E \cap A^c} && \mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \\ & && \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A^c \cap B^c). \end{aligned}$$

  But $A \cup B \subset (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$. Therefore

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A \cap B^c) + \mu^*(E \cap A^c \cap B) \geq \mu^*(E \cap (A \cup B))$$

  and

$$\begin{aligned} \mu^*(E) &\geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap \underbrace{(A^c \cap B^c)}_{=(A \cup B)^c}) \\ &= \mu^*(E \cap (A \cup B)) + \mu^*(E \cap ((A \cup B)^c)). \end{aligned}$$

  So $A \cup B \in \mathcal{A}$. Indeed $\mathcal{A}$ is an algebra.

- **Step 2: $\mathcal{A}$ is a σ-algebra.**
  This step is longer than Step 1, but it is the same idea (for details see [1, Proof of Theorem 4.6]). In the process of carrying out this step, we show that

$$\mu^*(E \cap \bigcup_{j=1}^{\infty} A_j) = \sum_{j\geq 1} \mu(E \cap A_j), \quad \forall (A_j)_{j\geq 1} \subset \mathcal{A} \text{ with } A_j \text{ p.w. disjoint and for } \ E \subset X. \tag{5}$$

- **Step 3: $\mu^*|_{\mathcal{A}}$ is a measure.**
  μ^* is an outer measure on σ-algebra $\mathcal{A}$, so using (5), we find

$$\mu^*(E) = \sum_{j\geq 1} \mu(E \cap A_j) + \mu^*(E \cap B^c) \qquad \text{for } B = \bigcup_{j=1}^{\infty} A_j.$$

  We need to show that μ^* is countably additive. Take $B = E$. Then

$$\mu^*\Big(\bigcup_{j=1}^{\infty} A_j\Big) = \sum_{j\geq 1} \mu^*(E \cap A_j) + \mu^*(\emptyset) \qquad \text{for } B = \bigcup_{j=1}^{\infty} A_j,$$

  completing Step 3.

- **Step 4: All μ^*-nullsets are in $\mathcal{A}$.**
  Let A be a μ^*-nullset, so $\mu^*(A) = 0$. Note that $\forall\ E \subset X$

$$\mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*(E \cap A^c) \leq \mu^*(E),$$

  so (4) holds, as required. □

Importance of Theorem 5.3: In the previous lecture we have considered

$$\mu^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \ell(A_j) : A_j \in \mathcal{D}\ \forall\, j \text{ and } E \subset \cup_j A_j \right\}, \tag{6}$$

where $\mathcal{D}$ is any collection of subsets of X such that $\mathcal{D} \supset \{\emptyset, X\}$. We have seen that μ^* defined by (6) on $\mathcal{D} = \{I \subset \mathbb{R} : I = (a,b]\}$ is an outer measure, but **not** a measure on $\mathcal{P}(\mathbb{R})$. However, by Theorem 5.3, $Leb = m = \mu^*|_{\mathcal{A}}$ with $\mathcal{A}$ as in Theorem 5.3 is a measure and $m(I) = b - a$.

5.1 Classical examples of null sets

Example 5.4 *Let $m := Leb$. Then $m(\{x\}) = 0$, $\forall\, x \in \mathbb{R}$. This is because* $\{x\} = \cap_{n=1}^{\infty} \underbrace{(x - \frac{1}{n}, x]}_{\in \mathcal{B}(\mathbb{R})}$
So $\{x\}$ is the countable intersection of sets in $\mathcal{B}(\mathbb{R})$, so $\{x\}$ is a Borel set. Next note that

$$m(\{x\}) = \lim_{n\to\infty} m((x - \frac{1}{n}, x]) = \lim_{n\to\infty} \big(x - (x - \frac{1}{n})\big) = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Also $\forall a, b \in \mathbb{R},\ a < b : m([a,b]) = m((a,b]) + m(\{a\}) = b - a$. So we can define Leb on closed intervals $[a,b]$: $m([a,b]) = b - a$.

Conclusion: Countable sets in $\mathbb{R}$ (as countable unions of nullsets) are in $\mathcal{B}(\mathbb{R})$ and Leb(countable set) $= 0$.

Example 5.5 ***(Middle Third Cantor set)*** *Start with* $F_0 := [0,1]$.

Remove the middle third: $F_1 = F_0 \setminus (\frac{1}{3}, \frac{2}{3}) = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$.

Continue this way: $F_2 = F_1 \setminus ((\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9})) = [0\frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$*, etc.*

Then $\mu(F_1) = 2\frac{1}{3} = \frac{2}{3}$, $m(F_2) = 4\frac{1}{9} = (\frac{2}{3})^2$ *and* $m(F_n) = (\frac{2}{3})^n$.

Now the ***Middle Third Cantor set*** *is defined as* $C := \cap_n F_n$*. We have* $m(C) = 0$.

Example 5.6 ***(Cantor-Lebesgue function)*** *Let*

$$f_0(x) = \begin{cases} \frac{1}{2}, & x \in (\frac{1}{3}, \frac{2}{3}), \\ \frac{1}{4}, & x \in (\frac{1}{9}, \frac{2}{9}), \\ \frac{3}{4}, & x \in (\frac{7}{9}, \frac{8}{9}), \\ \frac{1}{2} + \frac{1}{4} + \frac{3}{4}, & x \in \text{ next removed interval}, \\ \dots \end{cases}$$

Define

$$f(x) = \begin{cases} \inf\{f_0(y) : y > x\}, & x < 1, \\ 1, & x = 1. \end{cases}$$

So $f \equiv f_0$ *on* $[0,1] \setminus C$*. Although* f *increases only on the Cantor set (which is of measure zero),* f *is actually continuous.*

6 Construction of measures (III)

We have seen that one of Caratheodory's theorems, namely Theorem 5.3, allowed us to define the Lebesgue measure (on a σ-algebra smaller than $\mathcal{P}(\mathbb{R})$ but larger than $\mathcal{D} = \{I = [a,b)\}$). In Lecture 5 we have seen that $Leb = m = \mu^*|_{\mathcal{A}}$ with $\mathcal{A}$ the collection μ^*-measurable sets as in Definition 5.1 is a measure and $m(I) = b - a$ for all $I = [a,b)$.

In what follows we are interested to see whether there exists a procedure that allows to lift or to extend a measure from an algebra to a larger σ-algebra.

Definition 6.1 *Let* X *be a set and let* $\mathcal{A}_0$ *be an algebra on* X*. A* ***pre-measure*** *on* $(X, \mathcal{A}_0)$ *is a set function* $\ell : \mathcal{A}_0 \to [0, \infty]$ *such that*

i) $\ell(\emptyset) = 0$;

ii) If $A_j \in \mathcal{A}_0,\ j \geq 1$ *are pairwise (p.w.) disjoint such that* $\cup_{j=1}^{\infty} A_j \in \mathcal{A}_0$*, we have*

$$\ell(\cup_{j=1}^{\infty} A_j) = \sum_{j=1}^{\infty} \ell(A_j).$$

Remark 6.2 *The definition of a pre-measure is exactly the same as that of a measure, except for the fact that measures are defined on* σ*-algebras, whereas pre-measures are defined on less powerful collections of subsets such as algebras.*

Example 6.3 *We recall our previous Example 1.10. Given $X = (0, 1]$, we considered*

$$\mathcal{A}_0 = \{ \text{ finite unions of semi-open subintervals of } X\} \cup \emptyset.$$

