Project 2 :10001, 10101, 10034

Problem 1

a)

1. First of all, we can consider the UCC-system a queueing system, since $X(t)$ is a birth-death process, where each person must be in "service" before leaving the system, and each individual stands in queue to wait for their turn to be expedited. We have three requirements for the $M/M/1$-queue. First, we need the interarrival times to be independent and identically distributed exponentially with a parameter λ. The arrival of patients are stated as iid, and distributed as $\exp(\lambda)$, so this condition is satisfied. Second, we need the service times to be independent and identically distributed exponentially as well. We can consider treatments as the service, and these are given as independent and identically exponentially distributed with parameter μ. Thirdly, there must be one server, and the service times must be independent of the arrival process, which is given as an assumption. Hence, this can be modeled as a $M/M/1$-queue.

2. We know that due to the property of $M/M/1$ queues that the interarrival times and treatment times are memoryless giving $X(t)$ the Markov property that makes this a continuous-time Markov chain. Each state $i, i \geq 1$ has only two possible transitions, $i \to i+1$ or $i \to i-1$. Thus, the number of patients in UCC, $X(t)$, satisfies the properties of a birth-death process. Furthermore we obtain the birth rate as the arrival rate, λ. The death rate will be the rate at which each customer leaves the system, μ.

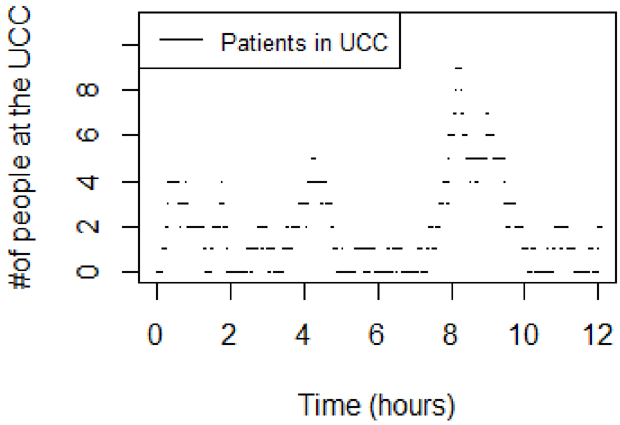
3. The long term average time in the system, W, can be calculated by Little's law, $W = \frac{L}{\lambda}$. The parameter L can be calculated as the expectation of the geometrically distributed limiting distribution $\pi_k = (\lambda/\mu)^k(1-\lambda/\mu)$. This is a geometric distribution with success probability $p = 1-\lambda/\mu$ and expected value $L = E[\text{Geom}(p)] = \lambda/(\mu-\lambda)$. We get

$$W = \frac{L}{\lambda} = \frac{1}{\lambda}\frac{\lambda}{\mu-\lambda} = \frac{1}{\mu-\lambda}. \tag{1}$$

b)

We are first supposed to determine the expected time a patient spends in the UCC. For each sojourn time (time between events) we add up a weighted average, by multiplying the value of $X(t), t \in [t_i, t_{i+1}]$ with $S_i = t_{i+1} - t_i$, where S_i represent sojourn time number i. We finally devide the sum by the total time $T_{max} = 50$days to obtain our result for the average number of patients in the UCC. We further compute an estimate for the expected time a patient stays in the UCC (the average time spent in the UCC) which is easily obtained by applying Little's law. We simply divide our mean number of patients with λ, which gives us the wanted result. The result was

$$\overline{W} = \frac{\overline{L}}{\lambda} = 0.9772\,\text{hours}. \tag{2}$$



For the CI we need to estimate the mean of 30 different simulation means in order to apply the central limit theorem, which tells us that the realizations of $\overline{W}$ are normally distributed around the true value of W. Since both our expected value and our variance of W is unknown, we must use the T-distribution. After we have found the sample mean of all the sample means,

$$\tilde{W} = 1/30 \sum_{k=1}^{30} \overline{W}_k = 0.9923 \text{ hours.} \tag{3}$$

we find the sample variance of each simulation by the usual formula

$$S^2 = \frac{1}{29} \sum_{k=1}^{30} (\overline{W}_k - \tilde{W})^2 = 0.0167$$

We use the marginal for the T-distribution,

$$t_{29,0.025} \sqrt{\frac{S^2}{n}} = 2.045 \sqrt{\frac{0.0167}{30}} = 0.0482 \tag{4}$$

giving a 95% CI of

$$W : [0.9440, 1.041] \tag{5}$$

measured in hours. The theoretical value from 1a) is

$$W = \frac{1}{\mu - \lambda} = \frac{1}{6 - 5} = 1 \text{ hour.} \tag{6}$$

As expected, we observe that the confidence interval consists of the theorical value.

