TMA4212 Num.diff.
Spring 2022

Project 2

Norwegian University of Science and Technology
Department of Mathematical Sciences

Practical information

- *Deadline and hand-in:* Sunday March 27 23:59. Hand in the project in ovsys (pdf file (max 10 pages long) + Jupyter Notebook (should run/compile without errors)).
- *Report:* The report can be written as a pdf-document together with your python code as Jupyter file. More details are present on the wiki page under "Exercises and projects". Write the report as a scientific report, not as a solution to an exercise. Meaning: Describe the problem you want to solve, describe the method you use, write math results as math statements, and make sure there is a connection between theoretical and numerical results etc. Use (readable!) plots when appropriate and explain clearly what you observe and whether the results are expected or not.
- *Grading:* Max 4 points for the report, max 8 for problem 1, max 4 for problem 2, max 4 points for problem 3. Total max score 20 points. (No bonus problem).
- *Learning objectives:*
  - develop and implement finite difference schemes for elliptic 2-D problems in complex domains and variable coefficients hyperbolic problems in 1-D,
  - make test problems to test the code, numerically estimate and present graphically convergence rates,
  - perform theoretical error analysis in some cases,
  - identify and solve potential deficiencies of a scheme,
  - communicate the results in a scientific manner.

Some advice:

- *Implementation:* Divide the coding of your solvers into smaller parts, test each part of the code before moving to the next.
- *Writing:* Imagine you are writing for fellow students that have not seen the project description. Try to make them understand and be interested in your results. Writing takes time, so start early and rewrite parts if necessary, that is part of the process.
- *Time organization:* Plan how much time you are willing to invest. Start early. If you are stuck at some point, it could be an idea to drop it and concentrate on writing a good report instead.

Part 1: Heat distribution in anisotropic materials

The Poisson equation is used to model the stationary temperature distribution T of a solid Ω. If the heat conductivity is κ, and the solid has internal heat sources h (an energy density), then conservation of energy, Fourier's law for the heat flux, and stationarity ($\partial_t T = 0$), give the following model

$$-\nabla \cdot (\kappa \nabla T) = h \qquad \text{in} \qquad \Omega. \tag{1}$$

In anisotropic materials the heat flows faster in some directions than others, and this means that the κ is a matrix. We will focus on two dimensional models with two distinguished directions for the heat flow:

$$\vec{d_1} = (1,0) \qquad \text{and} \qquad \vec{d_2} = (1,r) \quad \text{where} \quad r \in \mathbb{R}.$$

After normalisation this gives a heat conductivity of the form

$$\kappa = \begin{pmatrix} a+1 & r \\ r & r^2 \end{pmatrix},$$

where $a > 0$ is a constant and

$$R := \frac{a}{|\vec{d_2}|^2} = \frac{a}{1+r^2}$$

is the relative strength of the conductivity in the $\vec{d_1}$ versus $\vec{d_2}$ direction. In this case

$$\nabla \cdot (\kappa \nabla T) = (a+1)\partial_x^2 u + 2r\partial_x\partial_y u + r^2\partial_y^2 u = a\partial_x^2 u + (\vec{d_2} \cdot \nabla)^2 u. \tag{2}$$

Note in particular the second order directional derivative!

$\boxed{1}$ Let $\Omega = [0,1]^2$ (the unit square) with Dirichlet boundary conditions $u = g$ on $\partial\Omega$.

a) Let $r = 1$. Discretise and solve the problem numerically using second order central differences in the directions $\vec{d_1}$ and $\vec{d_2}$. Write the program experimenting with some simple choices of g and h.

OBS: Do NOT use the form with mixed derivatives. Discretised mixed derivatives are unstable. Discretise the 2nd order directional derivatives instead.

b) Show that the scheme in part (a) is monotone. Write down its stencil. Use the discrete max principle and the comparison function $\phi = \frac{1}{2}x(1-x)$ to show L^∞ stability. Derive an error bound. What is the rate of convergence for smooth solutions?

c) Test the scheme, check the convergence rate, plot the solution and log-log plots of the errors in some cases.