Recall that $\mathcal{A}_0$ is an algebra but it is not a σ-algebra. We argue directly (without the use of Theorem 5.3) that defining $m(I) = |I|$ for any semi-open interval and

$$m(A) = \sum_{j=1}^{n} m(I_j), \text{ for any } A = \cup_{j=1}^{n} I_j \in \mathcal{A}_0 \text{ with } I_j \text{ disjoint sets,} \tag{7}$$

will give us a pre-measure on the algebra $\mathcal{A}_0$. To show that this is the case, we just need to show that m is countably additive. For $k \geq 1$ define $A_k \in \mathcal{A}_0$ such that the A_k's are pairwise disjoint and $A := \cup_{k=1}^{\infty} A_k$ is in $\mathcal{A}_0$. Then $A = \cup_{j=1}^{n} I_j$ and $A_k = \cup_{i=1}^{m_k} J_{k_i}$ are disjoint unions of semi-open subintervals of X and hence of sets in $\mathcal{A}_0$. Using (7),

$$m(A) = \sum_{j=1}^{n} m(I_j) = \sum_{j=1}^{n} \sum_{k=1}^{\infty} \sum_{i=1}^{m_k} m(I_j \cap J_{k_i}) = \sum_{k=1}^{\infty} \sum_{i=1}^{m_k} m(J_{k_i}) = \sum_{k=1}^{\infty} m(A_k),$$

as required.

The third equality in the previous diplayed equation can be justified as follows. Since $\mathcal{A}_0$ is an algebra (so, closed under finite intersection) and thus given that J_i are such that $A = \cup_{i=1}^{m} J_i$ (since A does not have a unique representation as a finite union of elements of $\mathcal{A}_0$),

$$\sum_{j=1}^{n} m(I_j) = \sum_{j=1}^{n} \sum_{i=1}^{m} m(I_j \cap J_i) = \sum_{i=1}^{m} m(J_i).$$

as required.

6.1 Caratheodory's theorem: improved version of Theorem 5.3

Generalizing Example 6.3 we obatin the first part of what is known as **Caratheodory's theorem**. Prior to the statement we introduce a distintion between finite and infinite, but σ-finite measure spaces that will be used throughout the rest of the course.

Definition 6.4 *Let (X, A, μ) be a measure space. A measure μ is* ***finite*** *if $\mu(X) < \infty$.*
A measure μ is ***σ-finite*** *if there exist sets $E_i \in \mathcal{A}$ for $i = 1, 2, \ldots$ such that $\mu(E_i) < \infty$ for each i and $X = \cup_{i=1}^{\infty} E_i$. This is more general because it does still allow $\mu(X) = \infty$.*
If μ is a finite measure, then (X, A, μ) is called a ***finite measure space****, and similarly, if μ is a σ-finite measure, then (X, A, μ) is called a* ***σ-finite measure space.***

Theorem 6.5 *Suppose that $\mathcal{A}_0$ is an algebra and let $\ell : \mathcal{A}_0 \to [0, \infty]$ be a pre-measure on $\mathcal{A}_0$. Define for any $E \subset X$,*

$$\mu^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \ell(A_j) : A_j \in \mathcal{A}_0, \ E \subset \bigcup_{j=1}^{\infty} A_j \right\}, \tag{8}$$

then

a) μ^* *is an outer measure;*

b) $\mu^*(A) = \ell(A) \quad \forall\, A \in \mathcal{A}_0$*;*

c) Every set in $\mathcal{A}_0$ *is* μ^**-measurable;*

d) If ℓ *is* σ*-finite then there exists an extension of* ℓ *to a measure on* $\sigma(\mathcal{A}_0)$*.*

Proof.

(a) This is precisely Proposition 4.2.

(b) Let $E \in \mathcal{A}_0$. We want: $\mu^*(E) = \ell(E)$ with μ^* as in (8).
" $\leq$ ": In (8), take $A_1 = E$ and $A_2, A_3, \cdots = \emptyset$. Clearly, $\mu^*(E) \leq \ell(E)$.

" $\geq$ ": If $E \subset \cup_{i=1}^{\infty} A_i$ with $A_i \in \mathcal{A}_0$, let $B_n = E \cap (A_n \setminus (\cup_{i=1}^{n-1} A_i))$. Then $B_n \in \mathcal{A}_0$ for each n, B_n are p.w. disjoint, and $\cup_n B_n \in E$. Therefore,

$$\ell(E) = \sum_{n=1}^{\infty} \ell(B_n) \leq \sum_{i=1}^{\infty} \ell(A_i).$$

For (c) and (d) see [1, Proof of Theorem 4.16]. $\square$

There is one more item of importance in the previous theorem, which can enlarge the information given by item d) by giving sufficient conditions for uniqueness.

Theorem 6.6 *Assume the setup and notation of Theorem 6.5. If* ℓ *is* σ*-finite, then there exists a unique extension of* ℓ *to a measure on* $\sigma(\mathcal{A}_0)$*.*

We will not discuss the proof here, but interested students can read it in [1, Proof of Theorem 4.16].
Theorems 6.5 and 6.6 allow us to extend a measure from an algebra to a σ-algebra, which is unique under the condition of σ-finiteness. In fact, this extension procedure from Theorems 6.5 and 6.6 even holds for more general collections of subsets containing less structure than algebras.

Definition 6.7 *A collection* $\mathcal{H}$ *of subsets of* X *is called a* ***semiring*** *if*

- $\emptyset \in \mathcal{H}$*;*
- $A, B \in \mathcal{H}$ *implies that* $A \cap B \in \mathcal{H}$*;*
- *For all* $A, B \in \mathcal{H}$ *with* $A \subset B$ *there exist pairwise disjoints sets* $C_1, C_2, \ldots C_n \in \mathcal{H}$*, such that* $B \setminus A = \cup_{k=1}^{n} C_k$*.*

Example 6.8 *The class of bounded intervals in* $\mathbb{R}$ *is a semiring. The class of subintervals of* $(0, 1]$ *is a semiring as well. Every algebra (and hence every* σ*-algebra) is a semiring.*

For these less structured semirings we state the following extension theorem:

Theorem 6.9 *Let* $\mathcal{H}$ *be a semiring on* X*. Suppose that* $\mu : \mathcal{H} \to [0, \infty]$ *is a set function with* $\mu(\emptyset) = 0$*. Assume that* μ *is finitely additive and* σ*-subadditive. Then* μ *extends to a measure on the generated* σ*-algebra* $\sigma(\mathcal{H})$*. If* μ *is* σ*-finite then this extension is unique.*

7 Measurable functions

Throughout we let $(\mathcal{A}, X)$ be a measurable space. We call the elements of $\mathcal{A}$ **measurable** sets.

Definition 7.1 *A function $f : X \to \mathbb{R}$ is **($\mathcal{A}$-)measurable** if $\{x : f(x) > a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$.*
*A complex-valued function is **measurable** if both its real and imaginary parts are measurable.*

More generally:

Definition 7.2 *Let $(\mathcal{A}, X)$ and $(\mathcal{F}, Y)$ be measurable spaces. A function $f : X \to Y$ is called **($\mathcal{A}$-$\mathcal{F}$-)measurable** if $f^{-1}(F) := \{x \in X : f(x) \in F\} \in \mathcal{A}$ for all $F \in \mathcal{F}$.*

In this course, we will mainly work with real-valued measurable functions.