c)

1. We again check the three requirements for the $M/M/1$-queue. First, we need the interarrival times to be independent and identically distributed exponentially. We will explain why this true by the following reasoning (which will also answer the next problem).

Each arrival is Poisson distributed with parameter λ. Let the arrival of an urgent patient denote success with probability p, and a normal patient failure with probability $1-p$. Since each arrival is independent, has only two outcomes, and the probability of success is equal in each arrival, this satisfies the conditions of a Bernoulli trial. The arrivals as a whole then represents a sequence of Bernoulli trials and the arrivals of urgent patients sum to the number of successes in n trials. This has a binomial distribution conditional on the total arrivals. Referring to the book of this course, Pinsky, Karlin, "Introduction to Stochastic Modelling", and the proof of theorem **5.2** on page 225 we know that a binomial distribution with parameters n, p conditional on a Poisson distribution with parameter λ is an unconditional Poisson distribution with parameter λp.

Second, we need the service times to be independent and identically distributed exponentially as well. Since the urgent patients are uninfluenced by the normal patients in the UCC, the situation is unchanged when it comes to the service-part of the process, and thus the service times are still independent and exponentially distributed with parameter μ. Thirdly, there is still one server, and the service times are still independend of the arrival process. Hence, $\{U(t) : t \geq 0\}$ is an $M/M/1$-queue.

2. We refer to the derivation in the previous bullet point, and the result was λp.

3. The treatment time will remain unchanged, as we assume the urgent patient's treatment time is equal to that of a non-urgent person. Since no patient has priority above the urgent patient, they will certainly experience unchanged service time as in 1a).

4. Since it is still an $M/M/1$ queue for urgent patients with birth rate $\lambda_U = \lambda p$, we can find $W_U = (\mu - \lambda_U)^{-1} = (\mu - \lambda p)^{-1}$ we apply Little's law with our new quantities to obtain L_U

$$L_U = W_U \lambda_U = \frac{\lambda p}{\mu - \lambda p}. \tag{7}$$

As we see L_U has a similiar expression as L_{tot}, with the only difference being replacing λ with λp.

d)

1. Since the arrival of an urgent patient would cause a normal patient to get their treatment interrupted, one can not say that the individual service time for a normal patient is exponentially distributed. Hence, one of the requirements fail, and thus it is not an

M/M/1 queue.

2. The total long run mean number of patients must be the sum of long run mean number of urgent and normal patients

$$L_U + L_N = L_{tot}. \tag{8}$$

The treatment of each patient still follows an exponential distribution and is independent of the order in which treatments are given. That means the long run average number of patients stays the same. So the sum is equal to that of an $M/M/1$ queue without priorities

$$L_{tot} = \frac{\lambda}{\mu - \lambda}. \tag{9}$$

Hence we get the equation

$$L_N = L_{tot} - L_U = \frac{\lambda}{\mu - \lambda} - \frac{\lambda p}{\mu - \lambda p} = \frac{\lambda(\mu - \lambda p) - (\mu - \lambda)\lambda p}{(\mu - \lambda)(\mu - \lambda p)} \tag{10}$$

$$= \frac{\mu\lambda(1-p)}{(\mu - \lambda)(\mu - \lambda p)}. \tag{11}$$

e)

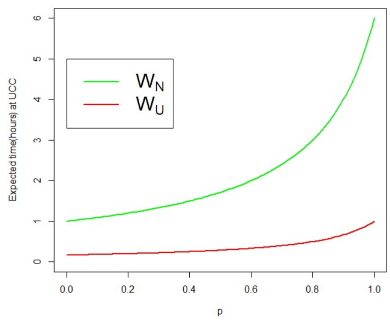
1. As we have L_U and L_N and the rates λp and $\lambda(1-p)$ for $U(t)$ and $N(t)$ respectively, it directly follows from Little's law that

$$W_U = \frac{L_U}{\lambda p} = \frac{1}{\mu - \lambda p} \tag{12}$$

2.
$$W_N = \frac{L_N}{\lambda(1-p)} = \frac{\mu}{(\mu - \lambda)(\mu - \lambda p)} \tag{13}$$

f)

1.