Hint: To find an exact solution of the problem, you may modify the right hand side f and the initial data. Then as explained in the lectures, any nice function will be a solution for a suitable choice of f and the initial data.

So far we were lucky since the directional derivatives could be computed on the naive/standard grid. Let us now take a bad direction so that this is no longer possible: Let $r \neq 1$ and irrational. We now test one idea to resolve this new problem.

d) Introduce a new grid with step sizes $h = \frac{1}{M}$ and $k = |r|h$ in the x and y directions. Adapt the scheme of part a) to this grid. Explain that the grid now will miss the upper boundary ($y = 1$) and how to overcome this problem. Implement the scheme and experiment. Test the scheme and check the convergence rate numerically. Plot the solution in one interesting case.

OBS: We ask for no analysis in problem d).

$\boxed{2}$ Let Ω be the part of the unit disk that is in the first quadrant.

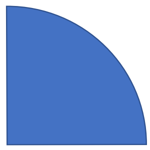

Figure 1: The domain Ω

The boundary then consists of the curves

$$\gamma_1 = [0,1] \times \{0\}, \quad \gamma_2 = \{0\} \times [0,1], \quad \text{and} \quad \gamma_3 = \big\{(x, \sqrt{1-x^2}) \ : \ x \in [0,1]\big\}.$$

Solve numerically the Dirichlet boundary value problem for (1) in the isotropic case when when $\kappa = I$ and

$$\nabla \cdot (\kappa \nabla T) = \Delta T.$$

Implement the scheme and experiment. Test both strategies suggested in the lectures for handling irregular boundaries: (i) modify the discretisation near the boundary, and (ii) fatten the boundary. Which one is easier/faster?

Hint: You may want to use normal extensions when programming fattening the boundary. For any point $\vec{x}$ in the first quadrant, the normal projection of onto the unit circle is $\frac{\vec{x}}{|\vec{x}|}$.

Part 2: A variable coefficient transport equation

In this part of the project, we will solve a variable coefficient transport equation on a bounded interval:

$$\text{(3)} \qquad \begin{cases} u_t + a(x,t)u_x = 0 & \text{in} \quad (0,1) \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad [0,1], \\ u(0,t) = g(t) & \text{in} \quad [0,T]. \end{cases}$$

For this problem to be well-posed, we assume that a is nice enough:

$$\text{(A)} \qquad a \in C^1([0,1] \times [0,T]), \qquad a(0,t) > 0 \quad \text{and} \quad a(1,t) > 0.$$

That $a > 0$ at the boundary implies that $x = 0/x = 1$ is inflow/outflow boundary, and that boundary conditions can only be imposed (are only needed) at $x = 0$. A simple example of such a function is $a = (x - \frac{1}{3})(x - \frac{2}{3})$.

$\boxed{3}$ Let a satisfy (A) and be sign changing.

a) Discretise (3) with the Upwind scheme (as explained in the lectures and not in the note). Implement/program the scheme and experiment with some simple choices of a, g, and u_0. Test the scheme, check the convergence rate, plot the solutions and log-log plots of the errors.

b) Show that the Upwind scheme is monotone and von Neuman stable under a CFL condition. Check if the scheme is disspative and dispersive.

c) Let $a = a(x)$, with no t-dependency to simplify. Discretise (3) with the Lax-Wendroff scheme. Implement/program the scheme and experiment with some simple choices of a, g, and u_0. Test the scheme, check the convergence rate, plot the solutions and log-log plots of the errors. What are the (best) convergence rates in time and space?

Note: The Lax-Wendroff scheme requires boundary conditions also at $x_M = 1$ (outflow). One way to overcome this problem is to set $u_M = u_{M-1}$, i.e. a constant extension from the interior. Higher order extensions can also be used.

OBS: Remember to implement the CLF condition in your code (see the lectures).