7.1 Some examples.

Example 7.3 *Suppose f is real-valued and $f \equiv c$ for some real constant c (that is, f is identically constant). Then the set $\{x : f(x) > a\}$ equals either $\emptyset$ or X and f is measurable.*

Example 7.4 *Let $A \in \mathcal{A}$ and let $f : X \to \mathbb{R}$ such that $f(x) = 1$ for $x \in A$ and $f(x) = 0$ for $x \notin A$. Such a function f is called the **indicator function** of the set A, often denoted $\mathbf{1}_A$. Then $\{x : f(x) > a\}$ equals either A, $\emptyset$ or the whole X, so f is measurable.*

Example 7.5 *Let $X = \mathbb{R}$ and $\mathcal{A} = \mathcal{B}(\mathbb{R})$. Let f be identity function, that is $f(x) = x$ for all $x \in \mathbb{R}$. Then $\{x : f(x) > a\} = (a, \infty) \subset \mathcal{B}(\mathbb{R})$, as required for f to be measurable.*

7.2 Properties of measurable functions

Proposition 7.6 *Let $f : X \to \mathbb{R}$. Then the following are equivalent:*

a) $\{x : f(x) > a\} \in \mathcal{A}$, for all $a \in \mathbb{R}$.

b) $\{x : f(x) \leq a\} \in \mathcal{A}$, for all $a \in \mathbb{R}$.

c) $\{x : f(x) < a\} \in \mathcal{A}$, for all $a \in \mathbb{R}$.

d) $\{x : f(x) \geq a\} \in \mathcal{A}$, for all $a \in \mathbb{R}$.

Proof. Because $A^c \in \mathcal{A}$ when $A \in \mathcal{A}$, a) is equivalent to b) and c) is equivalent to d). We will show that d) is equivalent to a). Note that

$$\{x : f(x) \geq a\} = \cap_{n=1}^{\infty}\{x : f(x) > a - \tfrac{1}{n}\}.$$

Hence, d) implies a). For the reverse implication, note that

$$\{x : f(x) > a\} = \cup_{n=1}^{\infty}\{x : f(x) > a + \tfrac{1}{n}\}.$$

The equivalence of b) and c) follows by a similar argument. □

Proposition 7.7 *Let $c \in \mathbb{R}$. If $f, g : X \to \mathbb{R}$ are measurable, then the functions $f+g$, $-f$, cf, fg, $\max\{f,g\}$ and $\min\{f,g\}$ are also measurable.*

Proof.

- $\underline{f+g}$:
  We start from $\{x : f(x) < a\} \in \mathcal{A}$ and $\{x : g(x) < a\} \in \mathcal{A}$ and want to show that $\{x : f(x) + g(x) < a\} \in \mathcal{A}$. Note that $f(x) + g(x) < a$ is equivalent to $f(x) < a - g(x)$, which further implies that there exists some $q \in \mathbb{Q}$ such that $f(x) < q < a - g(x)$. This is true because $\mathbb{Q}$ is dense in $\mathbb{R}$. Then

  $$\{x : f(x) + g(x) < a\} = \bigcup_{q \in \mathbb{Q}} \left(\{x : f(x) < q\} \cap \{x : g(x) < a - q\}\right),$$

  which is a countable union, and is hence in $\mathcal{A}$. Therefore $f + g$ is measurable.

- $\underline{-f}$:
  Note that

  $$\{x : -f(x) \geq a\} = \{x : f(x) < -a\},$$

  but the latter set is in $\mathcal{A}$ by Proposition 7.6, as required. Hence $-f$ is measurable.

- $\underline{cf}$:
  If $c > 0$ then

  $$\{x : cf(x) > a\} = \{x : f(x) > \tfrac{a}{c}\},$$

  which is in $\mathcal{A}$. If $c = 0$, then $cf \equiv 0$, so then this is a special case of Example 7.3 with the constant being set to 0 and again the conclusion follows. Lastly, if $c < 0$, set $d = -c > 0$. Then $cf = -df$ and measurability then follows from the cases above. Hence cf is measurable.

- $\underline{fg}$:
  We treat the case $f = g$, so f^2, first. For $a \geq 0$,

  $$\{x : f(x)^2 > a\} = \{x : f(x) > \sqrt{a}\} \cap \{x : f(x) < -\sqrt{a}\} \in \mathcal{A}.$$

  To treat fg, note that $fg = \frac{1}{2}\left((f+g)^2 - f^2 - g^2\right)$ and use the results on f^2 and $f + g$ to conclude that fg is measurable.

- $\underline{\max\{f,g\},\ \min\{f,g\}}$:
  Note that $\{x : \max(f(x), g(x)) > a\} = \{x : f(x) > a\} \cup \{x : g(x) > a\} \in \mathcal{A}$, as a union of two elements of $\mathcal{A}$. Finally, note that $\min\{f,g\} = -\max\{f,g\}$. Hence both are measurable.

$\square$

Recall that given a sequence $(a_j)_{j\geq 1}$ of real numbers we have $\limsup_{j\to\infty} a_j = \inf_j \sup_{\ell \geq j} a_\ell$ and $\liminf_{j\to\infty} a_j = \sup_j \inf_{\ell\geq j} a_\ell$. For a quick example consider

$$a_j = 1 \text{ if } j \text{ is odd}, \ a_j = -\frac{1}{j} \text{ if } j \text{ is even}.$$

Then $\limsup_{j\to\infty} a_j = 1$ and $\liminf_{j\to\infty} a_j = 0$.

Proposition 7.8 *If $(f_j)_{j\geq 1}$ is a sequence of $\mathcal{A}$-measurable functions $f_j : X \to \mathbb{R}$ then $\sup_j f_j, \inf_j f_j$, $\limsup_{j\to\infty} f_j$ and $\limsup_{j\to\infty} f_j$ are also $\mathcal{A}$-measurable.*

Proof. By definition the result for lim sup and lim inf will follow directly once we deal with sup and inf. For sup, write

$$\{x : \sup_j f_j(x) \leq a\} = \cap_{j=1}^{\infty} \{x : f_j(x) \leq a\}$$

and note that this is a countable intersection of sets in $\mathcal{A}$, and is therefore in $\mathcal{A}$ itself. Furthermore, $\{x : \sup_j f_j(x) > a\} = \{x : \sup_j f_j(x) \leq a\}^c$ and hence also in $\mathcal{A}$. Substituting sup with inf will yield a similar result. □

For further examples and properties of measurable functions see [1, Section 5.1].

8 The Lebesgue integral

Definition 8.1 *Let $(X, \mathcal{A})$ be a measurable space. For $E \in \mathcal{A}$, define the* ***indicator function*** *of the set E by*

$$\chi_E(x) := \mathbf{1}_E(x) := \begin{cases} 1, & x \in E \\ 0, & x \notin E. \end{cases}$$

A ***simple function*** *is a function of the form*

$$s(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{E_i}(x),$$

where the a_i's are real numbers and E_i's are measurable sets (i.e. $E_i \in \mathcal{A}$).

Recall from the previous lecture that indicator functions of measurable sets are measurable. Hence, by Proposition 7.7, simple functions are measurable as well.