2. $p \approx 0$ means that we have almost no urgent patients. Therefore the situation is approaching the situation we had in problem a) and b). Almost every patient is normal and we can view it as the first $M/M/1$ queue. $p \approx 1$ would imply that almost every patient that enters is urgent. This pushes the very rare normal patient out of the queue and they must eventually wait a long time, as the frequency of urgent patients increase. The urgent patients would then not benefit from the priority as everyone is urgent.

3. Note that $\mu = \frac{1}{10min} = 6$ patients per hour.
$p \approx 0$

$$W_N = \frac{\mu}{(\mu-\lambda)(\mu-\lambda p)} = \frac{1}{\mu-\lambda} = 1 \text{ hour.} \tag{14}$$

$p \approx 1$

$$W_N = \frac{\mu}{(\mu-\lambda)(\mu-\lambda p)} = \frac{\mu}{(\mu-\lambda)^2} = 6 \text{ hours.} \tag{15}$$

4.

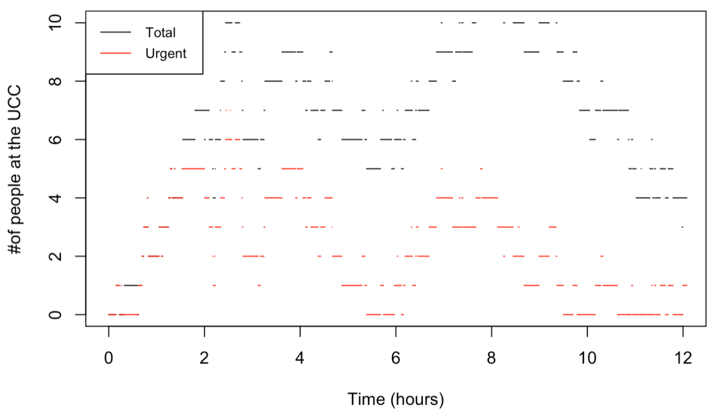
$$W_N = \frac{\mu}{(\mu-\lambda)(\mu-\lambda p)} = 2 \iff 2\mu - 2p\lambda = \frac{\mu}{\mu-\lambda} \tag{16}$$

$$2p\lambda = 2\mu - \frac{\mu}{\mu-\lambda} \iff p = \frac{\mu}{\lambda} - \frac{\mu}{2\lambda(\mu-\lambda)} = \frac{3}{5} = 0.60. \tag{17}$$

As we see, the results from **3.** and **4.** correspond with the with the figure above.

g)

3.



4. We were ought to estimate two quantities, namely the expected time for an urgent patient to stay at the UCC and the expected time a normal patient would stay at the UCC. The procedure for both cases were identical, and equal to the procedure we did in 1b); we first simulate joint realizations for a period of 50 days. For each of the simulations, we find the sample mean of patients in the UCC for both U and N, by a weighted average, as described in 1b) (further details are also in the code). We find the mean of all the 30 sample means of both quantities and we further calculate the sample variance. The 95% CI are obtained as the following:

$$W_U : \Big[0.49, 0.52\Big], \quad W_N : \Big[2.91, 3.24\Big].$$

Let us now compare these results to our theoretical results. First, let us find the theoretically true value for both;

$$W_U = \frac{1}{\mu - \lambda p} = \frac{1}{6 - 5 \cdot 0.8} = \frac{1}{2},$$

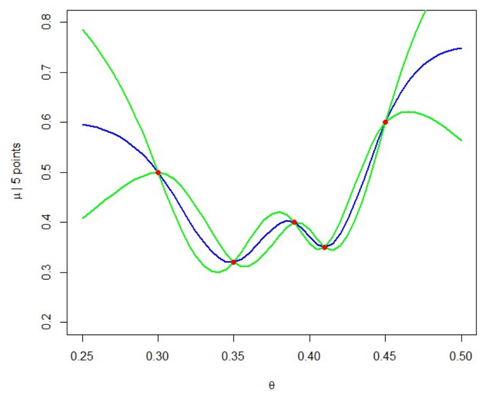
$$W_N = \frac{\mu}{(\mu - \lambda)(\mu - \lambda p)} = \frac{6}{(6-5)(6 - 5 \cdot 0.8)} = 3.$$

We see that both of the true values fall into our considerably narrow confidence intervals, which is as expected.