Definition 8.2 *Let $(X, \mathcal{A}, \mu)$ be a measure space. If $s(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{E_i}(x)$ is a non-negative simple function, define the* ***integral*** *w.r.t. the measure μ by*

$$\int_X s\, d\mu = \sum_{i=1}^{n} a_i \mu(E_i). \tag{9}$$

In particular, $\int_X \mathbf{1}_E\, d\mu = \mu(E)$ for all $E \in \mathcal{A}$. If μ is the Lebesgue measure (i.e. $(X, \mathcal{A}, \mu) \equiv (\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$), this is called the ***Lebesgue integral****.*

In this definition, if $a_i = 0$ and $\mu(E_i) = \infty$, we use the convention that $a_i \mu(E_i) = 0$. We now extend the definition of the integral to general measurable functions:

If $f \geq 0$ is a non-negative measurable function, we set

$$\int_X f\, d\mu := \sup \left\{ \int_X s\, d\mu \ : \ 0 \leq s \leq f, \ s \text{ is simple} \right\}. \tag{10}$$

If f is **not** non-negative, we can write $f = f^+ - f^-$, where

$$\underbrace{f^+ = \max\{f, 0\}}_{\text{the positive part}} \quad \text{and} \quad \underbrace{f^- = \max\{-f, 0\}}_{\text{the negative part}}$$

are both non-negative measurable functions. Assuming $\int_X f^+\, d\mu < \infty$ and $\int_X f^-\, d\mu < \infty$, we define

$$\int_X f\, d\mu := \int_X f^+\, d\mu - \int_X f^-\, d\mu.$$

If f is a **complex** function (write $f = u + iv$), and $\int (|u| + |v|)\, d\mu < \infty$, define

$$\int_X f\, d\mu := \int_X u\, d\mu + i \int_X v\, d\mu.$$

Notation (standard) For measurable sets $A \in \mathcal{A}$ we write $\int_A f d\mu := \int_X f \cdot \mathbf{1}_A d\mu$. If $m =$ Leb, then

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,dm(x) = \int_{[a,b]} f\,dm.$$

When integrating over the full space X we may omit the subscript X and write $\int f d\mu = \int_X f d\mu$. When the measure is clear, we may even write $\int f$ instead of $\int f\,d\mu$.

Remark 8.3

1. *For any simple function s the notation $s = \sum_{i=1}^n a_i \mathbf{1}_{E_i}$ is not unique. For example, if $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, then we have $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$. Luckily, the definition of $\int s\,d\mu$ is **not** affected by such a change of notation:*

$$\text{If} \quad s(x) = \sum_{i=1}^m a_i \mathbf{1}_{A_i} = \sum_{j=1}^n b_j \mathbf{1}_{B_j}(x), \quad \text{then} \quad \int s\,d\mu = \sum_{i=1}^m a_i \mu(A_i) = \sum_{j=1}^n b_j \mu(B_j).$$

2. *Note that the definitions given in* (9) *and* (10) *coincide for simple functions.*

Definition 8.4 *We say that f is **integrable** (w.r.t. μ) if $\int |f|\,d\mu < \infty$.*

8.1 Basic properties of integrals

Proposition 8.5

(a) If f is a real-valued function with $-\infty < a \le f(x) \le b < \infty$ for all $x \in X$, and $\mu(X) < \infty$, then

$$a\mu(X) \le \int f\,d\mu \le b\mu(X).$$

(b) If $f, g : X \to \mathbb{R}$ are both integrable and $f(x) \le g(x)$ for all $x \in X$, then

$$\int f\,d\mu \le \int g\,d\mu.$$

(c) If $f : X \to \mathbb{R}$ is integrable, then

$$\int \alpha f\,d\mu = \alpha \int f\,d\mu \qquad \forall\, \alpha \in \mathbb{C}.$$

(d) If $\mu(A) = 0$ and $f : X \to \mathbb{R}$ is integrable, then

$$\int_A f d\mu = \int f \cdot \mathbf{1}_A\,d\mu = 0.$$

Remark 8.6 *If f, g are measurable then fg is measurable. However, if f, g are integrable then fg need not to be integrable. For an easy example consider $X = \mathbb{N}$, $\mu(n) = \frac{1}{n^{p+1}}$ for $1 < p < 2$ and $f(n) = n$. Then $\int f\, d\mu = \sum \frac{n}{n^{p+1}} < \infty$, but $\int f^2\, d\mu = \sum \frac{n^2}{n^{p+1}} = \infty$.*

Proposition 8.7 *If f is integrable, then*

$$\left|\int f\, d\mu\right| \le \int |f|\, d\mu.$$

Proof. If f is real-valued, then $f \le |f|$ and therefore (by Proposition 8.5(b)) we have

$$\int f\, d\mu \le \int |f|\, d\mu.$$

(We could also split f into f^+ and f^- and the argument is the same.) If f is complex-valued, then $\int f\, d\mu$ is also complex. If $f = 0$, then the proposition is clear. If $f \neq 0$, then $\int f\, d\mu = re^{i\theta}$, $r > 0$, $\theta \in (0, 2\pi]$. So,

$$\left|\int f\, d\mu\right| = r = e^{-i\theta}\int f\, d\mu = \int e^{-i\theta} f\, d\mu. \tag{11}$$

Next, note that $\int \text{Re } f = \text{Re}(\int f)$ by definition. Since $|\int f|$ is real,

$$\left|\int f\, d\mu\right| =_{\text{by (11)}} \text{Re}\Big(\int e^{-i\theta} f\, d\mu\Big) = \int \text{Re } e^{-i\theta} f\, d\mu \le \int |f|\, d\mu.$$

□

To look at a few examples we introduce the following definitions.

Definition 8.8 *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f, g be measurable functions. We say that $f = g$* ***μ-almost everywhere (μ-a.e.)*** *if $\mu(\{x : f(x) \neq g(x)\}) = 0$.*

Example 8.9 *Take $m =$ Leb, and consider the measure space $(\mathbb{R}, \mathcal{B}, m)$. Suppose we have $-\infty < a_0 \le a_1 \le \cdots \le a_n < \infty$. Let f be a function such that*

$$f(x) = \begin{cases} x_i > 0, & x \in (a_{i-1}, a_i],\ i = 1, \ldots, n, \\ 0, & x \in (-\infty, a_0) \cup (a_n, \infty). \end{cases}$$

Then $f(x) = \sum_{i=1}^n x_i \mathbf{1}_{(a_{i-1}, a_i]}$, m-a.e. and thus,

$$\int f\, dm = \sum_{i=1}^n x_i (a_i - a_{i-1}), \tag{12}$$

Note that (12) is okay because by convention we take $0 \cdot \infty = 0$. If $f = 1_{(a,\infty)}$, then $\int f\, dm = \infty = m(a, \infty)$.

We would like to have some general criteria for a function to equal zero μ-a.e.

8.1.1 Criteria for a function to be zero a.e.

Proposition 8.10 *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable. Suppose that for any $A \in \mathcal{A}$ we have $\int_A f\, d\mu = 0$. Then $f = 0$ μ-almost everywhere.*

Proof. Let $A = \{x : f(x) \geq \varepsilon\}$ for some $\varepsilon > 0$. Then

$$0 = \int_A f\, d\mu = \int f \cdot 1_A\, d\mu \geq \int_A \varepsilon\, d\mu = \varepsilon\mu(A).$$

Thus $\varepsilon\mu(A) \leq 0$, which can only be true if $\mu(A) = 0$. Now write $A_n := \{x : f(x) \geq \frac{1}{n}\}$. Applying the previous argument to the case $\varepsilon = \frac{1}{n}$ yields $\mu(A_n) = 0$ for any n. However,

$$\begin{aligned} \mu(\{x : f(x) > 0\}) &= \mu\left(\bigcup_{n=1}^{\infty} \left\{x : f(x) \geq \frac{1}{n}\right\}\right) \\ &\leq \sum_{n=1}^{\infty} \mu(A_n) \\ &= 0. \end{aligned}$$

So $\mu(\{x : f(x) > 0\}) = 0$. Similarly, we have $\mu(\{x : f(x) < 0\}) = 0$. Put together, we have $f = 0$ everywhere except possibly on a set of measure zero, meaning we have $f = 0$ μ-a.e. $\square$

Proposition 8.11 *Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \mathbb{R}$ be $\mathcal{A}$-measurable and non-negative. If $\int f\, d\mu = 0$, then $f = 0$ μ-a.e.*

Proof. For any $A \in \mathcal{A}$ we have

$$0 \leq \int_A f d\mu = \int_X f \cdot \mathbf{1}_A\; d\mu \leq \int_X f d\mu = 0,$$

so by Proposition 8.10 we have $f = 0$ μ-a.e. $\square$

Corollary 8.12 *Let $m = Leb$ and $a \in \mathbb{R}$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is integrable and that $\int_a^x f(y)\, dm(y) = 0$ for any $x \in \mathbb{R}$. Then $f = 0$ m-a.e.*

9 Convergence Theorems

The theorems in these sections are all about under what conditions the order of taking an integral and a limit can be swapped. The first limit property we consider is the Monotone Convergence Theorem (MCT). Throughout we let $(X, \mathcal{A}, \mu)$ be a measure space, in particular $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m = Leb)$.

Theorem 9.1 (Monotone Convergence Theorem) *Let f_n be a sequence of non-negative measurable functions such that $f_1(x) \leq f_2(x) \leq \dots$ and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$. Then*

$$\int f_n \, d\mu \longrightarrow \int f \, d\mu \qquad \text{as } n \to \infty.$$

Proof. By Proposition 8.5 from the previous lecture, $(\int f_n \, d\mu)_{n\geq 1}$ is an increasing sequence of real numbers. Let $L = \lim_{n\to\infty} \int f_n \, d\mu$. But $f_n(x) \leq f(x)$ for all $n \in \mathbb{N}$ and $x \in X$, so $L \leq \int f \, d\mu$.

We want to show the other direction: $L \geq \int f \, d\mu$. Let $s = \sum_{i=1}^m a_i \mathbf{1}_{E_i}(x)$ with $a_i > 0$ and $E_i \in \mathcal{A}$ be a simple function. Suppose $s(x) \leq f(x)$ for all $x \in X$. Let $\alpha \in (0,1)$ and let $A_n = \{x : f_n(x) \geq \alpha \, s(x)\}$. Since $f_n(x) \uparrow f(x)$ for all $x \in X$ and $\alpha < 1$, we have

$$A_n \uparrow \bigcup_{n=1}^{\infty} A_n = \{x : f(x) > 0\}.$$

Also, for every $n \in \mathbb{N}$, we have

$$\int f_n \, d\mu \geq \int_{A_n} f_n \, d\mu \geq \alpha \int_{A_n} s \, d\mu = \alpha \int_{A_n} \sum_{i=1}^m a_i \mathbf{1}_{E_i} \, d\mu = \alpha \sum_{i=1}^m a_i \mu(E_i \cap A_n).$$

Using the previous two dispalyed equations,

$$\lim_{n\to\infty} \int f_n \, d\mu \geq \alpha \lim_{n\to\infty} \sum_{i=1}^m a_i \mu(E_i \cap A_n) = \alpha \sum_{i=1}^m a_i \mu(E_i) = \alpha \int s \, d\mu.$$

Hence $L \geq \alpha \int s \, d\mu$. But $\alpha \in (0,1)$ is arbitrary: we can take it as close to 1 as desired. Therefore $L \geq \int s \, d\mu$, and hence

$$L \geq \sup \left\{ \int s \, d\mu : 0 < s < f \right\} = \int f \, d\mu.$$

Hence we have $L \geq \int f \, d\mu$, so in conclusion $\lim_{n\to\infty} \int f_n d\mu = \int f d\mu$. $\square$

Example 9.2 (and warnings)

1. *$X = [0, \infty)$, $f_n(x) = -\frac{1}{n}$ for all $x \in X$. For each fixed n, we have $\int f_n \, d\mu = -\frac{1}{n} \int_X 1 \, d\mu = -\frac{\mu(X)}{n} = -\infty$. But $f := \lim_{n\to\infty} f_n(x) = 0$, so $f_n \uparrow f$ and $\int f \, d\mu = 0 \neq \lim_{n\to\infty} \int f_n \, d\mu$. Here the MCT does not apply because f_n are* **not** *non-negative functions.*

2. $f_n(x) = n\mathbf{1}_{(0,\frac{1}{n})}$, *so* $f_n \geq 0$ *and* $f_n(x) \to 0$ *for all* $x \in X = [0,\infty)$. *But* $\int f_n \, d\mu = n \int \mathbf{1}_{(0,\frac{1}{n})} \, d\mu = 1$. *This is no contradiction to MCT because* $f_n \not\uparrow f$.

Theorem 9.3 *Let* $f, g : X \to \mathbb{R}$ *be measurable functions. Assume either that* f, g *are non-negative or that* f, g *are integrable. Then*

$$\int (f+g) \, d\mu = \int f \, d\mu + \int g \, d\mu.$$

Proof. Those that are interested can read the proof of Theorem 7.4 in Chapter 7 of [1]. $\square$

Using the MCT and Theorem 9.3 we obtain the following stronger form of Theorem 9.3.

Proposition 9.4 *Suppose that* f_n *are non-negative measurable functions. Then*

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

Proof. Let $F_N = \sum_{n=1}^{N} f_n$ for any $N \in \mathbb{N}$. Note that $0 \leq F_N(x) \uparrow \sum_{n=1}^{\infty} f_n(x)$ as $N \to \infty$. Hence

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \int \lim_{N\to\infty} \sum_{n=1}^{N} f_n \, d\mu = \int \lim_{N\to\infty} F_N \, d\mu.$$

But $(F_N)_{N\geq 1}$ satisfies the assumption of the MCT. Thus

$$\int \lim_{N\to\infty} F_N \, d\mu =_{\text{MCT}} \lim_{N\to\infty} \int F_N \, d\mu =_{\text{Thm 9.3}} \lim_{N\to\infty} \sum_{n=1}^{N} \int f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu.$$

$\square$

10 Convergence Theorems (continued)

Fatou's Lemma:

Theorem 10.1 *[Fatou's Lemma] Suppose that f_n are non-negative and measurable. Then*

$$\int \liminf_{n\to\infty} f_n \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu.$$

Proof. The idea is to apply the MCT to the sequence $g_n := \inf_{j\ge n} f_j$. The functions g_n are non-negative and measurable (see the propositions in Lecture 6). Next, since $g_n \le f_j$ for all $j \ge n$ and $g_n \uparrow \liminf_{n\to\infty} f_n$, we have

$$\int g_n \, d\mu \le \int f_j \, d\mu \quad \text{for all } j \ge n,$$

and therefore $\int g_n \, d\mu \le \inf_{j\ge n} \int f_j \, d\mu$. Hence

$$\int \liminf_{n\to\infty} f_n \, d\mu = \int \lim_{n\to\infty} g_n \, d\mu =_{\text{MCT}} \lim_{n\to\infty} \int g_n \, d\mu \le \lim_{n\to\infty} \inf_{j\ge n} \int f_j \, d\mu = \liminf_{n\to\infty} \int f_n \, d\mu.$$

$\square$

Corollary 10.2 *Let f_n be measurable functions (allowed to be negative). Suppose that for all $x \in X$ we have $\lim_{n\to\infty} f_n(x) = f(x)$ and $\int |f_n| \, d\mu \le M < \infty$ for all $n \in \mathbb{N}$. Then $|f_n| \to |f|$ (clear) and $\int |f| \, d\mu \le M$ (consequence of Fatou's Lemma).*

Recall that we are interested in finding sufficient conditions to write

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu.$$

So far we have seen the MCT (Theorem 9.1), Fatou's lemma (Theorem 10.1) and linearity of the integral (Theorem 9.3). There is yet another very useful convergence theorem.