Problem 2

a)

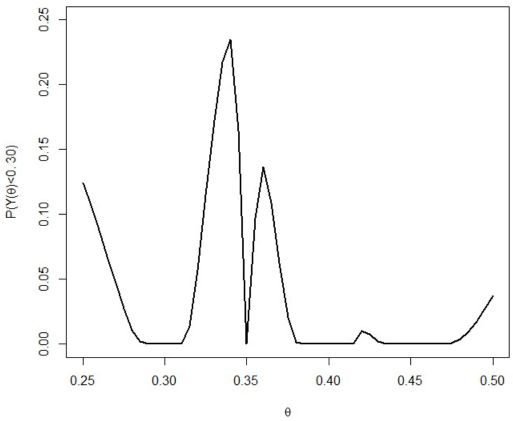
We plot the parameter $Y(\theta)$ as a function of θ and as expected our confidence interval is tight when the datapoints are close. This plot help us as we wish to find a θ such that the accuracy of the climate model is as good as possible We observe that the function



looks continuously differentiable. This is what we expect of a Matérn-like covariance function.

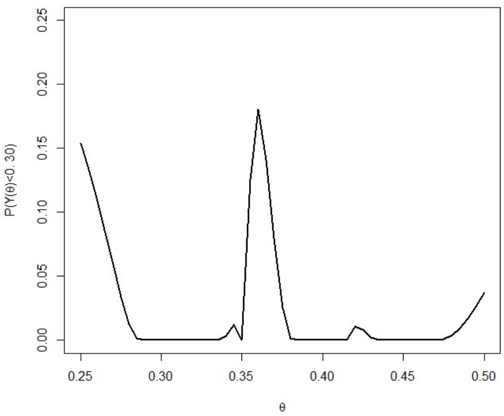
b)

By inspecting the plot below, we notice that it has a clear peak between $0.33 < \theta < 0.35$. As we want to achieve $Y(\theta) < 0.3$ we should select a θ-value on this interval. The probability as function of θ can be explained as a combination of the value of measured data and distance between measured points. Obviously the probability of measuring a low value will be very low close to datapoints with a relatively high value. So we expect the probability to rise when measured data is close to the desired value. Furthermore if the distance between measured data is relatively large, the variance is greater and allows for a greater range of values. So from the graph in a) we can note that between datapoints $0.30 \le \theta \le 0.35$ and $0.35 \le \theta \le 0.39$ we have a relatively high distance between the measured values. This combined with the value of datapoint 2, $y(0.35) = 0.32$, makes room for peaks on both sides of this measured point. Also note that near the endpoints our CI grows big, impliyng that also here we have a large variance. However the measured values closest to the endpoints are high enough to not give a peak near the peripheries of the interval.

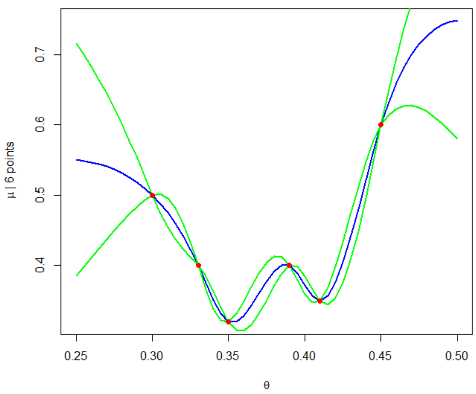


c)

When given an additional datapoint $Y(0.33) = 0.40$ near the previous peak, it disappears since it is certainly greater than 0.30. As we obtain more information, our model becomes more stable and variance is smaller. In this case that means that since the measured value was sufficiently larger than 0.30, the variance is now small enough to conclude that it will not drop to such a low value near this point. This way we notice that the optimal value of θ would be $\theta \simeq 0.36$.



(a) $P\{Y(\theta) < 0.30\}$



(b) The climate model given 6 datapoints