Dominated Convergence Theorem (DCT):

Theorem 10.3 *[Dominated Convergence Theorem] Suppose that $f_n : X \to \mathbb{R}$ are measurable functions such that $f_n(x) \to f(x)$ for all $x \in X$. Assume that there exists a non-negative* **integrable** *function $g : X \to \mathbb{R}$ such that $|f_n(x)| \le g(x)$ for all $x \in X$. Then*

$$\lim_{n\to\infty} \int f_n \, d\mu = \int f \, d\mu.$$

Proof. Using Theorem 10.1, we want to show that

$$\int f \, d\mu \le \liminf_{n\to\infty} \int f_n \, d\mu \tag{13}$$

$$\int f \, d\mu \ge \limsup_{n\to\infty} \int f_n \, d\mu. \tag{14}$$

Clearly (13) and (14) then lead to the desired result since they imply that

$$\int f\,d\mu \le \liminf_{n\to\infty} \int f_n\,d\mu \le \limsup_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu,$$

which forces $\int f\,d\mu = \liminf_{n\to\infty} \int f_n\,d\mu = \limsup_{n\to\infty} \int f_n\,d\mu$.

The assumptions of the theorem imply that f is necessarily integrable since

$$\begin{cases} f_n(x) \to f(x)\ \forall\, x \in X \\ \int |f_n|\,d\mu \le \int g \end{cases} \Rightarrow \begin{cases} |f_n(x)| \to |f(x)|\ \forall\, x \in X, \\ \int |f|\,d\mu \le \int g. \end{cases} \tag{15}$$

It remains to show that (13) and (14) are true.

- <u>(13)</u>: Since $|f_n| \le g$ everywhere, and g is non-negative, $f_n + g \ge 0$. Therefore

$$\begin{aligned} \int f\,d\mu + \int g\,d\mu \ &=_{\text{linearity}} \ \int (f+g)\,d\mu = \int (\liminf_{n\to\infty} f_n + g)\,d\mu \\ &\le_{\text{Fatou}} \ \liminf_{n\to\infty} \int (f_n + g)\,d\mu \\ &=_{\text{linearity}} \ \liminf_{n\to\infty} \int f_n\,d\mu + \int g\,d\mu. \end{aligned} \tag{16}$$

  But g and f are both integrable (by (15)). Also g is non-negative, so we can subtract $\int g\,d\mu$ in (16), which yields

$$\int f\,d\mu \le \liminf_{n\to\infty} \int f_n\,d\mu,$$

  giving us Equation (13).

- <u>(14)</u>:
  By a similar argument $g - f_n \ge 0$. Hence,

$$\begin{aligned} \int g - \int f\,d\mu\,d\mu \ &=_{\text{linearity}} \ \int (g-f)\,d\mu = \int \liminf_{n\to\infty} (g - f_n)\,d\mu \\ &\le_{\text{Fatou}} \ \liminf_{n\to\infty} \int (g - f_n)\,d\mu \\ &=_{\text{linearity}} \ \int g\,d\mu + \liminf_{n\to\infty} \int (-f_n)\,d\mu \int g\,d\mu - \limsup_{n\to\infty} \int f_n\,d\mu. \end{aligned} \tag{17}$$

  As before, subtract $\int g\,d\mu$ from (17). Thus

$$-\int f\,d\mu \le -\limsup_{n\to\infty} \int f_n\,d\mu \quad \Rightarrow \limsup_{n\to\infty} \int f_n\,d\mu \le \int f\,d\mu,$$

  which gives (13).

$\square$

Exercises:

Exercise 10.4 *Let $m = Leb$ and replace the condition* $\lim_{n\to\infty} f_n(x) = f(x)\ \forall\, x \in X$ *with*

$$f_n\ to f \quad m\text{-}a.e. \tag{18}$$

Does this new assumption change the conclusion of the DCT?

Solution: No! Let $A = \{x : f_n(x) \to f(x)\}$. By (18) we have $m(A^c) = 0$ and $h_n := \mathbf{1}_A f_n \to \mathbf{1}_A f =: h$. DCT applied to h_n, h gives $\lim_{n\to\infty} \int \mathbf{1}_A f_n\, dm = \int \mathbf{1}_A f\, dm$. But

$$\int f_n\, dm = \int \mathbf{1}_A f_n\, dm \tag{19}$$

which gives the conclusion. To see (19) note the following. If G is non-negative and $G = 0$ a.e., then $\int G\, dm = 0$. This further implies that decomposing G into G^+ and G^- yields

$$G = 0\ m\text{-a.e.} \quad \Rightarrow \quad \int G\, dm = 0. \tag{20}$$

Applying (20) to $G = f_n - \mathbf{1}_A f_n$ we obtain (19) and with it the following more general statement: If f, g are two funtions so that $f = g$ a.e. then $\int f = \int g$.

Exercise 10.5 *Suppose that $\mu(X) < \infty$ and that $f_n : X \to \mathbb{R}$ are uniformly bounded, i.e. there is an $M < \infty$ such that $|f_n| \le M$ for all $n \in \mathbb{N}$. Suppose*

$$\lim_{n\to\infty} f_n = f \ \ a.e.$$

Show that $\lim_{n\to\infty} \int f_n\, d\mu = \int f\, d\mu$.

Solution: The above is the so-called Bounded Convergence Theorem (BCT). It is an immediate consequence of Exercise 10.4, that is: DCT with "everywhere" replaced by "a.e".

11 On Riemann/Lebesgue integration

11.1 What is the Riemann integral?

Maybe you have seen the Riemann integral already in another course, but let us recall the construction. Our notation comes from [1, Chapter 9]. Let $f : [a, b] \to \mathbb{R}$ be real-valued function on a compact interval in $\mathbb{R}$. Like Lebesgue integral, we will approximate f with simple functions $\sum_{i=1}^{N} a_i 1_{E_i}$, both from above and from below, but the sets E_i must be intervals $[x_{i-1}, x_i]$. That is, we call $P = \{x_0, \dots, x_N\}$ a **partition** of $[a, b]$ if $a = x_0 < x_1 < \dots < x_N = b$ and $N \in \mathbb{N}$. W.r.t. this partition P we have the **upper** and **lower Riemann sum** of f:

$$U(P, f) = \sum_{i=1}^{N} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$

and

$$L(P, f) = \sum_{i=1}^{N} \left(\inf_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}).$$

To get the **upper** and **lower Riemann integral**, we take the infimum resp. supremum over all partitions:

$$\overline{R}(f) = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$$

and

$$\underline{R}(f) = \sup\{L(P, f) : P \text{ is a partition of } [a, b]\}.$$

A function f is **Riemann integrable** on $[a, b]$ if $\overline{R}(f) = \underline{R}(f)$, and in this case he common value is the **Riemann integral** $\int_a^b f(x)\, dx$.

Using this definition, unbounded functions are not Riemann integrable (or at least cannot have a finite integral), and also we have not yet defined the Riemann integral over unbounded intervals. However, we can often take limits of the end-points of the interval.

Example: Let $f(x) = \frac{\sin x}{x}$. The Riemann integral $\int_0^\infty f(x)\, dx$ is called an **improper integral** because the limit

$$\lim_{b \to \infty} \int_{1/b}^{b} f(x)\, dx.$$

exists. Although we cannot compute it by primitive functions (as we do in calculus), we say that $\int_0^\infty f(x)\, dx$ is this limit (in fact, it can be shown that $\int_0^\infty f(x)\, dx = \pi/2$).

11.2 When do the Riemann and Lebesgue integrals coincide?

The following theorem provides necessary and sufficient conditions for a function to be Riemann integrable and for the Riemann integral to equal the Lebesgue integral as in [1, Theorem 9.1].

Theorem 11.1 *A bounded real-valued function f on $[a, b]$ is Riemann integrable if and only if the set of points at which f is discontinuous has Lebesgue measure* 0*. In that case, f is Lebesgue measurable and the Riemann integral of f is equal in value to the Lebesgue integral of f.*

Example: Let $[a, b] = [0, 1]$ and $f = 1_A$, where A is the set of irrational numbers in $[0, 1]$. If $x \in [0, 1]$, every neighbourhood of x contains both rational and irrational points. Hence, f is discontinuous at every point of $[0, 1]$. Therefore f is not Riemann integrable.

Example: Define $f(x)$ on $[0, 1]$ to be 0 if x is irrational and to be $1/q$ if x is rational and equal to $x = p/q$ in its reduced form (i.e. $\gcd(p, q) = 1$). The function f is discontinuous at every rational. If x is irrational and $\epsilon > 0$, there are only finitely many rationals r for which $f(r) \geq \epsilon$, so taking δ less than the distance from x to any of this finite collection of rationals shows that $|f(y) - f(x)| < \epsilon$ if $|y - x| < \delta$. Hence f is continuous at x. Therefore the set of discontinuities is a countable set, hence of measure 0, so f is Riemann integrable

Example: The function $f(x) = \frac{1}{x} \cos \pi x$ is a Riemann integrable function on $[1, \infty)$ that is not Lebesgue integrable. The proof why f is Riemann integrable is not required here (note that $[1, \infty)$ is not a compact interval, so Theorem 11.1 cannot be applied immediately). It is, however, not hard to see why f is not Lebesgue integrable, namely $\int_1^\infty |f(x)|\, dx$ is not finite. To see this, note that $g(x) = \sum_{n \geq 1} \frac{1}{10n} \chi_{[n, n+0.1]}$ satisfies $|f| \geq g$ and

$$\int_1^\infty g(x)\, dx \geq \sum_{n=1}^\infty \frac{1}{10n} \frac{1}{10} = \frac{1}{100} \sum_{n=1}^\infty \frac{1}{n} = \infty.$$

Below we consider another example of a bounded measurable function which is not equal a.e. to any Riemann integrable function.

Example: see [2, Example 17.2]. First, we claim (and prove below) that there exist Borel subsets A of $(0, 1]$ so that given $m = Leb$,

$$m(A) \in (0, 1) \text{ and } m(A \cap I) > 0 \text{ for any subinterval } I \subset (0, 1).$$

Define $f = 1_A$. We will now show that there is no g so that g Riemann integrable and $f = g$, a.e. Suppose the contrary ($f = g$, a.e. with g Riemann integrable) and let $\{I_j\}$ be a decomposition of $(0, 1]$ into subintervals. Since

$$m(A \cap I_i \cap \{f = g\}) = m(A \cap I_i) > 0,$$

we have $g(y_i) = f(y_i) = 1$ for some $y_i \in I_i \cap A$. Hence,

$$\sum_i g(y_i)\, m(I_i) = 1 > m(A).$$

If g was Riemann integrable then its Riemann integral would coincide with Leb integral, so $\int g dx = \int f dx = m(A)$, in contradiction with the previous displayed equation.

It remains to prove the claim. Let $\{r_1, r_2, \ldots\}$ be an enumeration of rationals in $(0, 1)$. Suppose that there exists a small $\varepsilon > 0$ and choose $I_n = (a_n, b_n)$ so that $r_n \in I_n \subset (0, 1)$ and so that $m(I_n) = b_n - a_n < \varepsilon 2^{-n}$. Set $A = \cup_{n=1}^\infty I_n$. By subadditivity (mononicity) of m, we have $0 < m(A) < \varepsilon$. Since A contains all the rationals in $(0, 1)$, it is dense in $(0, 1)$. Thus, A is an open, dense set with measure close to 0. Therefore, if I is an open interval of $(0, 1)$ then I must intersect one of the I_n and as a consequence, $m(A \cap I) > 0$.

The previous example was meant to tell us that Lebesgue integration cannot be reduced to Riemann integration. The next theorem shows that we can still approximate every Leb intergable function by a Riemann integrable function (in a sense described below).

Theorem 11.2 *[2, Theorem 17.1] Suppose that* $\int |f|\,dx < \infty$ *and let* $\varepsilon > 0$. *Then*

i) There is a step function $g = \sum_{i=1}^{k} x_i 1_{A_i}$ *with bounded intervals* A_i *such that* $\int |f - g|\,dx < \varepsilon$.

ii) There is a continuous integrable function h *with bounded support so that* $\int |f - h|\,dx < \varepsilon$.

The proof will be discussed in class.

12 Types of convergence

Here we are mainly following [1, Chapter 10], but everything you need to know for this course is included in the notes below.

So far we have seen a.e.-convergence: $f_n \to f$ μ-a.e. if $\mu(\{x : f_n(x) \not\to f(x)\}) = 0$.

Definition 12.1 (Convergence in measure) *We say that f_n* **converges in measure** *to f if for each $\epsilon > 0$,*

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) \to 0 \qquad \text{as } n \to \infty.$$

All we need to know about L^p-spaces in this part is:

Definition 12.2 *For a measurable function f and $1 \le p < \infty$ define the so-called L^p**-norm** $\|f\|_p := \left(\int |f|^p \, d\mu\right)^{1/p}$. We say $f \in L^p$ if $\|f\|_p < \infty$.*

Definition 12.3 (Convergence in L^p) *Let $1 \le p < \infty$. We say that f_n* **converges in L^p** *to f if*

$$\|f_n - f\|_p \to 0 \qquad \text{as } n \to \infty.$$

Proposition 12.4 *Suppose that μ is a finite measure. If $f_n \to f$ μ-a.e., then $f_n \to f$ in measure (convergence a.e. implies convergence in measure.)*

Proof. Let $\epsilon > 0$ and let $f_n \to f$ μ-a.e. Let $A_n = \{x : |f_n(x) - f(x)| > \varepsilon\}$. Clearly $1_{A_n} \to 0$ μ-a.e. as $n \to \infty$. By DCT (in fact, BCT),

$$\mu(A_n) = \int 1_{A_n} \, d\mu \to 0,$$

so $\mu(A_n) \to 0$ as required. $\square$

Remark 12.5 *If $\mu(X) = \infty$, Proposition 12.4 is not true. For example, take $f_n = 1_{[n,\infty)}$. Then $f_n \to 0$ a.e., but $\int f_n(x) \, dx \not\to 0$.*

Proposition 12.6 (Chebyshev's Inequality) *Let $f \in L^p$ for some $1 \le p < \infty$. Then*

$$\mu(\{x : |f(x)| \ge a\}) \le \frac{\int |f|^p \, d\mu}{a^p}$$

for all $a > 0$.

Proof. Let $A = \{x : |f(x)| \ge a\}$. Note that $\mathbf{1}_A \le \frac{|f|^p}{a^p}$. Therefore

$$\mu(A) \le \int_A \frac{|f|^p}{a^p} \, d\mu = \frac{1}{a^p} \int |f|^p \, d\mu,$$

as claimed. $\square$

Remark 12.7 *Convergence in L^p implies convergence in measure. The other direction is* **not** *true. Take $f_n : [0,1] \to \mathbb{R}$, $f_n = n1_{[0,\frac{1}{n})}$. Then $f_n \to 0$ a.e., and $f_n \to 0$ in measure, but $f_n \not\to 0$ in L^p for every $p \geq 1$.*

Proposition 12.8 *If $f_n \to f$ in L^p, then $f_n \to f$ in measure.*

Proof. Let $\epsilon > 0$ be arbitrary. By Proposition 12.6 (the Chebyshev inequality)

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) = \mu(\{x : |f_n(x) - f(x)|^p > \epsilon^p\}) \leq_{\text{Chebyshev}} \frac{1}{\epsilon^p} \int |f_n - f|^p \, d\mu \to 0$$

as $n \to \infty$. $\square$

13 Additional material on L^p spaces

Items covered:
1.) Read the entire [1, Sections 15.1]
2.) Read the statement of the theorems in [1, Sections 15.1]; proofs are only optional and not subject for the exam.

13.1 Comments on [1, Sections 15.1] and useful examples

You need to understand the statement and proof of [1, Proposition 15.1, Lemma 15.2, Proposition 15.3]. These notes contain additional explanations in proofs and additional examples.

Addition to [1, Proof of Lemma 15.2]: Note that the given proof doesn't quite work for $p = 1$, because $f(0) = 0$ rather than $f(0) > 0$ in this case. However, $f(x) = 0$ for all $x \in \mathbb{R}$ for $p = 1$, so we do have $f(x) \geq 0$ as needed.

Example: Let $f(x) = x^{-\alpha}$ for some $\alpha > 0$.

- As function $f : (0, 1] \to \mathbb{R}$, it belongs to L^p for all $1 \leq p < 1/\alpha$. Indeed

$$\int_0^1 |f(x)|^p\, dx = \int_0^1 x^{-\alpha p}\, dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-\alpha p}\, dx = \lim_{\epsilon \to 0} \left[\frac{1}{1 - \alpha p} x^{1-\alpha p} \right]_\epsilon^1 < \infty$$

  only if $1 - \alpha p > 0$, so $p < 1/\alpha$.

  For $\alpha = 1/p$, the integration goes a bit differently:

$$\int_0^1 |f(x)|^p\, dx = \int_0^1 x^{-\alpha p}\, dx = \lim_{\epsilon \to 0} \int_\epsilon^1 x^{-1}\, dx = \lim_{\epsilon \to 0} [\log x]_\epsilon^1 = \infty.$$

  Note also that $f(x)$ is not bounded on $(0, 1]$, so $f \notin L^\infty$ either.

- As function $f : [1, \infty) \to \mathbb{R}$, it belongs to L^p for all $p > 1/\alpha$. Indeed

$$\int_1^\infty |f(x)|^p\, dx = \int_1^\infty x^{-\alpha p}\, dx = \lim_{y \to \infty} \int_1^y x^{-\alpha p}\, dx = \lim_{y \to \infty} \left[\frac{1}{1 - \alpha p} x^{1-\alpha p} \right]_1^y < \infty$$

  only if $1 - \alpha p < 0$, so $p > 1/\alpha$.

  For $\alpha = 1/p$, the integration again works a bit differently:

$$\int_1^\infty |f(x)|^p\, dx = \int_1^\infty x^{-\alpha p}\, dx = \lim_{y \to \infty} \int_1^y x^{-1}\, dx = \lim_{y \to \infty} [\log x]_1^y = \infty.$$

13.2 Useful examples for [1, Sections 15.2] and

Here you need to understand the statement of [1, Theorem 15.4] (proof is only optional) and the statement of [1, Proposition 15.5] (proof is again optional but still recommended to go through, since some techniques used might be useful for solving exercises). Step 1 of the proof of [1, Theorem 15.4] is very short and very useful for understanding the notion of *Cauchy sequences* *in L^p spaces.*

Example: Define $f, f_n : [0,2] \to \mathbb{R}$ as

$$f_n(x) = \begin{cases} x^n & x \in [0,1) \\ 1 & x \in [1,2] \end{cases} \qquad f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x \in [1,2]. \end{cases}$$

Then

$$\|f_n - f\|_p = \left(\int_0^2 |f_n - f|^p \, dx \right)^{1/p} = \left(\int_0^1 x^{pn} \, dx \right)^{1/p} = \left(\frac{1}{np+1} \right)^{1/p}.$$

For $p \in [1, \infty)$, this converges to 0, so $f_n \to f$ in L^p. In particular, the L^p-limit of continuous functions need not be continuous. For $p = \infty$, it works differently:

$$\|f_n - f\|_\infty = \sup_{x \in [0,2]} |f_n(x) - f(x)| = \sup_{x \in [0,1)} x^n = 1.$$

Hence $f_n \not\to f$ (and in fact $(f_n)_{n \in \mathbb{N}}$ doesn't converge) in L^∞. However, if $(g_n)_{n \in \mathbb{N}}$ is a sequence of continuous functions, and $g_n \to g$ in L^∞, then g is continuous too. This is the theorem saying that the uniform limits of continuous functions are again continuous. In other words, the space of continuous function $C(X)$ is complete in L^∞.

Example: The proof of Proposition 5.5 in [1] uses that indicator functions of intervals can be approximated in L^p by continuous functions, provided $p \in [1, \infty)$. Let us do this explicitly for the indicator function $f(x) = \mathbf{1}_{[a,b]}$:

$$f_n(x) = \begin{cases} 1 & x \in [a + \frac{1}{n}, b - \frac{1}{n}], \\ n(x-a) & x \in [a, a + \frac{1}{n}), \\ n(b-x) & x \in (b - \frac{1}{n}, b], \\ 0 & \text{otherwise.} \end{cases}$$

Then f and f_n differ only on the intervals $(a, a + \frac{1}{n})$ and $(b - \frac{1}{n}, b)$. Due to symmetry, we get

$$\|f - f_n\|_p^p = \int_a^{a+\frac{1}{n}} |n(x-a)|^p \, dx + \int_{b-\frac{1}{n}}^b |n(b-x)|^p \, dx = 2 \int_a^{a+\frac{1}{n}} |n(x-a)|^p \, dx.$$

Use a change of coordinates $u = n(x-a)$. This gives

$$\|f - f_n\|_p^p = \frac{2}{n} \int_0^1 u^p \, du = \frac{2}{n(p+1)}.$$

Therefore $\|f - f_n\|_p = (\frac{2}{n(p+1)})^{1/p}$, which converges to 0 as $n \to \infty$. However, this does not work if $p = \infty$: Indicator functions cannot be approximated by continuous functions in L^∞.

References

[1] R. Bass, Real analysis for graduate students, Online: http://bass.math.uconn.edu/rags010213.pdf

[2] P. Billingsley, Convergence of Probability Measures, 1999, J. Wiley and Sons.